MAXIMAL BRILL–NOETHER LOCI VIA K3 SURFACES

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Abstract. The Brill–Noether loci $\mathcal{M}^r_{g,d}$ parameterize curves of genus g admitting a linear system of rank r and degree d; when the Brill–Noether number is negative, they sit as proper subvarieties of the moduli space of genus g curves. We explain a strategy for distinguishing Brill–Noether loci by studying the lifting of linear systems on curves in polarized K3 surfaces, which motivates a conjecture identifying the maximal Brill–Noether loci. Via an analysis of the stability of Lazarsfeld–Mukai bundles, we obtain new lifting results for line bundles of type g^3_d which suffice to prove the maximal Brill–Noether loci conjecture in genus 3–19, 22, and 23 and infinitely many cases.

Introduction

Given a smooth projective complex curve C of genus g, classical Brill–Noether theory concerns the geometry of the variety $W^r_d(C)$, parameterizing the space of line bundles of type g^r_d, i.e., having degree d and at least $r+1$ linearly independent global sections on C. Specifically, the expected dimension of $W^r_d(C)$ is the *Brill–Noether number* $\rho(g,r,d) := g-(r+1)(g-d+r)$. In particular, when $\rho(g,r,d) \geq 0$, every smooth curve of genus g admits a line bundle of type g^r_d. If $\rho(g,r,d) < 0$, then a curve admitting such a g^r_d is called Brill–Noether special, and the *Brill–Noether locus* $\mathcal{M}^r_{g,d}$ parametrizing smooth curves of genus g admitting a line bundle of type g^r_d is a proper subvariety of the moduli space $\mathcal{M}_g$ of smooth curves of genus g, see [2].

In general, the geometry of Brill–Noether loci is complicated by the existence of multiple components with some that are non-reduced or not of the expected dimension. Indeed, while the Brill–Noether locus $\mathcal{M}^r_{g,d}$ has expected codimension $-\rho$ in $\mathcal{M}_g$, the actual codimension of its components is bounded above by $-\rho$ when $\rho < 0$, see e.g., [16], but it could be lower, and known examples with lower than expected codimension exist when $-\rho > g-3$, see [45]. On the other hand, when $\rho(g,r,d) = -1$, Eisenbud and Harris [13] show that $\mathcal{M}^r_{g,d}$ is irreducible of codimension 1. More generally, when $-3 \leq \rho \leq -1$, any component of $\mathcal{M}^r_{g,d}$ has codimension $-\rho$, see [11, 13, 49]. The Brill–Noether divisors were used by Harris, Mumford, and Eisenbud [12, 20, 21] in their investigation of the Kodaira dimension of $\mathcal{M}_g$ when $g \geq 23$.

A question of interest is then to determine the stratification of $\mathcal{M}_g$ by Brill–Noether loci and, in particular, to identify those loci that are maximal with respect to containment. For Brill–Noether divisors, this is equivalent to having distinct support, a property that is crucially used by Eisenbud and Harris [12], and further developed by Farkas [14], to give lower bounds on the Kodaira dimension of $\mathcal{M}_{23}$. There are various trivial containments among the Brill–Noether loci, e.g., $\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \cdots \subseteq \mathcal{M}^1_{g,k} = \mathcal{M}_g$, where $k \geq \lfloor \frac{g+3}{2} \rfloor$ is at least the generic gonality of a curve of genus g. Likewise, we have $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ by adding a base point to a g^r_d on C. Similarly, by subtracting a point not in the base locus, $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1) < 0$, see [15, 37]. Modulo these trivial containments, the *expected maximal Brill–Noether loci* are the $\mathcal{M}^r_{g,d}$, where for fixed r, with $2r \leq d \leq g-1$, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g,r-1,d-1) \geq 0$. Hence, every Brill–Noether locus is contained in an expected maximal one, and we conjecture that the expected maximal loci are distinct.

Conjecture 1. In every genus $g \geq 3$, the maximal Brill–Noether loci are the expected ones, except when $g = 7, 8, 9$.

The conjecture states that at least one component of each expected maximal Brill–Noether locus is not contained in any other Brill–Noether locus, hence the expected maximal loci are indeed the maximal elements in the containment lattice of all Brill–Noether loci. Concretely, this means that given any two expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$, there exists a genus g curve admitting a g^r_d but not a $g^{r'}_{d'}$.

In each genus $g = 7, 8, 9$, there is an unexpected containment between the two expected maximal Brill–Noether loci. In genus 8, Mukai [41, Lemma 3.8] proved the unexpected containment $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, see Proposition 6.3. In genus 7 and 9, Hannah Larson pointed out the unexpected containments $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$ and $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$, see Proposition 6.2 and Proposition 6.4.

Recently, there have been several breakthroughs in the study of Brill–Noether special curves of fixed gonality [9, 15, 24, 30, 31, 43, 44], from which one can deduce that the expected maximal $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ is not contained in any of the other expected maximal loci and hence is maximal, see Section 1. Additionally, following the work of Farkas [15] in genus 23, there has been recent focus on showing that Brill–Noether loci of codimension 1 and 2 are distinct, and showing various non-containments of Brill–Noether loci of codimension 2, see [6, 7, 8, 26]; in fact, for $g \geq 34$ and not divisible by 3, one can deduce that there are at least 2 maximal Brill–Noether loci. These results are proved using a mix of tropical, combinatorial, and limit linear series methods.

On the other hand, our approach is to use K3 surfaces to construct curves admitting a g^r_d, but not a $g^{r'}_{d'}$, thus distinguishing the Brill–Noether loci. This idea was introduced by Farkas [15], and further developed by Lelli-Chiesa [34, 36], who can produce curves on a K3 surface admitting a g^1_d or g^2_d, but not a $g^r_{d'}$. We further extend this technique to curves that admit a g^3_d, which suffices to prove our main theorem.

Theorem 1. Conjecture 1 holds in genus 3–19, 22, and 23.

In genus 23, Eisenbud and Harris [12], and Farkas [15], prove the part of this conjecture concerning the Brill–Noether divisors in their work on the birational geometry of the moduli space of curves. We also remark that known results on Brill–Noether loci of codimension 1 or 2 imply Conjecture 1 for infinitely many genera g, in particular, for any g such that $g + 1$ or $g + 2$ is of the form $\mathrm{lcm}(1, 2, \ldots, n)$ for some $n \geq 3$.

Recently, there has been a surge of progress on Conjecture 1. Indeed, using the gonality stratification, the genus 20 case was recently proven in [4]. Moreover, various non-containments of Brill–Noether loci have recently been proven by Teixidor i Bigas in [39]. Furthermore, the genus 21 case was proven by Bud in [5]. Recently, Conjecture 1 has been proven incomplete generality by the two authors together with Knutsen in [3], via the study of degenerations of linear series on curves on K3 surfaces, see Remark 1.

