CORRELATION LENGTH OF THE TWO-DIMENSIONAL RANDOM FIELD ISING MODEL VIA GREEDY LATTICE ANIMAL

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Abstract

For the two-dimensional random field Ising model where the random field is given by independent and identically distributed mean zero Gaussian variables with variance ε^2*, we study (one natural notion of) the correlation length, which is the critical size of a box at which the influences of the random field and of the boundary condition on the spin magnetization are comparable. We show that as* $\varepsilon \to 0$*, at zero temperature the correlation length scales as* $e^{\Theta(\varepsilon^{-4/3})}$ *(and our upper bound applies for all positive temperatures).*

1. Introduction

Let $\{h_v : v \in \mathbb{Z}^2\}$ be independent and identically distributed (i.i.d.) Gaussian random variables with mean zero and variance 1. For $N \geq 1$, let $\Lambda_N = \{v \in \mathbb{Z}^2 : |v|_\infty \leq N\} \subset \mathbb{Z}^2$ be the box of side length $2N$ centered at the origin o. For $u, v \in \mathbb{Z}^2$ with $|u - v| = 1$ (where $|\cdot|$ denotes the Euclidean norm), we say u and v are adjacent and write $u \sim v$. For $\varepsilon \geq 0$, the random field Ising model (RFIM) Hamiltonian $H^\pm$ on the configuration space $\{-1,1\}^{\Lambda_N}$ with plus (respectively, minus) boundary condition and external field $\{\varepsilon h_v : v \in \Lambda_N\}$ is defined to be

$$H^\pm(\sigma, \Lambda_N, \varepsilon h) = -\Big(\sum_{u\sim v, u,v\in\Lambda_N} \sigma_u\sigma_v \pm \sum_{u\sim v, u\in\Lambda_N, v\notin\Lambda_N} \sigma_u + \sum_{u\in\Lambda_N} \varepsilon h_u \sigma_u\Big), \quad (1)$$

for $\sigma \in \{-1,1\}^{\Lambda_N}$, where in the first sum each unordered edge appears once. For $\beta \geq 0$, let $\mu^\pm_{\beta,\Lambda_N,\varepsilon h}$ be the Gibbs measure on $\{-1,1\}^{\Lambda_N}$ at inverse-temperature β, defined as

$$\mu^\pm_{\beta,\Lambda_N,\varepsilon h}(\sigma) = \frac{1}{Z} e^{-\beta H^\pm(\sigma,\Lambda_N,\varepsilon h)}, \quad (2)$$

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where Z is the partition function so that $\mu^{\pm}_{\beta,\Lambda_N,\varepsilon h}(\sigma)$ is a probability measure. Note that $\mu^{\pm}_{\beta,\Lambda_N,\varepsilon h}$ is a random measure which itself depends on $\{h_v\}$. To clearly separate the two different sources of randomness, we will use $\mathbb{P}$ and $\mathbb{E}$ to refer to the probability measure with respect to the external field $\{h_v\}$; we use $\mu^{\pm}_{\beta,\Lambda_N,\varepsilon h}$ to denote the Ising measures and $\langle\cdot\rangle_{\mu^{\pm}_{\beta,\Lambda_N,\varepsilon h}}$ to denote the expectations with respect to the Ising measures. For instance, $\langle\sigma_o^+\rangle_{\mu^{+}_{\beta,\Lambda_N,\varepsilon h}}$ denotes the average value of the spin at the origin when we sample $\sigma^+ \in \{-1,1\}^{\Lambda_N}$ according to $\mu^{+}_{\beta,\Lambda_N,\varepsilon h}$. We are interested in the following quantity which measures the influence of the boundary condition:

$$\mathtt{m}_{\beta,\Lambda_N,\varepsilon} = \frac{1}{2}\mathbb{E}\big[\langle\sigma_o^+\rangle_{\mu^{+}_{\beta,\Lambda_N,\varepsilon h}} - \langle\sigma_o^-\rangle_{\mu^{-}_{\beta,\Lambda_N,\varepsilon h}}\big]. \tag{3}$$

For $\mathtt{m} \in (0,1)$, we consider the following notion of correlation length,

$$\psi(\beta,\mathtt{m},\varepsilon) = \min\{N : \mathtt{m}_{\beta,\Lambda_N,\varepsilon} \le \mathtt{m}\}, \tag{4}$$

which (for large β) amounts to the critical scale where the random field has a comparable influence as the boundary condition on the spin at the origin. Here we use the convention that $\min\emptyset = \infty$.

THEOREM 1.1
For every $\mathtt{m} \in (0,1)$*, there exists* $C = C(\mathtt{m}) > 0$ *such that* $\psi(\beta,\mathtt{m},\varepsilon) \le e^{C\varepsilon^{-4/3}}$ *for all* $\beta \ge 0$ *(including* $\beta = \infty$*), and that* $\psi(\infty,\mathtt{m},\varepsilon) \ge e^{C^{-1}\varepsilon^{-4/3}}$ *for* $\beta = \infty$.

Remark 1.2
The emergence of the $4/3$ exponent is somewhat unexpected, and it is reminiscent of the $4/3$-exponent in upper bounds on distances for Liouville quantum gravity at high temperatures (see [22]): the $4/3$-exponent arises from "back-of-the-envelope" computations that are similar in spirit for both scenarios (an interested reader may compare [22, Section 2] with Section 2.2). However, the random field Ising model and Liouville quantum gravity are two drastically different models, and as a result their mathematical treatments are different except that they both employ a framework of multi-scale analysis.

Remark 1.3
During the submission of this paper, the lower bound here was extended to low temperatures (i.e., to large finite β) by [25], which takes the result at $\beta = \infty$ as an input. The key idea in [25] is to extend the Peierls argument in the construction of the Peierls mapping, where the additional novelty is to also flip the signs of the disorder when flipping the signs of spins on a simply connected component. In addition, by [23,

Corollary 1.6] there is an exponential decay for $\beta < \beta_c$, and, moreover, the decaying rate is upper-bounded by that for $\varepsilon = 0$. Furthermore, the behavior for moderate $\beta > \beta_c$ seems rather challenging, and currently we have a weak belief that the $e^{\varepsilon^{-4/3}}$-scaling for the correlation length holds for all $\beta > \beta_c$. Ultimately, it would be very interesting to completely understand the phase diagram of the mapping from (β, ε) to the rate of exponential decay (as proved in [2], [24]), but this seems out of reach for now, and we do not have any intuition beyond what has been discussed.

Remark 1.4
More than one year after the arXiv post of this paper, and more than half a year after the arXiv post of [25], the paper [7] was posted, which proved an upper bound of $\exp(e^{O(\varepsilon^{-2})})$ and a lower bound of (the type of) $e^{\varepsilon^{-2/3}}$ for the correlation length. In addition, we note that the notion for the correlation length in the upper bound of [7] governs the rate of exponential decay, and thus in terms of upper bound it is a stronger notion than the one used in this paper.

Remark 1.5
A very natural question is whether one can prove the scaling of $e^{\varepsilon^{-4/3}}$ for the correlation length that governs the rate of exponential decay. As far as we can tell, to this end one needs to combine the techniques from [24] and [2] (see also [7]) with methods in this paper. This does not seem to be trivial since the key point of [24] and [2] is to prove that the boundary influence has a polynomial decay with a large power, while in this paper in order to derive a contradiction currently it seems inevitable to assume in the contradiction hypothesis that the boundary influence is lower-bounded by a constant. Maybe a vaguely plausible approach is to show that once the side length exceeds $e^{\varepsilon^{-4/3}}$, the boundary influence will start seeing a decay and that also the tortuosity assumption employed in [24] and [2] for disagreement percolation would hold. But, by all means, this is a highly nontrivial task, and we feel better to leave it for future study and advise an interested researcher to keep their mind open.

This result lies under the umbrella of the general Imry–Ma [37] phenomenon on the effect of disorder on phase transitions in two-dimensional physical systems. We next give a brief review of the development in the particular case of the random field Ising model (RFIM). In the limit of $N \to \infty$ with small fixed $\varepsilon > 0$, it was shown in [4] and [5] that $\mathtt{m}_{\beta,\Lambda_N,\varepsilon}$ decays to 0 for all $\beta \geq 0$, which also implies the uniqueness of the Gibbs state. The decay rate was then improved to $1/\sqrt{\log\log N}$ in [18], to N^{-c} (for some small $c > 0$) in [3], and finally to e^{-cN} in [24] and [2] (previously, exponential decay was shown in [8], [16], [30], [54] for large ε).

In three dimensions and above, however, the behavior is drastically different from that in two dimensions: it was shown in [36] that long range order exists at zero temperature with weak disorder; that is, $\mathtt{m}_{\infty,\Lambda_N,\varepsilon}$ does not vanish as N grows; later an analogous result was proved in [14] (see also [11, Chapter 7]) at low temperatures. A heuristic explanation for the different behaviors is as follows: in two dimensions the fluctuation of (the sum of) the random field in a box is of the same order as the size of the boundary, while in three dimensions and above the fluctuation of the random field is substantially smaller than the size of the boundary.

In the limit as $\varepsilon \to 0$, the scaling of the correlation length in both two dimensions and three dimensions (at some "critical" temperature) has remained largely elusive even from the point of view of physics predictions despite extensive studies. Previous works include (a partial list of) numeric studies [55], [48], [31], [46], [45], [47], [43], and [49] and nonrigorous derivations [44], [33], [9], [34], [21], [12], and [13]. It is worth noting that most of the studies in two dimensions were at zero temperature, but even in this case there was no consensus on the scaling of the correlation length: while a common belief seemed to be that it scales like $e^{\varepsilon^{-2}}$ (or upper-bounded by $e^{O(\varepsilon^{-2})}$) as argued in [33], [9], [13], [48], and [47], there were also other predictions including a scaling of $e^{\varepsilon^{-1}}$ in a more recent work [49]. (We note that some of these papers studied our notion of correlation length, and some studied the notion which is the inverse of the rate of exponential decay, and some were not very careful in distinguishing these two notions.) Prior to our work, the only mathematical result on the correlation length was (as far as we know) an upper bound of $e^{e^{O(\varepsilon^{-2})}}$ from [18] and [3].

Our proof method for the upper bound on the correlation length shares the underlying philosophy of "using the fluctuation of the random field to fight against the influence from the boundary" with the previous works [5], [18], [3], [24], and [2], and, in particular, in the sense that the proof strategy shares some similarity with [5] for deriving a contradiction for lower and upper bounds on difference of free energies. However, our strategy of deriving the lower bound on the difference of free energies (which is the key point for both [5] and our proof for upper bound on the correlation length) is very different from that in [5]. The proof of the lower bound of the correlation length is completely different from [5], [18], [3], [24], and [2] since this is a bound in a different direction from these works. In fact, it shares some similarity with [17] and [28] in terms of a connection to greedy lattice animals, as we elaborate in what follows. Let $\mathcal{A}_N$ be the collection of all connected subsets of Λ_N (i.e., lattice animals) that contain the origin, and let $\mathfrak{A}_N \subset \mathcal{A}_N$ be the collection of all simply connected subsets in $\mathcal{A}_N$. We define (the value of) the greedy lattice animal normalized by its boundary size as

$$\mathcal{S}_N = \max_{A \in \mathcal{A}_N} \frac{\sum_{v \in A} h_v}{|\partial A|} \quad \text{and} \quad \mathfrak{S}_N = \max_{A \in \mathfrak{A}_N} \frac{\sum_{v \in A} h_v}{|\partial A|}, \tag{5}$$

where $|\partial A|$ is the number of edges with exactly one endpoint in A. Theorem 1.1 is deeply connected to the following result (see Section 2 for an extensive discussion).

