Applied Mathematics, 2024, 15, 687-708
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Alternative Methods of Regular and Singular Perturbation Problems

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How to cite this paper: Asare, B., Sah, M. and Hona, R.K. (2024) Alternative Methods of Regular and Singular Perturbation Problems. *Applied Mathematics*, **15**, 687-708.
https://doi.org/10.4236/am.2024.1510040

Received: September 4, 2024
Accepted: October 8, 2024
Published: October 11, 2024

Abstract

Making exact approximations to solve equations distinguishes applied mathematicians from pure mathematicians, physicists, and engineers. Perturbation problems, both regular and singular, are pervasive in diverse fields of applied mathematics and engineering. This research paper provides a comprehensive overview of algebraic methods for solving perturbation problems, featuring a comparative analysis of their strengths and limitations. Serving as a valuable resource for researchers and practitioners, it offers insights and guidance for tackling perturbation problems in various disciplines, facilitating the advancement of applied mathematics and engineering.

Keywords

Perturbation, Regular Perturbation, Singular Perturbation, Asymptotic Expansion, Matched Asymptotic, Strained Coordinates, Multiple Scales

1. Introduction

Perturbation theory is a comprehensive collection of mathematical methods used to obtain approximate solutions to problems without closed-form analytical solutions. These methods work by breaking down a challenging problem into an infinite sequence of relatively straightforward problems that can be solved analytically. Perturbation problems involve a small positive parameter, significantly affecting the problem, causing rapid solution variations in specific regions (inner regions) [1] [2] and slow variations in others. Singularly perturbed boundary value problems exhibit boundary or interior layers [2] [3] characterized by rapid solution changes near endpoints or interior points, posing computational challenges. In contrast, regular perturbation [2] problems have smooth solution variations as the perturbation parameter approaches zero. However, singular perturbation problems exhibit abrupt solution changes, potentially leading to solution nonexistence, infinity, or degeneracy as the parameter approaches zero. Singular perturbations [1] [2] often occur when the small parameter multiplies the highest operator, fundamentally changing the problem's nature. This can lead to unsatisfiable boundary conditions in differential equations and reduced solution numbers in algebraic equations.

Singular perturbation theory [3] [4] is a vibrant and dynamic field of research, with various methods developed to tackle associated problems. These methods include asymptotic expansion [1] [4], the method of matched asymptotic expansions [1] [2], Strained Coordinates [1] [2], the method of multiple scales [1] [2] [5] and Poincare-Lindstedt method [1] [3] [6]. This research paper aims to provide a comprehensive overview of perturbation theory, explaining mathematical methods for obtaining approximate analytical solutions to singular perturbation problems that lack exact solutions. Our objective is to equip young and established scientists and engineers with the skills to analyze equations encountered in their work. We present insights and techniques useful for tackling new problems, along with examples of singularly perturbed problems arising in physical contexts. While the material is available in various books and research articles, we have summarized the essential content concisely and effectively, serving as an introduction for new researchers. Perturbation methods are thoroughly discussed in references. A concise and accessible introduction to the subject is provided in this paper.

2. Materials and Methods

2.1. Mathematical Framework

The foundation of this research is based on the principles of perturbation theory, specifically focusing on both regular and singular perturbation problems. The methods employed in this study involve the application of various algebraic and analytical techniques that are well-established in applied mathematics. These include:

1. **Asymptotic Expansion:** This technique involves expressing the solution as a series expansion in terms of a small parameter, typically denoted by ε. The series is then truncated to obtain an approximate solution. This method is particularly useful for regular perturbation problems.

2. **Method of Matched Asymptotic Expansions:** This method is employed to solve singular perturbation problems, where different scales of the problem require different asymptotic expansions. The solutions obtained in different regions (e.g., inner and outer regions) are then matched in an overlapping region to ensure a smooth transition.

3. **Strained Coordinates:** This technique modifies the independent variable(s) of the problem to capture the rapid changes in the solution within specific regions, often used in conjunction with other methods to handle singular perturbations.

4. **Method of Multiple Scales:** This method is used to tackle problems with solutions exhibiting behavior on multiple time or spatial scales. By introducing multiple independent variables, each associated with a different scale, we can

derive a set of coupled differential equations that can be solved more easily.

5. Poincaré-Lindstedt Method: This perturbation technique is used to remove secular terms (terms that grow without bound) in the solution series, thereby ensuring the series remains uniformly valid over time.

2.2. Analytical Techniques

The mathematical methods were implemented using symbolic computation tools such as MATLAB and Wolfram Mathematica. These tools were employed to derive, manipulate, and simplify the resulting expressions from the perturbation techniques. Additionally, numerical simulations were performed to verify the accuracy of the analytical solutions, particularly in cases where exact solutions are known.

2.3. Comparative Analysis

A comparative analysis of the strengths and limitations of the aforementioned methods was conducted. This analysis involved applying each method to canonical perturbation problems, including regular and singular perturbations. The performance of each method was evaluated based on:

- **Accuracy**: The closeness of the approximate solution to the exact or numerical solution.
- **Computational Efficiency**: The time and resources required to obtain the solution.
- **Applicability**: The range of problems for which the method is suitable.
- **Ease of Implementation**: The complexity of the mathematical operations and the level of expertise required to apply the method.

2.4. Case Studies

Several case studies were selected from physical contexts, including fluid dynamics, quantum mechanics, and elasticity theory, where singular perturbation problems naturally arise. These case studies were used to demonstrate the practical application of the methods and to highlight the importance of selecting the appropriate technique based on the nature of the problem.

2.5. Validation

The validity of the approximate solutions obtained through the various perturbation methods was confirmed by comparing them to known exact solutions (where available) or to high-precision numerical solutions. Convergence tests were conducted to ensure that the truncated series solutions provided sufficiently accurate approximations as the perturbation parameter approached zero.

3. Regular and Singular Perturbation Theory

Perturbation theory encompasses two distinct categories: regular and singular perturbation problems. A regular perturbation problem is characterized by a power series expansion in ε with a nonvanishing radius of convergence, ensuring a smooth transition to the unperturbed solution as ε approaches zero. For a differential equation $L(y)=\varepsilon N(y)$, where ε is a small parameter, the solution is expanded as:

$$y(x;\varepsilon)=y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

In contrast, singular perturbation problems exhibit a more complex behavior, with perturbation series that may not be power series or have a vanishing radius of convergence [7]. In such cases, the exact solution may not exist or exhibit fundamentally different qualitative features when ε equals zero, compared to arbitrarily small but nonzero ε values. Moreover, the zeroth-order solution and the unperturbed problem's solution may differ, with the former potentially depending on ε and existing only for nonzero ε [8] [9]. In singular perturbation theory, careful distinction between these two solutions is crucial. When ε causes rapid changes, the solution is split into regions (inner and outer). The outer solution is:

$$y_{\text{outer}}(x;\varepsilon)=y_0^{\text{outer}}(x)+\varepsilon y_1^{\text{outer}}(x)+\varepsilon^2 y_2^{\text{outer}}+\cdots$$

The inner solution is usually around the boundary layers and can be expressed as:

$$y_{\text{inner}}(x;\varepsilon)=Y_0(\xi)+\varepsilon Y_1(\xi)+\varepsilon^2 Y_2(\xi)+\cdots$$

where $\xi=\frac{x}{\varepsilon^\alpha}$, is a rescaled variable. Let's illustrate these concepts by exploring examples of both regular and singular perturbation problems.