The geometry of polarized K3 surfaces is intimately related to the Brill–Noether theory of curves C in the polarization class, see e.g., [25, 29, 38, 40, 47, 48]. Foundational to this is Green and Lazarsfeld's celebrated result that the Clifford index $\gamma(C)$ is constant as C moves in its linear system [18]. Donagi and Morrison [10, Theorem 5.1'] proved that if A is a complete basepoint free Brill–Noether special g^1_d on a non-hyperelliptic smooth curve $C \in |H|$, then $|A|$ is contained in the restriction of $|M|$ for a line bundle $M \in \mathrm{Pic}(S)$. In fact, they conjectured that this is always true, with some slight modifications due to Lelli-Chiesa.

Conjecture 2 (Donagi–Morrison Conjecture, [35] Conjecture 1.3)**.** Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g - 1$ and $\rho(g, r, d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C and $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

For further details and definitions, such as the notion of *adapted*, see Section 1.1. Lelli-Chiesa has verified the Donagi–Morrison conjecture for linear systems of type g^2_d under some mild hypotheses [34], and more recently [35], has proven the conjecture if the pair (C, A) does not have unexpected secant varieties up to deformation. Importantly, in [35, Appendix A], Lelli-Chiesa and Knutsen construct explicit examples that show Conjecture 2 is in general false for linear systems rank 3, see Remark 6.19. The proofs of these results use Lazarsfeld–Mukai bundles $E_{C,A}$ associated to the pair (C, A), and the fact that when the vector bundle $E_{C,A}$ has a nontrivial maximal destabalizing sub-line bundle $N \in \mathrm{Pic}(S)$, then $|A|$ is contained in the restriction of $|H \otimes N^\vee|$. For rank 2 linear systems, a case-by-case analysis of the Jordan–Hölder and Harder–Narasimhan filtrations of $E_{C,A}$ is used. This technique becomes much more difficult in higher rank. In general, Lelli-Chiesa [35, Theorem 4.2] proves that A does lift when it computes the Clifford index $\gamma(C)$. However, in genus $g \geq 14$, except for $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$, all of the expected maximal Brill–Noether loci correspond to *non-computing* Brill–Noether special linear systems, i.e., linear systems $|A|$ with $\rho(A) < 0$ and $\gamma(A) > \lfloor \frac{g-1}{2} \rfloor$ so that A cannot compute the Clifford index, as $\gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$.

Since one cannot hope to prove Conjecture 2 in general, our main lifting result is a proof of the Donagi–Morrison conjecture for linear systems of rank 3 and bounded degree.

Theorem 2. Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$ and $C \in |H|$ a smooth irreducible curve of Clifford index $\gamma(C)$. Suppose that S has no elliptic curves and $d < \frac{5}{4}\gamma(C) + 6$, then Conjecture 2 holds for any g^3_d on C with $d \leq g - 1$. Moreover, one has $c_1(M).C \leq \frac{3g-3}{2}$.

We prove a slightly more refined version, replacing the hypothesis on non-existence of elliptic curves with an explicit dependence on the Picard lattice of S, see Theorem 5.1.

With this lifting result in hand, Theorem 1 is proved by considering K3 surfaces (S, H) with a prescribed Picard group so that curves $C \in |H|$ have a g^r_d, and then proving that if C had a $g^3_{d'}$, its Donagi–Morrison lift would not be compatible with the Picard group. This latter argument involves some elementary lattice theory. More generally, we explain how a Donagi–Morrison type result together with some lattice theory imply Conjecture 1. As the Donagi–Morrison conjecture is not known in rank 4 and above, we cannot currently use these techniques to show that some of the expected maximal Brill–Noether loci are not contained in the $\mathcal{M}^4_{g,d}$ in genus 20, 21, and ≥ 24. In genus 22 and 23, known results about the codimension of components of Brill–Noether loci and non-containments of codimension 2 loci, together with our results, suffice to distinguish the expected maximal loci.

Remark 1. We note that the proof of Conjecture 1 in [3] uses different ideas than in this work, though both use K3 surface techniques. Here, we study the destabilizing filtrations of Lazarsfeld–Mukai bundles to directly verify new cases of the Donagi–Morrison conjecture (Conjecture 2), and develop the link between this conjecture and Conjecture 1. However, in [3], the proof of Conjecture 1 follows from an analysis of numerical conditions forced by nonsimplicity of Lazarsfeld–Mukai bundles and by rigidity of effective decompositions of the polarization class, and the subsequent constraints imposed on degenerations of linear series on curves on K3 surface.

Outline. In Section 1, we briefly analyze some constraints on lifting line bundles and find that in genus ≥ 14 the expected maximal Brill–Noether loci correspond to line bundles that cannot compute the Clifford index of the curve, and summarize how Conjecture 2 implies Conjecture 1. The following two sections, Section 2 and Section 3, provide some background on the notion of stability of coherent sheaves on K3 surfaces and on Lazarsfeld–Mukai bundles and their relation to lifting line bundles. We also briefly recall some useful facts about generalized Lazarsfeld–Mukai bundles which are needed in particular arguments. At the end of Section 3, we motivate our proof strategy in Proposition 3.17. In Section 4, we first reduce the problem to finding a bound for each terminal filtration of the Lazarsfeld–Mukai bundle associated to the g^3_d, a filtration obtained by taking the Harder–Narasimhan and Jordan–Hölder filtrations of the Lazarsfeld–Mukai bundle. We then find a bound on the degree of the g^3_d for each filtration. In Section 5, after having obtained bounds for every terminal filtration that does not have a maximal destabilizing sub-line bundle, we give the proof of Theorem 2. Finally, in Section 6, we use known results about dimensions of components of Brill–Noether loci and other lifting results to prove Theorem 1. In Section 6.3, we prove the results in genus 3–13, and the following sections summarize genus 14–23, with Section 6.11 devoted to showing how the conjecture is true for infinitely many genera.

Acknowledgments. The authors would like to thank Gabi Farkas for elucidating connections between Brill–Noether loci and the birational geometry of $\mathcal{M}_g$, David Jensen and Nathan Pflueger for very helpful conversations about Brill–Noether theory, Hannah Larson for pointing out the exceptional case of genus 9, Margherita Lelli-Chiesa for explaining technical aspects of her work, Andreas Leopold Knutsen for helpful comments on the Donagi–Morrison conjecture, Isabel Vogt for comments on a draft of the paper and pointing out recent relevant literature, and John Voight for helpful computations and comments on a draft of the paper. The first author is partially supported by Simons Foundation Collaboration Grant 712097 and National Science Foundation Grant DMS-2200845.

1. Maximal Brill–Noether Loci

In this section, we take a look at the analytic geometry of various Brill–Noether theory conditions on linear systems. We find simple bounds on the maximal Clifford index of Brill–Noether special linear systems and for linear systems that can potentially lift to a K3 surface without contradicting the Hodge index theorem. Furthermore, we find that all non-computing linear systems are always potentially liftable to K3 surfaces. We end with a discussion of how Conjecture 2 and lattice theory can imply Conjecture 1. We work with a fixed genus g throughout this section.