THEOREM 1.6
There exists a constant $C > 0$ such that for all $N \geq 3$ we have

$$C^{-1}(\log N)^{3/4} \leq \mathbb{E}[\mathfrak{S}_N] \leq \mathbb{E}[\mathcal{S}_N] \leq C(\log N)^{3/4}.$$

Remark 1.7
In Theorem 1.6 we described the maxima over both connected subsets and simply connected subsets for the following reasons: (1) both upper and lower bounds can be obtained for simply connected subsets first and then it is relatively easy to translate the bound to connected subsets; (2) while it is easier to prove the lower bound on the correlation length using the upper bound for the maximum over connected subsets, fundamentally what governs the behavior seems to be the maximum over simply connected subsets as we see in three dimensions (see also the proof in [25], where the maximum over simply connected subsets plays a fundamental role).

There is an interesting historical development on Theorem 1.6. The formulation of the statement immediately reminded the authors of the greedy lattice animal normalized by its volume (either normalized by the volume of the animal or by the volume of the box which contains the animal); this has been extensively studied for general disorder distributions (see [19], [20], [32], [35], [38]–[40], [42]). In particular, a rather precise description was obtained for the greedy lattice animal in [35], including that for rather general distributions (including the Gaussian distribution) the greedy lattice animal in a d-dimensional box of side length N normalized by N^d converges to a fixed constant (where the limiting constant depends on the distribution and the dimension). Despite a high degree of similarity in the definitions between the greedy lattice animal normalized by its boundary size and the version normalized by its volume, their behaviors seem to be quite different and the mathematical proofs in these two scenarios are largely different too: in some sense, such differences are suggested in the $(\log N)^{3/4}$ growth of $\mathcal{S}_N$, whereas in the version normalized by its volume, this was known to converge to a constant.

In three dimensions and higher, it was shown in [17] and [28] that the simply connected greedy lattice animal normalized by its boundary size (i.e., the analogue of $\mathfrak{S}_N$ in higher dimensions) is $O(1)$, which played a useful role in the proof for the existence of long range order at zero temperature in [36] and [14]. The $O(1)$ bound in three dimensions and higher and the $(\log N)^{3/4}$ growth in two dimensions for $\mathfrak{S}_N$ can be seen as a stronger version of the intuition underlying the Imry–Ma argument for the transition in dimension for statistical physics models with random field. Finally, we remark that in retrospect the proof in [17] and [28] amounts to a nontrivial application of Dudley's integral bound in [26] (note that the actual proof was implemented in a self-contained manner).

Initially, the authors thought that Theorem 1.6 was new and as a result provided a self-contained proof (for a slightly weaker version of Theorem 1.6) in the first version of this paper. During the submission, we discovered *in the literature* a nonobvious but deep connection between the greedy lattice animal normalized by its boundary size and the matching problem in Euclidean spaces. A fundamental problem is to match i.i.d. uniform points $X_1, \dots, X_{N^d}$ in a d-dimensional box containing N^d lattice points $y_1, \dots, y_{N^d}$ (i.e., to find a bijection π between these two set of points) in a certain optimal way. A classic result of [6] proved that $\mathbb{E}[\min_\pi \frac{1}{N^d} \sum_{i=1}^{N^d} |X_{\pi(i)} - y_i|] = \Theta(\sqrt{\log N})$ for $d = 2$. Since [6] there has been extensive work on matching problems, and one is encouraged to see [52] for an excellent account on the topic, which presents a unified proof via the majorizing measure theory. Of particular relevance to Theorem 1.6 is the celebrated work of [41] which showed that $\mathbb{E}[\min_\pi \max_{1 \le i \le N^d} |X_{\pi(i)} - y_i|] = O((\log N)^{3/4})$ for $d = 2$. The power of $3/4$ is deeply connected to the power in Theorem 1.6 via Hall's marriage lemma as we next explain.

Putting Halls's marriage lemma into the context of the matching problem, it states that if for each lattice point y_i there exists a collection of random points A_i such that

$$\left|\bigcup_{i \in I} A_i\right| \ge |I| \quad \text{for all } I \subset \{1, \dots, N^d\}, \tag{6}$$

then there exists a bijection π such that $X_{\pi(i)} \in A_i$. In light of this, a natural choice of A_i is the collection of all random points in a ball of radius r centered at y_i. As such, the result of [41] essentially reduces to showing that (6) holds for $r = O((\log N)^{3/4})$. It is plausible that in order to verify (6) one *essentially* only needs to consider I when I is the set of lattice points in a simply connected subset $\mathsf{I} \subset \mathbb{R}^d$. Since the union of the balls centered at I is an expansion of I, that is, the union of I and all points with distance at most r from I, a moment of thinking should lead to that with high probability for typical I (which turns out to be the ones we care most)

$$\lambda(\mathsf{I}) + cr\,\mathrm{length}(\partial\mathsf{I}) \le \left|\bigcup_{i \in I} A_i\right| \le \lambda(\mathsf{I}) + Cr\,\mathrm{length}(\partial\mathsf{I}),$$

where $c, C > 0$ are constants, $\lambda(\mathsf{I})$ is the number of random points in I, and $\mathrm{length}(\partial\mathsf{I})$ is the length of the boundary curve for I. Since $\lambda(\mathsf{I}) - |I|$ is a mean-zero random variable, which can be roughly regarded as a Gaussian variable, and thus in spirit

$\{\lambda(\mathsf{I}) - |I|\}_I$ resembles the lattice animal process. In light of this discussion, heuristically the result of [41] reduces to $\max_I \frac{\lambda(\mathsf{I})-|I|}{\mathrm{length}(\partial \mathsf{I})} = O((\log N)^{3/4})$, which resembles the upper bound in Theorem 1.6. Indeed, this connection was nicely explained in [52], which also nicely explains the conceptual difference for the behavior between $d = 2$ and $d \geq 3$.

Having explained the connection to the matching problem, we come back to what is most relevant to us, namely, the proof of Theorem 1.6. It turns out that a proof of Theorem 1.6 was essentially contained in [52], and in Section 4 we present this in a more explicit manner without claiming any credit. In addition to that, in the arXiv version of this paper (arXiv:2011.08768), we still keep our "original" proof since we feel that our proof seems to explain some of the geometric intuition in an arguably more intuitive way and thus we feel that this framework of multi-scale analysis may turn out to be useful in some related problems (e.g., random metric of Liouille quantum gravity).

We conclude the introduction by some discussions on future research. As a natural question, one may ask what is the correlation length for the random field Potts model. We expect that the same scaling of $e^{\varepsilon^{-4/3}}$ should occur. The nontrivial part is the upper bound, for which our proof uses monotonicity properties of the Ising model in a substantial manner.

2. Overview of the proof

In this section we introduce the main idea behind the proof of Theorem 1.1, and in particular we give some intuition for the exponent $4/3$. We will then discuss the obstacles that arise in making this proof sketch rigorous.

2.1. Notation

For a real (or integer-valued) vector $\mathbf{x}$ (in any dimension), we denote its Euclidean norm by $|\mathbf{x}|$. For a finite set A, we denote its cardinality by $|A|$. For $A \subset \mathbb{R}^2$ we denote the Lebesgue measure of A by $\lambda(A)$. For a curve η, we denote its length by $l(\eta)$. We use A^c to denote the complement of the set (or event) A. If A is an event, we denote its indicator by $\mathbf{1}_A$.

In what follows, we let $c, c', c'', C, C', C'' > 0$ be arbitrary constants whose values may change each time they appear, and may depend on $\mathtt{m}$ but not on ε or N. Numbered constants $c_1, c_2, \ldots$ may still depend on $\mathtt{m}$ but their values will be fixed throughout the paper.

We say two points $u, v \in \mathbb{Z}^2$ are adjacent to each other if $|u - v| = 1$, in which case we write $u \sim v$. When convenient, we will think of $\mathbb{Z}^2$ as being embedded in $\mathbb{R}^2$ in the obvious way. For any set $A \subset \mathbb{Z}^2$, we let $\partial A = \{(u, v) : u \sim v, u \in A, v \in \mathbb{Z}^2 \setminus A\}$ denote the edge boundary of A in the nearest neighbor graph on $\mathbb{Z}^2$.

2.2. Emergence of the 3/4 *exponent*

Let $\sigma^{\pm}(\Lambda_N, \varepsilon h)$ be the ground states with respect to the plus and minus boundary conditions; that is, they are minimizers of the Hamiltonians $H^{\pm}(\Lambda_N, \varepsilon h)$, respectively. (Since our field h has a continuous distribution, the ground state with respect to each boundary condition is unique with probability 1.) Suppose $\sigma_o^-(\Lambda_N, \varepsilon h) = 1$ and S is the connected component of $\{v \in \Lambda_N : \sigma_v^-(\Lambda_N, \varepsilon h) = 1\}$ that contains o. Then necessarily we have $\sum_{v\in S} \varepsilon h_v \geq |\partial S|$, because, otherwise, flipping spins on S would decrease the Hamiltonian and contradict the definition of the ground state. In other words,

$$\sigma_o^-(\Lambda_N, \varepsilon h) = 1 \quad \text{implies that} \max_{A\in\mathcal{A}_N} \frac{\sum_{v\in A} \varepsilon h_v}{|\partial A|} \geq 1. \tag{7}$$

This explains why the greedy lattice animal normalized by its boundary size is connected to the random field Ising model. From the discussion above, an upper bound on the greedy lattice animal directly gives a lower bound on the correlation length for $\beta = \infty$. In what follows, we will sketch an argument leading to the emergence of 3/4-exponent in the lower bound of Theorem 1.6.

For convenience of exposition, we will pass to the continuum. To each vertex $v \in \mathbb{Z}^2$ we can associate the axis-aligned unit square R_v centered at v, and to each subset $A \subset \mathbb{Z}^2$ the set $\mathtt{A} = \bigcup_{v\in A} R_v$. Notice that the perimeter of $\mathtt{A}$ (which we denote by $l(\partial\mathtt{A})$) is equal to the boundary size $|\partial A|$. Next, we let W be a standard white noise on $\mathbb{R}^2$ such that $W(R_v) = h_v$ for each $v \in \mathbb{Z}^2$. In particular, for any $A \subset \mathbb{Z}^2$ we have $\sum_{v\in A} h_v = W(\mathtt{A})$. We will sketch a procedure to construct a polygon $P \subset [-N, N]^2$ (for $N \geq e^{C\varepsilon^{-4/3}}$) such that each side of P has length at least 1 (we will refer to this as a *polygon animal* in what follows) and $\varepsilon W(P) > l(\partial P)$. The idea is to recursively expand P by possibly joining to it a triangle T such that the standard deviation of $\varepsilon W(T)$ is of the same order as $l(\partial(P \cup T)) - l(\partial P)$. We remark that we choose to add triangles instead of rectangles for the reason that adding a triangle with the same area results in a substantially smaller increase in the perimeter.