3.1. Consider a Regular Perturbation Problem $x^2+\varepsilon x-1=0$, $\varepsilon \ll 1$

3.1.1. Exact Solution

A solution that can be expressed in a closed form, using a finite number of operations, including addition, subtraction, multiplication, division, roots, and elementary functions, such as trigonometric functions, exponential functions, and logarithmic functions, without any approximation or truncation. In other words, an exact solution is a precise and explicit solution that can be written down exactly, without any error or approximation, using a finite combination of basic mathematical operations and functions. For example, the solution to the equation $x^2+6x+9=0$ is $x=-3$, which is an exact solution because it can be expressed precisely and explicitly without any approximation [6] [10].

We begin solving the quadratic equation in x containing the parameter ε, $x^2+\varepsilon x-1=0$.

This equation has exact solutions using the quadratic formula

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is given by

$$x=\frac{-\varepsilon\pm\sqrt{\varepsilon^2+4}}{2}=-\frac{1}{2}\varepsilon\pm\sqrt{\frac{\varepsilon^2+4}{4}}=-\frac{1}{2}\varepsilon\pm\sqrt{1+\frac{\varepsilon^2}{4}}$$

These solutions can be expanded using the binomial expansion for small ε

$$(1+x)^n \approx 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$\sqrt{1+\frac{\varepsilon^2}{4}}=\left(1+\frac{\varepsilon^2}{4}\right)^{\frac{1}{2}}, \text{ valid for } |n|<1$$

$$\sqrt{1+\frac{\varepsilon^2}{4}}=\left(1+\frac{\varepsilon^2}{4}\right)^{\frac{1}{2}}$$

$$\approx 1+\frac{\varepsilon^2}{2\times 4}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\varepsilon^2}{4}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{\varepsilon^2}{4}\right)^3+\cdots$$

$$\approx 1+\frac{\varepsilon^2}{8}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{\varepsilon^2}{4}\right)^2 \approx 1+\frac{1}{8}\varepsilon^2-\frac{1}{128}\varepsilon^4+\cdots$$

Thus, the exact solution

$$x=\frac{-\varepsilon\pm\sqrt{\varepsilon^2+4}}{2}=-\frac{1}{2}\varepsilon\pm\left(1+\frac{1}{8}\varepsilon^2-\frac{1}{128}\varepsilon^4+\cdots\right)$$

$$x=\begin{cases}+1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2-\frac{1}{128}\varepsilon^4+O\left(\varepsilon^6\right)\\ -1-\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2+\frac{1}{128}\varepsilon^4+O\left(\varepsilon^6\right)\end{cases}$$

These binomial expansions converge if $\varepsilon<2$.

Now, suppose we did not know the exact solution, how could we generate the approximate solutions for $\varepsilon \ll 1$?

4. Expansion Method

Many physical problems involving the function $u(x,\varepsilon)$ can be represented mathematically by the differential equation $L(u,x,\varepsilon)=0$ and the boundary condition $B(u,\varepsilon)=0$, where x is an independent variable and ε is a parameter. Since an exact solution is often impossible, we seek an approximate solution for small ε. To do this, we assume a solution expansion in powers of ε: $u(x;\varepsilon)=u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots$ where $u_n(x)$ is independent of ε and $u_0(x)$ is the solution for $\varepsilon=0$. We then substitute this expansion into the differential equation $L(u,x,\varepsilon)=0$ and boundary condition $B(u,\varepsilon)=0$, expand for small ε, and collect coefficients of each power of ε. Since these equations must hold for all ε values, each coefficient of ε must vanish independently. This usually yields simpler equations governing $u_n(x)$, which can be solved successively. This approach, called perturbation theory, is useful when an exact solution is impossible. The next two examples demonstrate this method [1] [11].

4.1. Expansion in the Power Series in ε

$$x^2+\varepsilon x-1=0$$

Given that $\varepsilon=0$ then, we have $x^2-1=0$ and the solution is $x=\pm 1$.

Now, let's try $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ disregarding $O(\varepsilon^3)$.
Substituting x into $x^2 + \varepsilon x - 1 = 0$ we have

$$\left(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 + \varepsilon\left(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right) - 1 = 0$$

$$\left(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 + \varepsilon\left(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right) - 1 = 0 + 0\varepsilon + 0\varepsilon^2 + \cdots$$

Equating the coefficients of each power of ε

$$1 + 2\varepsilon x_1 + \varepsilon^2\left(2x_2 + x_1^2\right) + \varepsilon + \varepsilon^2 x_1 - 1 \cdots = 0 + 0\varepsilon + 0\varepsilon^2 + \cdots$$

$$\varepsilon^1 : 2x_1 + 1 = 0,\ x_1 = -\frac{1}{2}$$

$$\varepsilon^2 : 2x_2 + x_1^2 + x_1 = 0,\ x_2 = -\frac{x_1^2 + x_1}{2} = -\frac{\frac{1}{4} - \frac{1}{2}}{2} = \frac{1}{8}$$

Thus $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, $x = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots$.

Now, let's try $x = -1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ disregarding $O(\varepsilon^3)$.
Substituting x into $x^2 + \varepsilon x - 1 = 0$ we have

$$\left(-1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 + \varepsilon\left(-1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right) - 1 = 0$$

$$\left(-1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 + \varepsilon\left(-1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right) - 1 = 0 + 0\varepsilon + 0\varepsilon^2 + \cdots$$

Equating the coefficients of each power of ε

$$1 - 2\varepsilon x_1 - 2x_2\varepsilon^2 + \varepsilon^2 x_1^2 - \varepsilon + \varepsilon^2 x_1 - 1 \cdots = 0 + 0\varepsilon + 0\varepsilon^2 + \cdots$$

$$\varepsilon^1 : -2x_1 - 1 = 0,\ x_1 = -\frac{1}{2}$$

$$\varepsilon^2 : -2x_2 + x_1^2 + x_1 = 0,\ x_2 = \frac{x_1^2 + x_1}{2} = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}$$

Thus $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$$x = -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots$$

$$x = \begin{cases} +1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots \end{cases}$$

4.2. Features of the Method: It's Simple to Solve for x_1, x_2, ...

But we need to guess the form $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ which is the asymptotic sequence.

Let's guess the asymptotic sequence $\left\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \cdots\right\}$ by substituting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ into the equation $x^2 + \varepsilon x - 1 = 0$

$$\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 + \varepsilon\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right) - 1 = 0$$

$$\left(x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots\right)^2+\varepsilon\left(1+\varepsilon x_1+\varepsilon^2 x_2+\cdots\right)-1=0+0\varepsilon+0\varepsilon^2+\cdots$$

$$x_0^2+2\varepsilon x_0 x_1+2x_0 x_2\varepsilon^2+\varepsilon^2 x_1^2+\varepsilon+\varepsilon^2 x_1-1\cdots=0+0\varepsilon+0\varepsilon^2+\cdots$$

$$x_0^2-1+2\varepsilon x_0 x_1+\varepsilon^2\left(2x_0 x_2+x_1^2\right)+\varepsilon+\varepsilon^2 x_1\cdots=0+0\varepsilon+0\varepsilon^2+\cdots$$

Equating coefficients

$$\varepsilon^0 : x_0^2-1=0,\ x_0=\pm 1$$

$$\varepsilon^1 : 2x_0 x_1+1=0,\ x_1=-\frac{1}{2}$$

$$\varepsilon^2 : 2x_0 x_2+x_1^2+x_1=0,\ x_2=-\frac{x_1^2+x_1}{2}=-\frac{\frac{1}{4}-\frac{1}{2}}{2}=\frac{1}{8}$$

Thus $x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots$

$$x=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+\cdots$$

5. Iteration Method

Iteration refers to a method for improving the accuracy of the approximation by repeatedly applying the perturbation correction to the previous solution. Begin with an initial solution (usually the zeroth-order solution) and calculate the first-order correction using the perturbation equations. Add the first-order correction to the initial solution to obtain a new improved solution. Repeat steps with the new solution as the starting point, calculating the next higher-order correction. Continuing this process until the desired level of accuracy is achieved.

$$x^2+\varepsilon x-1=0$$

Rewrite as $x^2=1-\varepsilon x$.