Let (S, H) be a polarized K3 surface of genus g. In the moduli space $\mathcal{K}_g^\circ$ of polarized K3 surfaces of genus g, the Noether–Lefschetz (NL) locus parameterizes K3 surfaces with Picard rank > 1. By Hodge theory, the NL locus is a union of countably many irreducible divisors, which we call NL divisors. In [19], Greer, Li, and Tian study the Picard group of $\mathcal{K}_g^\circ$ using Noether–Lefschetz theory and the locus of Brill–Noether special K3 surfaces in $\mathcal{K}_g^\circ$ is identified as a union of NL divisors. More generally, it is convenient to work with the moduli space of primitively quasi-polarized K3 surfaces, denoted $\mathcal{K}_g$ where $\mathcal{K}_g \setminus \mathcal{K}_g^\circ$ is a divisor parameterizing K3 surfaces containing a (-2)-exceptional curve. We define the NL divisor $\mathcal{K}_{g,d}^r$ to be the locus of polarized K3 surfaces $(S, H) \in \mathcal{K}_g$ such that the lattice

$$\Lambda_{g,d}^r = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

admits a primitive embedding in $\mathrm{Pic}(S)$ preserving H. We note that the $\mathcal{K}_{g,d}^r$ are each irreducible by [42]. As we'll show in Lemma 6.7, polarized K3 surfaces $(S, H) \in \mathcal{K}_{g,d}^r$ should be thought of as those having a curve $C \in |H|$ such that $L \otimes \mathcal{O}_C$ is a line bundle of type g_d^r, and we say that the lattice $\Lambda_{g,d}^r$ is *associated* to g_d^r. Specifically, we have the following lemma, which we prove in Section 6.

Lemma 1.1 (See Lemma 6.7)**.** *Let $(S, H) \in \mathcal{K}_{g,d}^r$ and let $C \in |H|$ be a smooth irreducible curve. If L and $H - L$ are basepoint free, $r \geq 2$, and $1 \leq d \leq g - 1$, then $L \otimes \mathcal{O}_C$ is a g_d^r.*

Conversely, one is interested in the question of when a given g_d^r on a curve in a K3 surface is the restriction of a line bundle from the K3; in this case, we say that the line bundle is a *lift* of the g_d^r. Lifting of line bundles on curves on K3 surfaces is considered in [10, 18, 34, 35, 38, 47]. In lifting Brill–Noether special linear systems on $C \in |H|$ to a line bundle $L \in \mathrm{Pic}(S)$, we are naturally led to considering two constraints. First, we have $\rho(g, r, d) < 0$ as the linear system is Brill–Noether special. We call the constraint $\rho(g, r, d) < 0$ the *Brill–Noether constraint.* If a g_d^r on a curve $C \in |H|$ on a polarized K3 surface (S, H) has a suitable lift (see Corollary 3.14), then $\mathrm{Pic}(S)$ admits a primitive embedding of $\Lambda_{g,d}^r$ preserving H, and in particular

$$\mathrm{disc}\big(\Lambda_{g,d}^r\big) := 4(g-1)(r-1) - d^2 < 0$$

by the Hodge index theorem. Thus we define

$$\Delta(g, r, d) := \mathrm{disc}\left(\Lambda_{g,d}^r\right) = 4(g-1)(r-1) - d^2 = 4(g-1)(r-1) - (\gamma(r, d) + 2r)^2,$$

where $\gamma(r,d) := d-2r$. We thus call the constraint $\Delta(g,r,d)<0$ the *Hodge constraint* as the inequality stems from the Hodge index theorem. We remark that when $\Delta(g,r,d)<0$, the Torelli theorem for polarized K3 surfaces implies that a very general K3 surface in $\mathcal{K}^r_{g,d}$ has $\mathrm{Pic}(S)=\Lambda^r_{g,d}$.

Remark 1.2. When considering the lifting of linear systems to K3 surfaces, it is more convenient to consider the Brill–Noether and Hodge constraints for fixed g in the (r,γ)-plane as opposed to the (r,d)-plane (where $\gamma=d-2r$), in particular, because the Clifford index of curves on K3 surfaces remains constant in their linear system [18]. In the (r,γ)-plane the Brill–Noether and Hodge constraints determine regions that are bounded by the curves $\rho(g,r,d)=0$ and $\Delta(g,r,d)=0$, which we call the *Brill–Noether hyperbola* and *Hodge parabola*, respectively. Simple calculations show that the maximum γ on the Brill–Noether hyperbola is obtained at $r=\sqrt{g}-1$ and $\gamma=g-2\sqrt{g}+1$, the intersection with the line $d=g-1$. Hence, taking $\gamma\le\lfloor g-2\sqrt{g}+1\rfloor$ suffices to bound Brill–Noether special linear systems. Similarly, the maximum γ on the Hodge parabola is given by $\gamma=\frac{g-5}{2}$, and obtained at the intersection with the line $d=g-1$ at $r=\frac{g+3}{4}$. Thus if $\gamma>\frac{g-5}{2}$ then $\Delta(g,r,d)<0$. Trivially $\lfloor\frac{g-4}{2}\rfloor\ge\frac{g-5}{2}$, and in fact the bound $\gamma\ge\lfloor\frac{g-4}{2}\rfloor \implies \Delta<0$ is the best possible as seen in genus 9, 13, and 17. As an example, we show the bounds in genus 100, as graphed on the (r,γ)-plane in Figure 1.

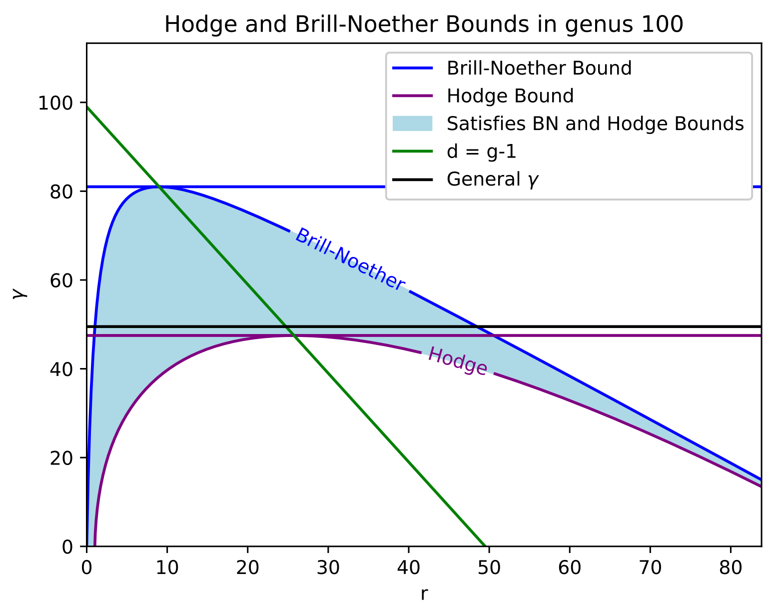

FIGURE 1. The Brill–Noether hyperbola ($\rho(g,r,d)=0$) and the Hodge parabola ($\Delta(g,r,d)=0$) in genus 100. The shaded area satisfies both $\rho(g,r,d)<0$ and $\Delta(g,r,d)<0$.

