SCIENCE CHINA
Mathematics

• ARTICLES •

May 2023 Vol. 66 No. 5: 1053–1072
https://doi.org/10.1007/s11425-021-1983-0

Up-to-constants comparison of Liouville first passage percolation and Liouville quantum gravity

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Received September 19, 2021; accepted May 23, 2022; published online November 17, 2022

Abstract Liouville first passage percolation (LFPP) with the parameter $\xi > 0$ is the family of random distance functions $\{D_h^\epsilon\}_{\epsilon>0}$ on the plane obtained by integrating $\mathrm{e}^{\xi h_\epsilon}$ along paths, where $\{h_\epsilon\}_{\epsilon>0}$ is a smooth mollification of the planar Gaussian free field. Recent works have shown that for all $\xi > 0$, the LFPP metrics, appropriately re-scaled, admit non-trivial subsequential limiting metrics. In the case $\xi < \xi_{\mathrm{crit}} \approx 0.41$, it has been shown that the subsequential limit is unique and defines a metric on γ-Liouville quantum gravity (LQG) $\gamma = \gamma(\xi) \in (0, 2)$. We prove that for all $\xi > 0$, each possible subsequential limiting metric is nearly bi-Lipschitz equivalent to the LFPP metric D_h^ϵ when ϵ is small, even if ϵ does not belong to the appropriate subsequence. Using this result, we obtain bounds for the scaling constants for LFPP which are sharp up to polylogarithmic factors. We also prove that any two subsequential limiting metrics are bi-Lipschitz equivalent. Our results are an input in subsequent works which shows that the subsequential limits of LFPP induce the same topology as the Euclidean metric when $\xi = \xi_{\mathrm{crit}}$ and that the subsequential limit of LFPP is unique when $\xi \geqslant \xi_{\mathrm{crit}}$.

Keywords Liouville quantum gravity, Gaussian free field, LQG metric, Liouville first passage percolation, supercritical LQG

MSC(2020) 60D05, 60G60

Citation: Ding J, Gwynne E. Up-to-constants comparison of Liouville first passage percolation and Liouville quantum gravity. Sci China Math, 2023, 66: 1053–1072, https://doi.org/10.1007/s11425-021-1983-0

1 Introduction

1.1 Liouville first passage percolation

Let h be the whole-plane Gaussian free field (GFF), normalized so that its average over the unit circle is zero. This means that h is the Gaussian process on $\mathbb{C}$ with covariances given by

$$\mathrm{Cov}(h(z), h(w)) = \log \frac{\max\{|z|, 1\} \max\{|w|, 1\}}{|z - w|}, \quad \forall\, z, w \in \mathbb{C}$$

(see [20, Subsection 2.1.1]). The GFF does not make sense as a random function, but it can be defined as a random generalized function, meaning that we can define its integral against a smooth test function with sufficiently fast decay at ∞. We refer to the expository articles [3, 18, 21] for more on the GFF.

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Recent works have shown that for $\xi > 0$, one can construct a random metric on $\mathbb{C}$ which is heuristically obtained by "weighting lengths of paths by $e^{\xi h}$, and then taking an infimum". The motivation for considering such a metric comes from the theory of Liouville quantum gravity (LQG). The reason for the quotations is that h is a generalized function, not a true function, so $e^{\xi h}$ does not make literal sense. Consequently, to construct this metric one needs to take a limit of a family of approximating metrics called *Liouville first passage percolation* (LFPP), which we discuss just below. The goal of this paper is to prove a quantitative estimate for how close the LFPP metrics are to the limiting metric, which neither implies nor is implied by the convergence. As consequences of this estimate, we deduce several other estimates for LFPP and LQG which are needed in future works.

Let us now discuss the definition of LFPP. For $t > 0$ and $z \in \mathbb{C}$, we define the heat kernel $p_t(z) := \frac{1}{2\pi t} e^{-|z|^2/2t}$ and we denote its convolution with h by

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z-w)dw, \quad \forall\, z \in \mathbb{C}, \tag{1.1}$$

where the integral is interpreted in the sense of distributional pairing. The reason for integrating against $p_{\varepsilon^2/2}(z,w)$ instead of $p_\varepsilon(z,w)$ is so that the variance of $h_\varepsilon^*(z)$ is of order $\log \varepsilon^{-1} + O_\varepsilon(1)$.

For a parameter $\xi > 0$, we define the *ε-Liouville first passage percolation* metric associated with h by

$$D_h^\varepsilon(z,w) := \inf_P \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)| dt, \quad \forall\, z, w \in \mathbb{C}, \tag{1.2}$$

where the infimum is over all the piecewise continuously differentiable paths $P : [0,1] \to \mathbb{C}$ from z to w. We are interested in (subsequential) limits of the re-normalized metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$, where the normalizing constant is defined by

$$\mathfrak{a}_\varepsilon := \text{median of } \inf\left\{ \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)| dt : P \text{ is a left-right crossing of } [0,1]^2 \right\}. \tag{1.3}$$

Here, by a left-right crossing of $[0,1]^2$ we mean a piecewise continuously differentiable path $P : [0,1] \to [0,1]^2$ joining the left and right boundaries of $[0,1]^2$.

The scaling constants $\mathfrak{a}_\varepsilon$ are not known explicitly, but it is shown in [8, Proposition 1.1] that for each $\xi > 0$, there exists a $Q = Q(\xi) > 0$ such that

$$\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)} \quad \text{as } \varepsilon \to 0. \tag{1.4}$$

We call Q the *LFPP distance exponent*. The existence of Q is proven by using a subadditivity argument, so its value is not known except that $Q(1/\sqrt{6}) = 5/\sqrt{6}$ [1)]. However, reasonably good rigorous upper and lower bounds for Q in terms of ξ are available [1,6,16].

LFPP undergoes a phase transition at the critical parameter value

$$\xi_{\text{crit}} := \inf\{\xi > 0 : Q(\xi) = 2\}. \tag{1.5}$$

We do not know ξ_{crit} explicitly, but the bounds from [16, Theorem 2.3] give the approximation $\xi_{\text{crit}} \in [0.4135, 0.4189]$.

Definition 1.1. We refer to LFPP with $\xi < \xi_{\text{crit}}$, $\xi = \xi_{\text{crit}}$ and $\xi > \xi_{\text{crit}}$ as the *subcritical*, *critical* and *supercritical* phases, respectively.

We now briefly discuss what happens in each of the three phases. We refer to the survey article [5] for a more detailed exposition. In the subcritical phase, it was shown by Ding et al. [4] that the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ admit subsequential limits in probability with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$. Moreover, every possible subsequential limit is a metric which induces the same topology on $\mathbb{C}$ as the Euclidean metric. Later, it was shown by Gwynne and Miller [15]

1) The case $\xi = 1/\sqrt{6}$ corresponds to Liouville quantum gravity with $\gamma = \sqrt{8/3}$, and the relation $Q(1/\sqrt{6}) = 5/\sqrt{6}$ is equivalent to the statement that the Hausdorff dimension of $\sqrt{8/3}$-LQG is equal to 4 (see [6] for more details).

(building on [10,13,14]) that the subsequential limit is uniquely characterized by a certain list of axioms, so $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converges as $\varepsilon \to 0$, not just subsequentially.

The limiting metric can be viewed as the distance function associated with γ-*Liouville quantum gravity* (LQG) for an appropriate value of $\gamma = \gamma(\xi) \in (0,2)$. In other words, we interpret the limit of subcritical LFPP as the Riemannian distance function for the Riemannian metric tensor $\mathrm{e}^{\gamma h(x,y)}(dx^2 + dy^2)$, where $dx^2 + dy^2$ is the Euclidean metric tensor. The relationship between γ and ξ is given by either of two equivalent, but non-explicit, formulas:

$$Q(\xi) = \frac{2}{\gamma} + \frac{\gamma}{2} \quad \text{or, equivalently,} \quad \gamma = \xi d(\xi), \tag{1.6}$$

where $d(\xi) > 2$ is the Hausdorff dimension of $\mathbb{C}$, equipped with the limiting metric. Neither of the formulas (1.6) gives an explicit relationship between ξ and γ since neither $Q(\xi)$ nor $d(\xi)$ is known explicitly (see [6] for further discussion).

In the supercritical and critical phases, we showed in [8] that the metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ admit non-trivial subsequential limits with respect to the topology on lower semicontinuous functions on $\mathbb{C} \times \mathbb{C}$, which we recall in Definition 1.2 below. Every subsequential limit is a metric (not just a pseudometric). In [9], we showed that the subsequential limit is unique, using some of the results of this paper.

In the supercritical case, if D_h is a subsequential limit of LFPP, then D_h does not induce the Euclidean topology on $\mathbb{C}$. Rather, there is an uncountable, dense, Lebesgue measure zero set of *singular points* $z \in \mathbb{C}$ such that

$$D_h(z,w) = \infty, \quad \forall\, w \in \mathbb{C} \setminus \{z\}. \tag{1.7}$$

Roughly speaking, the singular points correspond to the points $z \in \mathbb{C}$ for which (see [17, Proposition 1.11])

$$\limsup_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log \varepsilon^{-1}} > Q,$$

where $h_\varepsilon(z)$ is the average of h over the circle of the radius ε centered at z.

The (subsequential) limits of supercritical LFPP are related to Liouville quantum gravity with *matter central charge* $\mathbf{c}_{\mathrm{M}} \in (1,25)$. LQG with $\gamma \in (0,2)$ corresponds to $\mathbf{c}_{\mathrm{M}} = 25 - 6(2/\gamma + \gamma/2)^2 \in (-\infty, 1)$. The case $\mathbf{c}_{\mathrm{M}} \in (1,25)$ is much less understood, even from a physical perspective (see [8,12] for further discussion).

In the critical case $\xi = \xi_{\mathrm{crit}}$, there are no singular points and the subsequential limiting metrics induce the Euclidean topology. We proved this in [7], using the results of the present paper. This case corresponds to γ-LQG with $\gamma = 2$ or equivalently $\mathbf{c}_{\mathrm{M}} = 1$.

The goal of this paper is to prove, for each $\xi > 0$, up-to-constants bounds comparing $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ to any possible subsequential limit of $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{\varepsilon>0}$ (see Theorem 1.8). This comparison holds even at nearly microscopic scales, so it is not implied by the convergence of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$. As consequences of our bounds, we deduce estimates for the LFPP scaling constants $\{\mathfrak{a}_\varepsilon\}_{\varepsilon>0}$ which are new even in the subcritical case (see Theorem 1.11). We also prove that the scaling constants $\{\mathfrak{c}_r\}_{r>0}$ for the subsequential limiting metric, as defined in Axiom (V) of Definition 1.5 below, can be taken to be equal to $r^{\xi Q}$ (see Theorem 1.9). In the subcritical case, this was previously proven in [15] as a consequence of the uniqueness of the subsequential limit. This paper gives the first proof that one can take $\mathfrak{c}_r = r^{\xi Q}$ in the critical and supercritical cases. This fact helps to simplify the proof of the uniqueness of the critical and supercritical LQG metrics in [9].

1.2 Weak LQG metrics

The subsequential limits of LFPP satisfy a list of axioms which define a *weak LQG metric*. In this subsection, we state the axiomatic definition of a weak LQG metric from [17]. We first define the topology on the space of metrics that we will work with.

Definition 1.2. Let $X \subset \mathbb{C}$. A function $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *lower semicontinuous* if whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z,w)$, we have $f(z,w) \leqslant \liminf_{n\to\infty} f(z_n, w_n)$. The *topology on lower semicontinuous functions* is the topology whereby a sequence of such functions $\{f_n\}_{n\in\mathbb{N}}$ converges to another such function f if and only if

(i) whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z, w)$, we have $f(z, w) \leqslant \liminf_{n\to\infty} f_n(z_n, w_n)$;

(ii) for each $(z, w) \in X \times X$, there exists a sequence $(z_n, w_n) \to (z, w)$ such that $f_n(z_n, w_n) \to f(z, w)$.