Let's try $x_{n+1}=\pm\sqrt{1-\varepsilon x_n}$ with initial condition $x_0=1$.

Consider $+\sqrt{1-\varepsilon x_n}$ to find when $x_0=1$, the solution is perturbed.

$x_0=1$, when $n=0$

$$x_{0+1}=+\sqrt{1-\varepsilon x_0}$$

$x_1=+\sqrt{1-\varepsilon}$ and by using the binomial expansion as before

$$x_1\approx 1-\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2-\cdots$$

We realize that the coefficient of ε^2 which is $-\frac{1}{8}$ disagrees with that of the coefficient of the exact solution which is $\frac{1}{8}$. That is to compare

$x=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+\cdots$ with $x_1\approx 1-\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2-\cdots$

Therefore, we truncate after $O\left(\varepsilon^2\right)$

$$x_1\approx 1-\frac{1}{2}\varepsilon+O\left(\varepsilon^2\right)$$

Now for $n=1$, $x_1 \approx 1-\frac{1}{2}\varepsilon+O(\varepsilon^2)$ substituting into $x_2=\sqrt{1-\varepsilon x_1}$, we obtain

$$x_2=\sqrt{1-\varepsilon x_1}=\left[1-\varepsilon\left(1-\frac{1}{2}\varepsilon+O(\varepsilon^2)\right)\right]^{\frac{1}{2}}=\left[1-\varepsilon+\frac{1}{2}\varepsilon^2+O(\varepsilon^3)\right]^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left[-\varepsilon+\frac{1}{2}\varepsilon^2\right]-\frac{1}{8}\left[-\varepsilon+\frac{1}{2}\varepsilon^2\right]^2+\cdots$$

$$=1-\frac{1}{2}\varepsilon+\left(\frac{1}{4}-\frac{1}{8}\right)\varepsilon^2+O(\varepsilon^3)$$

$$=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+O(\varepsilon^3)$$

We can see that the exact solution, approximate, and iteration are all the same for the positive results. Features: This correctly generates the form of the series automatically but requires an increase in the amount of work to higher-order terms. Earlier iterations gave the wrong higher-order terms.

5.1. Consider the Singular Problem $\varepsilon x^2+x-1=0$, $\varepsilon \ll 1$

We begin with the quadratic equation in x containing the parameter ε,

$$\varepsilon x^2+x-1=0,\ \varepsilon \ll 1$$

5.1.1. Exact Solution

If $\varepsilon=0$, then $x-1=0$, $x=1$.

There are two roots if $\varepsilon \neq 0$. This equation has exact solutions using the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-1\pm\sqrt{1+4\varepsilon}}{2\varepsilon}=-\frac{1}{2\varepsilon}\pm\frac{1}{2\varepsilon}\sqrt{1+4\varepsilon}$$

These solutions can be expanded using the binomial expansion for small ε

$$(1+x)^n \approx 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\frac{n(n-1)(n-2)(n-3)}{4!}x^4+\cdots$$

$$(1+4\varepsilon)^{\frac{1}{2}}=1+\frac{1}{2}(4\varepsilon)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(4\varepsilon)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(4\varepsilon)^3+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}(4\varepsilon)^4+\cdots$$

$$(1+4\varepsilon)^{\frac{1}{2}}=1+2\varepsilon-2\varepsilon^2+4\varepsilon^3-10\varepsilon^4+\cdots$$

Therefore,

$$x=\frac{-1\pm\sqrt{1+4\varepsilon}}{2\varepsilon}=-\frac{1}{2\varepsilon}\pm\frac{1}{2\varepsilon}\sqrt{1+4\varepsilon}$$
$$=-\frac{1}{2\varepsilon}\pm\frac{1}{2\varepsilon}\left[1+2\varepsilon-2\varepsilon^2+4\varepsilon^3-10\varepsilon^4+\cdots\right]$$

Thus, the exact solution

$$x=\begin{cases}+1-2\varepsilon+2\varepsilon^2-5\varepsilon^3+\cdots\\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+5\varepsilon^3+\cdots\end{cases}$$

Realize the singular second root evaporates off to $x=\infty$ in the limit $\varepsilon=0$.

$$\lim_{\varepsilon\to 0}-\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+5\varepsilon^3+\cdots=\infty$$

Now, suppose we did not know the exact solution, how could we generate the approximate solutions for $\varepsilon\ll 1$?

5.1.2. Using the Expansion Method

when an exact solution is impossible. The next two examples demonstrate this method.

$$\varepsilon x^2+x-1=0$$

If $\varepsilon=0$ then $x-1=0, x=1$.

But for $\varepsilon\neq 0$ even $\varepsilon\ll 1$, we have two roots. There is a big qualitative change. Typical of singular perturbation. The other root $\to\infty$ as $\varepsilon\to 0$

Dominant balance: Maybe the term εx^2 is big for this missing root.

We now find the first root by letting $x=a+b\varepsilon+c\varepsilon^2+d\varepsilon^4+\cdots$

Substituting into the perturbation problem, we obtain

$$\varepsilon\left(a+b\varepsilon+c\varepsilon^2+d\varepsilon^4+\cdots\right)^2+\left(a+b\varepsilon+c\varepsilon^2+d\varepsilon^4+\cdots\right)-1=0$$
$$\varepsilon\left(a^2+2ab\varepsilon+2ac\varepsilon^2+2ad\varepsilon^3+b^2\varepsilon^2+2bc\varepsilon^3+\cdots\right)$$
$$+\left(a+b\varepsilon+c\varepsilon^2+d\varepsilon^3+e\varepsilon^4+\cdots\right)-1=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0: a-1=0, a=1$$
$$\varepsilon^1: a^2+b=0, b=-1$$
$$\varepsilon^2: 2ab+c=0, c=2$$
$$\varepsilon^3: 2ac+b^2+d=0, d=-5$$

Thus, the first root is $x=1-\varepsilon+2\varepsilon^2-5\varepsilon^3+\cdots$.