We recall that the *Clifford index* of a line bundle A on a smooth projective curve C is the integer $\gamma(A)=\deg(A)-2\,r(A)$ where $r(A)=h^0(C,A)-1$ is the rank of A. The Clifford index of C is

$$\gamma(C) := \min\{\gamma(A)\mid h^0(C,A)\ge 2 \text{ and } h^1(C,A)\ge 2\},$$

and a line bundle is said to *contribute to the Clifford index* if $h^0(C,A),\ h^1(C,A)\ge 2$. We say that a line bundle A on C *computes* the Clifford index of C if $\gamma(A)=\gamma(C)$. Clifford's theorem states that $0\le\gamma(C)\le\lfloor\frac{g-1}{2}\rfloor$, and when C is a general curve of genus g, $\gamma(C)=\lfloor\frac{g-1}{2}\rfloor$.

Definition 1.3. Let A be a Brill–Noether special g^r_d on a curve C of genus g, i.e. $\rho(g,r,d) < 0$. We say A is *non-computing* if $\gamma(r,d) > \lfloor \frac{g-1}{2} \rfloor$, that is, A is a Brill–Noether special g^r_d that cannot compute the Clifford index of C.

Lemma 1.4. *Let $g \geq 14$, $r \geq 2$, and $2r \leq d \leq g-1$. If $\mathcal{M}^r_{g,d}$ is an expected maximal Brill–Noether locus, then $\gamma(r,d) = d - 2r > \lfloor \frac{g-1}{2} \rfloor$. When $g < 14$, there are no non-computing g^r_d's.*

Proof. One can easily check that if $d - 2r \leq \lfloor \frac{g-1}{2} \rfloor$, then $\rho(g,r,d+1) < 0$, and hence $\mathcal{M}^r_{g,d}$ is not an expected maximal Brill–Noether locus. When $g < 14$, this is a simple computation enumerating all g^r_d's with Clifford index $\leq \lfloor \frac{g-1}{2} \rfloor + 1$. □

Thus for genus $g \geq 14$, except for $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$, all the maximal Brill–Noether loci are those associated to non-computing g^r_ds. If lifting results are able to distinguish between maximal Brill–Noether loci, there should not be an obvious obstruction to lifting the associated linear systems. In particular, the Hodge index theorem implies that the lattices obtained by lifting should have negative discriminant, which we show is true for non-computing g^r_ds below.

Proposition 1.5. *Let g, r, d be natural numbers with $2 \leq d \leq g-1$ and $1 \leq r \leq g-1$. Then the Hodge parabola lies under the Brill–Noether hyperbola. In particular, all non-computing linear systems, and all expected maximal Brill–Noether loci, satisfy $\Delta < 0$.*

Proof. For fixed $g \geq 2$, and for each constraint ($\rho = 0$ or $\Delta = 0$), we solve for γ as a function of r and g. On the curve $\rho(g,r,d) = 0$, $\gamma_\rho(r) = g - r - \frac{g}{r+1}$ expresses γ as a function of r. Likewise on the curve $\Delta(g,r,d) = 0$, we have $\gamma_\Delta(r) = 2\sqrt{(g-1)(r-1)} - 2r$ which expresses γ as a function of r. Observe that $\gamma_\rho = \gamma_\Delta$ has no solutions in the given range (solve for r in terms of g, and note that $g \geq 2$). Finally, since $\gamma_\rho(1) > 0$ and $\gamma_\Delta(1) < 0$, we see by continuity that $\gamma_\rho(r) - \gamma_\Delta(r) > 0$.

The bound $\gamma \geq \lfloor \frac{g-4}{2} \rfloor$ implies that $\Delta < 0$, as in the remark above. Since this is below the general Clifford index ($\lfloor \frac{g-1}{2} \rfloor$), we see that any lattice associated to a non-computing linear system will have negative discriminant. In particular, by Lemma 1.4 above, this applies to the expected maximal linear systems. □

We thus conjecture (Conjecture 1) that the maximal Brill–Noether loci are exactly the *expected maximal Brill–Noether loci*, which are Brill–Noether loci $\mathcal{M}^r_{g,d}$ where for fixed r, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g,r-1,d-1) \geq 0$. Equivalently, the expected maximal Brill–Noether loci correspond to the maximal g^r_d lying under the Brill–Noether hyperbola for each r, up to the containments $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ when $\rho(g,r,d+1) < 0$ and $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1) < 0$.

One could imagine that if there are any unexpected containments among Brill–Noether loci, then some would come from containments of the form $\mathcal{M}^1_{g,d} \subset \mathcal{M}^r_{g,d'}$. However, we find that the expected maximal $\mathcal{M}^1_{g,d}$ is not contained in the other expected maximal loci.

Proposition 1.6. *Let $\rho(g,r,d) < 0$, and $\gamma(r,d) \geq \lfloor \frac{g-1}{2} \rfloor + 1$, e.g., for a non-computing g^r_d. Then $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \nsubseteq \mathcal{M}^r_{g,d}$. When $9 \leq g < 14$, $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \nsubseteq \mathcal{M}^r_{g,d}$ for an expected maximal Brill–Noether locus with $r \geq 2$.*

Proof. Let $k = \frac{g+1}{2}$, and $r' = \min\{r, g-d+r-1\}$. We compute, using Plueger's Brill–Noether number,

$$\begin{aligned}
\rho_k(g,r,d) &= \max_{\ell \in \{0,\dots,r'\}} \rho(g,r-\ell,d) - \ell k = \rho(g,r,d) + (g - k - \gamma(r,d) + 1)\ell - \ell^2 \\
&\leq \max_{\ell \in \{0,\dots,r'\}} \rho(g,r,d) + \ell \left(g - \left\lfloor \frac{g-1}{2} \right\rfloor - \left\lfloor \frac{g+1}{2} \right\rfloor \right) - \ell^2 \\
&< \max_{\ell \in \{0,\dots,r'\}} \rho(g,r,d) + 2\ell - \ell^2 \leq \rho(g,r,d) + 1 \leq 0.
\end{aligned}$$

Therefore $\rho_k(g,r,d) < 0$. From [44, Theorem 1.1], for a general k-gonal curve C, we have $\dim W^r_d(C) \leq \rho_k(g,r,d)$, and $W^r_d(C)$ is empty if its dimension is negative, thus the above computation shows that a general k-gonal curve does not admit a g^r_d. Hence $\mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor} \nsubseteq \mathcal{M}^r_{g,d}$.