It follows from [2, Lemma 1.5] that the topology of Definition 1.2 is metrizable (see [8, Subsection 1.2]). Furthermore, [2, Theorem 1(a)] shows that this metric can be taken to be separable.

Definition 1.3. Let (X, d) be a metric space, with d allowed to take on infinite values.

• For a curve $P : [a, b] \to X$, the *d-length* of P is defined by

$$\text{len}(P; d) := \sup_T \sum_{i=1}^{\#T} d(P(t_i), P(t_{i-1})),$$

where the supremum is over all the partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$. Note that the d-length of a curve may be infinite.

• We say that (X, d) is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of d-length at most $d(x, y) + \varepsilon$ from x to y. A curve from x to y of d-length *exactly* $d(x, y)$, if such a curve exists, is called a *geodesic*.

• For $Y \subset X$, the *internal metric of d on Y* is defined by

$$d(x, y; Y) := \inf_{P\subset Y} \text{len}(P; d), \quad \forall\, x, y \in Y, \tag{1.8}$$

where the infimum is over all the paths P in Y from x to y. Note that $d(\cdot, \cdot; Y)$ is a metric on Y, except that it is allowed to take infinite values.

• If $X \subset \mathbb{C}$, we say that d is a *lower semicontinuous metric* if the function $(x, y) \to d(x, y)$ is lower semicontinuous with respect to the Euclidean topology. We equip the set of lower semicontinuous metrics on X with the topology on lower semicontinuous functions on $X \times X$, as in Definition 1.2, and the associated Borel σ-algebra.

Definition 1.4. Let d be a length metric on $\mathbb{C}$. For a region $A \subset \mathbb{C}$ with the topology of a Euclidean annulus, we write d (across A) for the d-distances between the inner and outer boundaries of A and d (around A) for the infimum of the d-lengths of paths in A which disconnect the inner and outer boundaries of A.

Distances around and across Euclidean annuli play a similar role to "hard crossings" and "easy crossings" of 2×1 rectangles in percolation theory. One can get a lower bound for the d-length of a path in terms of the d-distances across the annuli that it crosses. On the other hand, one can "string together" paths around Euclidean annuli to produce longer paths.

The following is (almost) a re-statement of [17, Definition 1.6].

Definition 1.5 (Weak LQG metric)**.** Let $\mathcal{D}'$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For $\xi > 0$, a *weak LQG metric with the parameter ξ* is a measurable function $h \mapsto D_h$ from $\mathcal{D}'$ to the space of lower semicontinuous metrics on $\mathbb{C}$ with the following properties. Let h be a *GFF plus a continuous function* on $\mathbb{C}$, i.e., h is a random distribution on $\mathbb{C}$ which can be coupled with a random continuous function f in such a way that $h - f$ has the law of the whole-plane GFF. Then the associated metric D_h satisfies the following axioms:

(I) (Length space) Almost surely, $(\mathbb{C}, D_h)$ is a length space.

(II) (Locality) Let $U \subset \mathbb{C}$ be a deterministic open set. The D_h-internal metric $D_h(\cdot, \cdot; U)$ is a.s. given by a measurable function of $h|_U$.

(III) (Weyl scaling) For a continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(\mathrm{e}^{\xi f} \cdot D_h)(z, w) := \inf_{P:z\to w} \int_0^{\text{len}(P;D_h)} \mathrm{e}^{\xi f(P(t))} dt, \quad \forall\, z, w \in \mathbb{C}, \tag{1.9}$$

where the infimum is over all the D_h-continuous paths from z to w in $\mathbb{C}$ parametrized by the D_h-length. Then a.s. $\mathrm{e}^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

(IV) (Translation invariance) For each deterministic point $z \in \mathbb{C}$, a.s. $D_{h(\cdot+z)} = D_h(\cdot+z, \cdot+z)$.

(V) (Tightness across scales) Suppose that h is a whole-plane GFF and let $\{h_r(z)\}_{r>0, z\in\mathbb{C}}$ be its circle average process. There are constants $\{\mathfrak{c}_r\}_{r>0}$ such that the following is true. Let $A \subset \mathbb{C}$ be a deterministic Euclidean annulus. In the notation of Definition 1.4, the random variables

$$\mathfrak{c}_r^{-1} \mathrm{e}^{-\xi h_r(0)} D_h \text{ (across } rA) \quad \text{and} \quad \mathfrak{c}_r^{-1} \mathrm{e}^{-\xi h_r(0)} D_h \text{ (around } rA)$$

and the reciprocals of these random variables for $r > 0$ are tight.

For $\xi < \xi_{\mathrm{crit}}$, it is shown in [15] (as a consequence of the uniqueness of weak LQG metrics) that every weak LQG metric satisfies an exact spatial scaling property which is stronger than Axiom (V). More precisely, for every $r > 0$, a.s.

$$D_{h(r\cdot)+Q\log r}(z, w) = D_h(rz, rw), \quad \forall\, z, w \in \mathbb{C}. \tag{1.10}$$

A metric which satisfies Axioms (I)–(IV) together with (1.10) is called a *strong LQG metric*.

In the case $\xi \geqslant \xi_{\mathrm{crit}}$, we do not yet know that every weak LQG metric satisfies (1.10). We can think of Axiom (V) as a substitute for (1.10). It allows us to get estimates which are uniform across different Euclidean scales, even though we do not have an exact scale invariance property (see [10,15,17] for further discussion of this point).

Definition 1.5 is slightly more general than the definition of a weak LQG metric used in other works [10, 15, 17]. The reason is that the aforementioned papers require a (rather weak) *a priori* bound for the scaling constants $\mathfrak{c}_r$ from Axiom (V). It follows from Theorem 1.9 below that Definition 1.5 is equivalent to the definition in [17], so the *a priori* bounds for $\mathfrak{c}_r$ are unnecessary. We emphasize that our proof of Theorem 1.9 does not use the results of [17].

Remark 1.6. The scaling constants $\{\mathfrak{c}_r\}_{r>0}$ from Axiom (V) are not uniquely determined by the law of D_h. If $\{\widetilde{\mathfrak{c}}_r\}_{r>0}$ is another sequence of non-negative real numbers and there is a constant $C > 0$ such that $C^{-1}\mathfrak{c}_r \leqslant \widetilde{\mathfrak{c}}_r \leqslant C\mathfrak{c}_r$ for each $r > 0$, then Axiom (V) holds with $\widetilde{\mathfrak{c}}_r$ instead of $\mathfrak{c}_r$. Conversely, if m_r defines the median of the random variable $\mathrm{e}^{-\xi h_r(0)} D_h$ (around $\mathbb{A}_{r,2r}(0)$), then Axiom (V) implies that there is a constant $C > 0$ depending only on the law of D_h such that $C^{-1}\mathfrak{c}_r \leqslant m_r \leqslant C\mathfrak{c}_r$. In particular, any two possible choices for $\{\mathfrak{c}_r\}_{r>0}$ are comparable up to constant multiplicative factors.

The following theorem is proven as [17, Theorem 1.7], building on the tightness result from [8].

Theorem 1.7 (See [17])**.** *Let $\xi > 0$. For every sequence of ε's tending to zero, there is a weak LQG metric D with the parameter ξ and a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ for which the following is true. Let h be a whole-plane GFF, or more generally a whole-plane GFF plus a bounded continuous function. Then the re-scaled LFPP metrics $\mathfrak{a}_{\varepsilon_n}^{-1} D_h^{\varepsilon_n}$, as defined in* (1.2) *and* (1.3)*, converge in probability to D_h with respect to the metric on lower semicontinuous functions on $\mathbb{C} \times \mathbb{C}$.*

Theorem 1.7 implies in particular that for each $\xi > 0$, there exists a weak LQG metric with the parameter ξ. In the case $\xi < \xi_{\mathrm{crit}}$, the convergence occurs with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}\times\mathbb{C}$ and the subsequential limit has been shown to be unique [4,15]. For $\xi \geqslant \xi_{\mathrm{crit}}$, the subsequential limit is shown to be unique in [9], using some of the results of the present paper.

1.3 Main results

Throughout this subsection, we fix $\xi > 0$, and let h be the whole-plane GFF and D_h be a weak LQG metric with the parameter ξ. We recall the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ with the parameter ξ from (1.2) and (1.3).

Our main result says that, roughly speaking, D_h is bi-Lipschitz equivalent to the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$. We cannot say that these metrics are literally bi-Lipschitz equivalent since D_h has a fractal structure whereas $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ is smooth. So it is not possible to get an up-to-constants comparison of these metrics at Euclidean scales smaller than ε. We get around this problem by looking at distances between Euclidean balls of the radius slightly larger than ε.

Theorem 1.8. *For each $\zeta \in (0,1)$, there exists a $C_0 > 0$ depending only on ζ and the law of D_h such that the following is true. Let $U \subset \mathbb{C}$ be a deterministic, connected, bounded open set and recall the notation for internal metrics on U from Definition* 1.3. *With probability tending to* 1 *as $\varepsilon \to 0$,*

$$\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_0 D_h(z,w;U), \quad \forall\, z,w \in U \tag{1.11}$$

and

$$D_h(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_0 \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(z,w;U), \quad \forall\, z,w \in U. \tag{1.12}$$

Theorem 1.8 is a new result even in the subcritical case, where we already know that $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \to D_h$ in probability with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$. Indeed, the convergence $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \to D_h$ only allows us to estimate the ratio of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(z,w)$ and $D_h(z,w)$ when $|z-w|$ is of constant order, whereas Theorem 1.8 gives a non-trivial estimate when $|z-w|$ is as small as $3\varepsilon^{1-\zeta}$.

In the critical and supercritical cases, we as of yet only have subsequential limits for $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$. In these cases, an important consequence of Theorem 1.8 is the following. If $\varepsilon_k \to 0$ is a sequence along which $\mathfrak{a}_{\varepsilon_k}^{-1} D_h^{\varepsilon_k} \to D_h$ in probability, then D_h is a weak LQG metric [17]. Therefore, Theorem 1.8 gives an up-to-constants comparison between $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ and D_h even if ε is not part of the sequence $\{\varepsilon_k\}$.

We now record several estimates which are consequences of Theorem 1.8. Our first estimate gives up-to-constants bounds for the scaling constants in Axiom (V).

Theorem 1.9. *Let $\{\mathfrak{c}_r\}_{r>0}$ be the scaling constants from Axiom* (V)*. There is a constant $C_1 > 0$ depending only on the law of D_h such that for each $r > 0$,*

$$C_1{}^{-1} r^{\xi Q} \leqslant \mathfrak{c}_r \leqslant C_1 r^{\xi Q}. \tag{1.13}$$

Due to Remark 1.6, Theorem 1.9 is equivalent to the statement that one can take $\mathfrak{c}_r = r^{\xi Q}$ in Axiom (V). In the subcritical case $\xi < \xi_{\mathrm{crit}}$, the exact scaling relation (1.10), which was proven in [15], already implies that $\mathfrak{c}_r = r^{\xi Q}$. However, this fact was deduced as a consequence of the uniqueness of weak LQG metrics, whereas the present paper gives a much simpler and more direct proof.

In the critical and supercritical cases, Theorem 1.9 is a new result. This result was used in [7] to show that every subsequential limit of critical LFPP induces the Euclidean topology. It was also used in the proof of the uniqueness of weak LQG metrics in the critical and supercritical cases [9].

Another consequence of Theorem 1.8 is the fact that any two weak LQG metrics are bi-Lipschitz equivalent.