We begin with the polygon $P_1 = [-N/2, N/2]^2$. Having constructed P_k, we construct P_{k+1} as follows. For each side s of P_k, we consider the isosceles triangle T_s with base given by the "middle" segment of s of length $l(s)/2$ and of height $\varepsilon^{2/3}l(s)/8$ that points out of P_k. We add T_s to the polygon if $W(T_s) > 0$ (which occurs with probability 1/2). If we do not add T_s, we split s into four sides of equal length. We let P_{k+1} be the polygon obtained by applying this procedure to each side of P_k. See Figure 1 for an illustration of the process.

Next, we let $a_k = \mathbb{E}[\varepsilon W(P_k) - l(\partial P_k)]$. Our goal is to lower-bound $a_{k+1} - a_k$. For each side s of P_k, we have $\lambda(T_s) = \varepsilon^{2/3}l(s)^2/32$ (recall that λ denotes the Lebesgue measure on $\mathbb{R}^2$), and an elementary calculation shows that adding T_s to P_k increases the perimeter by $\Delta_s < \varepsilon^{4/3}l(s)/16$. If we ignore the potential overlap between the triangles corresponding to different iterations of the scheme, then we would have $\mathbb{E}[\varepsilon W(T_s) \mid W(T_s) > 0] > 2\Delta_s$. Summing over all sides of P_k, we get that

$$a_{k+1} - a_k \geq \frac{1}{16}\varepsilon^{4/3}\mathbb{E}\big[l(\partial P_k)\big] \geq \frac{1}{16}\varepsilon^{4/3} l(\partial P_1).$$

Further, since at each step each side s is split into four sides of length at least $l(s)/4$, we see that for $k^* = \lfloor \log_{16} N \rfloor$ each side of P_{k^*} has length at least 1 deterministically. This implies that for $N \geq 10^5 \exp(10^5 \varepsilon^{-4/3})$, we have (noting that $a_1 = -l(\partial P_1)$)

$$a_{k^*} = a_1 + \sum_{k=1}^{k^*-1} (a_{k+1} - a_k) \geq \frac{1}{16}(k^* - 1)\varepsilon^{4/3} l(\partial P_1) - l(\partial P_1) \geq l(\partial P_1) = 4N.$$

The construction above captures the main idea of the proof for the lower bound in Theorem 1.6: while we ignored a number of technical details and we carried out the analysis in the continuum, it is straightforward to complete a formal argument. We will not do so since the proof of the upper bound on the correlation length contains a complete argument which is strictly more involved than the proof of the lower bound on the greedy lattice animal (formally, one can follow the proof in Section 3 with $\Gamma(A) = \sum_{v \in A} \varepsilon h_v$).

While the above construction suggests the emergence of the $4/3$ exponent in RFIM, it falls short of establishing either the upper or lower bound on the correlation length in Theorem 1.1. In the next two subsections, we will point out the main obstacles and describe at an overview level our approaches to address these challenges.

2.3. Upper bound on correlation length
Our goal is to prove that for every $\mathtt{m} \in (0, 1)$ there exists $C_1 = C_1(\mathtt{m}) > 0$ (independent of β) such that for all $\varepsilon \in (0, 1)$ and $N \geq \exp(C_1 \varepsilon^{-4/3})$,

$$\mathtt{m}_{\beta, \Lambda_{4N}, \varepsilon} \leq \mathtt{m}. \tag{8}$$

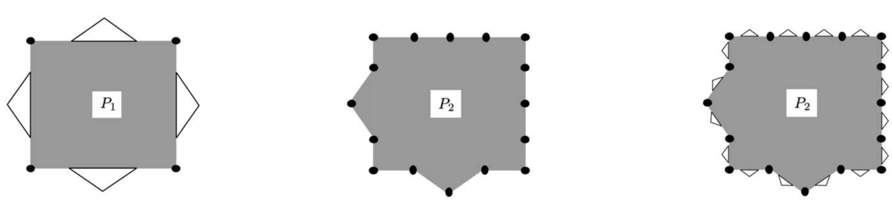

Figure 1. From left to right: P_1 with potential triangles to be added; P_1 with triangles added (i.e., P_2); P_2 with potential triangles to be added.

(We have used $4N$ instead of N in the above for later notational convenience.) While the construction in Section 2.2 hints at the emergence of the $4/3$ exponent, the following is a main obstacle in making this a rigorous proof for the upper bound on the correlation length even in the special case when $\beta = \infty$: the existence of $A \in \mathfrak{A}_{4N}$ such that $\varepsilon \sum_{v \in A} h_v > |\partial A|$ is not sufficient for $\sigma_o^-(\Lambda_{4N}, \varepsilon h) = 1$ (e.g., if $\varepsilon h_v = 20$ for some $v \sim o$ and $\varepsilon h_o = -5$, then $A = \{v, o\}$ satisfies the desired property but $\sigma_o^-(\Lambda_{4N}, \varepsilon h) = -1$; this is because when $|\varepsilon h_v| > 4$ the ground state at v agrees with the sign of h_v). To overcome this challenge, we will define a suitable Γ-function for general β, and in the special case of $\beta = \infty$ the function (very roughly speaking) can certify $\sigma_o^-(\Lambda_{4N}, \varepsilon h) = 1$ (the rigorous meaning of this is via an argument by contradiction). For $\Omega \subset \mathbb{Z}^2$ and an external field $f : \mathbb{Z}^2 \mapsto \mathbb{R}$, we define $H^{\pm}(\sigma, \Omega, f)$ and $\mu^{\pm}_{\beta,\Omega,f}$ as in (1) and (2) except replacing Λ_N, εh by Ω, f. Define the free energy

$$F^{\pm}(\Omega, f) = F^{\pm}(\beta, \Omega, f) = -\frac{1}{\beta} \log \sum_{\sigma \in \{-1,1\}^{\Omega}} e^{-\beta H^{\pm}(\sigma, \Omega, f)}. \tag{9}$$

For $A \subset \Omega$, our Γ-function is defined to be the difference of the free energies on $\Omega \setminus A$ with respect to the positive and negative boundary conditions, as follows:

$$\Gamma(A, \Omega, f) = \Delta F(\Omega \setminus A, f)$$
$$\text{where } \Delta F(B, f) = F^{+}(B, f) - F^{-}(B, f). \tag{10}$$

Before proceeding, we make a few remarks about why we choose the Γ-function as the difference of free energies on $\Omega \setminus A$ instead of A. In our analysis, we will let the reference domain be $\Omega = \Lambda_{2N}$ and construct a sequence $(A_n)_{n \geq 1}$ with increasing (expected) value of Γ. To this end, we need the increment $\Gamma(A \cup B) - \Gamma(A)$ to have nice monotonicity properties as a function of εh so that we can keep track of the probabilistic behavior of the increment when employing a recursive construction as in Section 2.2. The choice of $\Omega \setminus A$ gives the desired direction of monotonicity; see Lemma 3.1.

With Γ defined as in (10), our proof proceeds by demonstrating a contradiction if we assume (8) fails. On the one hand, we have the following upper bound (cf. [5, Proposition 5.2(iii)]).

LEMMA 2.1
$|\Gamma(A, \Omega, f)| \leq 2|\partial(\Omega \setminus A)|$ *for all* (A, Ω, f) *with* $A \subset \Omega$.

The proof in the case of the Ising model is elementary. It follows from the fact that $|H^{+}(\sigma, B, f) - H^{-}(\sigma, B, f)| \leq 2|\partial B|$ and

$$\Delta F(\beta, B, f) = -\frac{1}{\beta} \log\Big(\big\langle \exp\big(-\beta\big[H^{+}(\sigma, B, f) - H^{-}(\sigma, B, f)\big]\big)\big\rangle_{\mu^{-}_{\beta,B,f}}\Big).$$

On the other hand, assuming (8) fails, we will show that the variance of the increment $\Gamma(A \cup B, \Lambda_{2N}, \varepsilon h) - \Gamma(A, \Lambda_{2N}, \varepsilon h)$ is comparable to that of $\sum_{v \in B} \varepsilon h_v$ and then we can hope to follow the argument in Section 2.2 to construct a set whose Γ-function value is larger than its boundary size. As mentioned, a crucial feature we use in proving this is a monotonicity property for the increment of the Γ-function, as incorporated in Lemma 3.1.

With all these intuitions in place, the actual proof in Section 3 is written in a way that both fills in the gaps left by the heuristics from Section 2.2 and addresses the challenges from the random field Ising model. For the former, for instance, Figure 2 illustrates how we address the gap from correlations between different rounds of recursive constructions by making the decision for the triangle $T_{1,i}$ only based on disorder in the smaller blue triangle $T^*_{1,i}$. For the latter, Lemma 3.11 manifests the power of Lemma 3.1 and says that the correlation through the Ising measure is in our desirable direction and Lemma 3.12 says that the marginal effect from the disorder in a triangle to our observable is similar to the white noise value of this triangle.

2.4. *Lower bound on correlation length*

In light of (7), the lower bound on the correlation length for $\beta = \infty$ can be proved via an upper bound on the greedy lattice animal. This is an example of the classic question of computing the (expected) supremum of a Gaussian process. This has been well understood in general, culminating in Talagrand's majorizing measure theorem in [51], which improved previous results in [26] and [27]: as a highlight, an up-to-constant estimate for the supremum of a general Gaussian process was provided in terms of the (so-called) γ_2-functional associated with this process. Specifically for the example of our lattice animal process, the upper bound was already hinted in [41], as we explained earlier, whose proof together with proofs for various results on matching problems were unified and streamlined in [52]. In particular, the following result was essentially contained in [52].

PROPOSITION 2.2
Let $\mathfrak{B}_N$ be the collection of simply connected lattice animals contained in Λ_N. There exists a constant $C_1 > 0$ such that for $N > 1$ we have

$$\mathbb{P}\Big(\max_{B \in \mathfrak{B}_N} \frac{\sum_{v \in B} h_v}{|\partial B|} > C_1 (\log N)^{3/4} + u\Big) \leq \exp(-u^2/2) \quad \forall u > 0.$$

To conclude this section, we prove the lower bound in Theorem 1.1 and the upper bound in Theorem 1.6 using Proposition 2.2.