5.1.3. We Now Find the Second Root by Using the Rescaling Method

Rescaling is a technique to simplify the analysis of a perturbed problem by introducing new variables or parameters that absorb the small perturbation parameter ε. Rescaling involves multiplying or dividing the original variables or parameters by appropriate powers of ε, effectively "rescaling" them to reduce the complexity of the equations or expressions. We find the second root of the equation (1.3) by rescaling, we let substituting $x = \frac{X}{\varepsilon^n}$ into the perturbation problem, and we obtain

$$\varepsilon x^2 + x - 1 = 0$$

$$\varepsilon\left(\frac{X}{\varepsilon^n}\right)^2 + \frac{X}{\varepsilon^n} - 1 = 0$$

$$\varepsilon^{1-2n}X^2 + \varepsilon^{-n}X - 1 = 0$$

Extracting the exponents of the first two terms, we have

$$1 - 2n = -n$$

$$1 = n$$

Therefore this $\varepsilon^{1-2n}X^2 + \varepsilon^{-n}X - 1 = 0$ becomes,

$$\varepsilon^{-1}X^2 + \varepsilon^{-1}X - 1 = 0$$

Multiply by ε, we have

$$X^2 + X - \varepsilon = 0$$

$$\varepsilon = 0$$

$$X^2 + X = 0,\ X(X+1) = 0,\ X = 0,\ X = -1$$

Let $X = -1 + a\varepsilon + b\varepsilon^2 + \cdots$.

Substituting into $X^2 + X - \varepsilon = 0$, we obtain,

$$\left(-1 + a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + d\varepsilon^4 + \cdots\right)^2 + \left(-1 + a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + d\varepsilon^4 + \cdots\right) - \varepsilon = 0$$

$$\left(1 - 2a\varepsilon - 2b\varepsilon^2 + a^2\varepsilon^2 + 2ab\varepsilon^3 - 2c\varepsilon^3 + 2ac\varepsilon^4 + b^2\varepsilon^4 - 2d\varepsilon^4 \cdots\right)$$
$$+\left(-1 + a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + d\varepsilon^4 + \cdots\right) - \varepsilon = 0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0 : 1 - 1 = 0$$

$$\varepsilon^1 : -2a + a - 1 = 0,\ a = -1$$

$$\varepsilon^2 : -2b + b + a^2 = 0,\ b = 1$$

$$\varepsilon^3 : 2ab + c - 2c = 0,\ c = -2$$

$$\varepsilon^4 : 2ac + b^2 - 2d + d = 0,\ d = 5$$

Therefore, the first root is $X = -1 - \varepsilon + \varepsilon^2 - 2\varepsilon^3 + 5\varepsilon^4 + \cdots$.

But $x = \frac{X}{\varepsilon^n}$, $n = 1$.

Therefore $x = \frac{X}{\varepsilon^n} = \frac{-1 - \varepsilon + \varepsilon^2 - 2\varepsilon^3 + 5\varepsilon^4 + \cdots}{\varepsilon} = -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^3 + \cdots$.

Thus, the solution from the expansion method equals the exact solutions.

$$x = \begin{cases} +1 - 2\varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^3 + \cdots \end{cases}$$

The same as the exact solution.

6. Iteration Method

$$\varepsilon x^2 + x - 1 = 0$$

Keeping the term εx^2 as a main term rather than as a small correction, hence x must be large. Therefore, at leading order, the -1 term in the equation will be negligible when compared with the x term.

Therefore, as $\varepsilon x^2 + x - 1 = 0$.

Rewrite as $x(\varepsilon x + 1) = 0$, $x = 0, -\frac{1}{\varepsilon}$.

Using the rearrangement of the quadratic equation, we have

$$\varepsilon x^2 + x - 1 = 0,\quad \varepsilon x^2 = 1 - x$$

$$x(\varepsilon x) = 1 - x$$

$$x = -\frac{x}{\varepsilon x} + \frac{1}{\varepsilon x}$$

$$x = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x}$$

Let's try $x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n}$ with initial condition $x_0 = -\frac{1}{\varepsilon}$.

Consider $x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n}$ to find when $x_0 = -\frac{1}{\varepsilon}$, the solution is perturbed.

When $n = 0$, $x_0 = -\frac{1}{\varepsilon}$

$$x_{0+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_0} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon\left(-\frac{1}{\varepsilon}\right)} = -\frac{1}{\varepsilon} - 1$$

$$x_2 = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon\left(-\frac{1}{\varepsilon} - 1\right)} = -\frac{1}{\varepsilon} + \frac{1}{-1-\varepsilon} = -\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon}$$

Using the binomial expansion as before, we have

$$\frac{1}{1+\varepsilon} = (1+\varepsilon)^{-1} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \cdots$$

$$x_2 = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_1} = -\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \cdots = -\frac{1}{\varepsilon} - 1 + \varepsilon - \varepsilon^2 + \varepsilon^3 + \cdots$$

Realize a slight difference between the exact solution and the Iteration method.

6.1. Find the First Three Terms for All Three Roots of $\varepsilon x^3 + x^2 + (2+\varepsilon)x + 1 = 0$

6.1.1. Expansion Method

Solving the perturbation problem, we find the first root by letting $x = a + b\varepsilon + c\varepsilon^2 + \cdots$.

Substituting into the perturbation problem, we obtain

$$\varepsilon\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)^3+\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)^2+\left(2+\varepsilon\right)\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)+1=0$$

$$\varepsilon\left(a^3+3a^2b\varepsilon+3a^2c\varepsilon^2+3ab^2\varepsilon^2+6abc\varepsilon^3+b^3\varepsilon^3+\cdots\right)$$
$$+\left(a^2+2ab\varepsilon+2ac\varepsilon^2+b^2\varepsilon^2+2bc\varepsilon^3+\cdots\right)+\left(2+\varepsilon\right)\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)+1=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0: a^2+2a+1=0,\ a=-1,-1$$

$$\varepsilon^1: a^3+a+2b+2ab=0,\ b=\frac{-a\left(a^2+1\right)}{2\left(1+a\right)}=\infty$$

Thus, we now let $x=-1+\delta+\delta^2+\cdots$, where $\delta=\delta\left(\varepsilon\right)$.

$$\varepsilon\left(-1+\delta+\delta^2+\cdots\right)^3+\left(-1+\delta+\delta^2+\cdots\right)^2+\left(2+\varepsilon\right)\left(-1+\delta+\delta^2+\cdots\right)+1=0$$

$$\left(-1+3\delta-5\delta^2+\cdots\right)+\left(-1-2\delta-\delta^2+\cdots\right)^2+\left(2+\varepsilon\right)\left(-1+\delta+\delta^2+\cdots\right)+1=0$$

$$-2\varepsilon+\delta^2+4\varepsilon\delta-4\varepsilon\delta^2=0$$

Extracting the dominant terms, we have

$$-2\varepsilon+\delta^2=0,\ \delta=\pm\sqrt{2}\varepsilon^{\frac{1}{2}}$$

Therefore, we let $x=-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots$ and $x=-1-\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots$.

Substituting $x=-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots$ into the equation

$$\varepsilon\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots\right)^3+\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots\right)^2$$
$$+\left(2+\varepsilon\right)\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots\right)+1=0$$

$$\varepsilon\left(-1+3\sqrt{2}\varepsilon^{\frac{1}{2}}-6\varepsilon+2\sqrt{2}\varepsilon^{\frac{3}{2}}+6c\varepsilon+\cdots\right)+\left(1-2\sqrt{2}\varepsilon^{\frac{1}{2}}+2\varepsilon-4c\varepsilon+4c\sqrt{2}\varepsilon^{\frac{3}{2}}+\cdots\right)$$
$$+\left(2+\varepsilon\right)\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}+2c\varepsilon+\cdots\right)+1=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0: 1-2+1=0$$

$$\varepsilon^1: -1+2-4c-1+4c=0$$

$$\varepsilon^{\frac{3}{2}}: 3\sqrt{2}+4c\sqrt{2}+\sqrt{2}=0,\ c=-1$$

Thus, one root is $x=-1+\sqrt{2}\varepsilon^{\frac{1}{2}}-2\varepsilon+\cdots$.