Theorem 1.10. *Let D_h and $\widetilde{D}_h$ be two weak LQG metrics with the parameter $\xi > 0$. There is a deterministic constant $C_2 > 1$ such that a.s.*

$$C_2{}^{-1} D_h(z,w) \leqslant \widetilde{D}_h(z,w) \leqslant C_2 D_h(z,w), \quad \forall\, z,w \in \mathbb{C}. \tag{1.14}$$

Previously, Theorem 1.10 was established for all $\xi > 0$, under the additional hypothesis that the constants $\mathfrak{c}_r$ for the two metrics D_h and $\widetilde{D}_h$ are the same (see [13, Theorem 1.6] and [17, Lemma 2.23]). In the subcritical case, this was an important input of the proof of the uniqueness in [15]. Theorem 1.10 was used in the proof of the uniqueness of the critical and supercritical LQG metrics in [9]. Essentially, the proof of the uniqueness proceeded by showing that the optimal upper and lower bi-Lipschitz constants coincide.

Our last estimate gives bounds for the LFPP scaling constants $\mathfrak{a}_\varepsilon$. To put our result in context, we note that the previous works have shown that $\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)}$ (see [15, Theorem 1.5] for the subcritical case and [8, Proposition 1.9] for the supercritical and critical cases). In the subcritical case, the convergence of LFPP implies that $\mathfrak{a}_\varepsilon = \phi(\varepsilon)\varepsilon^{1-\xi Q}$, where ϕ is slowly varying [15, Corollary 1.11], but one does not get any better bound than $\varepsilon^{1-\xi Q + o_\varepsilon(1)}$ for a fixed value of ε. We improve the $\varepsilon^{o_\varepsilon(1)}$ error to a polylogarithmic error.

Theorem 1.11. *Let $\{\mathfrak{a}_\varepsilon\}_{\varepsilon>0}$ be the LFPP scaling constants from* (1.3)*. There are constants $C_3 > 0$ and $b > 0$ depending only on ξ such that for each $\varepsilon \in (0, 1/2]$,*

$$C_3{}^{-1}(\log \varepsilon^{-1})^{-b}\varepsilon^{1-\xi Q} \leqslant \mathfrak{a}_\varepsilon \leqslant C_3(\log \varepsilon^{-1})^{b}\varepsilon^{1-\xi Q}. \tag{1.15}$$

We expect that $\mathfrak{a}_\varepsilon = (c + o_\varepsilon(1))\varepsilon^{1-\xi Q}$ for some $c > 0$, but the techniques of the present paper are not strong enough to give this (see also Remark 3.10).

2 Preliminaries

2.1 Notational conventions

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b$, we define the discrete interval $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

If $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (resp. $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (resp. tends to zero) as $\varepsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

Let $\{E^\varepsilon\}_{\varepsilon>0}$ be a one-parameter family of events. We say that E^ε occurs with *polynomially high probability* as $\varepsilon \to 0$ if there is a $p > 0$ (independent of ε and possibly of other parameters of interest) such that $\mathrm{P}[E^\varepsilon] \geqslant 1 - O_\varepsilon(\varepsilon^p)$.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the open Euclidean ball of the radius r centered at z. More generally, for $X \subset \mathbb{C}$ we write $B_r(X) = \bigcup_{z\in X} B_r(z)$. We also define the open Euclidean annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}, \quad \forall\, 0 < r_r < r_2 < \infty. \tag{2.1}$$

2.2 Independence across concentric annuli

As is the case for many papers involving LQG distances, a key tool in our proof is the following estimate, which is a consequence of the fact that the restrictions of the GFF h to disjoint concentric annuli are nearly independent (see [13, Lemma 3.1] for a proof of a slightly more general result).

Lemma 2.1 (See [13])**.** *Fix $0 < \mu_1 < \mu_2 < 1$. Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{k+1}/r_k \leqslant \mu_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k\in\mathbb{N}}$ be events such that $E_{r_k} \in \sigma((h - h_{r_k}(0))\,|_{\mathbb{A}_{\mu_1 r_k,\mu_2 r_k}(0)})$ for each $k \in \mathbb{N}$ (where we use the notation for Euclidean annuli from Subsection* 2.1*). For each $a > 0$, there exist $p = p(a, \mu_1, \mu_2) \in (0, 1)$ and $c = c(a, \mu_1, \mu_2) > 0$ such that if*

$$\mathrm{P}[E_{r_k}] \geqslant p, \quad \forall\, k \in \mathbb{N}, \tag{2.2}$$

then

$$\mathrm{P}[\exists\, k \in [1, K]_{\mathbb{Z}} \textit{ such that } E_{r_k} \textit{ occurs}] \geqslant 1 - c\mathrm{e}^{-aK}, \quad \forall\, K \in \mathbb{N}. \tag{2.3}$$

2.3 Localized approximation of LFPP

A somewhat annoying feature of our definition of LFPP is that the mollified process $\{h^*_\varepsilon\}_{\varepsilon>0}$ does not depend locally on h. This is because the heat kernel $p_{\varepsilon^2/2}(z)$ is non-zero on all of $\mathbb{C}$. In [10, Subsection 2.1], Dubédat et al. got around this difficulty by introducing a truncated version $\widehat{h}^*_\varepsilon$ of h^*_ε which depends locally on h. They then showed that LFPP defined by using $\widehat{h}^*_\varepsilon$ instead of h^*_ε itself is a good approximate for D^ε_h.

The truncated version of LFPP used in [10] is not quite good enough for our purposes since with the definitions used there, $\widehat{h}^*_\varepsilon(z)$ depends on $h\,|_{B_{\varepsilon^{1/2}}(z)}$. We need a range of dependence which is smaller than $\varepsilon^{1-\zeta}$ for every $\zeta > 0$. So in this subsection, we introduce a truncated version of LFPP with a smaller range of dependence. We follow closely the exposition in [10, Subsection 2.1], but some of our estimates are sharper.

Fix $q > 0$. For $\varepsilon > 0$, let $\psi_\varepsilon : \mathbb{C} \to [0,1]$ be a deterministic, smooth, radially symmetric bump function which is identically equal to 1 on $B_{\varepsilon(\log \varepsilon^{-1})^q/2}(0)$ and vanishes outside $B_{\varepsilon(\log \varepsilon^{-1})^q}(0)$. We can choose ψ_ε in such a way that $(z,\varepsilon) \mapsto \psi_\varepsilon(z)$ is smooth. Recalling that $p_s(z)$ denotes the heat kernel, we define

$$\widehat{h}^*_\varepsilon(z) := Z_\varepsilon^{-1} \int_{\mathbb{C}} \psi_\varepsilon(z-w) h(w) p_{\varepsilon^2/2}(z-w) dw, \tag{2.4}$$

where the integral interpreted in the sense of the distributional pairing and normalizing constant is given by

$$Z_\varepsilon := \int_{\mathbb{C}} \psi_\varepsilon(w) p_{\varepsilon^2/2}(w) dw. \tag{2.5}$$

Let us now note some properties of $\widehat{h}^*_\varepsilon$.

Since ψ_ε vanishes outside $B_{\varepsilon(\log \varepsilon^{-1})^q}(0)$, we see that $\widehat{h}^*_\varepsilon(z)$ is a.s. determined by $h|_{\varepsilon B_{(\log \varepsilon^{-1})^q}(z)}$. It is easy to see that $\widehat{h}^*_\varepsilon$ a.s. admits a continuous modification (see Lemma 2.2 below). We henceforth assume that $\widehat{h}^*_\varepsilon$ is replaced by such a modification.

If $c \in \mathbb{R}$ and we define $(\widehat{h+c})^*_\varepsilon$ as in (2.4) but with $h+c$ in place of h, then

$$(\widehat{h+c})^*_\varepsilon = \widehat{h}^*_\varepsilon + c. \tag{2.6}$$

This is because of the normalization by Z_ε^{-1} in (2.4).

As in (2.11), we define the localized LFPP metric

$$\widehat{D}^\varepsilon_h(z,w) := \inf_{P: z \to w} \int_0^1 \mathrm{e}^{\xi \widehat{h}^*_\varepsilon(P(t))} |P'(t)| dt, \tag{2.7}$$

where the infimum is over all the piecewise continuously differentiable paths from z to w. By the definition of $\widehat{h}^*_\varepsilon$,

$$\text{for any open } U \subset \mathbb{C}, \text{ the internal metric } \widehat{D}^\varepsilon_h(\cdot,\cdot;U) \text{ is a.s. determined by } h|_{B_{\varepsilon(\log \varepsilon^{-1})^q}(U)}. \tag{2.8}$$

Lemma 2.2. *Almost surely, $(z,\varepsilon) \mapsto \widehat{h}^*_\varepsilon(z)$ is continuous. Furthermore, for each bounded open set $U \subset \mathbb{C}$, a.s.*

$$\lim_{\varepsilon \to 0} \sup_{z \in \overline{U}} |h^*_\varepsilon(z) - \widehat{h}^*_\varepsilon(z)| = 0. \tag{2.9}$$

In particular, a.s.

$$\lim_{\varepsilon \to 0} \frac{\widehat{D}^\varepsilon_h(z,w;U)}{D^\varepsilon_h(z,w;U)} = 1, \quad \text{uniformly over all } z,w \in U \text{ with } z \neq w. \tag{2.10}$$

For the proof of Lemma 2.2, we re-use the following estimate, which is [10, Lemma 2.2].

Lemma 2.3. *For each $R > 0$ and $\zeta > 0$, a.s.*

$$\sup_{z \in B_R(0)} \sup_{r>0} \frac{|h_r(z)|}{\max\{\log(1/r), |\log r|^{1/2+\zeta}, 1\}} < \infty. \tag{2.11}$$

Proof of Lemma 2.2. **Step 1** (Polar coordinate representation)**.** The functions $w \mapsto \psi_\varepsilon(z-w)$ and $w \mapsto p_{\varepsilon^2/2}(z,w)$ are each radially symmetric about z, i.e., they depend only on $|z-w|$. Using the circle average process $\{h_r\}_{r>0}$, we may therefore write in polar coordinates

$$h^*_\varepsilon(z) = \frac{2}{\varepsilon^2} \int_0^\infty r h_r(z) \mathrm{e}^{-r^2/\varepsilon^2} dr \quad \text{and} \quad Z_\varepsilon \widehat{h}^*_\varepsilon(z) = \frac{2}{\varepsilon^2} \int_0^{\varepsilon(\log \varepsilon^{-1})^q} r h_r(z) \psi_\varepsilon(r) \mathrm{e}^{-r^2/\varepsilon^2} dr. \tag{2.12}$$

From this representation and the continuity of the circle average process [11, Proposition 3.1], we infer that $(z,\varepsilon) \mapsto \widehat{h}^*_\varepsilon(z)$ a.s. admits a continuous modification. The rest of the proof is an elementary, but somewhat tedious, calculation using (2.12) and Lemma 2.3.