Proof of lower bound in Theorem 1.1 and upper bound in Theorem 1.6
The main step of the proof is relating the bound on simply connected lattice animals to a bound on lattice animals. Let $\mathcal{B}_N$ be the collection of connected lattice animals contained in Λ_N. We claim that

$$\max_{B \in \mathfrak{B}_N} \frac{|\sum_{v \in B} h_v|}{|\partial B|} = \max_{B' \in \mathcal{B}_N} \frac{|\sum_{v \in B'} h_v|}{|\partial B'|}. \tag{11}$$

For any lattice animal B, let $\tilde{B}$ be the collection of vertices that is enclosed by B, that is, disconnected by B from ∞. Let $B_1, \dots, B_k$ be the connected components of $\tilde{B} \setminus B$. Note that $B_1, \dots, B_k$ are simply connected, since if v is separated from ∞ by B_i, then $v \in \tilde{B}$, and, in addition, since B is connected, it follows that $v \notin B$. Since ∂B is the disjoint union of $\partial \tilde{B}$ and $\partial B_1, \dots, \partial B_k$, we have

$$\begin{aligned} \frac{|\sum_{v \in B} h_v|}{|\partial B|} &\le \frac{1}{|\partial B|} \sum_{i=1}^{k} \Big| \sum_{v \in B_i} h_v \Big| \\ &= \sum_{i=1}^{k} \frac{|\partial B_i|}{|\partial B|} \frac{|\sum_{v \in B_i} h_v|}{|\partial B_i|} \\ &\le \max_{i=1,\dots,k} \frac{|\sum_{v \in B_i} h_v|}{|\partial B_i|}, \end{aligned}$$

where the last inequality follows from the fact that the coefficients $|\partial B_i|/|\partial B|$ sum up to 1. This completes the verification of (11). By Proposition 2.2 (and the fact that h is symmetric), the maximum on the left-hand side of (11) is of order $(\log N)^{3/4}$, which proves the upper bound in Theorem 1.6. This also shows that it is less than ε^{-1} with high probability as long as $N \le \exp(\varepsilon^{4/3}/C)$. By (7) (and a symmetric condition for $\sigma_o^+(\Lambda_N, \varepsilon h)$), this implies that $\sigma_o^\pm(\Lambda_N, \varepsilon h) = \pm 1$ with high probability and thus completes the proof of the lower bound in Theorem 1.1. □

3. Upper bound on correlation length

This section is devoted to the proof of the upper bound on the correlation length, as incorporated in (8). Recall the definition of Γ-function given in (10). Recall from Lemma 2.1 that $\Gamma(A, \Omega, f) \le 2|\partial(\Omega \setminus A)|$ for all (A, Ω, f). With this at hand, we use the bulk of this section to show that if (8) fails, then there exists a random subset $\mathsf{P}^* \subset \Lambda_{2N}$ such that

$$\mathbb{E}\big[\Gamma(\mathsf{P}^*, \Lambda_{2N}, \varepsilon h) - 2\big|\partial(\Lambda_{2N} \setminus \mathsf{P}^*)\big|\big] > 0, \tag{12}$$

which is a contradiction. As mentioned in Section 2.3, a key element of our analysis is a monotonicity property of the Γ-function which we incorporate in Lemma 3.1.

In Section 3.2, we construct P^* by enhancing the procedure in Section 2.2 in order to address additional complications due to the complexity of the Γ-function. In Section 3.3 we carry out the probabilistic analysis and prove (12) under the assumption that (8) fails.

3.1. Monotonicity property of the Γ-function

LEMMA 3.1
For disjoint subsets $A, B \subset \Omega$, we have that $(\Gamma(A \cup B, \Omega, f) - \Gamma(A, \Omega, f))$ is increasing in $\{f_v : v \in B\}$, decreasing in $\{f_v : v \notin A \cup B\}$, and does not depend on $\{f_v : v \in A\}$.

Proof
Recall the definition of ΔF in (10). Write

$$\Delta\langle\sigma_v\rangle_{\beta,\Omega,f} = \frac{1}{2}\big(\langle\sigma_v^+\rangle_{\mu^+_{\beta,\Omega,f}} - \langle\sigma_v^-\rangle_{\mu^-_{\beta,\Omega,f}}\big).$$

We compute partial derivatives and get that

$$\partial_{f_v}\Delta F(A') = -2\Delta\langle\sigma_v\rangle_{\beta,A',f}\mathbf{1}_{v\in A'} \tag{13}$$

for any $A' \subset \mathbb{Z}^2$ (where the minus sign inherits from that in the definition of free energy). Write

$$\begin{aligned} G(A, B, \Omega, f) &= \Gamma(A \cup B, \Omega, f) - \Gamma(A, \Omega, f) \\ &= \Delta F\big(\Omega \setminus (A \cup B), f\big) - \Delta F(\Omega \setminus A, f). \end{aligned} \tag{14}$$

Using (13) and the monotonicity of the Ising model (cf. [3, Section 2.2]) we get that for $v \in \Omega \setminus (A \cup B)$,

$$\partial_{f_v} G(A, B, \Omega, f) = 2\big(\Delta\langle\sigma_v\rangle_{\beta,\Omega\setminus A,f} - \Delta\langle\sigma_v\rangle_{\beta,\Omega\setminus(A\cup B),f}\big) \le 0;$$

for $v \in B$,

$$\partial_{f_v} G(A, B, \Omega, f) = 2\Delta\langle\sigma_v\rangle_{\beta,\Omega\setminus A,f} \ge 0;$$

and for $v \in A$, $\partial_{f_v} G(A, B, \Omega, f) = 0$. This completes the proof of the lemma. □

It is also worth noting that it follows from the expressions obtained for the partial derivatives of G that

$$\big|\partial_{f_v} G(A, B, \Omega, f)\big| \le 2 \quad \text{for all } A, B \subset \Omega \text{ and } v \in \Omega. \tag{15}$$

Figure 2. (Color online) P_1 with $(T_{1,i})_{i=1}^4$. The blue triangles are $(T_{1,i}^*)_{i=1}^4$.

3.2. Randomized geometric constructions

In this subsection we give the details of the construction of the random set P^* (following Section 2.2) and prove a few geometric lemmas.

3.2.1. Construction of P^*

In order to construct P^*, we will recursively construct a sequence of polygons $(P_n)_{n\geq 1}$ contained in $[-2N, 2N]^2$ and a corresponding sequence of subsets $(\mathsf{P}_n)_{n\geq 1}$ given by $\mathsf{P}_n = P_n \cap \mathbb{Z}^2$. Let $\mathtt{m} \in (0, 1)$, and let $\delta = 10^{-2}(\varepsilon\mathtt{m})^{2/3}$ (where 10^{-2} is chosen as a small but otherwise arbitrary constant). As an initialization for our procedure, we set $P_1 = [-N, N]^2$ and let $(S_{1,i})_{i=1}^4$ be the sides of P_1, numbered in counterclockwise order with $S_{1,1}$ being the bottom side. We next describe our recursive construction.

For $n \geq 1$, assume that P_n has been constructed and that P_n has 4^n sides $(S_{n,i})_{i=1}^{4^n}$ numbered in counterclockwise order. For each i, let $r_{n,i} = l(S_{n,i})/4$ and partition $S_{n,i}$ into four segments of length $r_{n,i}$. Let $T_{n,i}$ be the isosceles triangle with base given by the two middle segments of $S_{n,i}$ and height $\delta r_{n,i}$ such that $T_{n,i}$ points out from P_n (note that $T_{n,i}$ is measurable with respect to P_n; see Remark 3.2(ii)). Let $\mathsf{T}_{n,i} = T_{n,i} \cap \mathbb{Z}^2$. Further, let $T_{n,i}^* \subset T_{n,i}$ be the triangle consisting of all points in $T_{n,i}$ which have distance at least $2\delta r_{n,i}/3$ from the base, and let $\mathsf{T}_{n,i}^* = T_{n,i}^* \cap \mathbb{Z}^2$. See Figure 2 for an illustration. We will decide whether to add the triangle $T_{n,i}$ to the polygon based on the current polygon and the field in $T_{n,i}^*$ only (instead of the field in $T_{n,i}$); this ensures that our construction explores disjoint regions in different iterations (see Lemma 3.7).

In order to construct P_{n+1}, we will decide whether to add the triangle $T_{n,i}$ for $1 \leq i \leq 4^n$ depending on whether the expected increase to the value of the Γ-function is larger than the resulting increase in the boundary size of the polygon. To formalize this idea, we will recursively define a sequence of polygons $(P_{n,i})_{i=0}^{4^n}$ and their corresponding lattice subsets $\mathsf{P}_{n,i} = P_{n,i} \cap \mathbb{Z}^2$. For the base case, we let $P_{n,0} = P_n$. For $1 \leq i \leq 4^n$, let $\mathscr{F}_{n,i}$ be the σ-algebra generated by $P_{n,i-1}$ and $\{h_v : v \in \mathsf{T}_{n,i}^*\}$

(by definition, $T_{n,i}$ is measurable with respect to P_n as mentioned earlier, and thus from our recursive construction below $T_{n,i}$ is also measurable with respect to $P_{n,i-1}$, as elaborated in Remark 3.10). Note that $\mathcal{F}_{n,i}$ is not increasing. In particular, $\mathcal{F}_{n,i}$ contains information about $\{h_v : v \in \bigcup_{i'=1}^{i-1} \mathsf{T}^*_{n,i'}\}$ only via $P_{n,i-1}$. Define

$$\gamma_{n,i} = \mathbb{E}\big[\Gamma(\mathsf{P}_{n,i-1} \cup \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon h) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon h) \mid \mathcal{F}_{n,i}\big] \tag{16}$$

as the aforementioned expected increment of Γ (see Remark 3.2(iv)). Then we define

$$P_{n,i} = \begin{cases} P_{n,i-1} \cup T_{n,i}, & \text{if } \gamma_{n,i} \geq 10\delta^2 r_{n,i}, \\ P_{n,i-1}, & \text{if } \gamma_{n,i} < 10\delta^2 r_{n,i}. \end{cases}$$

We let $P_{n+1} = P_{n,4^n}$ and let $(S_{n+1,i})_{i=1}^{4^{n+1}}$ be the sides of P_{n+1}, numbered in counterclockwise order so that $S_{n+1,j} \subset S_{n,i} \cup T_{n,i}$ for $1 \leq i \leq 4^n$ and $4(i-1)+1 \leq j \leq 4i$. That is, for each i the sides of P_{n+1} that "come from" $S_{n,i}$ are $(S_{n+1,j})_{j=4(i-1)+1}^{4i}$. This concludes the construction of P_{n+1}.

Finally, we let $n^* = \lfloor \log_{16}(N) \rfloor$, $P^* = P_{n^*}$, and $\mathsf{P}^* = \mathsf{P}_{n^*}$. This choice of n^* ensures that $\delta r_{n,i}$ is large for all $n \leq n^*$, which will allow us to approximate $|\mathsf{T}^*_{n,i}|$ by the area of $T^*_{n,i}$.

Before proceeding, we make a few expository remarks on our construction.

Remark 3.2
(i) We have assumed that for each $n \geq 1$, the triangles $(T_{n,i})_{i=1}^{4^n}$ are disjoint and $T_{n,i} \cap P_n \subset S_{n,i}$ for all i. This is justified by Lemma 3.3. We also note that if $\gamma_{n,i} \leq 10\delta^2 r_{n,i}$ (i.e., $T_{n,i}$ is not included in P_{n+1}), then $S_{n,i}$ is split into four sides of P_{n+1} with internal angle π between them. These two assumptions ensure that P_{n+1} is a polygon with 4^{n+1} sides.

