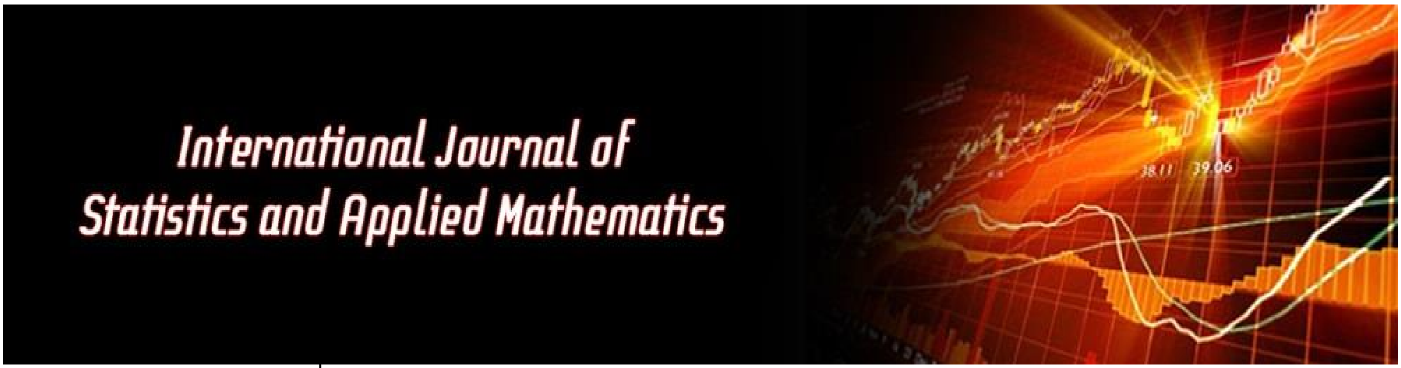
International Journal of Statistics and Applied Mathematics 2024; 9(6): 34-44



ISSN: 2456-1452
Maths 2024; 9(6): 34-44

https://www.mathsjournal.com
Received: 10-09-2024
Accepted: 05-10-2024

Boampong Asare
Department of Engineering, United Tribes Technical College, Bismarck, North Dakota, United States

Manohar Sah
Department of Engineering, United Tribes Technical College, Bismarck, North Dakota, United States

Ram Krishna Hona
Department of Environmental Science, United Tribes Technical College, Bismarck, North Dakota, United States

Corresponding Author:
Boampong Asare
Department of Engineering, United Tribes Technical College, Bismarck, North Dakota, United States

Epsilon variations and their impact on solution accuracy: A matched asymptotic approach

Boampong Asare, Manohar Sah and Ram Krishna Hona

DOI: https://dx.doi.org/10.22271/maths.2024.v9.i6a.1883

Abstract
This study investigates the method of matched asymptotic expansions (MAE) for solving singular perturbation problems, focusing on graphical data matching and simulation. We assess the accuracy, computational efficiency, and impact of the perturbation parameter epsilon (ε) on the agreement between exact and composite solutions across various problem types. Results demonstrate that MAE-derived composite solutions closely match exact solutions, with minimal deviations observed. As ε decreases, both solutions converge to 1, indicating asymptotic stability, but larger ε values lead to significant deviations, signaling divergence. The solutions exhibit consistent behavior, converging and diverging at similar points with high accuracy. This comprehensive analysis highlights MAE's effectiveness in solving small parameter problems and its utility in graphical simulations, offering valuable insights for future research in singular perturbation theory, particularly in visual representation and numerical validation of solutions.

Keywords: Solutions, particularly, representation

Introduction

Singular perturbation [1, 2] problems arise frequently in various fields of science and engineering, where small parameters can significantly influence the behavior of a system. These problems are characterized by the presence of different scales, leading to solutions that exhibit rapid variations in certain regions and more gradual changes elsewhere. The method of matched asymptotic expansions (MAE) [2, 3] is a widely used technique to address such problems, enabling the construction of approximate solutions by dividing the problem into multiple regions—typically referred to as "inner" [1, 2] and "outer" [2, 3, 4] regions—where different asymptotic expansions are applied. The solutions from these regions are then matched in an intermediate overlap region to ensure a smooth transition between them.

While much progress has been made in deriving exact and composite solutions for singular perturbation problems, the graphical matching and simulation of these solutions remain underexplored. The existing literature primarily focuses on the theoretical foundations and analytical results, often leaving a gap in the empirical validation of how closely these approximate solutions match the exact solutions across a range of problems.

This study aims to bridge that gap by providing a comprehensive evaluation of existing methods for solving canonical singular perturbation problems using MAE. We systematically assess each method based on four key criteria: accuracy, computational efficiency, applicability, and implementation simplicity. Through graphical data matching and numerical simulations, we investigate the extent to which composite solutions approximate exact solutions, with specific attention to small deviations in critical regions. Our findings reveal that the method of matched asymptotic expansions can achieve high levels of accuracy with minimal computational resources, making it a robust tool for solving a wide range of singular perturbation problems [4, 5, 6, 10].

In the following sections, we present a detailed comparison of exact and composite solutions, supported by graphical data for selected small parameter problems.

The results offer new insights into the practical application of MAE in both theoretical analysis and computational simulations.

Methodological Framework

Mathematical Structure

This research is grounded in perturbation theory, particularly singular perturbation problems. The methodology leverages established algebraic and analytical techniques from applied mathematics, including:

a. **Method of Matched Asymptotic Expansions**: This method is employed to solve singular perturbation problems, where different scales of the problem require different asymptotic expansions. The solutions obtained in different regions (e.g., inner and outer regions) are matched in an overlapping region to ensure a smooth transition.

This section provides an overview of the common methodologies employed in the existing literature to address singular perturbation problems. We outline the general approaches and techniques utilized to tackle these challenges.

Analytical Techniques

Symbolic computation tools, namely R-programming, MATLAB, and Wolfram Mathematica, were leveraged to implement mathematical methods, enabling the derivation, manipulation, and simplification of expressions obtained through perturbation techniques. Additionally, numerical simulations were performed on these platforms to validate the analytical solutions' accuracy, benchmarked against exact solutions where possible, thereby guaranteeing the precision and reliability of the results.

Comparative Analysis

A comprehensive evaluation of existing methods for canonical perturbation problems, including singular perturbations, was performed. The analysis assessed each method's effectiveness based on four critical metrics:

a. **Accuracy**: Proximity of approximate solutions to exact or numerical solutions.
b. **Computational Efficiency**: Computational time and resource requirements.
c. **Applicability**: Breadth of suitable problem types.
d. **Implementation Simplicity**: Complexity of mathematical operations and required expertise.