Step 2 (Comparing h^*_ε with $Z_\varepsilon \widehat{h}^*_\varepsilon$)**.** Since $\psi_\varepsilon \equiv 1$ on $B_{\varepsilon(\log \varepsilon^{-1})^q/2}(z)$ and ψ_ε takes values in $[0,1]$,

$$|h^*_\varepsilon(z) - Z_\varepsilon \widehat{h}^*_\varepsilon(z)| \leqslant \frac{2}{\varepsilon^2}\int_{\varepsilon(\log \varepsilon^{-1})^q/2}^{\infty} r|h_r(z)|\mathrm{e}^{-r^2/\varepsilon^2}dr. \tag{2.13}$$

By Lemma 2.3 (applied with $\zeta = 1/2$, say), a.s. there is a random $C = C(U) > 0$ such that $|h_r(z)| \leqslant C\max\{\log(1/r), \log r, 1\}$ for each $z \in U$ and $r > 0$. Plugging this into (2.13) shows that a.s.

$$\sup_{z\in U}|h^*_\varepsilon(z) - Z_\varepsilon \widehat{h}^*_\varepsilon(z)| \leqslant \frac{2C}{\varepsilon^2}\int_{\varepsilon(\log \varepsilon^{-1})^q/2}^{\infty} r\max\{\log(1/r), \log r, 1\}\mathrm{e}^{-r^2/\varepsilon^2}dr. \tag{2.14}$$

We claim that the right-hand side of (2.14) tends to zero as $\varepsilon \to 0$, which implies that a.s. $\sup_{z\in U}|h^*_\varepsilon(z) - Z_\varepsilon \widehat{h}^*_\varepsilon(z)| \to 0$. To prove the claim, we first substitute $r = \varepsilon u/\sqrt{2}$ and $dr = (\varepsilon/\sqrt{2})du$ to get

$$\begin{aligned}&\frac{2}{\varepsilon^2}\int_{\varepsilon(\log \varepsilon^{-1})^q/2}^{\infty} r\max\{\log(1/r), \log r, 1\}\mathrm{e}^{-r^2/\varepsilon^2}dr\\ &\quad = \int_{(\log \varepsilon^{-1})^q/\sqrt{2}}^{\infty} u\max\left\{\log\frac{\sqrt{2}}{\varepsilon u}, \log\frac{\varepsilon u}{\sqrt{2}}, 1\right\}\mathrm{e}^{-u^2/2}du.\end{aligned} \tag{2.15}$$

We now split the right-hand side of (2.15) into the integrals over $[\sqrt{2}/\varepsilon, \infty)$ and over $[(\log \varepsilon^{-1})^q/\sqrt{2}, \sqrt{2}/\varepsilon]$. The integral over $[\sqrt{2}/\varepsilon, \infty)$ is bounded above by

$$\int_{\sqrt{2}/\varepsilon}^{\infty}[\log u + 1]u\mathrm{e}^{-u^2/2}du, \tag{2.16}$$

which clearly tends to zero as $\varepsilon \to 0$. The integral over $[(\log \varepsilon^{-1})^q/\sqrt{2}, \sqrt{2}/\varepsilon]$ is bounded above by

$$\begin{aligned}&\left[\log\left(\frac{2}{\varepsilon(\log \varepsilon^{-1})^q}\right) + 1\right]\int_{(\log \varepsilon^{-1})^q/\sqrt{2}}^{\sqrt{2}/\varepsilon} u\mathrm{e}^{-u^2/2}du\\ &\quad \leqslant \left[\log\left(\frac{2}{\varepsilon(\log \varepsilon^{-1})^q}\right) + 1\right]\int_{(\log \varepsilon^{-1})^q/\sqrt{2}}^{\infty} u\mathrm{e}^{-u^2/2}du\\ &\quad = \left[\log\left(\frac{2}{\varepsilon(\log \varepsilon^{-1})^q}\right) + 1\right]\mathrm{e}^{-(\log \varepsilon^{-1})^{2q}/4},\end{aligned} \tag{2.17}$$

which also tends to zero as $\varepsilon \to 0$. Hence, the right-hand side of (2.14) tends to zero.

Step 3 (Z_ε close to 1)**.** To eliminate the normalizing factor Z_ε in (2.14), we first note that $\int_{\mathbb{C}} p_{\varepsilon^2/2}(w)dw = 1$. From this and (2.5), we have

$$\begin{aligned}0 \leqslant 1 - Z_\varepsilon &\leqslant \frac{2}{\varepsilon^2}\int_0^{\infty} r(1-\psi_\varepsilon(r))\mathrm{e}^{-r^2/\varepsilon^2}dr\\ &\leqslant \frac{2}{\varepsilon^2}\int_{\varepsilon(\log \varepsilon^{-1})^q/2}^{\infty} r\mathrm{e}^{-r^2/\varepsilon^2}dr\\ &= \int_{(\log \varepsilon^{-1})^q/\sqrt{2}}^{\infty} u\mathrm{e}^{-u^2/2}du \ \text{(by substituting } r = \varepsilon u/\sqrt{2})\\ &= \mathrm{e}^{-(\log \varepsilon^{-1})^{2q}/4}.\end{aligned} \tag{2.18}$$

Furthermore, using Lemma 2.3 and (2.12), we get a.s.

$$\begin{aligned}\sup_{z\in U}|Z_\varepsilon \widehat{h}^*_\varepsilon(z)| &\leqslant \frac{2}{\varepsilon^2}\int_0^{\varepsilon(\log \varepsilon^{-1})^q} r|h_r(z)|\mathrm{e}^{-r^2/\varepsilon^2}dr \ \text{(by (2.12) and the fact that } |\psi_\varepsilon(r)| \leqslant 1)\\ &= \int_0^{(\log \varepsilon^{-1})^q/\sqrt{2}} |h_{\varepsilon u/\sqrt{2}}(z)|u\mathrm{e}^{-u^2/2}du \ \text{(by substituting } r = \varepsilon u/\sqrt{2})\end{aligned}$$

$$\begin{aligned}
&\preceq \int_0^{(\log \varepsilon^{-1})^q/\sqrt{2}} \left[\log \frac{\sqrt{2}}{\varepsilon} + \log u^{-1}\right] u e^{-u^2/2} du \ \text{ (by Lemma 2.3)}\\
&\preceq \left[\log \frac{\sqrt{2}}{\varepsilon}\right] + \int_0^{(\log \varepsilon^{-1})^q/\sqrt{2}} [\log u^{-1}] u e^{-u^2/2} du\\
&\preceq \left[\log \frac{\sqrt{2}}{\varepsilon}\right] + (\log \varepsilon^{-1})^q \ \text{ (since } (\log u^{-1}) u e^{-u^2/2} \leqslant 1 \text{ on } [0,1])\\
&\preceq (\log \varepsilon^{-1})^{\max\{q,1\}}, \qquad (2.19)
\end{aligned}$$

where $\preceq$ denotes the inequality up to a possibly random constant factor which does not depend on ε.

By the triangle inequality,

$$\sup_{z\in U} |h_\varepsilon^*(z) - \widehat{h}_\varepsilon^*(z)| \leqslant \sup_{z\in U} |h_\varepsilon^*(z) - Z_\varepsilon \widehat{h}_\varepsilon^*(z)| + (1 - Z_\varepsilon) Z_\varepsilon^{-1} \sup_{z\in U} |Z_\varepsilon \widehat{h}_\varepsilon^*(z)|. \qquad (2.20)$$

By (2.14), the first term on the right-hand side of (2.20) tends to zero a.s. as $\varepsilon \to 0$. By (2.18) and (2.19), the second term tends to zero a.s. as $\varepsilon \to 0$ as well. We thus obtain (2.9). The relation (2.10) is immediate from (2.9) and the definitions of D_h^ε and $\widehat{D}_h^\varepsilon$. □

3 Proofs

Throughout this section, we fix $\xi > 0$, and let h be a whole-plane GFF and D_h be a weak LQG metric as in Definition 1.5. We prove Theorem 1.8 in Subsections 3.2–3.4, and then deduce our other main theorems from Theorem 1.8 in Subsection 3.5.

3.1 Outline of the proof

The idea of the proof of Theorem 1.8 is as follows. In Subsection 3.2, we use a basic scaling calculation together with the tightness of LFPP (see [4, 8]) and the tightness across scales condition for D_h (see Axiom (V)) to get the following. There is a constant $C > 0$ such that for each $r \in [\varepsilon, 1]$ and each $z \in \mathbb{C}$, it holds with high probability that

$$\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \text{ (around } \mathbb{A}_{2r,3r}(z)) \leqslant C \frac{r \mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} D_h \text{ (across } \mathbb{A}_{r,2r}(z)), \qquad (3.1)$$

where we use the notation for distances across and around annuli from Definition 1.4. Moreover, one has an analogous inequality with the roles of D_h and $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ interchanged. Actually, for technical reasons we mostly work with the localized LFPP metric $\widehat{D}_h^\varepsilon$ from Subsection 2.3 instead of D_h^ε itself.

In Subsection 3.3, we restrict attention to values of r in the set

$$\mathcal{R}^\varepsilon := \left\{10^{-j} \varepsilon^{1-\zeta} : j = 1, \ldots, \left\lfloor \frac{\zeta}{2} \log_{10} \varepsilon^{-1} \right\rfloor - 11\right\} \subset [\varepsilon, \varepsilon^{1-\zeta}]. \qquad (3.2)$$

We use a multi-scale argument based on Lemma 2.1 to say that if $\widetilde{\mathcal{R}}^\varepsilon \subset \mathcal{R}^\varepsilon$ is a subset with $\#\widetilde{\mathcal{R}}^\varepsilon \geqslant \#\mathcal{R}^\varepsilon/100$, then the following is true. For each open set $U \subset \mathbb{C}$, it holds with high probability that we can cover U by balls $B_r(z)$ for $z \in U$ and $r \in \widetilde{\mathcal{R}}^\varepsilon$ for which the event in (3.1) occurs. By stringing together paths around annuli of the form $\mathbb{A}_{2r,3r}(z)$ whose D_h^ε-lengths are under control, we then deduce that with high probability,

$$\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C \left(\max_{r\in\widetilde{\mathcal{R}}^\varepsilon} \frac{r \mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon}\right) D_h(z, w; U), \quad \forall\, z, w \in U \qquad (3.3)$$

for a possibly larger constant C. One also has an analogous inequality with the roles of D_h and $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ interchanged. The argument leading to (3.3) is similar to the proof of the bi-Lipschitz equivalence of two metrics coupled with the GFF in [13, Section 4].

The inequality (3.3) implies Theorem 1.8 if we have up-to-constants bounds for the ratios $\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon}$ for $r \in \widetilde{\mathcal{R}}^\varepsilon$. In Subsection 3.4, we obtain such up-to-constants bounds via the following bootstrap argument based on (3.3). If we fix distinct points $z, w \in U$, then $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(z,w)$ and $D_h(z,w)$ are each typically of constant order. The inequality (3.3) therefore implies that $\max_{r\in\widetilde{\mathcal{R}}^\varepsilon} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon} \geqslant A^{-1}$ for some constant A, which does not depend on r or ε. Using the analog of (3.3) with the roles of D_h and $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ interchanged and possibly increasing A, we also see that $\min_{r\in\widetilde{\mathcal{R}}^\varepsilon} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon} \leqslant A$. Since $\widetilde{\mathcal{R}}^\varepsilon$ is an arbitrary subset of $\mathcal{R}^\varepsilon$ of cardinality at least $\#\mathcal{R}^\varepsilon/100$, this implies that all but $(2/100)\#\mathcal{R}^\varepsilon$ values of $r \in \mathcal{R}^\varepsilon$ are "good" in the sense that $\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon} \in [A^{-1}, A]$. We then obtain Theorem 1.8 by applying (3.3) (and its analog with $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ and D_h interchanged) with $\widetilde{\mathcal{R}}^\varepsilon$ equal to the set of "good" values of $r \in \mathcal{R}^\varepsilon$.

3.2 Event for LFPP and LQG distances

Fix $q > 0$ and for $\varepsilon > 0$, let $\widehat{D}_h^\varepsilon$ be the localized LFPP metric of Subsection 2.3. For $z \in \mathbb{C}$, $\varepsilon \in (0,1)$, $r \in [\varepsilon, 1]$ and $C > 0$, let $E_r^\varepsilon(z;C)$ be the event that the following are true:

$$\begin{aligned}
&D_h \text{ (across } \mathbb{A}_{r,2r}(z)) \geqslant C^{-1}\mathfrak{c}_r \mathrm{e}^{\xi h_r(z)},\\
&D_h \text{ (around } \mathbb{A}_{2r,3r}(z)) \leqslant C\mathfrak{c}_r \mathrm{e}^{\xi h_r(z)},\\
&\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon \text{ (across } \mathbb{A}_{r,2r}(z)) \geqslant C^{-1}\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon}\mathrm{e}^{\xi h_r(z)},\\
&\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon \text{ (around } \mathbb{A}_{2r,3r}(z)) \leqslant C\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon}\mathrm{e}^{\xi h_r(z)}.
\end{aligned} \tag{3.4}$$

We eventually apply the independence of the GFF across disjoint concentric annuli (see Lemma 2.1) and a union bound to show that if C is large enough, then with high probability there are many points z and radii r for which $E_r(z)$ occurs. Let us now explain why the event $E_r(z;C)$ fits into the framework of Lemma 2.1.