The statement for $9 \leq g < 14$ is obtained simply by calculating ρ_k explicitly, and noting that in each case $\rho_k < 0$. □

Remark 1.7. In [15], Farkas asks the general question of when does a general k-gonal curve of genus g have no other linear series g^r_d with $\rho(g,r,d) < 0$? The above proposition answers the case when $k = \lfloor\frac{g+1}{2}\rfloor$, when the curve has maximal sub-general gonality. If a curve has a Brill–Noether special $g^{r'}_{d'}$, then it has a g^r_d for an expected maximal Brill–Noether locus, and the above shows this is not the case. In general, this question is answered by recent breakthroughs in Brill–Noether theory for curves of fixed gonality, see e.g. [9, 15, 24, 30, 31, 43, 44].

In Lemma 6.7, we show that under mild assumptions, the curves $C \in |H|$ on a polarized K3 surface (S,H) with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ associated to an expected maximal locus with $r \geq 2$, all have general Clifford index. Thus the $\mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}$ does not contain other expected maximal loci in many genera. Similar results have been proven by Fakras and Lelli-Chiesa [15, 34].

A natural question is whether lattices corresponding to g^r_ds can be contained as sublattices in each other. In general, the answer is yes. Already in genus 14, we see that $\Lambda^2_{14,10}$ could be embedded as a sublattice of $\Lambda^2_{14,8}$. However, these are not associated to expected maximal loci. In particular, we would like to show that lattices associated to expected maximal loci cannot contain any lattices associated to other g^r_d. This turns out to be false (see Section 1.1). However, we can prove that lattices associated to Brill–Noether special linear systems with lower than general Clifford index cannot be contain in lattices associated to expected maximal loci, and that any containments between lattices associated to an expected maximal loci and those associated to non-computing g^r_ds must be equalities.

Proposition 1.8. *Let $\Lambda^r_{g,d}$ be associated to an expected maximal g^r_d.*

(i) Any lattice $\Lambda^{r'}_{g,d'}$ associated to a special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) < \lfloor\frac{g-1}{2}\rfloor$ for any r' or $\gamma(g^{r'}_{d'}) = \lfloor\frac{g-1}{2}\rfloor$ if $r' \neq 1$ cannot be contained in $\Lambda^r_{g,d}$.

(ii) Let $d' \leq g-1$. Any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $g^{r'}_{d'}$ is not contained in $\Lambda^r_{g,d}$, unless the lattices are isomorphic. Similarly, any lattice associated to a non-computing $g^{r'}_{d'}$ with $d' \leq g-1$ is not contained in the lattice associated to an expected maximal g^r_d unless they are isomorphic.

Proof. To simplify notation, we write Δ for the discriminant of a lattice Λ. To emphasize a sublattice of $\Lambda^r_{g,d}$, we write Λ_{exp} for $\Lambda^r_{g,d}$ and Λ_{sub} for a sublattice.

To prove (i), we recall that if $\Lambda_{sub} \subset \Lambda_{exp}$ is a finite index sublattice, then we have $\Delta_{sub} = [\Lambda_{exp} : \Lambda_{sub}]^2\Delta_{exp}$. We calculate that the ratio $\frac{\Delta_{sub}}{\Delta_{exp}}$ is never a square for the lattices considered. Specifically, we show that the largest negative discriminant $-\Delta_{sub}$ among lattices with $\gamma < \lfloor\frac{g-1}{2}\rfloor$, divided by the negative discriminant $-\Delta_{exp}$ of any lattice associated to an expected maximal linear system, is not an integer. Because $\Delta(g,r,d) = \mathrm{disc}(-\Lambda^r_{g,d}) = d^2 - 4(g-1)(r-1)$, it is clear that for fixed γ this decreases as r increases until $d = g-1$. It follows that none of the lattices considered can be contained in $\Lambda^1_{g,\lfloor\frac{g+1}{2}\rfloor}$, the expected maximal loci with $r = 1$. From now on, we assume $r > 1$. Furthermore, we can take

- $\max(-\Delta_{sub}) = d^2$ with $d = \frac{g+1}{2}$ when $\gamma = \frac{g-1}{2} - 1$; or
- $\max(-\Delta_{sub}) = d^2 - 4(g-1)$ with $d = \frac{2}{3}g + 2$ when $\gamma = \frac{g-1}{2}$.

We also note that $-\Delta$ increases when r and γ both increase by 1, and increases as γ increases for fixed r. Thus if $r' \geq r$, then clearly $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. If $r' < r$, then moving from $g^{r'}_{d'}$ to g^r_d, we take steps increasing r' and γ by 1 until we hit r (and then take steps increasing γ) or hit the line $d = g-1$ and we take steps increasing γ by 1 and decreasing r' by 1. Since each of these steps increase $-\Delta$, we again see that $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. We can always take these steps since we may assume we start at $r = 1$ or $r = 2$, and the expected maximal g^r_d lie far above. Thus (i) is proved.

To prove (ii), we similarly bound $\max(-\Delta)$ and $\min(-\Delta)$ for non-computing g^r_ds. It can be verified that the ratio $\frac{\min(-\Delta)}{\max(-\Delta)} > \frac{1}{4}$ for $r < \sqrt{g}$, and hence $\max(-\Delta) < 4\min(-\Delta)$, thus the discriminants of lattices associated to the expected maximal Brill–Noether loci cannot differ by a square greater than 1. Hence if the lattices associated to expected maximal loci are contained, they must be the same lattice. Since $-\Delta$ increases as r decreases and as γ increases until $d = g-1$, this argument in fact shows that any lattice associated to a non-expected maximal non-computing $g^{r'}_{d'}$ cannot be contained in the lattice of an expected maximal g^r_d unless they have the same discriminant. □

Remark 1.9. In fact, computation up to large genus shows that the lattices associated to expected maximal loci do not contain any lattices associated to other expected maximal loci. We conjecture that this is always true, though a proof of this is currently unknown.

1.1. **Program: Donagi–Morrison implies maximal Brill–Noether loci.** To verify Conjecture 1, our strategy is for fixed genus g and distinct expected maximal $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$ to prove that for a very general K3 surface $(S,H) \in \mathcal{K}^r_{g,d}$, a smooth curve $C \in |H|$ admits a g^r_d but not a $g^{r'}_{d'}$. We do this by combining three kinds of results: (i) a lifting result, (ii) showing that $C \in |H|$ has a g^r_d given by restricting $L \in \Lambda^r_{g,d}$, and (iii) a comparison result that distinguishes lattices. The latter two can be checked for any fixed genus. If all the lattices can be distinguished, a lifting result like the Donagi–Morrison conjecture (Conjecture 2) implies Conjecture 1.

We start by defining a few terms in Conjecture 2.

Definition 1.10. Let S be a K3 surface, $C \subset S$ be a curve, and $A \in \mathrm{Pic}(C)$ and $M \in \mathrm{Pic}(S)$ be line bundles. We say that the linear system $|A|$ is contained in the restriction of $|M|$ to C when for every $D_0 \in |A|$, there is some divisor $M_0 \in |M|$ such that $D_0 \subset C \cap M_0$.