Case Studies

Singular perturbation methods were applied to case studies from fluid dynamics, quantum mechanics, and elasticity theory, showcasing their practical relevance in physical contexts. These examples highlighted the methods' effectiveness and emphasized the need for careful technique selection based on the specific characteristics of each problem.

Validation

The accuracy of the approximate solutions derived via perturbation methods was verified through comparisons with available exact solutions or high-precision numerical solutions. Additionally, convergence tests were performed to guarantee that the truncated series solutions yielded sufficiently accurate approximations as the perturbation parameter tended to zero.

The Method of Matched Asymptotic Expansion

The Method of Matched Asymptotic Expansions (MMAE) is a mathematical technique for solving singular perturbation problems by matching asymptotic expansions in different regions. In the Method of Matched Asymptotic Expansions, the solution is divided into two regions an outer region, where a regular perturbation expansion is valid, and an inner region (boundary layer), where a singular perturbation expansion is necessary. The outer expansion is typically represented as $y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$ while the inner expansion is $Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \cdots$ where $X = \frac{x}{\varepsilon^a}$ is the stretched variable. The matching process involves equating the outer and inner expansions in an overlap region, $y(x;\epsilon) \sim Y(X;\epsilon)$ to determine the matching conditions, which enables the derivation of a uniformly valid solution. By systematically matching the expansions, MMAE overcomes the limitations of traditional perturbation methods, providing a powerful tool for analyzing complex phenomena in fluid dynamics, quantum mechanics, electromagnetism, and other fields [7, 8, 9, 10]. To illustrate the power and utility of asymptotic matching, let's consider the following equation.

$\boldsymbol{\varepsilon y'' + (1+\varepsilon)y' + y = 0}$**, and subject to the boundary condition** $\boldsymbol{y(0) = 0, y(1) = e^{-1}}$

Obtaining the Outer solution

Given the minimal value of ε, our initial approach is to find the zeroth-order solution (or outer solution) [7] by neglecting the ε-term in the equation, effectively simplifying it. This approximation is valid in the outer region, away from the boundary layer where the solution undergoes rapid change**.** Using the perturbation method, we let
$y(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ so that

$$y'(x) = y^1{}_0 + \varepsilon y^1{}_1 + \varepsilon^2 y^1{}_2 + \cdots$$

$$y'' = y''_0 + \varepsilon y''_1 + \varepsilon^2 y''_2 + \cdots$$

Substituting into the DE, $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, we obtain

$$\varepsilon\{y''_0 + \varepsilon y''_1 + \varepsilon^2 y''_2 + \cdots\} + \{1+\varepsilon\}\{y^1{}_0 + \varepsilon y^1{}_1 + \varepsilon^2 y^1{}_2 + \cdots\} + y_0 + \varepsilon y_1 + \varepsilon^2 y_1 + \cdots = 0$$

Equating coefficient of ε and applying the outer boundary condition $y(1) = e^{-1}$, we have

$$\varepsilon^0: {y^1}_0 + y_0 = 0, \frac{{y''}_0}{y_0} = -1$$

$$\ln(y_0) = -x + c,\ y_0 = ae^{-x},\ ae^{-1} = ae^{-1}, a =$$

Thus, $y_0 = e^{-x}$

Rescaling: Obtaining the inner solution

We let $\xi = \frac{x}{\varepsilon}$, so that $x = \varepsilon\xi$

$$\frac{dx}{d\xi} = \varepsilon, y' = \frac{dy}{d\xi} \bullet \frac{d\xi}{dx} = \frac{1}{\varepsilon}\frac{dy}{d\xi}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{dy}{d\xi}\right) = \frac{1}{\varepsilon}\frac{d}{d\xi}\left(\frac{dy}{dx}\right) = \frac{1}{\varepsilon}\frac{d}{d\xi}\left(\frac{1}{\varepsilon}\frac{dy}{d\xi}\right) = \frac{1}{\varepsilon^2}\frac{d^2y}{d\xi^2}$$

Substituting into the DE $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, and solving, we obtain

$$\varepsilon\left\{\frac{1}{\varepsilon^2}\frac{d^2y}{d\xi^2}\right\} + \{1+\varepsilon\}\left\{\frac{1}{\varepsilon}\frac{dy}{d\xi}\right\} + y(\xi) = 0$$

$$y_i(\xi) = b + de^{-\xi}$$

Applying the inner boundary condition $y_i(0) = 0$ we have

$$b + d = 0 \text{ or } b = -d,\ y_i(\xi) = b\{1 - e^{-\xi}\}$$

Matching the inner to the outer

$$\lim_{\xi\to\infty} y_i(\xi) = \lim_{x\to 0} y_0,\ \lim_{\xi\to\infty} b\{1 - e^{-\xi}\} = \lim_{x\to 0} e^{-x},\ b\{1 - e^{-\infty}\} = e^{-0}, b = 1$$

Thus, $y_i(\xi) = 1 - e^{-\xi}$

Composite solution/Uniform approximation

A composite solution is a combined solution that asymptotically matches the outer (zeroth order) solution and the inner (first order) solution, creating a single, uniformly valid approximation across the entire domain. The composite solution is formed by adding the outer and inner solutions, then subtracting the overlap region (where the two solutions are identical) to avoid double-counting. This ensures a smooth transition between the two regions providing an accurate approximation of the exact solution [1, 2,3, 10].

$$y_i(x) = y_i(\xi) + y_0(x) - \lim_{x\to 0} y_0(x)$$

$$y_i(x) = 1 - e^{-\xi} + e^{-x} - 1 = e^{-x} - e^{-\xi} = e^{-x} - e^{-\frac{x}{\varepsilon}}$$

The exact solution from equation (2.2) is $y(x) = \frac{e^{\frac{1}{\varepsilon}}\left(e^{-x} - e^{-\frac{x}{\varepsilon}}\right)}{e^{\frac{1}{\varepsilon}} - e}$ and the approximate solution is $y_i(x) = e^{-x} - e^{-\frac{x}{\varepsilon}}$. The graphs display the approximate and exact values of $y_i(x)$ and $y(x)$ for 101 simulated x values, with epsilon (ε) values of 0.001, 0.002, 0.01, and 0.1.

Figure 1.