If $r > 2\varepsilon(\log\varepsilon^{-1})^q$, then by the locality of D_h (see Axiom (II)) and (2.8), the event $E_r^\varepsilon(z;C)$ is a.s. determined by the restriction of h to the annulus $\mathbb{A}_{r/2,4r}(z)$. This locality property for $E_r^\varepsilon(z;C)$ is the reason why we use $\widehat{D}_h^\varepsilon$ instead of D_h^ε in the definition. Since D_h and $\widehat{D}_h^\varepsilon$ transform in the same way when we add a constant to h (see Axiom 1.5 and (2.6)), we see that in fact $E_r^\varepsilon(z;C)$ is a.s. determined by $h|_{\mathbb{A}_{r/2,4r}(z)}$ viewed modulo additive constants.

Using tightness across scales (see Axiom (V)) along with the tightness and scaling properties of LFPP, we can show that $E_r(z;C)$ occurs with high probability when C is large.

Lemma 3.1. *For each $p \in (0,1)$, there exists a $C > 0$ depending on p, ξ and the law of D_h such that*

$$\mathrm{P}[E_r^\varepsilon(z;C)] \geqslant p, \quad \forall\, z \in \mathbb{C}, \quad \forall\, 0 < \varepsilon \leqslant r \leqslant 1. \tag{3.5}$$

Proof. Since the event $E_r^\varepsilon(z;C)$ is determined by h, viewed modulo additive constants, the law of h is translation invariant modulo additive constants, and D_h and $\widehat{D}_h^\varepsilon$ depend on h in a translation invariant way, we see that $\mathrm{P}[E_r^\varepsilon(z;C)]$ does not depend on z. So we can restrict attention to the case $z = 0$. Let $\widetilde{E}_r^\varepsilon(C)$ be defined in the same manner as the event $E_r^\varepsilon(0;C)$ of (3.4), but with the ordinary LFPP metric D_h^ε instead of the truncated LFPP metric $\widehat{D}_h^\varepsilon$. The reason we want to look at D_h^ε is the scaling property in (3.8) below.

Due to the uniform comparison between the fields $\widehat{h}_\varepsilon^*$ and h_ε^* involved in the definitions of $\widehat{D}_h^\varepsilon$ and D_h^ε (see Lemma 2.2), it suffices to show that for each $p \in (0,1)$, there exists a $C = C(p,\xi) > 0$ such that

$$\mathrm{P}[\widetilde{E}_r^\varepsilon(C)] \geqslant p, \quad \forall\, 0 < \varepsilon \leqslant r \leqslant 1. \tag{3.6}$$

To prove (3.6), we first use tightness across scales (see Axiom (V)) to find $C > 0$ as in the lemma statement such that for each $r \in (0,1]$, it holds with probability at least $(1+p)/2$ that

$$D_h \text{ (across } \mathbb{A}_{r,2r}(0)) \geqslant C^{-1}\mathfrak{c}_r \mathrm{e}^{\xi h_r(0)} \quad \text{and} \quad D_h \text{ (around } \mathbb{A}_{2r,3r}(0)) \leqslant C\mathfrak{c}_r \mathrm{e}^{\xi h_r(0)}. \tag{3.7}$$

To lower-bound the probabilities of the conditions for $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ in the definition of $\widetilde{E}_r^\varepsilon(C)$, we write $h^r = h(r\cdot) - h_r(0)$, which has the same law of h. A simple scaling calculation using the definitions of h_ε^* and D_h^ε shows that

$$\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(ru, rv) = \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \mathrm{e}^{\xi h_r(0)} \times \mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r}(u, v), \quad \forall\, u, v \in \mathbb{C} \tag{3.8}$$

(see [10, Lemma 2.6]). Since $h^r \overset{d}{=} h$, we can use the tightness of crossing distances for LFPP [8, Proposition 4.1] to get that after possibly increasing C, for each $0 < \varepsilon \leqslant r \leqslant 1$, we can deduce with probability at least $(1+p)/2$ that

$$\mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r} \text{ (across } \mathbb{A}_{1,2}(0)) \geqslant C^{-1} \quad \text{and} \quad \mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r} \text{ (around } \mathbb{A}_{2,3}(0)) \leqslant C. \tag{3.9}$$

By (3.8) and (3.9), for each $0 < \varepsilon \leqslant r \leqslant 1$, it holds with probability at least $(1+p)/2$ that

$$\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \text{ (across } \mathbb{A}_{r,2r}(0)) \geqslant C^{-1} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \mathrm{e}^{\xi h_r(0)} \quad \text{and} \quad \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \text{ (around } \mathbb{A}_{2r,3r}(0)) \leqslant C \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \mathrm{e}^{\xi h_r(0)}. \tag{3.10}$$

Combining (3.7) with (3.10) gives (3.6). □

3.3 Comparing LFPP and LQG distances using distances at smaller scales

Let us now define the set of radii which we will consider when we apply Lemma 2.1. Fix $\zeta \in (0,1)$ and for $\varepsilon \in (0,1)$, let

$$N^\varepsilon := \left\lfloor \frac{\zeta}{2} \log_{10} \varepsilon^{-1} \right\rfloor - 10.$$

Let

$$\mathcal{R}^\varepsilon := \{10^{-j} \varepsilon^{1-\zeta} : j = 1, \ldots, N^\varepsilon - 1\} \subset \left[\varepsilon^{1-\zeta/2}, \frac{1}{100} \varepsilon^{1-\zeta}\right]. \tag{3.11}$$

We note that $\#\mathcal{R}^\varepsilon = N^\varepsilon$ and for any $r, r' \in \mathcal{R}^\varepsilon$ with $r < r'$, we have $r'/r \geqslant 10$.

The following lemma tells us that if $\widetilde{\mathcal{R}}^\varepsilon \subset \mathcal{R}^\varepsilon$ is a large enough subset, then with high probability we can compare D_h-distances and $\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon$-distances at scales larger than $\varepsilon^{1-\zeta}$, up to a factor depending on the ratios $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r \mathfrak{a}_\varepsilon)$ for $r \in \widetilde{\mathcal{R}}^\varepsilon$.

Lemma 3.2. *There exists a $C_4 > 0$ depending on ζ and the law of D_h such that the following is true. Let $U \subset \mathbb{C}$ be a deterministic, connected, bounded open set. Also let $\varepsilon \in (0,1)$ and $\widetilde{\mathcal{R}}^\varepsilon \subset \mathcal{R}^\varepsilon$ be a deterministic subset with $\#\widetilde{\mathcal{R}}^\varepsilon \geqslant N^\varepsilon/100$. It holds with polynomially high probability as $\varepsilon \to 0$, at a rate depending only on U, ζ and the law of D_h, that the following is true. For each $z, w \in U$, we have*

$$\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_4 \left(\max_{r \in \widetilde{\mathcal{R}}^\varepsilon} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} \right) D_h(z, w; U) \tag{3.12}$$

and

$$D_h(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_4 \left(\min_{r \in \widetilde{\mathcal{R}}^\varepsilon} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} \right)^{-1} \mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon(z, w; U). \tag{3.13}$$

The statement of Lemma 3.2 is similar to the statement of Theorem 1.8, except that in Lemma 3.2 our estimates have an extra factor which depends on the ratios $r\mathfrak{a}_{\varepsilon/r}(\mathfrak{c}_r \mathfrak{a}_\varepsilon)$ for $r \in \widetilde{\mathcal{R}}^\varepsilon$. In Subsection 3.4, we deduce Theorem 1.8 from Lemma 3.2 by finding a choice of $\widetilde{\mathcal{R}}^\varepsilon$ for which these ratios are of constant order.

The proof of Lemma 3.2 is similar to the proof in [13, Section 4] that two metrics coupled with the GFF which satisfy certain conditions are bi-Lipschitz equivalent. We first use Lemma 2.1 to find lots of points z and radii $r \in \widetilde{\mathcal{R}}^\varepsilon$ for which $E_r^\varepsilon(z; C)$ occurs (see Lemma 3.3). By the definition (3.4) of $E_r^\varepsilon(z; C)$, for each such z and r,

$$\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon \text{ (around } \mathbb{A}_{2r,3r}(z)) \leqslant C^2 \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} D_h \text{ (across } \mathbb{A}_{r,2r}(z)) \tag{3.14}$$

and a similar inequality holds with the roles of $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$ and D_h interchanged. To prove (3.12), we consider a D_h-geodesic P. We then string together paths around annuli of the form $\mathbb{A}_{2r,3r}(z)$ which intersect P in order to produce a path with approximately the same endpoints as P. Using (3.14), we can arrange that the $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$-length of this new path is bounded above in terms of the D_h-length of P. The bound (3.13) is proven via a similar argument with the roles of $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$ and D_h interchanged.

Lemma 3.3. *Assume the setup of Lemma 3.2. There exists a $C_5>0$ depending only on ζ and the law of D_h such that with polynomially high probability as $\varepsilon\to 0$, at a rate depending only on U, ζ and the law of D_h, the following is true. For each $u\in(\frac{\varepsilon}{100}\mathbb{Z}^2)\cap B_1(U)$, there exists an $r\in\widetilde{\mathcal{R}}^\varepsilon$ such that the event $E_r^\varepsilon(u;C_5)$ occurs.*

Proof. We have $\#\widetilde{\mathcal{R}}^\varepsilon\geqslant N^\varepsilon/100\geqslant\lfloor\frac{\zeta}{200}\log_{10}\varepsilon^{-1}\rfloor$. Moreover, by (3.11), if we list the elements of $\widetilde{\mathcal{R}}^\varepsilon$ in the numerical order, then the ratio of any two consecutive elements is at least 10. For each $r\in\widetilde{\mathcal{R}}^\varepsilon$, we have $r\geqslant\varepsilon^{1-\zeta/2}\geqslant\varepsilon(\log\varepsilon^{-1})^q$, so as explained just after (3.4), the event $E_r^\varepsilon(u;C)$ is a.s. determined by $h|_{\mathbb{A}_{r/2,4r}(u)}$, viewed modulo additive constants. By Lemma 3.1, for any $p\in(0,1)$ we can choose $C=C(p,\xi)>0$ such that $\mathrm{P}[E_r^\varepsilon(u;C)]\geqslant p$ for each $r\in\widetilde{\mathcal{R}}^\varepsilon$ and each $u\in\mathbb{C}$. From this and Lemma 2.1 (applied with $K=\#\widetilde{\mathcal{R}}^\varepsilon$, the radii r_k equal to the elements of $\widetilde{\mathcal{R}}^\varepsilon$, and a equal to a large constant times $1/\zeta$), we find that there exists a $C_5>0$ as in the lemma statement such that for each $u\in\mathbb{C}$,

$$\mathrm{P}[\exists\, r\in\widetilde{\mathcal{R}}^\varepsilon \text{ such that } E_r^\varepsilon(u;C_5) \text{ occurs}]=1-O_\varepsilon(\varepsilon^{100}).$$

We now conclude via a union bound over $O_\varepsilon(\varepsilon^2)$ elements of $(\frac{\varepsilon}{100}\mathbb{Z}^2)\cap B_1(U)$. □

We now turn our attention to the proof of Lemma 3.2. Let $C_5>0$ be as in Lemma 3.3. Throughout the proof, we assume that the event of Lemma 3.3 occurs, which happens with polynomially high probability as $\varepsilon\to 0$. We show (via a purely deterministic argument) that (3.12) holds. The proof of (3.13) is similar, with the roles of $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$ and D_h interchanged.