Definition 1.11. A line bundle M is *adapted* to $|H|$ when

(i) $h^0(S,M) \geq 2$ and $h^0(S, H \otimes M^\vee) \geq 2$; and
(ii) $h^0(S, M \otimes \mathcal{O}_C)$ is independent of the smooth curve $C \in |H|$.

Thus whenever M is adapted to $|H|$, condition (i) ensures that $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and condition (ii) ensures that $\gamma(M \otimes \mathcal{O}_C)$ is constant as C varies in its linear system and is satisfied if either $h^1(S,M) = 0$ or $h^1(S, H \otimes M^\vee) = 0$.

Definition 1.12. Let (S,H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g,r,d) < 0$. We call a line bundle M a *Donagi–Morrison lift* of A if M satisfies the conditions in Conjecture 2. That is,

- M is adapted to $|H|$,
- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

We call a line bundle M a *potential Donagi–Morrison lift* of A if M satisfies $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$ and $\deg(M \otimes \mathcal{O}_C) \geq \deg(A)$. Note that a Donagi–Morrison lift is a potential Donagi–Morrison lift. We say a (potential) Donagi–Morrison lift is of type g^s_e if $M^2 = 2s-2$ and $M.H = e$.

We summarize a few potential results distinguishing lattices, each of which would be useful in verifying Conjecture 1 given an appropriate lifting result.

- **(L1).** For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $\mathcal{M}^{r'}_{g,d'}$, one has $\Lambda^{r'}_{g,d'} \not\subseteq \Lambda^r_{g,d}$.
- **(L2).** For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ with $\lfloor \frac{g+1}{2} \rfloor \le \gamma(r',d') \le \lfloor g - 2\sqrt{g} + 1 \rfloor$ and $1 \le r' \le \lfloor \frac{g-1-\gamma(r',d')}{2} \rfloor$, one has $\Lambda^{r'}_{g,d'} \not\subseteq \Lambda^r_{g,d}$.
- **(L3).** For a pair of lattices $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$ both associated to expected maximal Brill–Noether loci, and any lattice $\Lambda^s_{g,e}$ such that $\lfloor \frac{g+1}{2} \rfloor \le \gamma(s,e) \le \gamma(r',d')$ and $1 \le s \le \lfloor \frac{g-1-\gamma(s,e)}{2} \rfloor$, one has $\Lambda^s_{g,e} \not\subseteq \Lambda^r_{g,d}$.

We note that L2 implies L1. Furthermore, for fixed r and d, L2 implies L3 for all r' and d'.

Remark 1.13. The bounds on $\gamma(s,e)$ and s in L3 include all lattices associated to a potential Donagi–Morrison lift of a $g^{r'}_{d'}$. Indeed, suppose M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, and say M is of type g^s_e. The lower bound on $\gamma(s,e)$ comes from Proposition 1.8 (i). Since M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, we have $\gamma(s,e) \le \gamma(r',d')$, which is the upper bound on $\gamma(s,e)$. Since $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, this forces $H \otimes M^\vee \otimes \mathcal{O}_C$ to be at least a g^1_{2g-2-e}, whereby $s \le \frac{g-1-\gamma(s,e)}{2}$ as $2s \le e$, which gives the upper bound on s.

Similarly, the bounds in L2 include all lattices associated to a potential Donagi–Morrison lift of an expected maximal linear system. $M \otimes \mathcal{O}_C$ must have Clifford index no bigger than the expected maximal g^r_d by Conjecture 2, the upper bound on $\gamma(r',d')$ comes from Remark 1.2. The other bounds are obtained in the same way as for L3.

Remark 1.14. As stated above, computations show that L1 holds for every expected maximal locus up to large genus.

We note that L2 and L3 do not always hold. The first genus where L3 fails is $g = 56$, where L3 fails for the lattices $\Lambda^r_{g,d} = \Lambda^2_{56,39}$ and $\Lambda^{r'}_{g,d'} = \Lambda^6_{56,49}$; indeed, in attempting to check whether $\mathcal{M}^2_{56,39}$ can be contained in $\mathcal{M}^3_{56,44}$, a g^3_{44} on a curve $C \in |H|$ for a very general $(S,H) \in \mathcal{K}^2_{56,39}$ has a potential Donagi–Morrison lift M of type g^6_{49}. However, $\Lambda^2_{56,39} \cong \Lambda^6_{56,49}$, and so L3 does not hold. In this case, because $\rho(56,2,39) = -1$ and $\rho(56,3,44) = -4$, we clearly have $\mathcal{M}^2_{56,39} \not\subseteq \mathcal{M}^3_{56,44}$. Hence the failure of L3 does not necessarily obstruct our program to prove that Conjecture 2 implies Conjecture 1.

The next genus where L3 fails is $g = 89$, where the locus $\mathcal{M}^3_{89,69}$ could possibly be contained in $\mathcal{M}^4_{89,75}$ or $\mathcal{M}^5_{89,79}$. This is because line bundles of type g^3_{36} and g^4_{75} have a potential Donagi–Morrison lift M of type g^{10}_{85}, and the lattice $\langle H, M \rangle = \Lambda^{10}_{89,85}$ is isomorphic to $\Lambda^3_{89,69}$, so that L3 does not hold. In this example, $\mathcal{M}^3_{89,69}$ has codimension 3 in $\mathcal{M}_{89}$, whereas $\mathcal{M}^4_{89,75}$ and $\mathcal{M}^5_{89,79}$ both have codimension 1, hence the codimensions of the loci do not rule out the possibility that $\mathcal{M}^3_{89,69}$ is not maximal. Thus in genus 89, Conjecture 2 together with L2 is not sufficient to imply Conjecture 1 without additional techniques.

We note that below genus 200, except for genus $56, 89, 91, 92, 145, 153$, and 190, L2 holds, and thus Conjecture 2 implies Conjecture 1.

Proposition 1.15. *Let $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$ be two expected maximal Brill–Noether loci. Suppose (S,H) is a polarized K3 surface with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$, and $L \otimes \mathcal{O}_C$ is a g^r_d. If the Donagi–Morrison conjecture (Conjecture 2) holds for $g^{r'}_{d'}$ on C and L3 holds for the pair $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$, then $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^{r'}_{g,d'}$. In particular, if Conjecture 2 and L2 hold for all expected maximal g^r_d in genus g, then Conjecture 1 holds in genus g.*

Proof. The condition L3 implies that $\mathrm{Pic}(S)$ cannot admit any potential Donagi–Morrison lift of the $g^{r'}_{d'}$. Hence the existence of a $g^{r'}_{d'}$ on C contradicts the Donagi–Morrison conjecture. Therefore C has no $g^{r'}_{d'}$, as was to be shown. □

To state a related question, we need a simple definition.