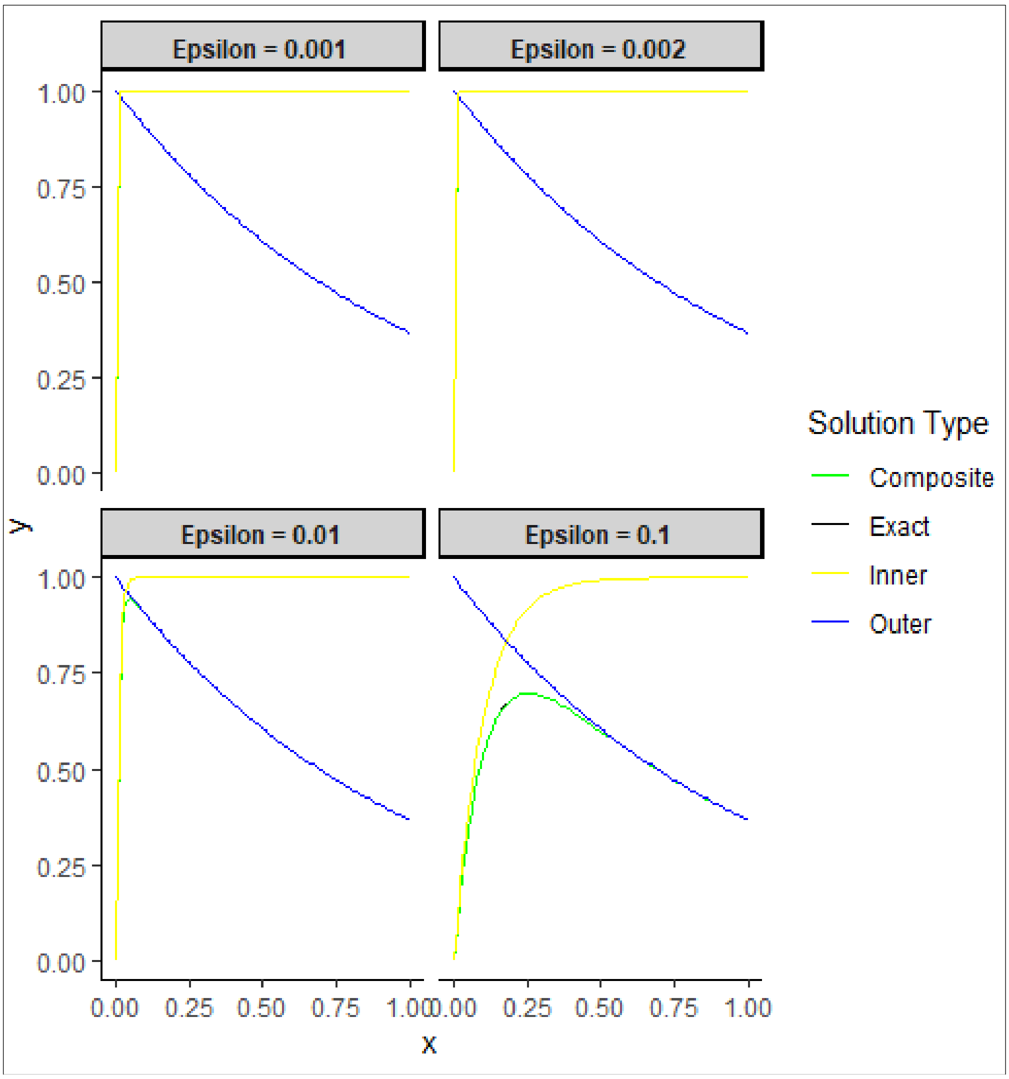


Fig 1: The above displays individual graphs for epsilon values of 0.001, 0.002, 0.01, and 0.1, each based on 101 simulated x values.

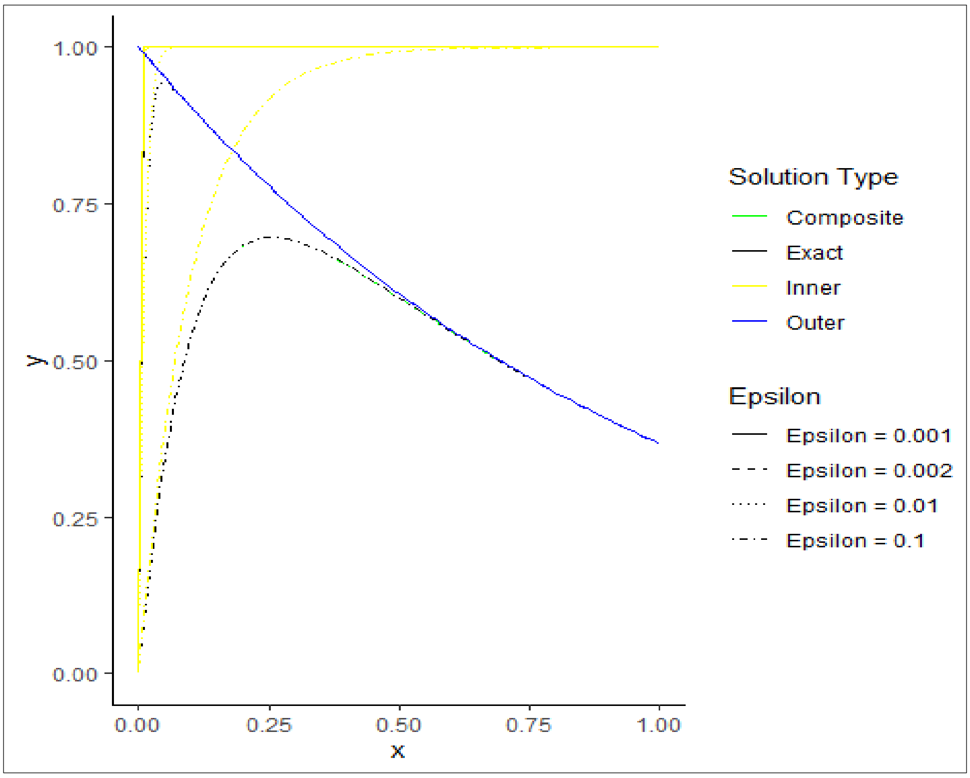


Fig 2: Above illustrates the combined graph of the inner, outer, composite, and exact solutions for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01, and 0.1.