(ii) It will be useful in our proof that the numbering of the sides of P_n is deterministic so that the sequence $(T_{n,i})_{i=1}^{4^n}$ is measurable with respect to P_n. The specific choice given in the construction is made for convenience.

(iii) Our choice of δ is based on similar considerations to those given in Section 2.2. The condition $\gamma_{n,i} > 10\delta^2 r_{n,i}$ is based on the following calculation. Since $l(\partial(P_{n,i-1} \cup T_{n,i})) - l(\partial P_{n,i-1}) \leq \delta^2 r_{n,i}$, if adding $T_{n,i}$ to $P_{n,i-1}$ increases Γ by $10\delta^2 r_{n,i}$, then the difference between Γ and $8l(\partial P)$ will increase (the constant 8 will be explained in Section 3.3.2).

(iv) Note that $\gamma_{n,i}$ depends on $\{h_v : v \in \bigcup_{(k,j)<(n,i)} \mathsf{T}^*_{k,j}\}$ only through $P_{n,i-1}$ due to our particular choice of $\mathcal{F}_{n,i}$ (here $(k,j) < (n,i)$ if $k < n$, or $k = n$ and $j < i$). The reason we choose $\mathcal{F}_{n,i}$ this way is that we can show the expected value of the derivative of the increment with respect to h_v for $v \in \mathsf{T}^*_{n,i}$ is bounded from below by $\mathtt{m}_{\beta,\Lambda_{4N},\varepsilon}$, which is at least $\mathtt{m}$ by our assumption that (8) fails. Therefore, the lower bound on the variance obtained this way is comparable to the upper bound on the variance obtained from general Gaussian concentration inequality, and this is a useful property in our analysis later. If instead $\gamma_{n,i}$ was defined by conditioning on the field in $T^*_{k,j}$ for $(k, j) \le (n, i)$, then the field in previously rejected triangles would affect $\gamma_{n,i}$ but potentially only very weakly. This would mean that our lower bound on the variance of $\gamma_{n,i}$ would be much smaller than the upper bound from Gaussian concentration inequality since now the upper bound would be from the field in a much larger region.

3.2.2. Geometric lemmas

In this subsection we prove a few lemmas which ensure that the polygons $(P_n)_{n=1}^{\infty}$ have desirable geometric properties.

LEMMA 3.3

For all $n \ge 1$, the triangles $(T_{n,i})_{n=1}^{4^n}$ are disjoint and $T_{n,i} \cap P_n \subset S_{n,i}$ for $1 \le i \le 4^n$.

LEMMA 3.4

$T_{n,i} \subset [-2N, 2N]^2$ for all $n \ge 1$ and $1 \le i \le 4^n$.

We first state and prove a lemma which easily implies Lemmas 3.3 and 3.4. We begin with some notation. Let $\mathcal{I} = \{(n, i) : n \ge 1, 1 \le i \le 4^n\}$, and let $\mathcal{G}$ be the directed forest with vertex set $\mathcal{I}$ and edge set

$$\big\{\big((n, i), (n + 1, j)\big) : 4(i - 1) + 1 \le j \le 4i\big\}.$$

That is, there is an edge from (n, i) to $(n + 1, j)$ if $S_{n+1,j} \subset S_{n,i} \cup T_{n,i}$. In this case, we say that $S_{n+1,j}$ is a child of $S_{n,i}$ (or $S_{n,i}$ is the parent of $S_{n+1,j}$). We let $\mathcal{G}_{n,i}$ be the subtree of $\mathcal{G}$ rooted at (n, i). That is, the subgraph of $\mathcal{G}$ on the vertices $(k, j) \in \mathcal{I}$ for which there exists a directed path from (n, i) to (k, j). If $(k, j) \in \mathcal{G}_{n,i}$ we call (k, j) a descendant of (n, i).

LEMMA 3.5

Let $(n, i) \in \mathcal{I}$ and $\mathcal{T}_{n,i}$ be the isosceles triangle with base $S_{n,i}$ and height $2\delta r_{n,i}$ that contains $T_{n,i}$. Then for every $(k, j) \in \mathcal{G}_{n,i}$, we have $\mathcal{T}_{k,j} \subset \mathcal{T}_{n,i}$.

See Figure 3 for an illustration of $(\mathcal{T}_{1,i})_{i=1}^{4}$.

Proof

It suffices to show that if $(n + 1, j)$ is a child of (n, i), then $\mathcal{T}_{n,i}$ contains $\mathcal{T}_{n+1,j}$. For concreteness, we take $i = 1$ and therefore $1 \le j \le 4$. It is immediate that $\mathcal{T}_{n+1,j} \subset \mathcal{T}_{n,1}$ for $j \in \{1, 4\}$ and that $\mathcal{T}_{n+1,j} \subset \mathcal{T}_{n,1}$ for $j \in \{2, 3\}$ if $T_{n,1}$ is not contained in

P_{n+1}. Assuming $T_{n,1} \subset P_{n+1}$, we can use the fact that $T_{n,1}$ is similar to $\mathcal{T}_{n,1}$ (and in fact their sides are parallel) to show that the distance between $\partial T_{n,1} \setminus S_{n,1}$ and $\partial \mathcal{T}_{n,1} \setminus S_{n,1}$ is given by $d_{n,1} = \frac{\delta}{\sqrt{1+\delta^2}} r_{n,1}$. See Figure 4 for an illustration. Further, the height of $\mathcal{T}_{n+1,2}$ and $\mathcal{T}_{n+1,3}$ is given by $\frac{\delta\sqrt{1+\delta^2}}{2} r_{n,1}$. Since $\delta < 1$, this height is strictly smaller than $d_{n,1}$ and therefore $\mathcal{T}_{n+1,2}$ and $\mathcal{T}_{n+1,3}$ are contained in $\mathcal{T}_{n,1}$ as claimed. □

Proof of Lemma 3.3

Let $\theta = \arctan(\delta)$, and note that θ is the internal angle (with respect to $\mathcal{T}_{n,i}$) between $S_{n,i}$ and the other sides of $\mathcal{T}_{n,i}$. The same holds for $T_{n,i}$. Since $\delta < 1$, we have $\theta < \pi/4$.

It suffices to show that for every $(n,i),(n,j) \in \mathcal{I}$ we have $\mathcal{T}_{n,i} \cap \mathcal{T}_{n,j} = S_{n,i} \cap S_{n,j}$. We prove this by induction. It clearly holds for $P_1 = [-N,N]^2$. By Lemma 3.5, if it holds for P_n, then $\mathcal{T}_{n+1,i} \cap \mathcal{T}_{n+1,j} = S_{n+1,i} \cap S_{n+1,j}$ when $(n+1,i)$ and $(n+1,j)$ are not siblings (i.e., they do not have the same parent). When $(n+1,i)$ and $(n+1,j)$ are siblings, it is immediate that $\mathcal{T}_{n+1,i} \cap \mathcal{T}_{n+1,j} = \emptyset$ unless $S_{n+1,i}$ and $S_{n+1,j}$ are adjacent (i.e., $|i-j|=1$). Assuming $S_{n+1,i}$ and $S_{n+1,j}$ are adjacent, we note that the external (with respect to P_{n+1}) angle between them is at least $\pi - \theta$. Recall that the internal (with respect to $\mathcal{T}_{n+1,i}$) angle between $S_{n+1,i}$ and the other sides of $\mathcal{T}_{n+1,i}$ is θ, and the same holds for j. Since $3\theta < \frac{3\pi}{4} < \pi$, we see that $\mathcal{T}_{n+1,i} \cap \mathcal{T}_{n+1,j} = S_{n+1,i} \cap S_{n+1,j}$ (see Figure 5 for an illustration of this argument). □

Proof of Lemma 3.4

Since $\delta < 1/2$, we have $\mathcal{T}_{1,j} \subset [-2N,2N]^2$ for $j = 1,2,3,4$, so the conclusion follows from Lemma 3.5. □

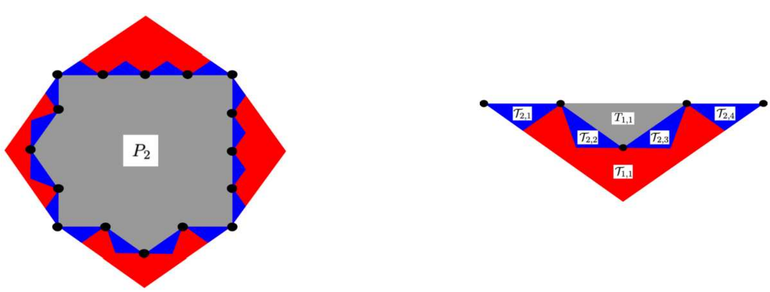

Figure 3. (Color online) P_2 with $\bigcup_{i=1}^{16} \mathcal{T}_{2,i}$ in blue and $\bigcup_{i=1}^{4} \mathcal{T}_{1,i} \setminus \bigcup_{i=1}^{16} \mathcal{T}_{2,i}$ in red.

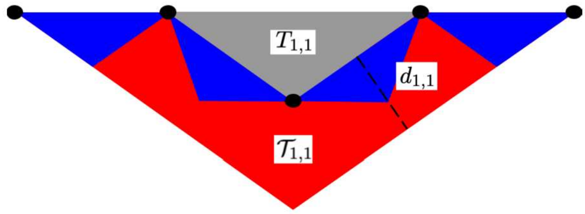

Figure 4. (Color online) $d_{1,1}$ is the distance between $\partial T_{1,1} \setminus S_{1,1}$ and $\partial \mathcal{T}_{1,1} \setminus S_{1,1}$.

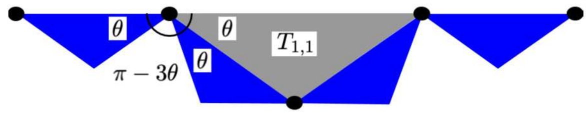

Figure 5. (Color online) The fact that $\theta < \pi/3$ ensures that $\mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$ intersect only at their common vertex.

We prove a few more lemmas that will be useful for probabilistic analysis in Section 3.3.

LEMMA 3.6
Let P be a polygon with q sides and $\mathsf{P} = P \cap \mathbb{Z}^2$. Then $|\partial \mathsf{P}| \le \sqrt{2} l(\partial P) + 2q$.

Proof
Note that $|\partial \mathsf{P}|$ is bounded above by the number of edges that intersect ∂P (if ∂P contains a vertex in $\mathbb{Z}^2$, then we count this as two intersections). In addition, the number of edges intersecting any line segment is upper-bounded by 2 plus the ℓ_1 distance between its endpoints, which is in turn bounded by 2 plus $\sqrt{2}$ times the Euclidean length of the segment. This yields the desired bound. □

LEMMA 3.7
Let (n,i), $(k,j) \in \mathcal{I}$ be such that $(n,i) \ne (k,j)$. Then $T^*_{n,i} \cap T^*_{k,j} = \emptyset$.