Definition 1.16. We define the *special Clifford index* of C as

$$\widetilde{\gamma}(C) := \min\{\gamma(A) \mid \rho(A) < 0,\ h^0(C,A) \geq 2, \text{ and } h^1(C,A) \geq 2\}.$$

We say a Brill–Noether special line bundle A on C *computes* the special Clifford index if $\gamma(A) = \widetilde{\gamma}(C)$.

Lelli-Chiesa's lifting result [35, Theorem 4.2] provides a lift of Brill–Noether special line bundles computing the Clifford index. A similar result for line bundles computing the special Clifford index of the curve together with L1 would imply that $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$ for $\gamma(r,d) \geq \gamma(r',d')$. We are left with three questions to which positive answers would imply parts of Conjecture 1.

Question 1. When L1 or L2 fail, can the Brill–Noether loci be distinguished in another way?

Question 2. Under what conditions does a line bundle computing the special Clifford index of a curve C lift to a line bundle on S?

Question 3. Does the Donagi–Morrison conjecture hold for expected maximal g^r_ds?

We note that the work on Brill–Noether theory for fixed gonality, if it were extended to higher rank, could provide another approach to distinguishing Brill–Noether loci that is complementary to the Donagi–Morrison lifting approach.

2. Stability of Sheaves on K3 Surfaces

We recall the notions of stability and slope stability of torsion free coherent sheaves on a polarized K3 surface (S,H) and Harder–Narasimhan (HN) and Jordan–Hölder (JH) filtrations. Let E be a torsion free coherent sheaf on (S,H) The *slope* of E is $\mu_H(E) := \frac{c_1(E).H}{\mathrm{rk}(E)}$. A torsion free coherent sheaf is called slope stable or μ-stable (μ-semistable) if $\mu_H(F) < \mu_H(E)$ (respectively, $\mu_H(F) \leq \mu_H(E)$) for all coherent sheaves $F \subseteq E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. We define the *normalized Hilbert polynomial* of E to be

$$p(E,n) := \frac{\chi(E \otimes H^n)}{\mathrm{rk}(E)} = \frac{H^2}{2!}n^2 + \mu(E)n + \frac{\chi(E)}{\mathrm{rk}(E)}$$

where the second equality follows from Riemann–Roch. We say E is (Gieseker) stable (semistable) if $p(F,n) < p(E,n)$ (respectively, $p(F,n) \leq p(E,n)$) for all proper subsheaves $F \subsetneq E$, where for two polynomials $f(n), g(n)$ we say $f(n) < g(n)$ ($f(n) \leq g(n)$) if this is true for $n \gg 0$.

We have the following implications for a torsion free coherent sheaf E

$$\mu\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \mu\text{-semistable}.$$

Every torsion free coherent sheaf E has a unique Harder–Narasimhan filtration, which is an increasing filtration

$$0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

such that the factors $gr_i^{HN}(E) = HN_i(E)/HN_{i-1}(E)$ for $i = 1, \ldots, \ell$ are torsion free semistable sheaves with normalized Hilbert polynomials $p_i = p(gr_i^{HN}(E), n)$ satisfying

$$p_{\max} = p_1 > \cdots > p_\ell = p_{\min}.$$

In particular, we see that $\mu(gr_1^{HN}(E)) > \mu(gr_2^{HN}(E)) > \cdots > \mu(gr_\ell^{HN}(E))$. If E is a vector bundle, the sheaves $HN_i(E)$ are locally free. We also have $\mu(HN_1(E)) > \mu(HN_2(E)) > \cdots > \mu(E)$.

Likewise, every (μ)-semistable sheaf E has a Jordan–Hölder filtration, which is an increasing filtration

$$0 = JH_0(E) \subset JH_1(E) \subset \cdots \subset JH_\ell(E) = E,$$

such that the factors $gr_i^{JH}(E) = JH_i(E)/JH_{i-1}(E)$ for $i = 1, \dots, \ell$ are torsion free stable sheaves with normalized Hilbert polynomial $p(E,n)$. In particular, $\mu(E) = \mu(gr_i^{JH}(E))$ for all i. The JH filtration is not uniquely determined, however the associated graded object $gr^{JH}(E) = \bigoplus_i gr_i^{JH}(E)$ is uniquely determined by E.

We also briefly recall some facts about the moduli space of stable and semistable sheaves on K3 surfaces from [23]. For a sheaf E on (S,H), the *Mukai vector* is given by

$$v(E) := ch(E)\sqrt{td(S)} = (\mathrm{rk}(E),\ c_1(E),\ ch_2(E) + \mathrm{rk}(E)) = (\mathrm{rk}(E),\ c_1(E),\ \chi(E) - \mathrm{rk}(E))\,,$$

considered as an element in $H^*(S,\mathbb{Z})$. For a fixed Mukai vector v, the moduli space of semistable sheaves with Mukai vector v is denoted $M(v)$, and the open (possibly empty) subscheme of stable sheaves is denoted $M(v)^s \subset M(v)$. The Mukai pairing is given by

$$\langle v(E), v(F)\rangle := -\chi(E,F) = -\sum_i (-1)^i \operatorname{Ext}^i(E,F) = -\int_S v(E)^* \wedge v(F),$$

where for $v(E) = v^0 + v^2 + v^4 \in H^*(S,\mathbb{Z})$ with $v^i \in H^i(S,\mathbb{Z})$, $v(E)^* := v^0 - v^2 + v^4$. We recall that the space of stable sheaves with Mukai vector v, $M(v)^s$ is either empty or a smooth quasi-projective variety of dimension $2 + \langle v, v\rangle$.

3. Lazarsfeld–Mukai Bundles and Lifting

We briefly recall some facts about Lazarsfeld–Mukai bundles (LM bundles) and state a few useful facts that motivate our proof. Let $\iota\colon C \hookrightarrow S$ be a smooth irreducible curve of genus g in S and A a basepoint free line bundle on C of type g^r_d. We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0 \longrightarrow F_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{ev} \iota_*(A) \longrightarrow 0.$$

Dualizing gives $E_{C,A} := F_{C,A}^\vee$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \iota_*(\omega_C \otimes A^\vee) \longrightarrow 0;$$

whereby the following facts about the LM bundle $E_{C,A}$ are readily proved.

Proposition 3.1. *Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g^r_d on $C \subset S$, then:*

- $\det E_{C,A} = c_1(E_{C,A}) = [C]$ *and* $c_2(E_{C,A}) = \deg(A)$*;*
- $\mathrm{rk}(E_{C,A}) = r+1$ *and $E_{C,A}$ is globally generated off the base locus of* $\iota_*(\omega_C \otimes A^\vee)$*;*
- $h^0(S,E_{C,A}) = h^0(C,A) + h^0(C,\omega_C \otimes A^\vee) = 2r+1+g-d = g-(d-2r)+1$*;*
- $h^1(S,E_{C,A}) = h^2(S,E_{C,A}) = 0$*,* $h^0(S,E_{C,A}^\vee) = h^1(S,E_{C,A}^\vee) = 0$*;*
- $\chi(F_{C,A} \otimes E_{C,A}) = 2(1-\rho(g,r,d))$*.*

A vector bundle E is called *simple* if $\mathrm{End}(E)$ is a division algebra. Over an algebraically closed field, this is equivalent to $h^0(E^\vee \otimes E) = 1$. Thus we see that $E_{C,A}$ is non-simple if $\rho(g,r,d) < 0$.