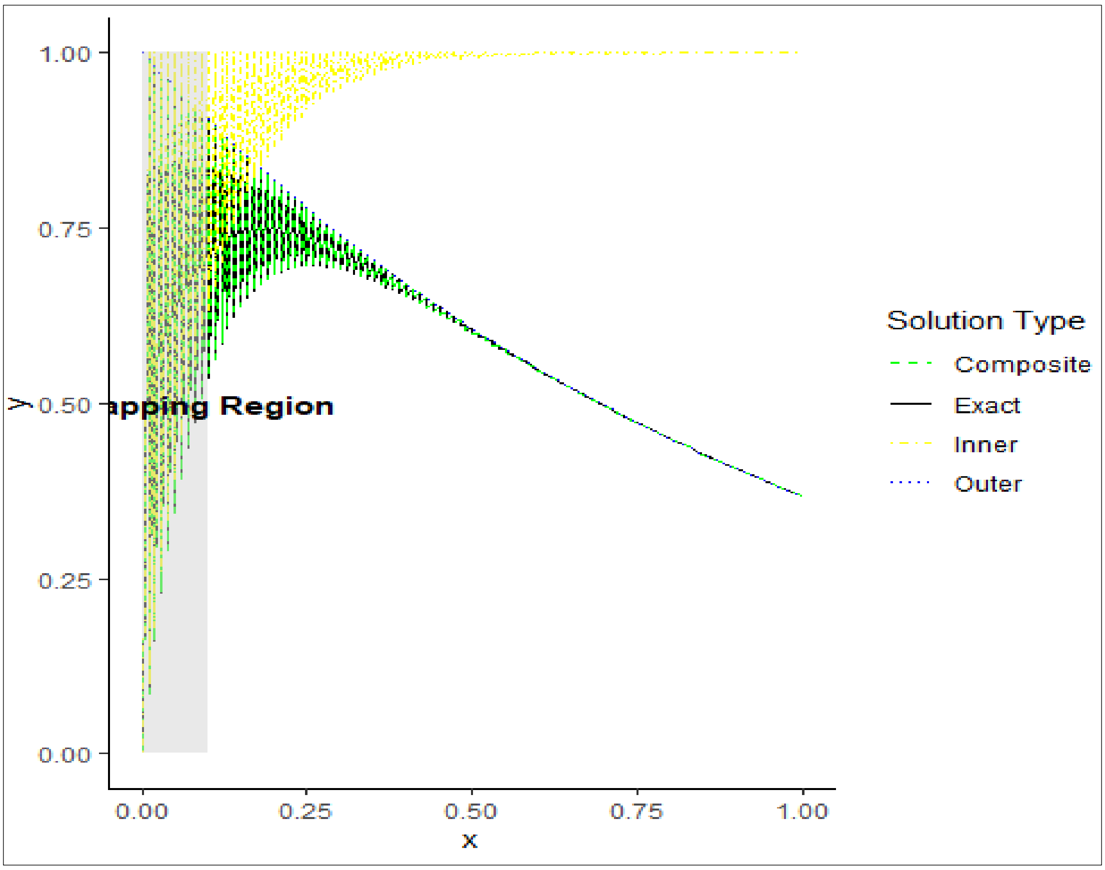


Fig 3: Above illustrates the overlap region of the inner (yellow), outer (blue), composite (green), and exact solutions (black) for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01 and 0.1

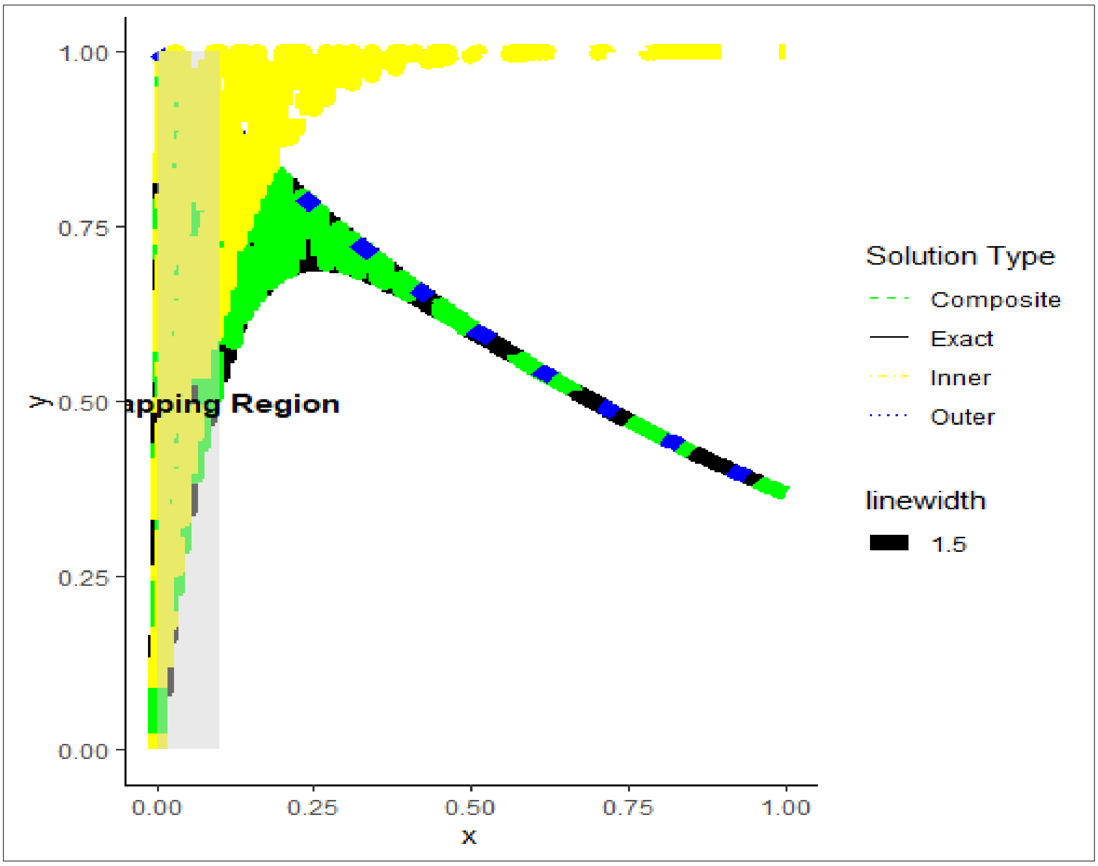


Fig 4: Similarly, figure 4 above clearly illustrates the regions of the inner (yellow), outer (blue), composite (green), exact solutions (black), and the overlap for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01, and 0.1.

The graphs and data reveal key insights into the behavior of the Exact Solution and Composite Solution. As ε decreases, both solutions converge to 1, indicating asymptotic stability, but larger ε values (e.g., $\varepsilon = 0.1$) lead to significant deviations, signaling divergence. The maximum x values shift rightward with increasing ε. The Composite Solution accurately approximates the Exact Solution for small ε values but slightly underestimates it for larger ε values. The analysis highlights ε-dependent stability, potential bifurcations, and sensitivity to ε changes. Recommendations include monitoring ε values for convergence, verifying Composite Solution accuracy, and analyzing x-value sensitivity. Overall, the graphs validate the perturbation analysis, underscoring the crucial role of ε values in solution stability and behavior.