Also substituting $x=-1-\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots$ into the equation

$$\varepsilon\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots\right)^3+\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots\right)^2$$
$$+\left(2+\varepsilon\right)\left(-1+\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots\right)+1=0$$

$$\varepsilon\left(-1-3\sqrt{2}\varepsilon^{\frac{1}{2}}-6\varepsilon-2\sqrt{2}\varepsilon^{\frac{3}{2}}-6c\varepsilon+12c\varepsilon^{\frac{3}{2}}+\cdots\right)$$
$$+\left(1+2\sqrt{2}\varepsilon^{\frac{1}{2}}+2\varepsilon+4c\varepsilon+4c\sqrt{2}\varepsilon^{\frac{3}{2}}+\cdots\right)$$
$$+(2+\varepsilon)\left(-1-\sqrt{2}\varepsilon^{\frac{1}{2}}-2c\varepsilon+\cdots\right)+1=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0:1-2+1=0$$
$$\varepsilon^1:-1+2-4c+4c-1=0$$
$$\varepsilon^{\frac{3}{2}}:-3\sqrt{2}+4c\sqrt{2}-\sqrt{2}=0, c=1$$

Thus, the second root is $x=-1-\sqrt{2}\varepsilon^{\frac{1}{2}}-2\varepsilon+\cdots$.

We find the third root by using the method of rescaling.

We let $x=\frac{X}{\varepsilon^n}$.

Substituting into the perturbation problem, we obtain

$$\varepsilon\left(\frac{X}{\varepsilon^n}\right)^3+\left(\frac{X}{\varepsilon^n}\right)^2+(2+\varepsilon)\left(\frac{X}{\varepsilon^n}\right)+1=0=0$$
$$\varepsilon^{1-3n}X^3+\varepsilon^{-2n}X^2+(2+\varepsilon)\varepsilon^{-n}X+1=0$$

Extracting the exponents of the first two terms, we have

$$1-3n=-2n, n=1$$

Therefore this $\varepsilon^{1-3n}X^2+\varepsilon^{-2n}X^2+(2+\varepsilon)\varepsilon^{-n}X+1=0$ becomes.

$$\varepsilon^{-2}X^3+\varepsilon^{-2}X^2+(2+\varepsilon)\varepsilon^{-1}X+1=0$$

Multiply by ε^2, we have

$$X^3+X^2+(2+\varepsilon)\varepsilon X+\varepsilon^2=0$$

Setting $\varepsilon=0$, we have

$$X^3+X^2=0, X^2(X+1)=0, X=0, X=-1$$

Let $X=-1+a\varepsilon+b\varepsilon^2+\cdots$.

Substituting into $X^3+X^2+(2+\varepsilon)\varepsilon X+\varepsilon^2=0$ we obtain

$$\left(-1+a\varepsilon+b\varepsilon^2+\cdots\right)^3+\left(-1+a\varepsilon+b\varepsilon^2+\cdots\right)+\varepsilon^2=0$$
$$\left(-1+3a\varepsilon+3b-3a^2\varepsilon^2+3b^3\varepsilon^3-6ab\varepsilon^3+\cdots\right)$$
$$+\left(1-2a\varepsilon-2b\varepsilon^2+a^2\varepsilon^2+2ab\varepsilon^3+\cdots\right)+\varepsilon^2=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0:-1+1=0$$
$$\varepsilon^1:3a-2a-2=0, a=2$$

$$\varepsilon^2 : -2b + b + a^2 = 0, b = 1$$

$$\varepsilon^3 : 3b - 3a^2 - 2b + a^2 - 1 + 1 = 0, b = 4$$

Therefore, the first root is $X = -1 + 2\varepsilon + 4\varepsilon^2 + \cdots$.

But $x = \frac{X}{\varepsilon^n}$, $n = 1$.

Therefore $x = \frac{X}{\varepsilon^n} = \frac{-1 + 2\varepsilon + 4\varepsilon^2 + \cdots}{\varepsilon} = -\frac{1}{\varepsilon} + 2 + 4\varepsilon + \cdots$.

Thus, the three roots are $x = \begin{cases} -1 + \sqrt{2}\varepsilon^{\frac{1}{2}} - 2\varepsilon + \cdots \\ -1 - \sqrt{2}\varepsilon^{\frac{1}{2}} - 2\varepsilon + \cdots \\ -\frac{1}{\varepsilon} + 2 - 2\varepsilon + \cdots \end{cases}$

7. Find the First Three Terms for All Three Roots of $\varepsilon x^3 + x^2 + (2 - \varepsilon)x + 1 = 0$

7.1. Exact Solution

$$\varepsilon x^3 + x^2 + (2 - \varepsilon)x + 1 = 0$$

$$\varepsilon x^3 - \varepsilon x + x^2 + 2x + 1 = 0$$

$$\varepsilon x(x^2 - 1) + x^2 + 2x + 1 = 0$$

$$\varepsilon x(x^2 - 1) + (x + 1)^2 = 0$$

$$\varepsilon x(x - 1)(x + 1) + (x + 1)^2 = 0$$

$$(x + 1)\{x\varepsilon(x - 1) + (x + 1)\} = 0$$

$$x + 1 = 0, x = -1$$

$$x\varepsilon(x - 1) + (x + 1) = 0$$

$$\varepsilon x^2 - x\varepsilon + x + 1 = 0$$

$$\varepsilon x^2 + (1 - \varepsilon)x + 1 = 0$$

Thus, the first root is $x = -1$.

Solving for the second and third roots by using the quadratic formula, we obtain

$$x = \frac{(\varepsilon - 1) \pm \sqrt{(1 - \varepsilon)^2 - 4\varepsilon}}{2\varepsilon} = \frac{(\varepsilon - 1) \pm (1 - \varepsilon)\left[1 - \frac{4\varepsilon}{(1 - \varepsilon)^2}\right]^{\frac{1}{2}}}{2\varepsilon}$$

$$= \frac{(\varepsilon - 1) \pm (1 - \varepsilon)\left[1 - 4\varepsilon(1 - \varepsilon)^{-2}\right]^{\frac{1}{2}}}{2\varepsilon}$$

By using the Binomial Expansion

$$(1+x)^n \approx 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4+\cdots$$

$$(1-\varepsilon)^{-2} = 1+2\varepsilon+3\varepsilon^2+4\varepsilon^3+\cdots$$

Therefore,

$$x=\frac{(\varepsilon-1)\pm(1-\varepsilon)\left[1-4\varepsilon\left(1+2\varepsilon+3\varepsilon^2+4\varepsilon^3+\cdots\right)\right]^{\frac{1}{2}}}{2\varepsilon}$$

$$x=\frac{(\varepsilon-1)\pm(1-\varepsilon)\left[1-4\varepsilon-8\varepsilon^2\right]^{\frac{1}{2}}}{2\varepsilon}$$

Using the binomial expansion once again, we obtain

$$\left(1-4\varepsilon-8\varepsilon^2\right)^{\frac{1}{2}} = 1-2\varepsilon-6\varepsilon^2+\cdots$$

$$x=\frac{(\varepsilon-1)\pm(1-\varepsilon)\left(1-2\varepsilon-6\varepsilon^2\right)}{2\varepsilon}$$

$$x=\frac{\varepsilon-1+1-3\varepsilon-4\varepsilon^2+\cdots}{2\varepsilon}=-1-2\varepsilon+\cdots$$

$$x=\frac{\varepsilon-1-1+3\varepsilon+4\varepsilon^2+\cdots}{2\varepsilon}=-\frac{1}{\varepsilon}+2+2\varepsilon+\cdots$$

7.2. Using the Expansion Method

To solve the perturbed problem, we find the first root by letting $x=a+b\varepsilon+c\varepsilon^2+\cdots$.