To this end, fix distinct points $z,w\in U$ and let $P:[0,T]\to U$ be a path in U from z to w of D_h-length at most $2D_h(z,w;U)$. We assume that P is parametrized by its D_h-length. We will build a path from $B_{\varepsilon^{1-\zeta}}(z)$ to $B_{\varepsilon^{1-\zeta}}(w)$ which approximates P and whose $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$-length is bounded above.

To do this, we first inductively define a sequence of times $\{t_k\}_{k\in\mathbb{N}_0}\subset[0,T]$ (see Figure 1 for an illustration of the definitions). Let $t_0=0$. Inductively, assume that $k\in\mathbb{N}_0$ and t_k has been defined. If $t_k=T$, we set $t_{k+1}=T$. Otherwise, we choose $u_k\in(\frac{\varepsilon}{100}\mathbb{Z}^2)\cap B_1(U)$ such that $P(t_k)\in B_\varepsilon(u_k)$. Since we are assuming that the event of Lemma 3.3 occurs, we can choose $r_k\in\widetilde{\mathcal{R}}^\varepsilon$ such that $E_{r_k}^\varepsilon(u_k;C_5)$ occurs. By the definition (3.4) of $E_{r_k}^\varepsilon(u_k;C_5)$, there exists a path $\pi_k\subset\mathbb{A}_{2r_k,3r_k}(u_k)$ which disconnects the inner and outer boundaries of this annulus such that

$$(\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon\text{-length of }\pi_k)\leqslant 2C_5\frac{r_k\mathfrak{a}_{\varepsilon/r_k}}{\mathfrak{a}_\varepsilon}\mathrm{e}^{\xi h_{r_k}(u_k)}. \tag{3.15}$$

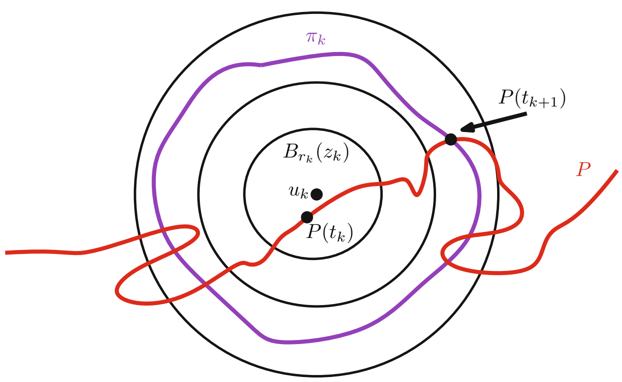

Figure 1 (Color online) Illustration of the objects involved in one step of the iterative construction of the times t_k. The red path is a segment of P and the two annuli in the figure are $\mathbb{A}_{r_k,2r_k}(u_k)$ and $\mathbb{A}_{2r_k,3r_k}(u_k)$

We can take π_k to be a Jordan curve (i.e., a homeomorphic image of the circle). Let t_{k+1} be the first time after t_k at which the path P hits π_k, or $t_{k+1} = T$ if no such time exists. Let

$$K := \max\{k \in \mathbb{N} : t_k < T\}.$$

By the definition (3.11) of $\mathcal{R}^\varepsilon$, we have $r_k \geqslant \varepsilon$ for each k, so by our choice of u_k we have $P(t_k) \in B_{r_k}(u_k)$. Since $\pi_k \subset \mathbb{A}_{2r_k,3r_k}(u_k)$, we see that if $k+1 \leqslant K$, then P must cross between the inner and outer boundaries of $\mathbb{A}_{r_k,2r_k}(u_k)$ between times t_k and t_{k+1}. Since P is parametrized by the D_h-length and by (3.4),

$$t_{k+1} - t_k \geqslant D_h \text{ (across } \mathbb{A}_{r_k,2r_k}(u_k)) \geqslant {C_5}^{-1} \mathfrak{c}_{r_k} \mathrm{e}^{\xi h_{r_k}(u_k)}, \quad \forall\, k \leqslant K-1. \tag{3.16}$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^{K-1} (\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon\text{-length of } \pi_k) &\leqslant \sum_{k=0}^{K-1} 2C_5 \frac{r_k \mathfrak{a}_{\varepsilon/r_k}}{\mathfrak{a}_\varepsilon} \mathrm{e}^{\xi h_{r_k}(u_k)} \text{ (by (3.15))} \\
&\leqslant \sum_{k=0}^{K-1} 2{C_5}^2 \frac{r_k \mathfrak{a}_{\varepsilon/r_k}}{\mathfrak{c}_{r_k} \mathfrak{a}_\varepsilon} (t_{k+1} - t_k) \text{ (by (3.16))} \\
&\leqslant 2{C_5}^2 \left(\sup_{r \in \widetilde{\mathcal{R}}^\varepsilon} \frac{r \mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} \right) \sum_{k=0}^{K-1} (t_{k+1} - t_k) \\
&\leqslant 4{C_5}^2 \left(\sup_{r \in \widetilde{\mathcal{R}}^\varepsilon} \frac{r \mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} \right) D_h(z, w; U) \text{ (by our choice of } P).
\end{aligned} \tag{3.17}$$

By definition, each of the paths π_k for $k = 0, \dots, K$ intersects P, which is contained in U, and has the Euclidean diameter at most $6r_k \leqslant \varepsilon^{1-\zeta}$. Therefore,

$$\bigcup_{k=0}^{K-1} \pi_k \subset B_{\varepsilon^{1-\zeta}}(U). \tag{3.18}$$

In light of (3.16) and (3.18), to conclude the proof of (3.12) (with $4{C_5}^2$ instead of C_4) it remains to prove the following topological lemma.

Lemma 3.4. *In the notation above, the union of the paths π_k for $k = 0, \dots, K-1$ contains a path from $B_{\varepsilon^{1-\zeta}}(z)$ to $B_{\varepsilon^{1-\zeta}}(w)$.*

Indeed, once Lemma 3.4 is established, (3.17) implies that the $\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon$-length of the path from the lemma is at most $2{C_5}^2 (\sup_{r \in \widetilde{\mathcal{R}}^\varepsilon} \frac{r \mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon}) D_h(z, w; U)$ and (3.18) implies that the path from the lemma is contained in $B_{\varepsilon^{1-\zeta}}(U)$. Hence (3.12) holds with $C_4 = 4{C_5}^2$.

Proof of Lemma 3.4. For $k = 0, \dots, K-1$, let V_k be the open region which is disconnected from ∞ by the path π_k. Since π_k is a Jordan curve, we have that V_k is bounded and $\partial V_k = \pi_k$. By construction, $P \subset \bigcup_{k=0}^{K-1} V_k$. Furthermore, the Euclidean diameter of each V_k is at most $6r_k \leqslant \varepsilon^{1-\zeta}$. Let $\mathcal{K} \subset [0, K-1]_{\mathbb{Z}}$ be a subset which is minimal in the sense that $P \subset \bigcup_{k \in \mathcal{K}} V_k$ and P is not covered by any proper subcollection of the sets V_k for $k \in \mathcal{K}$.

Since P is connected, it follows that $\bigcup_{k \in \mathcal{K}} V_k$ is connected. Indeed, if this set had two proper disjoint open subsets, then each would have to intersect P (by minimality) which would contradict the connectedness of P. Furthermore, by minimality, none of the sets V_k for $k \in \mathcal{K}$ is properly contained in a set of the form $V_{k'}$ for $k' \in \mathcal{K}$.

We claim that $\bigcup_{k \in \mathcal{K}} \pi_k$ is connected. Indeed, if this was not the case then we could partition $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ such that $\mathcal{K}_1$ and $\mathcal{K}_2$ are non-empty and $\bigcup_{k \in \mathcal{K}_1} \pi_k$ and $\bigcup_{k \in \mathcal{K}_2} \pi_k$ are disjoint. By the minimality of $\mathcal{K}$, it cannot be the case where any of the sets $V_{k'}$ for $k' \in \mathcal{K}_2$ is contained in $\bigcup_{k \in \mathcal{K}_1} V_k$. Furthermore, since $\bigcup_{k \in \mathcal{K}_1} \pi_k$ and $\bigcup_{k \in \mathcal{K}_2} \pi_k$ are disjoint, it cannot be the case where any set of the form $V_{k'}$ for $k' \in \mathcal{K}_2$ intersects both $\bigcup_{k \in \mathcal{K}_1} V_k$ and $\mathbb{C} \setminus \bigcup_{k \in \mathcal{K}_1} V_k$: indeed otherwise $V_{k'}$ would have to intersect $\partial V_k = \pi_k$ for some $k \in \mathcal{K}_1$, which would mean that either $V_{k'} \supset V_k$ or $\pi_{k'}$ intersects π_k. The first case is impossible by the preceding paragraph and the second case is impossible by our choices of $\mathcal{K}_1$ and $\mathcal{K}_2$. Hence $V_{k'}$ must be entirely contained in $\mathbb{C} \setminus \bigcup_{k\in\mathcal{K}_1} V_k$. Therefore, $\bigcup_{k\in\mathcal{K}_1} V_k$ and $\bigcup_{k\in\mathcal{K}_2} V_k$ are disjoint. This contradicts the connectedness of $\bigcup_{k\in\mathcal{K}} V_k$, and therefore gives our claim.

Since $\bigcup_{k\in\mathcal{K}} V_k$ contains P, each of the sets V_k has the Euclidean diameter at most $\varepsilon^{1-\zeta}$, and $\bigcup_{k\in\mathcal{K}} \pi_k$ is connected, it follows that $\bigcup_{k\in\mathcal{K}} \pi_k$ contains a path from $B_{\varepsilon^{1-\zeta}}(z)$ to $B_{\varepsilon^{1-\zeta}}(w)$, as required (recalling that $P(0) = z$ and $P(T) = w$). □

3.4 Up-to-constants comparison of LFPP and LQG distances

In order to deduce Theorem 1.8 from Lemma 3.2, we need to produce a large subset of $\mathcal{R}^\varepsilon$ such that the ratios $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r\mathfrak{a}_\varepsilon)$ for $r \in \mathcal{R}^\varepsilon$ are of constant order. The existence of such a subset turns out to be a consequence of Lemma 3.2. Indeed, for fixed distinct points $z, w \in \mathbb{C}$ we know *a priori* that $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon(z, w)$ and $D_h(z, w)$ are each typically of constant order. If there were a large number of scales $r \in \mathcal{R}^\varepsilon$ for which $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r\mathfrak{a}_\varepsilon)$ is very small, then Lemma 3.2 would imply that $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon(z, w)$ is typically much smaller than a small constant times $D_h(z, w)$, which is impossible. Similarly, there cannot be too many values of $r \in \mathcal{R}^\varepsilon$ for which $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r\mathfrak{a}_\varepsilon)$ is very large. Hence this ratio must be of constant order for "most" $r \in \mathcal{R}^\varepsilon$. Let us now make this reasoning precise.

Lemma 3.5. *There exists a $C_6 > 1$ depending only on ζ and the law of D_h such that for each $\varepsilon \in (0, 1)$, there are at least $N^\varepsilon/2$ values of $r \in \mathcal{R}^\varepsilon$ such that*

$$C_6{}^{-1} \leqslant \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon} \leqslant C_6. \tag{3.19}$$

Proof. For any $\varepsilon_0 > 0$, the scaling constants $\mathfrak{a}_\varepsilon$ for $\varepsilon \in [\varepsilon_0, 1]$ are bounded above and below by constants depending only on ε_0 and ξ and the constants $\mathfrak{c}_r$ for $r \in [\varepsilon_0, 1]$ are bounded above and below by constants depending only on ε_0 and the law of D_h. Hence, we can choose $C_6 > 1$ depending only on ε_0, ζ and the law of D_h such that (3.19) holds for all $\varepsilon \in [\varepsilon_0, 1]$ and all $r \in [\varepsilon, \varepsilon^{1-\zeta}]$. Therefore, it suffices to find $C_6 > 1$ as in the lemma statement such that the lemma statement holds for each small enough $\varepsilon > 0$ (depending on ζ and the law of D_h).