Proof
We assume without loss of generality that $n \le k$. Let j' be the unique integer such that (k,j) is a descendant of (n,j') (if $k = n$, then $j = j'$). By Lemma 3.5 we have $T^*_{n,i} \subset \mathcal{T}_{n,i}$ and $T^*_{k,j} \subset \mathcal{T}_{n,j'}$. We showed in the proof of Lemma 3.3 that if $j' \ne i$, then $\mathcal{T}_{n,i} \cap \mathcal{T}_{n,j'} = S_{n,i} \cap S_{n,j'}$, which implies $T^*_{n,i} \cap T^*_{k,j} = \emptyset$ since $T^*_{n,i} \cap S_{n,i} = \emptyset$. Therefore, we assume $j' = i$ (i.e., $S_{k,j}$ is a descendant of $S_{n,i}$). Note that this implies that $k > n$. To conclude the proof, we consider separately the case that $T_{n,i}$ is

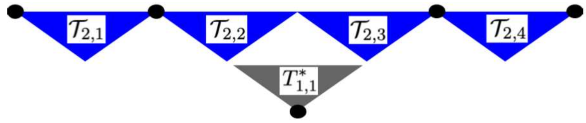

Figure 6. (Color online) If $Z_{1,1} = 0$, then $T^*_{1,1}$ is disjoint from $(\mathcal{T}_{2,i})_{i=1}^4$.

contained in P_{n+1} and the case that it is not. For concreteness, we let $i = 1$. If $T_{n,1}$ is not contained in P_{n+1}, then $T^*_{n,i}$ is disjoint from $\mathcal{T}_{n+1,a}$ for $a \in \{1, 2, 3, 4\}$ since the base of $\mathcal{T}_{n+1,a}$ is a subset of $S_{n,1}$, the height of $\mathcal{T}_{n+1,a}$ is $\delta r_{n,i}/2$, and $T^*_{n,i}$ consists of points with distance at least $2\delta r_{n,1}/3$ from $S_{n,1}$ (see Figure 6 for an illustration). By Lemma 3.5, $T^*_{k,j} \subset \mathcal{T}_{n+1,a}$ for some $a \in \{1, 2, 3, 4\}$ so it follows $T^*_{n,i}$ is disjoint from $T^*_{k,j}$. If $T_{n,1}$ is a subset of P_{n+1}, then so is $T^*_{n,1}$. By Lemma 3.3, $T^*_{k,j}$ is disjoint from P_k which contains P_{n+1} (because $k \geq n + 1$), so $T^*_{n,i}$ and $T^*_{k,j}$ are disjoint. □

For the next lemmas, we consider $\mathcal{I}$ to be ordered by lexicographical ordering (i.e., $(n', i') < (n, i)$ if $n' < n$, or $n' = n$ and $i' < i$). For $(n, i) \in \mathcal{I}$, let

$$Z_{n,i} = \mathbf{1}_{\gamma_{n,i} > 10\delta^2 r_{n,i}}. \tag{17}$$

LEMMA 3.8
Let $(k, j), (n, i) \in \mathcal{I}$. If $(k, j) \leq (n, i)$ and $Z_{k,j} = 0$, then $T^*_{k,j} \cap P_{n,i} = \emptyset$.

Proof
If $Z_{k,j} = 0$, then by Lemma 3.5 we have $P_{n,i} \cap \mathcal{T}_{k,j} \subset \bigcup_{a=4(j-1)+1}^{4j} \mathcal{T}_{k+1,a}$. Since $T^*_{k,j}$ is contained in $\mathcal{T}_{k,j}$, it suffices to show that if $(k + 1, a)$ is a child of (k, j) and $Z_{k,j} = 0$ then $T^*_{k,j} \cap \mathcal{T}_{k+1,a} = \emptyset$, which was shown in the proof of Lemma 3.7 (see Figure 6 for an illustration). □

LEMMA 3.9
For $(n, i) \in \mathcal{I}$ the collection $\{Z_{k,j} : (k, j) \leq (n, i)\}$ is measurable with respect to $P_{n,i}$.

Remark 3.10
Given $\{Z_{k,j} : (k, j) < (n, i)\}$, we can recover the construction up until the (n, i)th step, so we can recover $\{P_{k,j} : (k, j) < (n, i)\}$ and, in particular, we can recover $\{P_1, \ldots, P_n\}$. Since $T_{k,j}$ is measurable with respect to P_k, it follows from Lemma 3.9 that the collection $\{T_{k,j} : (k, j) \leq (n, i)\}$ is measurable with respect to $P_{n,i-1}$.

Proof of Lemma 3.9
First, we prove that $Z_{k,j} = \mathbf{1}_{T_{k,j} \subset P_{n,i}}$. By definition, if $Z_{k,j} = 1$, then $T_{k,j} \subset P_{k,j} \subset P_{n,i}$. By Lemma 3.8, if $Z_{k,j} = 0$, then $T_{k,j}$ is not contained in $P_{n,i}$.

Therefore, it suffices to show that $T_{k,j}$ is measurable with respect to $P_{n,i}$. We prove this by induction on (k, j). It clearly holds for $k = 1$ because $(T_{1,j})_{j=1}^{4}$ are deterministic. If $k \geq 2$ and $T_{s,a}$ is measurable with respect to $P_{n,i}$ for all $(s,a) < (k, j)$, then it follows that $\{Z_{s,a} : (s,a) < (k, j)\}$ is measurable with respect to $P_{n,i}$ and, in particular, P_k is measurable with respect to $P_{n,i}$. Since $T_{k,j}$ is measurable with respect to P_k, this concludes the proof. □

3.3. Probabilistic analysis of the geometric construction
In this subsection, we provide the probabilistic analysis of our randomized geometric construction. A key ingredient is a resampling inequality, leveraging the monotonicity of the increments of the Γ-function established in Lemma 3.1.

3.3.1. A resampling inequality
For $(n,i) \in \mathcal{I}$, we let

$$B_{n,i} = \bigcup_{(k,j)\in\mathcal{I},(k,j)\leq(n,i)} \mathsf{T}^{*}_{k,j}$$

be the set of vertices in $\mathbb{Z}^2$ where the external field is explored for the construction of $P_{n,i}$. By Lemma 3.7, $\mathsf{T}^{*}_{n,i} \cap B_{n,i-1} = \emptyset$.

LEMMA 3.11
For $(n,i) \in \mathcal{I}$, let g be a random field such that $g_v = h_v$ for $v \notin B_{n,i-1}$ and $\{g_v : v \in B_{n,i-1}\}$ is a collection of independent mean-zero Gaussian variables with variance 1 that is independent of h. Recall that $\mathcal{F}_{n,i}$ is the σ-algebra generated by $P_{n,i-1}$ and $\{h_v : v \in \mathsf{T}^{*}_{n,i}\}$. Let

$$\tilde{\gamma}_{n,i} = \mathbb{E}\big[\Gamma(\mathsf{P}_{n,i-1} \cup \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon g) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon g) \mid \mathcal{F}_{n,i}\big].$$

Then $\gamma_{n,i} \geq \tilde{\gamma}_{n,i}$ almost surely.

In words, the lemma states that if we resample the field on $B_{n,i-1}$ after constructing $P_{n,i-1}$, then the expected increment to Γ from adding $\mathsf{T}_{n,i}$ to $\mathsf{P}_{n,i-1}$ decreases.

Proof of Lemma 3.11
Let $C_{n,i-1} = B_{n,i-1} \cap P_{n,i-1}$ and $D_{n,i-1} = B_{n,i-1} \setminus P_{n,i-1}$. By Lemma 3.9, $B_{n,i-1}$ is measurable with respect to $P_{n,i-1}$. It follows that $C_{n,i-1}$ and $D_{n,i-1}$ are measurable with respect to $P_{n,i-1}$. By Lemma 3.8,

$$C_{n,i-1} = \bigcup_{(k,j)\in\mathcal{I},(k,j)<(n,i),Z_{k,j}=1} \mathsf{T}^*_{k,j},$$

$$D_{n,i-1} = \bigcup_{(k,j)\in\mathcal{I},(k,j)<(n,i),Z_{k,j}=0} \mathsf{T}^*_{k,j}.$$

Let Q be a polygon such that $\mathbb{P}(P_{n,i-1} = Q) > 0$. We let $B_{n,i-1}(Q)$ be the value of $B_{n,i-1}$ on the event $\{P_{n,i-1} = Q\}$, and similarly for $C_{n,i-1}(Q)$ and $D_{n,i-1}(Q)$. Note that the event $\{P_{n,i-1} = Q\}$ is measurable with respect to $h|_{B_{n,i-1}(Q)}$ (here $h|_A$ denotes the restriction of h to A).

We claim that the event $\{P_{n,i-1} = Q\}$ is decreasing with respect to $h|_{D_{n,i-1}(Q)}$ and increasing with respect to $h|_{C_{n,i-1}(Q)}$. That is, if f is a realization of the field such that $P_{n,i-1}(f) = Q$ and f' is a realization of the field such that $f'_v \geq f_v$ for all $v \in C_{n,i-1}(Q)$ and $f'_v \leq f_v$ for all $v \in D_{n,i-1}(Q)$, then $P_{n,i-1}(f') = Q$. To see this, we prove inductively that $P_{k,j}(f) = P_{k,j}(f')$ for each $(k, j) \leq (n,i)$. It clearly holds for $(k, j) = (1,0)$ since $P_{1,0} = [-N,N]^2$ deterministically. If $(k, j) \leq (n,i)$ and $P_{k,j-1}(f) = P_{k,j-1}(f')$, then $\gamma_{k,j}(f) \leq \gamma_{k,j}(f')$ if $Z_{k,j}(f) = 1$ and $\gamma_{k,j}(f) \geq \gamma_{k,j}(f')$ if $Z_{k,j}(f) = 0$ (this is because $\gamma_{k,j}$ is a function of $(P_{k,j-1}, h|_{\mathsf{T}^*_{k,j}})$ and is increasing in $h|_{\mathsf{T}^*_{k,j}}$ for fixed $P_{k,j-1}$). This implies that $Z_{k,j}(f) = Z_{k,j}(f')$, and, as a result, $P_{k,j}(f) = P_{k,j}(f')$, completing the proof by induction.