Table 1: Solutions for Epsilon = 0.001

0	Exact Solution	Composite Solution
0.00	0.000000	0.000000
0.01	0.990004	0.990004
0.02	0.980199	0.980199
0.03	0.970446	0.970446
0.04	0.960789	0.960789
0.05	0.951229	0.951229

Table 2: Solutions for Epsilon = 0.002

x	Exact Solution	Composite Solution
0.00	0.000000	0.000000
0.01	0.983312	0.983312
0.02	0.980153	0.980153
0.03	0.970445	0.970445
0.04	0.960789	0.960789
0.05	0.951229	0.951229

Table 3: Solutions for Epsilon = 0.01

x	Exact Solution	Composite Solution
0.00	0.000000	0.000000
0.01	0.622170	0.622170
0.02	0.844863	0.844863
0.03	0.920658	0.920658
0.04	0.942474	0.942474
0.05	0.944491	0.944491

Table 4: Solutions for Epsilon = 0.1

x	Exact Solution	Composite Solution
0.00	0.000000	0.000000
0.01	0.085223	0.085212
0.02	0.161488	0.161468
0.03	0.229656	0.229627
0.04	0.290505	0.290469
0.05	0.344741	0.344699

Tables 1 through 4 examine how solutions change with increasing values of epsilon (ε). For small ε values (0.001, 0.002), the composite solution converges closely to the exact solution. However, as ε increases (0.01, 0.1), the composite solution diverges significantly. The perturbation grows with ε, amplifying divergence. Specifically, ε values of 0.001 and 0.002 yield accurate composite solutions, 0.01 shows growing perturbation, and 0.1 results in large perturbation and significant divergence. This indicates the method's sensitivity to perturbations, losing accuracy with larger ε values.

Table 5: Exact Solution

ε	Maximum	x
0.001	0.990004	0.01
0.002	0.983312	0.01
0.01	0.944491	0.05
0.1	0.696864	0.26

Table 6: Composite Solution

ε	Maximum	x
0.001	0.990004	0.01
0.002	0.983312	0.01
0.01	0.944491	0.05
0.1	0.696778	0.26

Tables 5 and 6 show the maximum values for the Exact solution and the Composite solution, respectively, based on simulations of 101 x values with epsilon values of 0.001, 0.002, 0.01, and 0.1. The analysis reveals that as ε decreases, the maximum values of both exact and composite solutions converge to 1, indicating asymptotic stability. However, for larger ε values (e.g., ε = 0.1), significant deviation from 1 occurs, signaling divergence. Additionally, the x values corresponding to maximum values are sensitive to ε, shifting rightward as ε increases, and while the composite solution accurately approximates the exact solution for small ε values, it slightly underestimates it for larger ε values. Overall, the solutions exhibit asymptotic stability as ε approaches 0, but divergence occurs for larger ε values. To ensure convergence, it is recommended to use smaller ε values, monitor x values for sensitivity to ε, and verify the composite solution's accuracy for larger ε values.

$\varepsilon y'' + (1+\varepsilon)y' + y = 0$, and subject to the boundary condition $y(0) = 0, y(1) = 1$

Obtaining the Outer solution

Given the minimal value of ε, our initial approach is to find the zeroth-order solution (or outer solution) by neglecting the ε-term in the equation, effectively simplifying it. This approximation is valid in the outer region, away from the boundary layer where the solution undergoes rapid change**.** Using the perturbation method, we let

$y(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ so that

$y'(x) = y^1{}_0 + \varepsilon y^1{}_1 + \varepsilon^2 y^1{}_2 + \cdots$

$y'' = y''_0 + \varepsilon y''_1 + \varepsilon^2 y''_2 + \cdots$

Substituting into the DE, $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, we obtain

$$\varepsilon\{y''_0 + \varepsilon y''_1 + \varepsilon^2 y''_2 + \cdots\} + \{1+\varepsilon\}\{y^1{}_0 + \varepsilon y^1{}_1 + \varepsilon^2 y^1{}_2 + \cdots\} + y_0 + \varepsilon y_1 + \varepsilon^2 y_1 + \cdots = 0$$

Equating coefficient of ε and applying the outer boundary condition $y(1) = 1$ we have

$y^1{}_0 + y_0 = 0,$

$\frac{y''_0}{y_0} = -1, \ln(y_0) = -x + c, y_0 = ae^{-x}, 1 = ae^{-1}, a = e$

Thus, $y_0 = e^{1-x}$

Rescaling: Obtaining the inner solution

We let $\xi = \frac{x}{\varepsilon}$, so that $x = \varepsilon\xi$

$$\frac{dx}{d\xi} = \varepsilon, y' = \frac{dy}{d\xi}\bullet\frac{d\xi}{dx} = \frac{1}{\varepsilon}\frac{dy}{d\xi}, y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{dy}{d\xi}\right) = \frac{1}{\varepsilon}\frac{d}{d\xi}\left(\frac{dy}{dx}\right) = \frac{1}{\varepsilon}\frac{d}{d\xi}\left(\frac{1}{\varepsilon}\frac{dy}{d\xi}\right) = \frac{1}{\varepsilon^2}\frac{d^2y}{d\xi^2}$$

Substituting into the DE, $\varepsilon y'' + (1+\varepsilon)y' + y = 0$ and solving, we obtain

$\varepsilon y'' + (1+\varepsilon)y^1 + y = 0$

$\varepsilon\left\{\frac{1}{\varepsilon^2}\frac{d^2y}{d\xi^2}\right\} + \{1+\varepsilon\}\left\{\frac{1}{\varepsilon}\frac{dy}{d\xi}\right\} + y(\xi) = 0, y_i(\xi) = b + de^{-\xi}$

Applying the inner boundary condition $y_i(0) = 0$ we have

$b + d = 0$ or $b = -d, y_i(\xi) = b\{1 - e^{-\xi}\}$

Matching the inner to the outer

$\lim_{\xi\to\infty} y_i(\xi) = \lim_{x\to 0} y_0, \lim_{\xi\to\infty} y_i(\xi) = \lim_{x\to 0} y_0$

$\lim_{\xi\to\infty} b\{1 - e^{-\xi}\} = \lim_{x\to 0} e^{1-x}, b\{1 - e^{-\infty}\} = e^1, b = e$