For $T > 1$, let $\widetilde{\mathcal{R}}_{T,+}^\varepsilon$ (resp. $\widetilde{\mathcal{R}}_{T,-}^\varepsilon$) be the set of $r \in \mathcal{R}^\varepsilon$ such that $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r\mathfrak{a}_\varepsilon) > T$ (resp. $r\mathfrak{a}_{\varepsilon/r}/(\mathfrak{c}_r\mathfrak{a}_\varepsilon) < T^{-1}$). If the lemma statement does not hold with $C_6 = T$, then either $\#\widetilde{\mathcal{R}}_{T,+}^\varepsilon \geqslant N^\varepsilon/4$ or $\#\widetilde{\mathcal{R}}_{T,-}^\varepsilon \geqslant N^\varepsilon/4$. Assume that $\#\widetilde{\mathcal{R}}_{T,+}^\varepsilon \geqslant N^\varepsilon/4$ (while the other case is treated similarly with the roles of D_h and $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$ interchanged). We show that T is bounded above by a constant depending on ζ and the law of D_h.

By (3.13) of Lemma 3.2 applied with $\widetilde{\mathcal{R}}^\varepsilon = \widetilde{\mathcal{R}}_{T,+}^\varepsilon$ and $U = B_2(0)$, there exists a $C_4 > 0$ such that with polynomially high probability as $\varepsilon \to 0$,

$$D_h(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{2+\varepsilon^{1-\zeta}}(0)) \leqslant C_4T^{-1}\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon(z, w; B_2(0)), \quad \forall\, z, w \in B_2(0), \tag{3.20}$$

which implies that

$$D_h \text{ (across } \mathbb{A}_{1+\varepsilon^{1-\zeta},2-\varepsilon^{1-\zeta}}(0)) \leqslant C_4T^{-1}\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon \text{ (across } \mathbb{A}_{1,2}(0)). \tag{3.21}$$

By tightness across scales (see Axiom (V)), there exists an $S > 0$ depending only on the law of D_h such that whenever $\varepsilon < 1/100$, we have

$$\mathrm{P}[D_h \text{ (across } \mathbb{A}_{1+\varepsilon^{1-\zeta},2-\varepsilon^{1-\zeta}}(0)) \geqslant S^{-1}] \geqslant \frac{3}{4}. \tag{3.22}$$

By [8, Proposition 4.1] and Lemma 2.2, after possibly increasing S we can arrange that also

$$\mathrm{P}[\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon \text{ (across } \mathbb{A}_{1,2}(0)) \leqslant S] \geqslant \frac{3}{4}. \tag{3.23}$$

By combining (3.21)–(3.23), we obtain that with probability at least $1/4 - o_\varepsilon(1)$ (with the rate of convergence of $o_\varepsilon(1)$ depending only on ζ and the law of D_h), we have

$$S^{-1} \leqslant C_4T^{-1}S,$$

i.e., $T \leqslant C_4 S^2$. Hence, if ε is small enough so that $1/4 - o_\varepsilon(1) > 0$, then $T \leqslant C_4 S^2$. Therefore, the lemma statement holds with $C_6 = C_4 S^2$. □

As a consequence of Lemma 3.5, we obtain a version of Theorem 1.8 with $\widehat{D}_h^\varepsilon$ in place of D_h^ε.

Proposition 3.6. *For each $\zeta \in (0,1)$, there exists a $C_0 > 0$ depending only on ζ and the law of D_h such that the following is true. Let $U \subset \mathbb{C}$ be a deterministic, connected, bounded open set. With polynomially high probability as $\varepsilon \to 0$,*

$$\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_0 D_h(z,w;U), \quad \forall\, z,w \in U \tag{3.24}$$

and

$$D_h(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{\varepsilon^{1-\zeta}}(U)) \leqslant C_0 \mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon(z,w;U), \quad \forall\, z,w \in U. \tag{3.25}$$

Proof. Let C_6 be as in Lemma 3.5 and let $\widetilde{\mathcal{R}}^\varepsilon$ be the set of $r \in \mathcal{R}^\varepsilon$ for which (3.19) holds. By Lemma 3.5, we have $\#\widetilde{\mathcal{R}}^\varepsilon \geqslant N^\varepsilon/2$, so we can apply Lemma 3.2 to get that with polynomially high probability as $\varepsilon \to 0$, the bounds (3.12) and (3.13) hold for our above choice of $\widetilde{\mathcal{R}}^\varepsilon$. We then use (3.19) to bound the maximum and minimum appearing in (3.12) and (3.13) in terms of C_6. This gives the proposition statement with $C_0 = C_4 C_6$. □

Proof of Theorem 1.8. This is immediate from Lemma 2.2 and Proposition 3.6. □

3.5 Bounds for scaling constants and bi-Lipschitz equivalence

In this subsection, we prove Theorems 1.9–1.11.

Theorem 1.8 provides non-trivial bounds relating $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon(z,w)$ and $D_h(z,w)$ whenever $|z-w|$ is of larger order than $\varepsilon^{1-\zeta}$. From this and the scaling properties of LFPP, we get bounds for the ratios $\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon}$ whenever r is much larger than $\varepsilon^{1-\zeta}$. These bounds will be the main input in the proofs of Theorems 1.9 and 1.11.

Lemma 3.7. *There is a constant $C_7 > 1$ depending only on ζ and the law of D_h such that the following is true. For each $R \geqslant 1$, there exists an $\varepsilon_* = \varepsilon_*(R,\zeta) > 0$ such that for each $\varepsilon \in (0,\varepsilon_*]$ and each $r \in [100\varepsilon^{1-\zeta}, R]$,*

$$C_7^{-1} \leqslant \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon} \leqslant C_7. \tag{3.26}$$

We emphasize the distinction between Lemmas 3.5 and 3.7: the former gives bounds for the ratios $\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r \mathfrak{a}_\varepsilon}$ which hold for most $r \in \mathcal{R}^\varepsilon \subset [\varepsilon, \varepsilon^{1-\zeta}]$ whereas the latter gives bounds for all $r \in [100\varepsilon^{1-\zeta}, 1]$.

Proof of Lemma 3.7. We find C_7 and ε_* such that the upper bound in (3.26) holds. The lower bound is obtained via a similar argument with the roles of $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$ and D_h interchanged.

Fix $R \geqslant 1$. By Theorem 1.8 applied with $U = B_{3R}(0)$, there exists a $C_0 > 0$ depending only on ζ and the law of D_h such that with polynomially high probability as $\varepsilon \to 0$ (with the rate of convergence depending on R, ζ and the law of D_h), we have

$$\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon(B_{\varepsilon^{1-\zeta}}(z), B_{\varepsilon^{1-\zeta}}(w); B_{3R+\varepsilon^{1-\zeta}}(0)) \leqslant C_0 D_h(z,w;B_{3R}(0)), \quad \forall\, z,w \in B_{3R}(0).$$

By applying this last inequality to points on the inner and outer boundaries of $\mathbb{A}_{r-\varepsilon^{1-\zeta},2r+\varepsilon^{1-\zeta}}(0)$, we get that with polynomially high probability as $\varepsilon \to 0$ for each $r \in [100\varepsilon^{1-\zeta}, R]$,

$$\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon \text{ (across } \mathbb{A}_{r,2r}(0)) \leqslant C_0 D_h \text{ (across } \mathbb{A}_{r-\varepsilon^{1-\zeta},2r+\varepsilon^{1-\zeta}}(0)). \tag{3.27}$$

Using tightness across scales (see Axiom (V)) and tightness of LFPP crossing distances (see [8, Proposition 4.1]), as in the proof of Lemma 3.1, we find that there exists an $S > 0$ depending only on the law of D_h such that for each $r \in [100\varepsilon^{1-\zeta}, R]$,

$$\begin{aligned} &\mathrm{P}\left[\mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon \text{ (across } \mathbb{A}_{r,2r}(z)) \geqslant S^{-1} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \mathrm{e}^{\xi h_r(0)}\right] \geqslant \frac{3}{4}, \\ &\mathrm{P}\left[D_h \text{ (across } \mathbb{A}_{r-\varepsilon^{1-\zeta},2r+\varepsilon^{1-\zeta}}(z)) \leqslant S\mathfrak{c}_r \mathrm{e}^{\xi h_r(0)}\right] \geqslant \frac{3}{4}. \end{aligned} \tag{3.28}$$

By combining (3.27) and (3.28), we get that for each $r \in [100\varepsilon^{1-\zeta}, R]$, it holds with probability at least $1/4 - o_\varepsilon(1)$ (with the rate of convergence depending only on R, ζ and the law of D_h) that

$$S^{-1}\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \leqslant C_0 S\mathfrak{c}_r.$$

Hence, if ε is small enough so that $1/4 - o_\varepsilon(1) > 0$, then

$$\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{c}_r\mathfrak{a}_\varepsilon} \leqslant C_0 S^2.$$

This gives the upper bound in (3.26) with $C_7 = C_0S^2$. As noted above, the lower bound is proven similarly. □

We deduce our bounds for $\mathfrak{c}_r$ and $\mathfrak{a}_\varepsilon$ (see Theorems 1.9 and 1.11) from Lemma 3.7 together with elementary deterministic arguments. For the proof of Theorem 1.9, we need the following classical lemma, which tells us that if a sequence $\{x_n\}_{n\in\mathbb{N}}$ is both subadditive and superadditive, up to a constant additive error, then x_n/n converges to a limit and one can bound the rate of convergence.

Lemma 3.8 (Subadditive rate lemma). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and assume that there is a $c > 0$ such that*

$$x_n + x_m - c \leqslant x_{n+m} \leqslant x_n + x_m + c, \quad \forall\, n, m \in \mathbb{N}. \tag{3.29}$$

Then there is an $\alpha > 0$ such that

$$|x_n/n - \alpha| \leqslant c/n, \quad \forall\, n \in \mathbb{N}. \tag{3.30}$$

Lemma 3.8 follows from the proof of [19, Lemma 1.9.1] (applied with x_n/c in place of x_n). The statement of [19, Lemma 1.9.1] gives $|x_n/n - \alpha| \leqslant c$ instead of (3.30), but the proof shows that in fact (3.30) holds.

We also need a basic *a priori* estimate comparing the scaling constants $\mathfrak{c}_r$ for different values of r.