By the Fortuin–Kasteleyn–Ginibre (FKG) inequality for product measures (see [29]), we get that conditional on $\{P_{n,i-1} = Q\}$ we have the following: $(h|_{C_{n,i-1}(Q)}, -h|_{D_{n,i-1}(Q)})$ stochastically dominates $(g|_{C_{n,i-1}(Q)}, -g|_{D_{n,i-1}(Q)})$ (note the minus sign for the field on $D_{n,i-1}(Q)$). By construction, $h|_{\Lambda_{2N}\setminus B_{n,i-1}(Q)} = g|_{\Lambda_{2N}\setminus B_{n,i-1}(Q)}$ on $\{P_{n,i-1} = Q\}$. Therefore, conditional on $\{P_{n,i-1} = Q\}$ and on $h|_{\mathsf{T}^*_{n,i}(Q)}$ (thus also conditional on $g|_{\mathsf{T}^*_{n,i}(Q)}$ since $h|_{\mathsf{T}^*_{n,i}(Q)} = g|_{\mathsf{T}^*_{n,i}(Q)}$), the field $(h|_{\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q)}, -h|_{\Lambda_{2N}\setminus(\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q))})$ stochastically dominates the field $(g|_{\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q)}, -g|_{\Lambda_{2N}\setminus(\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q))})$. Let $\Delta_Q : \mathbb{R}^{\Lambda_{2N}} \to \mathbb{R}$ be the function given by

$$\Delta_Q(f) = \Gamma\big(\mathsf{P}_{n,i-1}(Q) \cup \mathsf{T}_{n,i}(Q), \Lambda_{2N}, f\big) - \Gamma\big(\mathsf{P}_{n,i-1}(Q), \Lambda_{2N}, f\big).$$

By Lemma 3.1, Δ_Q is increasing in $f|_{\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q)}$ and decreasing in $f|_{\Lambda_{2N}\setminus(\mathsf{P}_{n,i-1}(Q)\cup\mathsf{T}_{n,i}(Q))}$. It follows that given $\{P_{n,i-1} = Q\}$ and $h|_{\mathsf{T}^*_{n,i}}$, we have that $\Delta_Q(\varepsilon h)$ stochastically dominates $\Delta_Q(\varepsilon g)$. Since

$$\gamma_{n,i}\mathbf{1}_{P_{n,i-1}=Q} = \mathbb{E}\big[\Delta_Q(\varepsilon h) \mid \mathcal{F}_{n,i}\big]\mathbf{1}_{P_{n,i-1}=Q}$$

and

$$\tilde{\gamma}_{n,i}\mathbf{1}_{P_{n,i-1}=Q} = \mathbb{E}\big[\Delta_Q(\varepsilon g) \mid \mathcal{F}_{n,i}\big]\mathbf{1}_{P_{n,i-1}=Q},$$

this proves the lemma. □

3.3.2. Quantitative probabilistic analysis

We first show that each triangle $T_{n,i}$ has a decent probability to be included in P^*. Recall that $\delta = 10^{-2}(\mathtt{m}\varepsilon)^{2/3}$.

LEMMA 3.12

For $\mathtt{m} \in (0,1)$, there exist constants $C_2, c_2 > 0$ (depending on $\mathtt{m}$) such that the following holds. Suppose that (8) fails for some $N \geq e^{C_2\varepsilon^{-4/3}}$. Then for all $1 \leq n \leq \log_{16}(N)$ and $1 \leq i \leq 4^n$,

$$\mathbb{P}(\gamma_{n,i} \geq 10\delta^2 r_{n,i}) \geq c_2.$$

Proof

In light of Lemma 3.11, in order to prove the lemma it suffices to show that for all $(n,i) \in \mathcal{I}$ with $n \leq \log_{16}(N)$ we have

$$\mathbb{P}(\tilde{\gamma}_{n,i} \geq 10\delta^2 r_{n,i}) \geq c_2. \tag{18}$$

As in the proof of Lemma 3.11, we will work conditionally on $P_{n,i-1}$. We let g be as in Lemma 3.11. Note that g is a collection of independent standard Gaussian random variables. Recall the definition of G given in (14). We have

$$\tilde{\gamma}_{n,i} = \mathbb{E}\big[G(\mathsf{P}_{n,i-1}, \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon g) | P_{n,i-1}, g|_{\mathsf{T}^*_{n,i}}\big]. \tag{19}$$

Since $G(A, B, \Omega, f)$ is an odd function of $f|_{\Omega\setminus A}$ for all fixed (A, B, Ω), we see that $G(\mathsf{P}_{n,i-1}, \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon g)$ is an odd function of $g|_{\Lambda_{2N}\setminus\mathsf{P}_{n,i-1}}$ when $P_{n,i-1}$ is fixed. Since g is independent of $P_{n,i-1}$ (because $P_{n,i-1}$ is measurable with respect to $h|_{B_{n,i-1}}$) and g has a symmetric distribution, this implies that $\tilde{\gamma}_{n,i}$ is an odd function of $g|_{\mathsf{T}^*_{n,i}}$ when $P_{n,i-1}$ is fixed. In particular, we have

$$\mathbb{E}[\tilde{\gamma}_{n,i} \mid P_{n,i-1}] = 0. \tag{20}$$

Next, we give a lower bound on the variance of $\tilde{\gamma}_{n,i}$. By (19) and the formulas derived in the proof of Lemma 3.1 for the partial derivatives of the increment of Γ, we obtain that, for $v \in \mathsf{T}^*_{n,i}$,

$$\mathbb{E}[\partial_{g_v}\tilde{\gamma}_{n,i} \mid P_{n,i-1}] = 2\varepsilon\mathbb{E}\big[\Delta\langle\sigma_v\rangle_{\Lambda_{2N}\setminus\mathsf{P}_{n,i-1},\varepsilon g} \mid P_{n,i-1}\big].$$

Recall the definition of $\mathtt{m}_{\beta,\Lambda_N,\varepsilon}$ in (3). For $\Omega \subset \mathbb{Z}^2$, define $\mathtt{m}_{\beta,\Omega,\varepsilon}$ similarly by replacing Λ_N with Ω. By monotonicity of the Ising model (cf. [3, Section 2.2]), we have that $\mathtt{m}_{\beta,\Omega,\varepsilon}$ is decreasing in Ω, and therefore, for $v \in \mathsf{T}^*_{n,i}$,

$$\mathbb{E}[\partial_{g_v}\tilde{\gamma}_{n,i} \mid P_{n,i-1}] \geq 2\varepsilon \mathrm{m}_{\beta,\Lambda_{2N}-v,\varepsilon} \geq 2\varepsilon \mathrm{m}_{\beta,\Lambda_{4N},\varepsilon} \geq 2\varepsilon \mathrm{m},$$

where the last inequality follows from our assumption that (8) fails. It then follows from [15, Proposition 3.5] that

$$\mathbb{E}[\tilde{\gamma}_{n,i}^2 \mid P_{n,i-1}] = \mathrm{Var}[\tilde{\gamma}_{n,i} \mid P_{n,i-1}] \geq (2\varepsilon \mathrm{m})^2 |\mathsf{T}_{n,i}^*|. \tag{21}$$

In addition, by (15) we see that $\tilde{\gamma}_{n,i}$ is a Lipschitz function of $g|_{\mathsf{T}_{n,i}^*}$ with Lipschitz constant $2\varepsilon\sqrt{|\mathsf{T}_{n,i}^*|}$ (with respect to the ℓ_2 norm) for each fixed $P_{n,i-1}$. Therefore, by (20) and by the Gaussian concentration inequality (see [10], [50]; see also [1, Theorem 2.1], [53, Theorem 3.25]) we get that

$$\mathbb{E}[\tilde{\gamma}_{n,i}^4 \mid P_{n,i-1}] \leq 10^5 \varepsilon^4 |\mathsf{T}_{n,i}^*|^2. \tag{22}$$

A simple computation gives that, for any $t > 0$,

$$\begin{aligned}\mathbb{E}[\tilde{\gamma}_{n,i}^2 \mid P_{n,i-1}] &\leq t^2 + \mathbb{E}(\tilde{\gamma}_{n,i}^2 \mathbf{1}_{\tilde{\gamma}_{n,i}^2 \geq t^2} \mid P_{n,i-1}) \\ &\leq t^2 + \sqrt{\mathbb{E}(\tilde{\gamma}_{n,i}^4 \mid P_{n,i-1})}\sqrt{\mathbb{P}(\tilde{\gamma}_{n,i}^2 \geq t^2 \mid P_{n,i-1})}.\end{aligned}$$

Setting $t = \varepsilon \mathrm{m}\sqrt{|\mathsf{T}_{n,i}^*|}$ and combining with (22) and (21), we obtain that

$$\mathbb{P}(\tilde{\gamma}_{n,i} \geq \varepsilon \mathrm{m}\sqrt{|\mathsf{T}_{n,i}^*|} \mid P_{n,i-1}) = \frac{1}{2}\mathbb{P}(\tilde{\gamma}_{n,i}^2 \geq (\varepsilon \mathrm{m})^2 |\mathsf{T}_{n,i}^*| \mid P_{n,i-1}) \geq 10^{-5}\mathrm{m}^4, \tag{23}$$

where the first equality follows from the fact that, conditioned on $P_{n,i-1}$, the law of $\tilde{\gamma}_{n,i}$ is symmetric around 0. It is obvious that the number of lattice points in any isosceles triangle in $\mathbb{R}^2$ with base length and height larger than 100 is at least half of the area of the triangle. Since $N \geq e^{C_2\varepsilon^{-4/3}}$ and $1 \leq n \leq \log_{16}(N)$, we have that the base length and the height of $T_{n,i}^*$ (which are $\frac{2r_{n,i}}{3}$ and $\frac{\delta r_{n,i}}{3}$, respectively) are both larger than 100 as long as C_2 is a large enough constant. Therefore,

$$|\mathsf{T}_{n,i}^*| \geq 2^{-1}\lambda(T_{n,i}^*) = 18^{-1}\delta r_{n,i}^2.$$

Combined with (23) and $\delta = 10^{-2}(\varepsilon \mathrm{m})^{2/3}$, it completes the proof of (18). □

We are now ready to conclude the proof on the upper bound for the correlation length.

Proof of (8)
We will prove (12), provided that (8) fails for $N \geq e^{C_1\varepsilon^{-4/3}}$ for a large enough constant C_1, and thus obtain a contradiction with Lemma 2.1. This in turn proves (8), as required.

Since for each $n \geq 1$, P_n has 4^n sides and $n^* \leq \log_{16}(N)$, we see that P^* has at most $N^{1/2}$ sides. By construction, $l(\partial P^*) \geq l(\partial P_1) = 8N$. Therefore, by Lemma 3.6 we have

$$|\partial \mathsf{P}^*| \leq \sqrt{2} l(\partial P^*) + 2N^{1/2} \leq 2l(\partial P^*).$$

In addition, $|\partial(\Lambda_{2N} \setminus \mathsf{P}^*)| = |\partial \mathsf{P}^*| + 16N$. Therefore, it suffices to show that

$$\mathbb{E}\big[\Gamma(\mathsf{P}^*, \Lambda_{2N}, \varepsilon h) - 8l(\partial P^*)\big] > 0. \tag{24}$$

For $n \geq 1$, let $X_n = \Gamma(\mathsf{P}_n, \Lambda_{2N}, \varepsilon h) - 8l(\partial P_n)$. For $(n,i) \in \mathcal{I}$, let $X_{n,i} = \Gamma(\mathsf{P}_{n,i}, \Lambda_{2N}, \varepsilon h) - 8l(\partial P_{n,i})$ and $Y_{n,i} = X_{n,i} - X_{n,i-1}$. We assume from now on that $n < n^*$. We have

$$l\big(\partial(P_{n,i-1} \cup T_{n,i})\big) - l(\partial P_{n,i-1}) = 2\sqrt{1+\delta^2} r_{n,i} - 2r_{n,i} \leq \delta^2 r_{n,i}.$$

Recalling definition of $Z_{n,i}$ as in (17), we get that:

- if $Z_{n,i} = 0$, then $P_{n,i} = P_{n,i-1}$ and thus $Y_{n,i} = 0$;
- if $Z_{n,i} = 1$, then

$$\begin{aligned} Y_{n,i} = {} & \big(\Gamma(\mathsf{P}_{n,i-1} \cup T_{n,i}, \Lambda_{2N}, \varepsilon h) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon h)\big) \\ & - 8\big(l\big(\partial(P_{n,i-1} \cup T_{n,i})\big) - l(\partial P_{n,i-1})\big), \end{aligned}$$

  where the difference in the perimeter is bounded by $\delta^2 r_{n,i}$.