Thus, $y_i(\xi) = e(1 - e^{-\xi}) = e - e^{1-\xi}$

Composite solution/Uniform approximation

$y_i(x) = y_i(\xi) + y_0(x) - \lim_{x\to 0} y_0(x)$

$y_i(x) = e - e^{1-\xi} + e^{1-x} - e = e^{-x} - e^{1-\xi} = e^{1-x} - e^{1-\frac{x}{\varepsilon}}$

$y_i(x) = e\left(e^{-x} - e^{-\frac{x}{\varepsilon}}\right)$

The exact solution is $y(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}}$ and the approximate solution is $y_i(x) = e\left(e^{-x} - e^{-\frac{x}{\varepsilon}}\right)$

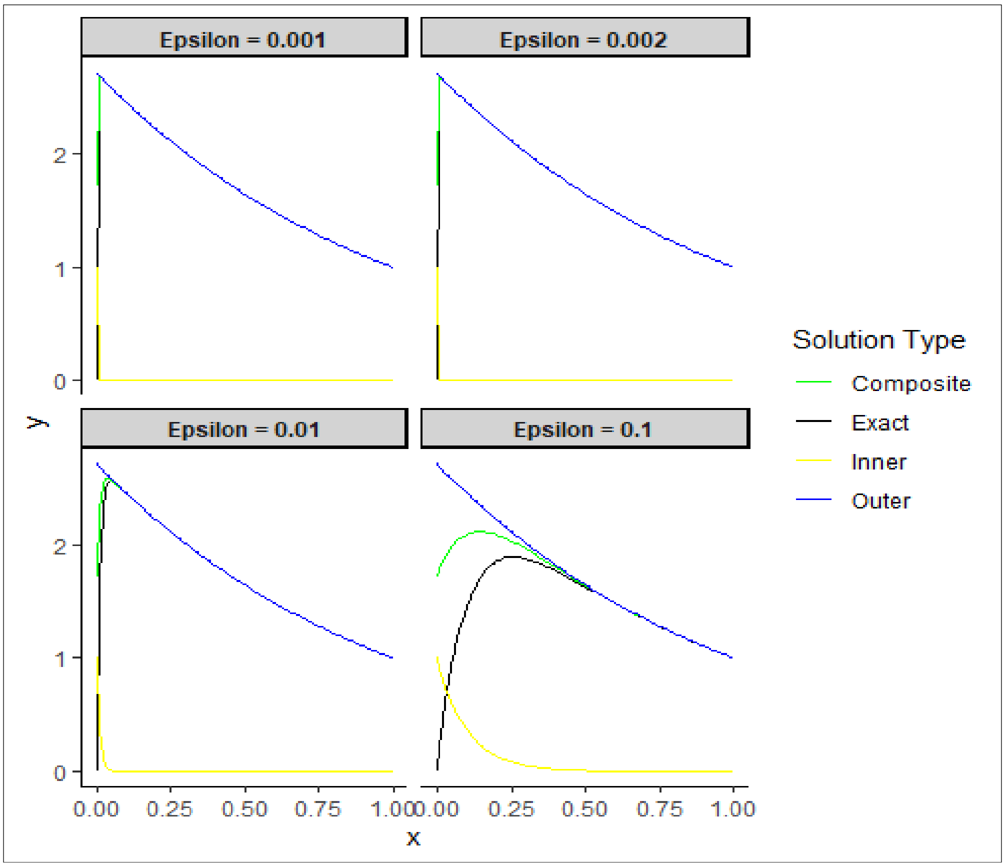


Fig 5: The above displays individual graphs for epsilon values of 0.001, 0.002, 0.01, and 0.1, each based on 101 simulated x values.

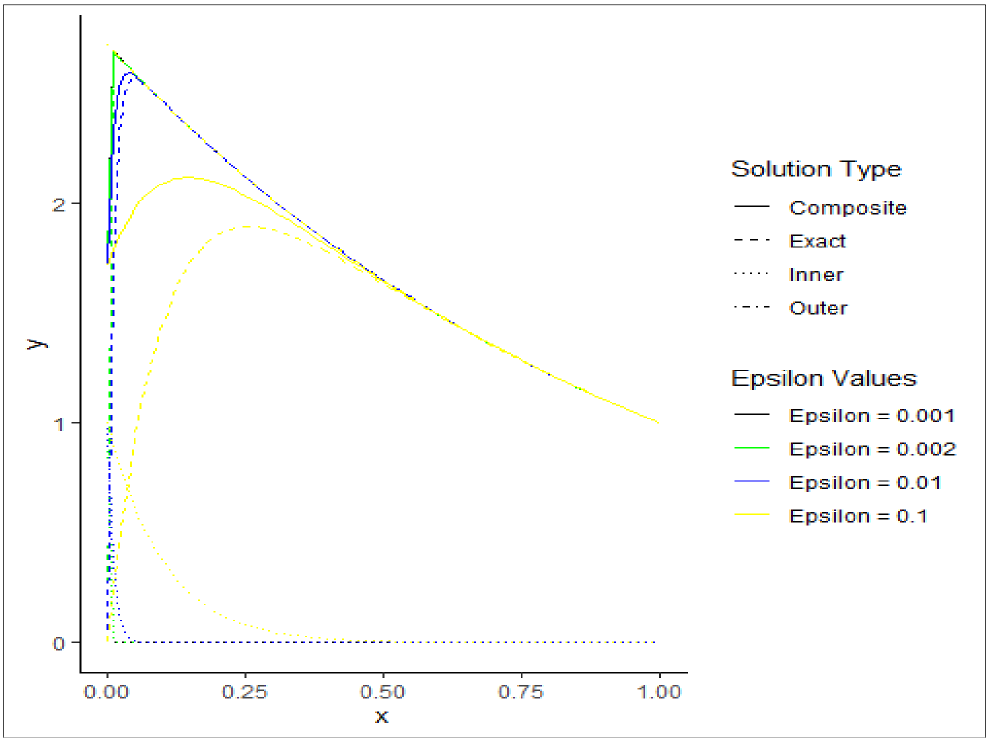


Fig 6: The figure above illustrates the combined graph of the inner, outer, composite, and exact solutions for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01, and 0.1.