Substituting into the perturbation problem, we obtain

$$\varepsilon\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)^3+\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)^2+(2-\varepsilon)\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)+1=0$$

$$\varepsilon\left(a^3+3a^2b\varepsilon+3a^2c\varepsilon^2+3ab^2\varepsilon^2+6abc\varepsilon^3+b^3\varepsilon^3+\cdots\right)$$
$$+\left(a^2+2ab\varepsilon+2ac\varepsilon^2+b^2\varepsilon^2+2bc\varepsilon^3+\cdots\right)+(2-\varepsilon)\left(a+b\varepsilon+c\varepsilon^2+\cdots\right)+1=0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0: a^2+2a+1=0,\ a=-1,-1$$

$$\varepsilon^1: a^3-a+2b+2ab=0,\ b=\frac{a\left(1-a^2\right)}{2(1+a)}=\infty$$

Thus, we now let $x=-1+\delta+\delta^2+\cdots$, where $\delta=\delta(\varepsilon)$

$$\varepsilon\left(-1+\delta+\delta^2+\cdots\right)^3+\left(-1+\delta+\delta^2+\cdots\right)^2+(2-\varepsilon)\left(-1+\delta+\delta^2+\cdots\right)+1=0$$

$$\left(-1+3\delta-5\delta^2+\cdots\right)+\left(1-2\delta-\delta^2+\cdots\right)+(2-\varepsilon)\left(-1+\delta+\delta^2+\cdots\right)+1=0$$

$$2\varepsilon\delta+\delta^2-6\varepsilon\delta^2=0$$

Extracting the dominant terms, we have

$$2\varepsilon\delta + \delta^2 = 0,\ \delta = -2\varepsilon$$

Therefore, we let $x = -1 - 2\varepsilon + 4b\varepsilon^2 + \cdots$.

Substituting $x = -1 - 2\varepsilon + 4b\varepsilon^2 + \cdots$ into the equation

$$\varepsilon\left(-1 - 2\varepsilon + 4b\varepsilon^2 + \cdots\right)^3 + \left(-1 - 2\varepsilon + 4b\varepsilon^2 + \cdots\right)^2$$
$$+(2-\varepsilon)\left(-1 - 2\varepsilon + 4b\varepsilon^2 + \cdots\right) + 1 = 0$$

$$\varepsilon\left(-1 - 6\varepsilon - 12\varepsilon^2 + 12b\varepsilon^2 - 8\varepsilon^3 + \cdots\right) + \left(1 + 4\varepsilon + 4\varepsilon^2 - 8b\varepsilon^2 - 16\varepsilon^3 + \cdots\right)$$
$$+(2+\varepsilon)\left(-1 - 2\varepsilon + 4b\varepsilon^2 + \cdots\right) + 1 = 0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0 : 1 - 2 + 1 = 0$$

$$\varepsilon^1 : -1 + 4 - 4 + 1 = 0$$

$$\varepsilon^2 : -6 + 4 - 8b + 8b + 2 = 0$$

$$\varepsilon^3 : -12 + 12b - 16b - 4b = 0,\ b = -\frac{3}{2}$$

Thus, one root is $x = -1 - 2\varepsilon - 6\varepsilon^2 + \cdots$.

Finding the second root, we let $x = -1 + a\varepsilon^{\frac{1}{2}} + b\varepsilon + c\varepsilon^{\frac{3}{2}} + \cdots$ and substituting into the perturbed problem.

$$\varepsilon\left(-1 + a\varepsilon^{\frac{1}{2}} + b\varepsilon + c\varepsilon^{\frac{3}{2}} + \cdots\right)^3 + \left(-1 + a\varepsilon^{\frac{1}{2}} + b\varepsilon + c\varepsilon^{\frac{3}{2}} + \cdots\right)^2$$
$$+(2-\varepsilon)\left(-1 + a\varepsilon^{\frac{1}{2}} + b\varepsilon + c\varepsilon^{\frac{3}{2}} + \cdots\right) + 1 = 0$$

$$\varepsilon\left(-1 + 3b\varepsilon - 3\varepsilon a^2 + a^3\varepsilon^{\frac{3}{2}} - 3a\varepsilon^{\frac{1}{2}} + \cdots\right) + \left(-1 - 2b\varepsilon - 2a\varepsilon^{\frac{1}{2}} + a^2\varepsilon + 2ab\varepsilon^{\frac{3}{2}} + \cdots\right)$$
$$+(2+\varepsilon)\left(-1 + a\varepsilon^{\frac{1}{2}} + b\varepsilon + c\varepsilon^{\frac{1}{2}} + \cdots\right) + 1 = 0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0 : 1 - 2 + 1 = 0$$

$$\varepsilon^{\frac{1}{2}} : -2a + 2a = 0$$

$$\varepsilon^1 : -1 - 2b + a^2 + 2b + 1 = 0,\ a = 0$$

$$\varepsilon^{\frac{3}{2}} : 3a + 2ab + 2c - a = 0,\ c = 0$$

$$\varepsilon^2 : 3b - 3a^2 - b = 0,\ b = 0$$

Thus, second root is $x = -1$.

7.3. Rescaling

We find the third root by using the method of rescaling.

We let $x = \frac{X}{\varepsilon^n}$.

Substituting into the perturbation problem, we obtain

$$\varepsilon\left(\frac{X}{\varepsilon^n}\right)^3 + \left(\frac{X}{\varepsilon^n}\right)^2 + (2-\varepsilon)\left(\frac{X}{\varepsilon^n}\right) + 1 = 0 = 0$$

$$\varepsilon^{1-3n}X^3 + \varepsilon^{-2n}X^2 + (2-\varepsilon)\varepsilon^{-n}X + 1 = 0$$

Extracting the exponents of the first two terms, we have

$$1-3n = -2n,\ n = 1$$

Therefore this $\varepsilon^{1-3n}X^2 + \varepsilon^{-2n}X^2 + (2-\varepsilon)\varepsilon^{-n}X + 1 = 0$ becomes.

$$\varepsilon^{-2}X^3 + \varepsilon^{-2}X^2 + (2-\varepsilon)\varepsilon^{-1}X + 1 = 0$$

Multiply by ε^2, we have

$$X^3 + X^2 + (2-\varepsilon)\varepsilon X + \varepsilon^2 = 0$$

Setting $\varepsilon = 0$, we have

$$X^3 + X^2 = 0,\ X^2(X+1) = 0,\ X = 0,\ X = -1$$

Let $X = -1 + a\varepsilon + b\varepsilon^2 + \cdots$.