Altogether, we have that

$$Y_{n,i} \geq Z_{n,i}\big[\Gamma(\mathsf{P}_{n,i-1} \cup \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon h) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon h) - 8\delta^2 r_{n,i}\big].$$

Recalling the definition of $\gamma_{n,i}$ as in (16), we obtain

$$\begin{aligned} & \mathbb{E}\big[Z_{n,i}\big(\Gamma(\mathsf{P}_{n,i-1} \cup \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon \cdot h) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon \cdot h)\big)\big] \\ & \quad = \mathbb{E}\big[\mathbb{E}\big(Z_{n,i}\big(\Gamma(\mathsf{P}_{n,i-1} \cup \mathsf{T}_{n,i}, \Lambda_{2N}, \varepsilon \cdot h) - \Gamma(\mathsf{P}_{n,i-1}, \Lambda_{2N}, \varepsilon \cdot h) \mid \mathcal{F}_{n,i}\big)\big)\big] \\ & \quad = \mathbb{E}[Z_{n,i}\gamma_{n,i}] \geq \mathbb{E}[10\delta^2 r_{n,i} Z_{n,i}], \end{aligned}$$

where we used the fact that $Z_{n,i}$ is measurable with respect to $\mathcal{F}_{n,i}$. Therefore,

$$\mathbb{E}[Y_{n,i}] \geq 2\delta^2 \mathbb{E}[r_{n,i} Z_{n,i}].$$

It follows from the construction of P_n that for every $(n,i) \in \mathcal{I}$ we have $l(S_{n,i}) \geq l(\partial P_1)4^{-n}$. Therefore, $r_{n,i} \geq l(\partial P_1)4^{-n-1}$. Plugging this into the previous display gives

$$\mathbb{E}[Y_{n,i}] \geq 2\delta^2 4^{-n-1} l(\partial P_1)\mathbb{P}(\gamma_{n,i} > 10\delta^2 r_{n,i}).$$

Finally, we will set $C_1 \geq C_2$ so we can apply Lemma 3.12 and get

$$\mathbb{E}[Y_{n,i}] \geq 2c_2\delta^2 4^{-n-1} l(\partial P_1).$$

Summing over i gives

$$\mathbb{E}[X_{n+1} - X_n] \geq 2^{-1} c_2 \delta^2 l(\partial P_1).$$

Since Γ is an odd function of h, we have $\mathbb{E}[X_1] = -l(\partial P_1)$. Therefore

$$\mathbb{E}[X_{n^*}] = \mathbb{E}[X_1] + \sum_{n=1}^{n^*-1} \mathbb{E}[X_{n+1} - X_n] \geq \big(c_2 \delta^2 (n^* - 1)/2 - 1\big) l(\partial P_1).$$

Plugging in $n^* = \lfloor \log_{16}(N) \rfloor$ and $\delta = 10^{-2}(\varepsilon \mathtt{m})^{2/3}$, we see that $\mathbb{E}[X_{n^*}] > 0$ (which is a rewrite of (24)) for $N \geq e^{C_1 \varepsilon^{-4/3}}$, provided that $C_1 \geq C_2$ is a large enough constant (depending on $\mathtt{m}$). □

4. Upper bound on greedy lattice animal

This section is devoted to the proof of Proposition 2.2, which is essentially [52, Theorem 4.4.2]. We make the connection between [52, Theorem 4.4.2] and Proposition 2.2 slightly more explicit, and we claim no credit for material in this section.

For a Gaussian process X indexed on a set T, define the canonical metric $d_X : T \times T \to [0, \infty)$ for (T, X) by

$$d_X(s,t) = \mathbb{E}\big[\big(X(s) - X(t)\big)^2\big]^{1/2}. \tag{25}$$

Next, we review the $\gamma_{\alpha,\beta}$-functionals which measure the size of a metric space in a way that can be used to control the maximum of a Gaussian process. We begin with an auxiliary definition.

Definition 4.1
Given a set T, an admissible sequence on T is an increasing sequence of partitions $(\Pi_n)_{n \geq 0}$ of T such that $|\Pi_0| = 1$ and $|\Pi_n| \leq 2^{2^n}$ for $n \geq 1$.

For a partition Π_n of a set T and an element $t \in T$, we will denote by $\pi_n(t)$ the element of Π_n that contains t. Now we are ready to define the $\gamma_{\alpha,\beta}$ functionals.

Definition 4.2
Given a set T, a metric d on T, and numbers $\alpha, \beta > 0$, let

$$\gamma_{\alpha,\beta}(T,d) = \inf_{(\Pi_n)} \sup_{t \in T} \Big[\sum_{n \geq 0} \big(2^{n/\alpha} \mathsf{diam}\big(\pi_n(t), d\big)\big)^{\beta}\Big]^{1/\beta},$$

where the infimum is taken over all admissible sequences and $\mathsf{diam}(\pi_n(t), d)$ is the d-diameter for $\pi_n(t)$. In addition, define $\gamma_2(T,d) = \gamma_{2,1}(T,d)$.

With this definition in place, we can state Talagrand's majorizing measure theorem (see [51] and [52, Theorem 2.2.22]) which gives a tight bound on the expectation of the supremum of a Gaussian process in terms of the γ_2-functional. Write $\|X\|_T = \sup_{t\in T} X_t$.

THEOREM 4.3
There exists a universal constant K such that the following holds. If T is a set and X is a centered Gaussian process indexed on T, then we have

$$\mathbb{E}\big[\|X\|_T\big] \le K\gamma_2(T, d_X).$$

Next, we state the Borell–Tsirelson–Ibragimov–Sudakov (Borell–TIS) inequality. For a set T and a Gaussian process $(X_t)_{t\in T}$ indexed on T, let $\sigma_X^2 = \sup_{t\in T}\mathrm{Var}(X_t)$. The Borell–TIS inequality says that the tails of the maximum of X behave roughly like those of a Gaussian random variable with variance σ_X^2 (see [10], [50], or [1, Theorem 2.1] for a proof):

$$\mathbb{P}\big(\big|\|X_T\| - \mathbb{E}\big[\|X_T\|\big]\big| > z\big) \le 2\exp\Big(-\frac{z^2}{2\sigma_X^2}\Big) \quad \text{for all } z > 0. \tag{26}$$

Note that we do not need to assume X is centered for (26). Note also that for any lattice animal A we have $\mathrm{Var}(\sum_{v\in A} h_v) = |A| \le |\partial A|^2$, so if T is a set of lattice animals and X_t is the sum of the Gaussian variables in the lattice animal t normalized by its boundary size, we have $\sigma_X^2 \le 1$.

Having introduced these tools, we turn to the proof of Proposition 2.2. It is more convenient to work with the unnormalized lattice animal processes, so we will partition the lattice animals by the lengths of their boundaries. The following lemma is the key to the proof of Proposition 2.2.

LEMMA 4.4
For a vertex $v \in \mathbb{Z}^2$ and an integer $k \ge 2$, let $\mathfrak{A}_{v,k}$ be the collection of simply connected lattice animals A such that $|\partial A| \le 2^k$, $v \in A$, and there exists $u \sim v$ such that u is not in A. For $A \in \mathfrak{A}_{v,k}$, let $Y_A = \sum_{v\in A} h_v$. Then there exists a constant $C > 0$ such that

$$\mathbb{P}\Big(\max_{A\in\mathfrak{A}_{v,k}} Y_A \ge Ck^{3/4}2^k + u2^k\Big) \le 2e^{-u^2/2} \quad \textit{for all } u > 0.$$

In Lemma 4.4, we restricted to A containing v *on its boundary* so that we have $|\mathfrak{A}_{k,v}| \le 2^{2^{k+1}}$ (as explained in the proof of Lemma 4.4 below). Lemma 4.4 can be deduced as a consequence of the following two lemmas in [52].

LEMMA 4.5 ([52, Lemma 4.4.6])
Let $n \geq 1$ and T be a set such that $|T| \leq 2^{2^n}$. Let d be a metric on T. Then ($\sqrt{d}$ is also a metric and)

$$\gamma_2(T, \sqrt{d}) \leq n^{3/4} \gamma_{1,2}(T, d)^{1/2}.$$

LEMMA 4.6 ([52, Proposition 4.4.5])
There exists a constant $C > 0$ such that

$$\gamma_{1,2}(\mathfrak{A}_{v,k}, d_Y^2) \leq C 2^{2k}.$$

Note that [52, Proposition 4.4.5] was stated in a slightly different context, but the metric space it applies to is easily seen to be isomorphic to $\mathfrak{A}_{v,k}$ with distance d_Y^2 since $d_Y^2(A, A') = \mathbb{E}[(Y_A - Y_{A'})^2]$ is simply the cardinality of the symmetric difference of A and A'.

Proof of Lemma 4.4
In order to apply Lemma 4.5, we need a bound on the cardinality of $\mathfrak{A}_{k,v}$. By considering a simply connected lattice animal A as the lattice points enclosed by a loop consisting of $|\partial A|$ edges of the dual lattice $(1/2, 1/2) + \mathbb{Z}^2$, it is easy to see that $|\mathfrak{A}_{k,v}| \leq 2^{2^{k+1}}$ (this is because one can construct a loop by starting an edge near v and adding new edges sequentially where each new edge has at most four choices). At this point, it is immediate from Lemmas 4.5 and 4.6 that $\gamma_2(\mathfrak{A}_{k,v}, d_Y) \leq C k^{3/4} 2^k$. Thus by Theorem 4.3, we have that $\mathbb{E}[\max_{A \in \mathfrak{A}_{v,k}} Y_A] \leq C k^{3/4} 2^k$. Therefore, we can obtain Lemma 4.4 by (26) and the fact that $\mathrm{Var}(Y_A) \leq 2^{2k}$ for all $A \in \mathfrak{A}_{v,k}$. □

Proof of Proposition 2.2
The proof is the same as the proof of [52, Theorem 4.4.2] using [52, Proposition 4.4.3]. Note that the total length of all edges in Λ_N is $2 \cdot (2N + 1) \cdot 2N$, and let $k^* = \min\{k : 2^k \geq 2 \cdot (2N + 1) \cdot 2N\}$. We have $k^* \leq C \log N$, and for any $A \in \mathfrak{A}_N$ there exists $2 \leq k \leq k^*$ such that $2^{k-1} \leq |\partial A| \leq 2^k$. Therefore, using Lemma 4.4 and a union bound over v and k we have for some constant $C > 0$,

$$\mathbb{P}\Big(\max_{A \in \mathfrak{A}_N} \frac{Y_A}{|\partial A|} \geq C(\log N)^{3/4} + x \Big) \leq C e^{\log N - x^2/2}.$$

Letting $x = C'(\log N)^{3/4} + u$ for a large enough constant C' concludes the proof. □

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