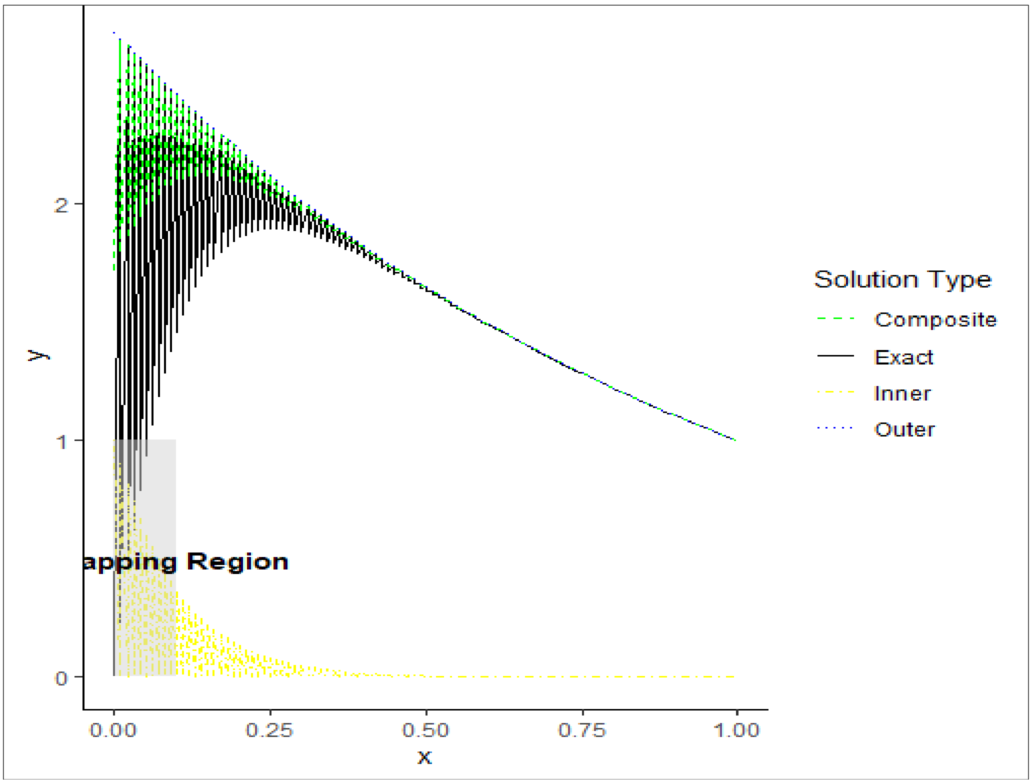


Fig 7: The figure above illustrates the overlap region of the inner (yellow), outer (blue), composite (green), and exact solutions (black) for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01 and 0.1

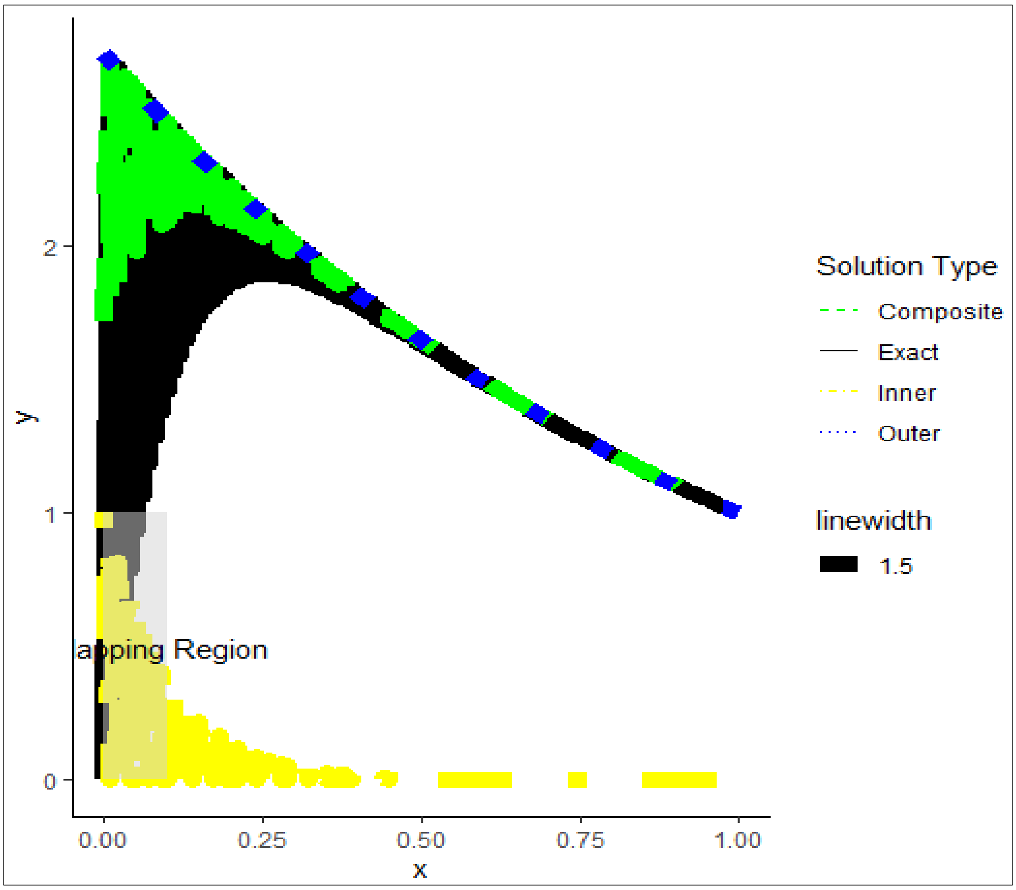


Fig 8: Similarly, the figure above clearly illustrates the regions of the inner (yellow), outer (blue), composite (green), exact solutions (black), and the overlap for 101 simulated x values, with epsilon values of 0.001, 0.002, 0.01, and 0.1.

The analysis of figures 1 through 4 reveals valuable insights into the behavior of the exact and composite solutions. Both graphs exhibit a decrease in maximum value as epsilon (ε) increases, with the exact solution graph showing a more pronounced shift in the location of the maximum value. The composite solution graph remains relatively flat for small ε values, indicating robustness to perturbations. The exact solution graph exhibits a smooth, continuous decrease, suggesting stability for small ε values.

The analysis indicates convergence for small ε values (0.001 and 0.002), with similar maximum values and occurrence points (x = 0.01) between the exact and composite solutions. However, as ε increases (0.01 and 0.1), the solutions diverge, with differing maximum values and occurrence points. This suggests that the solutions are stable and convergent for small epsilon but sensitive and divergent for larger epsilon. Overall, the results imply that the solutions exhibit: (1) convergence for small ε values and (2) divergence for larger ε values, highlighting the importance of considering perturbation size in solution analysis.