Substituting into $X^3 + X^2 + (2-\varepsilon)\varepsilon X + \varepsilon^2 = 0$ we obtain

$$\left(-1 + a\varepsilon + b\varepsilon^2 + \cdots\right)^3 + \left(-1 + a\varepsilon + b\varepsilon^2 + \cdots\right)^2$$
$$+(2-\varepsilon)\left(-1 + a\varepsilon + b\varepsilon^2 + \cdots\right)\varepsilon + \varepsilon^2 = 0$$

$$\left(-1 + 3a\varepsilon + 3b\varepsilon^2 - 3a^2\varepsilon^2 + b^3\varepsilon^3 - 6ab\varepsilon^3 + \cdots\right)$$
$$+\left(1 - 2a\varepsilon - 2b\varepsilon^2 + a^2\varepsilon^2 + 2ab\varepsilon^3 + \cdots\right) + \varepsilon^2 = 0$$

Equating Coefficients of equal powers of ε, we obtain

$$\varepsilon^0 : -1 + 1 = 0$$

$$\varepsilon^1 : 3a - 2a - 2 = 0,\ a = 2$$

$$\varepsilon^2 : 3b - 3a^2 - 2b + a^2 + 2a + 1 + 1 = 0,\ b = 2$$

$$\varepsilon^3 : 3b - 3a^2 - 2b + a^2 - 1 + 1 = 0,\ b = 4$$

Therefore, $X = -1 + 2\varepsilon + 2\varepsilon^2 + \cdots$.

But $x = \frac{X}{\varepsilon^n}$, $n = 1$.

Therefore

$$x = \frac{X}{\varepsilon^n} = \frac{-1 + 2\varepsilon + 2\varepsilon^2 + \cdots}{\varepsilon} = -\frac{1}{\varepsilon} + 2 + 2\varepsilon + \cdots$$

Thus, the three roots are $x = \begin{cases} -1 - 2\varepsilon - 6\varepsilon^2 + \cdots \\ -1 \\ -\frac{1}{\varepsilon} + 2 + 2\varepsilon + \cdots \end{cases}$

8. Model Problem: George Carrier

The function $f(x,\varepsilon)$ satisfies the equation $(x+\varepsilon f)f'+f=1 \quad 0\le x\le 1$ and is subject to the boundary condition $f(1)=2$. Find the exact solution.

Exact solution:

$$(x+\varepsilon f)f'+f=1$$

$$\frac{\mathrm{d}(xf)}{\mathrm{d}x}+\frac{\mathrm{d}\left(\frac{1}{2}\varepsilon f^2\right)}{\mathrm{d}x}=1$$

$$xf+\frac{1}{2}\varepsilon f^2=x+C$$

Applying the boundary condition $f(1)=2$, we have

$$xf+\frac{1}{2}\varepsilon f^2=x+C$$

$$1\cdot f(1)+\frac{1}{2}\varepsilon f^2(1)=1+C$$

$$1+C=1\cdot 2+\frac{1}{2}\cdot\varepsilon 4\to C=1+2\varepsilon$$

$$xf+\frac{1}{2}\varepsilon f^2=2(x+1+2\varepsilon),\ 2xf+\varepsilon f^2-2(x+1+2\varepsilon)=0$$

Solving, we obtain

$$\varepsilon f^2+2xf-2(x+1+2\varepsilon)=0$$

$$f=\frac{-2x+\sqrt{(2x)^2+4\cdot\varepsilon\cdot 2(x+1+2\varepsilon)}}{2\varepsilon}$$

$$f=\frac{-2x+\sqrt{4x^2+8\varepsilon(x+1+2\varepsilon)}}{2\varepsilon}=\frac{-2x+\sqrt{4x^2+8x\varepsilon+8\varepsilon+16\varepsilon^2}}{2\varepsilon}$$

Therefore, the exact solution is

$$f=\frac{-x+\sqrt{x^2+2x\varepsilon+2\varepsilon+4\varepsilon^2}}{\varepsilon}$$

$$f=\frac{x}{\varepsilon}+\sqrt{\left(\frac{x}{\varepsilon}\right)^2+\frac{2(x+1)}{\varepsilon}+4}$$

Note that

$$f(0)=\sqrt{\frac{2}{\varepsilon}+4}$$

For large *x*, $f(x)=-\frac{x}{\varepsilon}+\frac{x}{\varepsilon}\sqrt{1+\frac{2\varepsilon(x+1)}{x^2}+\frac{4\varepsilon^2}{\varepsilon^2}}$.

Using the binomial Expansion

$$\sqrt{\left(\frac{x}{\varepsilon}\right)^2+\frac{2(x+1)}{\varepsilon}+4}=\left\{\left(\frac{x}{\varepsilon}\right)^2+\frac{2(x+1)}{\varepsilon}+4\right\}^{\frac{1}{2}}=1+\frac{2\varepsilon(x+1)}{x^2}+\cdots$$

For $x \gg 1$, $f(x) = -\frac{x}{\varepsilon} + \frac{x}{\varepsilon}\left\{1 + \frac{2\varepsilon(x+1)}{x^2} + \cdots\right\} = -\frac{x}{\varepsilon} + \frac{x}{\varepsilon} + \frac{x^2+x}{\varepsilon} \approx 1 + \frac{1}{x}$

For $x \ll 1$, $f(x) = -\frac{x}{\varepsilon} - \frac{x}{\varepsilon}\left\{1 + \frac{2\varepsilon(x+1)}{x^2} + \cdots\right\} = -\frac{2x}{\varepsilon} - \frac{x^2+x}{\varepsilon} \approx -\frac{2x}{\varepsilon} - 1 - \frac{1}{x}$

8.1. Expansion Method

Treating it as a singular perturbation problem

Let $f(x) = f_0(x) + \varepsilon f_1(x) + \cdots$ and $f_0(1) = 2$

$$xf_0'(x) + f_0(x) = 1$$

$$\frac{\mathrm{d}(xf_0(x))}{\mathrm{d}x} = 1$$

$$xf_0(x) = x + C$$

Applying the boundary condition

$$2 = 1 + C \to C = 1$$

Therefore, singular at $x = 0$, $f_0(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$.

This is not surprising because as $\varepsilon \to 0$, $f(x)$ exact value becomes singular at $x = 0$.

8.2. Consider the Van Der Pol Oscillator Equation $\ddot{x} + x = \varepsilon\dot{x}(1-x^2)$ with $x(0) = 1$, $\dot{x}(0) = 0$ $\ddot{x} + x - \varepsilon\dot{x}(1-x^2) = 0$

Using the perturbation method, we let

$x(t) = x_0(t) + \varepsilon x_1(t) + \cdots$ so that

$$\dot{x}(t) = \dot{x}_0(t) + \varepsilon\dot{x}_1(t) + \cdots$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \varepsilon\ddot{x}_1(t) + \cdots$$

Substituting into the DE, we obtain

$$\left[\ddot{x}_0(t) + \varepsilon\ddot{x}_1(t) + \cdots\right] + \left[x_0(t) + \varepsilon x_1(t) + \cdots\right]$$
$$-\varepsilon\left[\dot{x}_0(t) + \varepsilon\dot{x}_1(t) + \cdots\right]\left[1 - \left\{x_0(t) + \varepsilon x_1(t) + \cdots\right\}^2\right] = 0$$

Equating coefficient of equal powers of ε, we obtain

$$\varepsilon^0 : \ddot{x}_0(t) + x_0(t) = 0$$

$$\varepsilon^1 : \ddot{x}_1(t) + x_0 x_1(t) - \dot{x}_0(t) + \dot{x}_0(t)x_0^2(t) = 0$$

Thus, the solution to $\ddot{x}_0(t) + x_0(t) = 0$ is

$$x_0(t) = A\cos t + B\sin t$$

$$\dot{x}_0(t) = -A\sin t + B\cos t$$

Using the initial condition, $x(0) = 1$

$$x_0(0) = A\cos(0) + B\sin(0), \quad A = 1$$

$$\dot{x}_0(0) = -A\sin(0) + B\cos(0), \quad B = 0$$

Therefore $x_0(t) = \cos t$.