Lemma 3.9. *For each $K > 1$, there exists a $C > 1$ depending on K and the law of D_h such that $C^{-1}\mathfrak{c}_r \leqslant \mathfrak{c}_{r'} \leqslant C\mathfrak{c}_r$ whenever $r > 0$ and $r' \in [K^{-1}r, Kr]$.*

Proof. Fix a Euclidean annulus $A \subset \mathbb{C}$. We can find finitely many Euclidean annuli $A_1, \dots, A_k$ satisfying that for each $s \in [K^{-1}, K]$, there exists a $j \in [1, k]_\mathbb{Z}$ such that sA is contained in A_j and disconnects the inner and outer boundaries of A_j (with the aspect ratios of A_j's larger than the aspect ratio of A). Similarly, we can find finitely many Euclidean annuli $A'_1, \dots, A'_k$ (whose aspect ratios will be smaller than the aspect ratio of A) satisfying that for each $s \in [K^{-1}, K]$, there exists a $j' \in [1, k]_\mathbb{Z}$ such that $A'_{j'}$ is contained in sA and disconnects the inner and outer boundaries of sA. We have

$$\begin{aligned} &D_h \text{ (around } A_j) \leqslant D_h \text{ (around } sA) \leqslant D_h \text{ (around } A'_{j'}),\\ &D_h \text{ (across } A'_{j'}) \leqslant D_h \text{ (across } sA) \leqslant D_h \text{ (across } A_j). \end{aligned}$$

From this and Axiom (V), applied to each of the annuli $A_1, \dots, A_k$ and $A'_1, \dots, A'_k$, we see that the random variables

$$\mathfrak{c}_r^{-1}\mathrm{e}^{-\xi h_r(0)} \sup_{r'\in[K^{-1}r, Kr]} D_h \text{ (around } r'A) \tag{3.31}$$

are tight, and the same holds if we replace the sup by an inf and take the reciporicals of the random variables, and/or we replace "across" by "around". Furthermore, since $t \mapsto h_{\mathrm{e}^{-t}}(0)$ is a standard Brownian motion (see the calculations in [11, Subsection 3.1]), we see that the random variables

$$\sup_{r'\in[K^{-1}r, Kr]} \exp(\xi|h_{r'}(0) - h_r(0)|) \tag{3.32}$$

are tight. Combining (3.31) and (3.32) shows that the random variables

$$\mathfrak{c}_r^{-1} \sup_{r'\in[K^{-1}r, Kr]} \mathrm{e}^{-\xi h_{r'}(0)} D_h \text{ (around } r'A)$$

are tight, and the same holds if we replace the sup by an inf and take the reciporicals of the random variables, and/or we replace "across" by "around". Consequently, Axiom (V) holds with $\mathfrak{c}_r$ replaced by the scaling factor $\widetilde{\mathfrak{c}}_r$ which equals $\mathfrak{c}_{K^n}$ whenever $r \in [K^n, K^{n+1}]$. By Remark 1.6, we see that there is a constant $C > 1$ such that $C^{-1}\mathfrak{c}_{K^n} \leqslant \mathfrak{c}_r \leqslant C\mathfrak{c}_{K^n}$ whenever $r \in [K^n, K^{n+1}]$. This implies the lemma statement with C^2 in place of C. □

Proof of Theorem 1.9. Throughout the proof, we assume that all the implicit constants in $\asymp$ depend only on the law of D_h. Let $r, s > 0$. By three applications of Lemma 3.7 applied with $\zeta = 1/2$, if ε is sufficiently small (depending on r and s), then

$$\mathfrak{c}_r\mathfrak{c}_s \asymp \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \times \frac{s\mathfrak{a}_{\varepsilon/(sr)}}{\mathfrak{a}_{\varepsilon/r}} = \frac{sr\mathfrak{a}_{\varepsilon/(sr)}}{\mathfrak{a}_\varepsilon} \asymp \mathfrak{c}_{sr}. \tag{3.33}$$

For $n \in \mathbb{N}$, write $x_n = \log \mathfrak{c}_{2^{-n}}$. By taking s and r to be powers of 2 and taking the log of both sides of (3.33), we conclude that $\{x_n\}_{n\in\mathbb{N}}$ satisfies (3.29). Therefore, Lemma 3.8 implies that there exists an $\alpha > 0$ such that $|x_n/n - \alpha| \leqslant c/n$ for all n, or equivalently,

$$\mathfrak{c}_{2^{-n}} \asymp 2^{-\alpha n}.$$

By [8, Proposition 4.2], for $0 < \varepsilon < r$, we have

$$\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} = r^{\xi Q + o_r(1)}$$

with the rate of convergence of $o_r(1)$ depending only on the law of D_h. By combining this with Lemma 3.7, we infer that $\alpha = \xi Q$. This gives (1.13) when r is a negative power of 2.

The case of a general choice of $r \in (0, 1]$ follows from the case $r = 2^{-n}$ together with Lemma 3.9. To treat the case $r > 1$, we apply (3.33) with $s = 1/r < 1$ to get $\mathfrak{c}_r \asymp \mathfrak{c}_1/\mathfrak{c}_{1/r} \asymp r^{\xi Q}$. □

Proof of Theorem 1.11. Let $\varepsilon_* > 0$ be as in Lemma 3.7 with $\zeta = 1/4$ and $R = 1$. By possibly shrinking ε_*, we can arrange that also $100\varepsilon^{3/4} \leqslant \varepsilon^{1/2}$ for each $\varepsilon \in (0, \varepsilon_*]$. Lemma 3.7 (applied with $\zeta = 1/4$) combined with Theorem 1.9 implies that there is a constant $A = A(\xi) > 1$ (in particular, $A = C_1C_7$) such that if $0 < \varepsilon \leqslant \varepsilon_*$ and $r \in [\varepsilon^{1/2}, 1]$, then

$$A^{-1}r^{\xi Q-1} \leqslant \frac{\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \leqslant Ar^{\xi Q-1}. \tag{3.34}$$

After possibly increasing A, we can remove the constraint that $\varepsilon \leqslant \varepsilon_*$.

For $k \in \mathbb{N}_0$, we apply (3.34) with $\varepsilon = 2^{-2^k}$ and $r = 2^{-2^k}/\delta$ to find that

$$A^{-1}\delta^{1-\xi Q}2^{(1-\xi Q)2^k} \leqslant \frac{\mathfrak{a}_\delta}{\mathfrak{a}_{2^{-2^k}}} \leqslant A\delta^{1-\xi Q}2^{(1-\xi Q)2^k}, \quad \forall\, \delta \in [2^{-2^k}, 2^{-2^{k-1}}]. \tag{3.35}$$

In particular, taking $\delta = 2^{-2^{k-1}}$ gives

$$A^{-1}2^{(1-\xi Q)2^{k-1}} \leqslant \frac{\mathfrak{a}_{2^{-2^{k-1}}}}{\mathfrak{a}_{2^{-2^k}}} \leqslant A2^{(1-\xi Q)2^{k-1}}. \tag{3.36}$$

We apply this inequality with j instead of k, and then multiply over all $j = 1, \ldots, k$ to get

$$A^{-k}2^{(1-\xi Q)(2^k-1)} \leqslant \frac{\mathfrak{a}_{1/2}}{\mathfrak{a}_{2^{-2^k}}} \leqslant A^k 2^{(1-\xi Q)(2^k-1)}. \tag{3.37}$$

Re-arranging the above inequalities shows that there is a constant $C = C(\xi) > 0$ such that

$$C^{-1}A^{-k}2^{-(1-\xi Q)2^k} \leqslant \mathfrak{a}_{2^{-2^k}} \leqslant CA^k 2^{-(1-\xi Q)2^k}, \quad \forall\, k \in \mathbb{N}, \tag{3.38}$$

where we absorbed $\mathfrak{a}_{1/2}$ into C.

For a given $\delta \in (0, 1/2]$, choose $k \in \mathbb{N}$ such that $\delta \in [2^{-2^k}, 2^{-2^{k-1}}]$. Note that

$$k \in [\log_2 \log_2 \delta^{-1}, \log_2 \log_2 \delta^{-1} + 1]. \tag{3.39}$$

By (3.35) and (3.38),

$$C^{-1}A^{-k-1}\delta^{1-\xi Q} \leqslant \mathfrak{a}_\delta \leqslant CA^{k+1}\delta^{1-\xi Q}. \tag{3.40}$$

By (3.39), A^{k+1} is bounded above by a ξ-dependent constant times $(\log\delta^{-1})^b$ for some $b = b(\xi) > 0$. Thus (3.40) implies (1.15). □

Remark 3.10. Our proof does not yield bounds better than polylogarithmic upper and lower bounds for $\mathfrak{a}_\varepsilon/\varepsilon^{1-\xi Q}$. Indeed, the estimate (3.34) for $r \in [\varepsilon^{1/2}, 1]$ is still satisfied, e.g., if $\mathfrak{a}_\varepsilon = (\log\varepsilon^{-1})^b\varepsilon^{1-\xi Q}$ for some $b \in \mathbb{R}$ (with the constant A depending on b). In order to get bounds better than polylogarithmic bounds, we would need to improve on (3.34) so that either it holds for all $r \in [\phi(\varepsilon)\varepsilon, 1]$, where $\lim_{\varepsilon\to 0}\log\phi(\varepsilon)/\log\varepsilon = 0$, or it holds with A replaced by something of order $1 + o_\varepsilon(1)$. Either of these improvements would require non-trivial new ideas.

Proof of Theorem 1.10. This can be deduced from Theorem 1.9 and a generalization of the bi-Lipschitz equivalence criterion from [13, Theorem 1.6]. However, we instead give a more self-contained proof.

Let $U \subset \mathbb{C}$ be a deterministic, connected, bounded open set. We apply Theorem 1.8 (with $\zeta = 1/2$) to compare each of D_h and $\widetilde{D}_h$ to $\mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon$. We obtain that there is a deterministic constant $C_2 > 0$ depending only on the laws of D_h and $\widetilde{D}_h$ such that with probability tending to 1 as $\varepsilon \to 0$,

$$\widetilde{D}_h(B_{\varepsilon^{1/2}}(z), B_{\varepsilon^{1/2}}(w); B_{\varepsilon^{1/2}}(U)) \leqslant C_2 D_h(z, w; U), \quad \forall\, z, w \in U \tag{3.41}$$

and

$$D_h(B_{\varepsilon^{1/2}}(z), B_{\varepsilon^{1/2}}(w); B_{\varepsilon^{1/2}}(U)) \leqslant C_2 \widetilde{D}_h(z, w; U), \quad \forall\, z, w \in U. \tag{3.42}$$

In particular, C_2 is the product of the constants appearing in Theorem 1.8 for D_h and $\widetilde{D}_h$, respectively. Shrinking ε makes the conditions (3.41) and (3.42) stronger. Since these conditions hold with probability tending to 1 as $\varepsilon \to 0$, we infer that a.s. there is a random $\varepsilon_* = \varepsilon_*(U) > 0$ such that (3.41) and (3.42) hold for each $\varepsilon \leqslant \varepsilon_*$.

Now let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing family of bounded open sets whose union is all of $\mathbb{C}$. From the preceding paragraph, we infer that a.s. there is a random sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, the conditions (3.41) and (3.42) hold with $U = U_n$ for each $\varepsilon \leqslant \varepsilon_n$.

Almost surely, every D_h-bounded set is Euclidean-bounded [17, Lemma 3.12]. Consequently, it is a.s. the case where for any two distinct points $z, w \in \mathbb{C}$ which are non-singular for D_h, there exists an $n \in \mathbb{N}$ such that every path from z to w whose D_h-length is at most $2D_h(z, w)$ is contained in U_n. This implies that $D_h(z, w; U_n) = D_h(z, w)$. By combining this with the preceding paragraph, we find that for each $\varepsilon \in (0, \varepsilon_n]$,

$$\widetilde{D}_h(B_{\varepsilon^{1/2}}(z), B_{\varepsilon^{1/2}}(w)) \leqslant \widetilde{D}_h(B_{\varepsilon^{1/2}}(z), B_{\varepsilon^{1/2}}(w); B_{\varepsilon^{1/2}}(U_n)) \leqslant C_2 D_h(z, w).$$

Since $\widetilde{D}_h$ is lower semicontinuous, if we take the liminf of the left-hand side of this inequality as $\varepsilon \to 0$, we obtain $\widetilde{D}_h(z, w) \leqslant C_2 D_h(z, w)$. This holds for any two points $z, w \in \mathbb{C}$ which are non-singular for D_h. If either z or w is a singular point for D_h, then $D_h(z, w) = \infty$ so $\widetilde{D}_h(z, w) \leqslant C_2 D_h(z, w)$ vacuously. We thus obtain the upper bound in (1.14). The lower bound is obtained similarly with the roles of D_h and $\widetilde{D}_h$ interchanged. □

Acknowledgements Jian Ding was supported by National Science Foundation of USA (Grant Nos. DMS-1757479 and DMS-1953848). Ewain Gwynne was supported by Clay Research Fellowship. The authors thank the referees for helpful comments on an earlier version of this paper.

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