Table 7: Solutions for Epsilon = 0.001

x	Exact Solution	Composite Solution
0.00	0.000000	0.000000
0.01	2.691111	2.691189
0.02	2.664456	2.664456
0.03	2.637944	2.637944
0.04	2.611696	2.611696
0.05	2.585710	2.585710

Table 8: Solutions for Epsilon = 0.002

x	Exact Solution	Composite Solution
0.00	0.000000	1.718282
0.01	2.672919	2.684497
0.02	2.664333	2.664411
0.03	2.637944	2.637944
0.04	2.611696	2.611696
0.05	2.585710	2.585710

Table 9: Solutions for Epsilon = 0.01

x	Exact Solution	Composite Solution
0.00	0.000000	1.718282
0.01	1.691234	2.323355
0.02	2.296577	2.529121
0.03	2.502609	2.588157
0.04	2.561909	2.593381
0.05	2.567394	2.578972

Table 10: Solutions for Epsilon = 0.1

x	Exact Solution	Composite Solution
0.00	0.0000000	1.718282
0.01	0.2316600	1.786397
0.02	0.4389695	1.845725
0.03	0.6242688	1.897126
0.04	0.7896751	1.941376
0.05	0.9371040	1.979179

Tables 7 to 10 show that the composite solution's accuracy and stability are highly dependent on the value of ε. For smaller values (ε = 0.001 and ε = 0.002), the composite solution closely matches the exact solution across all given values, demonstrating stability and accuracy. However, as ε increases (ε = 0.01 and ε = 0.1), the composite solution diverges significantly from the exact solution, particularly for smaller input values. This sensitivity to ε indicates that small changes in ε can lead to substantial changes in the solution. A perturbation analysis reveals that for small ε, the composite solution exhibits a first-order perturbation effect, accurately approximating the exact solution, but as ε increases, higher-order perturbation effects become significant, causing divergence. Overall, the results highlight the importance of carefully selecting ε to ensure the composite solution's accuracy and stability.

Table 11: Exact Solution

ε	Maximum	x
0.001	2.69111	0.01
0.002	2.67292	0.01
0.01	2.56739	0.05
0.1	1.89427	0.26

Table 12: Composite Solution

ε	Maximum	x
0.001	2.69119	0.01
0.002	2.68450	0.01
0.01	2.59338	0.04
0.1	2.11656	0.14

The analysis of Tables 11 and 12 reveals that the maximum values of the Exact and Composite solutions decrease as the epsilon (ε) value increases. The maximum values for small ε values (0.001 and 0.002) are similar for both solutions, occurring at x = 0.01. However, as ε increases to 0.01 and 0.1, the maximum values diverge, with the Composite solution gradually declining. Notably, the occurrence of the maximum value shifts to larger x values for the Exact solution (x = 0.05 and 0.26) but remains relatively consistent for the Composite solution (x = 0.04 and 0.14). These findings suggest that the solution's sensitivity to ε varies, with the composite solution demonstrating greater robustness to perturbations.

Conclusion

The analysis reveals that the maximum values of the Exact and Composite solutions decrease as the epsilon (ε) value increases. The maximum values for small ε values $(0.001\ and\ 0.002)$ are similar for both solutions, occurring at (x = 0.01). However, as ε increases to 0.01 and 0.1, the maximum values diverge, with the Composite solution gradually declining. Notably, the occurrence of the maximum value shifts to larger (x) values for the Exact solution (x = 0.05) and (x = 0.26) but remains relatively consistent for the Composite solution $(x = 0.04)$ and $(x = 0.14)$. These findings suggest that the solutions' sensitivity to ε varies, with the Composite solution demonstrating greater robustness to perturbations.

The Composite solution's accuracy and stability are highly dependent on the value of ε. For smaller values (ε = 0.001 and ε = 0.002), the Composite solution closely matches the Exact solution across all given values, demonstrating stability and accuracy. However, as ε increases (ε = 0.01 and ε = 0.1), the Composite solution diverges significantly from the Exact solution, particularly for smaller input values. This sensitivity to ε indicates that small changes in ε can lead to substantial changes in the solution. A perturbation analysis reveals that for small ε, the Composite solution exhibits a first-order perturbation effect, accurately approximating the Exact solution, but as ε increases, higher-order perturbation effects become significant, causing divergence.

Overall, the results highlight the importance of carefully selecting ε to ensure the Composite solution's accuracy and stability. The analysis of the graphs reveals valuable insights into the behavior of the Exact and Composite solutions. Both graphs exhibit a decrease in maximum value as ε increases, with the Exact solution graph showing a more pronounced shift in the location of the maximum value. The Composite solution graph remains relatively flat for small ε values, indicating robustness to perturbations. The Exact solution graph exhibits a smooth, continuous decrease, suggesting stability for small ε values.

Acknowledgments

This work is supported in part by National Science Foundation grant no. HRD-1839895

References

1. Nayfeh AH. Perturbation Methods. 1st ed. Johnson Wiley & Sons; c2000. ISBN: 9783527617609. Available from: https://doi.org/10.1002/9783527617609.fmatter
2. Dalla Riva M, Lanza de Cristoforis M, Musolino P. Singularly Perturbed Boundary Value Problems: A Functional Analytic Approach. Springer; c2021. ISBN: 978-3-030-76258-2. Available from: https://doi.org/10.1007/978-3-030-76259-9
3. Nayfeh AH. Introduction to Perturbation Techniques. Wiley; c2011. ISBN: 978-3-527-61845-3.
4. Kevorkian J, Cole JD. Perturbation Methods in Applied Mathematics. 1st ed. Springer Verlag; c1981. ISBN: 978-1-4757-4213-8. Available from: https://doi.org/10.1007/978-1-4757-4213-8
5. Hinch EJ. Perturbation Methods. Cambridge University Press; c1991. ISBN: 9781139172189. Available from: https://doi.org/10.1017/CBO9781139172189
6. Holmes M. Introduction to Perturbation Methods. Springer; c1995. ISBN: 978-1-4612-5347-1. Available from: https://doi.org/10.1007/978-1-4612-5347-1
7. Simmonds JG, Mann JE Jr. A First Look at Perturbation Theory. Dover Publications, Inc.; c1997. ISBN: 978-0486675510.
8. O'Malley RE. Singular Perturbation Methods for Ordinary Differential Equations. Springer-Verlag; c1991. ISBN: 978-0-387-97556-6. Available from: https://doi.org/10.1007/978-1-4612-0977-5
9. HJ. Asymptotic Analysis and Singular Perturbation Theory. University of California at Davis; c2004.
10. Asare B, Sah M, Hona R. Alternative Methods of Regular and Singular Perturbation Problems. Appl Math. 2024;15:687-708. DOI: 10.4236/am.2024.1510040