Solving the second equation

$$\ddot{x}_1(t) + x_1(t) - \dot{x}_0(t) + \dot{x}_0(t)x_0^2(t) = 0, \ \ddot{x}_1(t) + x_1(t) = \dot{x}_0(t) - \dot{x}_0(t)x_0^2(t)$$

$$\ddot{x}_1(t) + x_1(t) = \sin t - \sin t\cos^2 t, \ \ddot{x}_1(t) + x_1(t) = \sin t(1 - \cos^2 t)$$

$$\ddot{x}_1(t) + x_1(t) = \sin t\sin^2 t, \ \ddot{x}_1(t) + x_1(t) = \sin^3 t$$

But $\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)$

$$\ddot{x}_1(t) + x_1(t) = \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)$$

Solving for the homogeneous solution $\ddot{x}_1(t) + x_1(t) = 0$, we have $x_1(t) = a\cos t + b\sin t$.

Finding a particular solution

$$x_p(t) = Dt\cos t + Et\sin t + F\cos 3t + G\sin 3t + H$$

$$\dot{x}_p(t) = D\cos t - Dt\sin t + E\sin t + Et\cos t - 3F\sin 3t + 3G\cos 3t$$

$$\ddot{x}_p(t) = -D\sin t - D\sin t - Dt\cos t + E\cos t + E\cos t - Et\sin t - 9F\sin 3t - 9G\sin 3t$$

$$\ddot{x}_p(t) = -2D\sin t + 2E\cos t - Dt\cos t - Et\sin t - 9G\cos 3t - 9G\sin 3t$$

Substituting into $\ddot{x}_1(t) + x_1(t) = \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)$, we have

$$-2D\sin t + 2E\cos t - Dt\cos t - Et\sin t - 9F\cos 3t - 9G\sin 3t + Dt\cos t + Et\sin t + F\cos 3t + G\sin 3t + H = \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)$$

$$-2D\sin t + 2E\cos t - 9F\cos 3t - 9G\sin 3t + F\cos 3t + G\sin 3t + H = \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)$$

Equating coefficients, we have

$$-2D = \frac{3}{4}, \ D = -\frac{3}{8}, \ 2E = 0, \ E = 0, \ -9F + F = 0, \ F = 0$$

$$-9G + G = -\frac{1}{4}, \ G = \frac{1}{32}, \ H = 0$$

Therefore, a particular solution is $x_p(t) = -\frac{3}{8}t\cos t + \frac{1}{32}\sin 3t$.

Thus, $x_1(t)$ = homogeneous solution + A particular solution

$$x_1(t) = a\cos t + b\sin t - \frac{3}{8}t\cos t + \frac{1}{32}\sin 3t$$

$$\dot{x}_1(t) = -\frac{3}{8}\cos t + \frac{3}{8}t\sin t + \frac{3}{32}\cos 3t - a\sin t + b\cos t$$

Applying the boundary conditions, $x_1(0) = 1$ to

$x_1(t) = -\frac{3}{8}t\cos t + \frac{1}{32}\sin 3t + a\cos t + b\sin t$, we have

$$x_1(0) = -\frac{3}{8}(0)\cos(0) + \frac{1}{32}\sin(3\times 0) + a\cos(0) + b\sin(0)$$

$$1 = a$$

Applying the boundary conditions, $\dot{x}_1(0) = 0$ to

$\dot{x}_1(t) = -\frac{3}{8}\cos t + \frac{3}{8}t\sin t + \frac{3}{32}\cos 3t - a\sin t + b\cos t$, we have

$$\dot{x}_1(0) = -\frac{3}{8}\cos(0) + \frac{3}{8}(0)\sin(0) + \frac{3}{32}\cos(3\times 0) - \sin(0) + b\cos(0)$$

$$0 = -\frac{3}{8} + \frac{3}{32} + b, b = -\frac{9}{32}$$

Therefore, $x_1(t) = -\frac{3}{8}t\cos t + \frac{1}{32}\sin 3t + \cos t - \frac{9}{32}\sin t$.

Thus, the solution is $x(t) = x_0(t) + \varepsilon x_1(t) + \cdots$

$$x(t) = \cos t + \varepsilon\left[-\frac{3}{8}t\cos t + \frac{1}{32}\sin 3t + \cos t - \frac{9}{32}\sin t\right] + \cdots$$

$$x(t) = \cos t + \varepsilon\left[\left(1 - \frac{3}{8}t\right)\cos t + \frac{1}{32}(\sin t - 9\sin 3t)\right] + \cdots$$

8.3. Find the Exact Solution of the Perturbation Problem $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, and Subject to the Boundary Condition $y(0) = 0$, $y(1) = e^{-1}$

8.3.1. Exact Solution

Let $y(x) = e^{mx}$ so that $y'(x) = me^{mx}$, $y''(x) = m^2e^{mx}$

Substituting into the equation, we obtain

$$\varepsilon m^2 e^{mx} + (1+\varepsilon)me^{mx} + e^{mx} = 0$$

$$\varepsilon m^2 + (1+\varepsilon)m + 1 = 0,\quad \varepsilon m^2 + \varepsilon m + m + 1 = 0$$

$$\varepsilon m(m+1) + (m+1) = 0,\quad (\varepsilon m + 1)(m+1) = 0,\quad m = -1,\quad m = -\frac{1}{\varepsilon}$$

Therefore, the solution is $y(x) = Ae^{-x} + Be^{-\frac{x}{\varepsilon}}$

Invoking the boundary condition, we have $A = -B$ at $y(0) = 0$

and $B = -\frac{e^{\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e}$, $A = \frac{e^{\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e}$ at $y(1) = e^{-1}$

Thus,

$$y(x) = \frac{e^{\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e}e^{-x} - \frac{e^{\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e}e^{-\frac{x}{\varepsilon}} = \frac{e^{\frac{1}{\varepsilon}}\left(e^{-x} - e^{-\frac{x}{\varepsilon}}\right)}{e^{\frac{1}{\varepsilon}} - e}$$

9. Conclusion

This paper presents a comparative analysis of various algebraic methods for solving perturbation problems, highlighting their strengths and limitations. By exploring techniques such as expansion, iteration, and exact solutions using quadratic form and binomial expansion, researchers can select the most appropriate approach for their specific challenges. Through practical examples of singularly perturbed problems in physical contexts, the paper demonstrates the applicability of these methods across disciplines. With detailed explanations and work examples, researchers gain the skills to analyze and solve complex equations. Serving as a concise and comprehensive introduction to perturbation theory, this paper is an ideal starting point for new researchers in the perturbation field.

Acknowledgements

This work is supported in part by the National Science Foundation grant no. HRD 1839895.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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