In [35], generalized LM bundles are defined and prove useful in lifting special line bundles on a curve $C \in |H|$ to a line bundle on the polarized K3 surface (S,H).

Definition 3.2. Let C be a curve and $A \in \mathrm{Pic}(C)$. The linear system $|A|$ is called *primitive* if both A and $\omega_C \otimes A^\vee$ are basepoint free.

Definition 3.3 ([35] Definition 1)**.** A torsion free coherent sheaf E on S with $h^2(S,E) = 0$ is called a *generalized Lazarsfeld–Mukai bundle* (gLM bundle) of type (I) or (II), respectively, if

(I) E is locally free and generated by global sections off a finite set;
or
(II) E is globally generated.

Remark 3.4 ([35] Remark 1)**.** If conditions (I) and (II) of Definition 3.3 are both satisfied, then E is the LM bundle associated with a smooth irreducible curve $C \subset S$ and a primitive linear series (A, V) on C, i.e. $E = E_{C,(A,V)}$, where $E_{C,(A,V)}$ is the dual of the kernel of the evaluation map $V \otimes \mathcal{O}_S \to A$. Furthermore, $V = H^0(C, A)$ if and only if $h^1(S, E) = 0$, in which case E is just the LM bundle associated to A.

Definition 3.5. Let E be a gLM bundle. The *Clifford index of E* is:

$$\gamma(E) := c_2(E) - 2(\mathrm{rk}(E) - 1).$$

Remark 3.6. For the LM bundle $E_{C,A}$ for a smooth curve $C \subset S$ and A a g^r_d on C, one has $\gamma(E_{C,A}) = \gamma(A)$ by Proposition 3.1.

Lemma 3.7 ([35] Corollary 2.5)**.** *Let E be a gLM bundle of rank r and $c_1(E)^2 > 0$. Then, $\gamma(E) \geq 0$. Furthermore, $\gamma(E) = 0$ only in the following cases:*

(a) *$r = 1$ and E is a globally generated line bundle;*
(b) *$E = E_{C,\omega_C}$ for some smooth irreducible curve $C \subset S$ of genus $g = r \geq 2$;*
(c) *$r > 1$ and $E = E_{C,(r-1)g^1_2}$ for some smooth hyperelliptic curve $C \subset S$ of genus $g > r$.*

Lemma 3.8. *Let $N \in \mathrm{Pic}(S)$ be nontrivial and globally generated with $h^0(S, N) \neq 0$. Let $E = E_{C,A}$ and suppose we have a short exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

with E/N torsion free. Then E/N satisfies $h^1(S, E/N) = h^2(S, E/N) = 0$. If A is primitive, then E/N is a gLM bundle of type (II). If we further assume that E/N is locally free, then it is a LM bundle for a smooth irreducible curve $D \in |H - N|$. If A is not primitive and E/N is assumed locally free, then E/N is a gLM bundle of type (I). In any of the above cases, we have

- *$c_1(E/N) = H - N$;*
- *$c_2(E/N) = d + N^2 - H.N$;*
- *$\gamma(E/N) = \gamma(E_{C,A}) + N^2 - H.N + 2$.*

Proof. If A is primitive, we see that E/N is globally generated as E is globally generated. From the long exact sequence in cohomology, and noting that $h^2(S, N) = h^1(S, E) = h^2(S, E) = 0$, we see that $h^1(S, E/N) = h^2(S, E/N) = 0$. Thus E/N is a gLM bundle of type (II). If E/N is assumed to be locally free, then as in Remark 3.4, $E/N = E_{D,B}$ is the LM bundle associated to a smooth irreducible curve $D \subset S$ and a line bundle B on D. Finally, if A is not primitive, then E/N is globally generated off a finite set as it is the quotient of E, which is also globally generated off a finite subset. Thus E/N is a gLM of type (I).

Applying Whitney's formula to the exact sequence, we see that

$$1 + c_1(E) + c_2(E) = (1 + c_1(E/N) + c_2(E/N))\,(1 + N)\,,$$

hence $c_1(E/N) = H - N$ and $c_2(E/N) = d + N^2 - H.N$. Finally, as $\gamma(E/N) = c_2(E/N) - 2(\mathrm{rk}(E/N) - 1)$ and $\mathrm{rk}(E/N) = \mathrm{rk}(E) - 1 = (r + 1) - 1 = r$, it follows that

$$\gamma(E/N) = d + N^2 - H.N - 2(r - 1) = d - 2r + N^2 - H.N + 2 = \gamma(E) + N^2 - H.N + 2.$$

□

Remark 3.9. If A is of type g^r_d and $L = H - N$ is a lift of A with $L^2 = 2r - 2$, then the last equality gives $\gamma(E/N) = \gamma(A) + (2r - 2) - d + 2 = 0$.

Remark 3.10. The same argument as above shows that if A is primitive and $M \subset E = E_{C,A}$ is a subsheaf such that E/M is torsion free (e.g. obtained through a Harder–Narasimhan filtration), then E/M is a gLM bundle of type (II). Moreover, by [35, Proposition 2.7], if $c_1(E/M)^2 = 0$, then $c_2(E/M) = 0$. In the following sections, we will use the contrapositive of this when $c_2(E/M) > 0$.

We give a brief summary of gLM bundles of low Clifford index. Such a characterization can be useful in eliminating certain types of filtrations of Lazarsfeld–Mukai bundles, see Section 6.6.

Proposition 3.11. *Let $E = E_{C,A}$ be a LM bundle associated to a primitive linear system A on $C \subset S$. Suppose there is a globally generated saturated line bundle $N \subset E$ with $h^0(S,N) \geq 2$ and $\gamma(E/N) \leq 2$. Then either $c_1(E/N)^2 = 0$ in which case $E/N = \mathcal{O}_S(\Sigma)^{\oplus r(A)}$ for an irreducible elliptic curve $\Sigma \subset S$, or $c_1(E/N)^2 > 0$ and one of the following holds:*

(i) $\gamma(E/N) = 0$ *(hence E/N is a LM bundle);*
(ii) $(E/N)^{\vee\vee}$ *is a LM bundle of Clifford index* 0*;*
(iii) E/N *or* $(E/N)^{\vee\vee}$ *is a LM bundle of Clifford index* 1*;*
(iv) E/N *is a LM bundle of Clifford index* 2*.*

Proof. By Lemma 3.8, we see that E/N is a gLM bundle of type (II). If $c_1(E/N)^2 = 0$, [35, Proposition 2.7] gives $E/N = \mathcal{O}_C(\Sigma)^{\oplus r(A)}$, as stated.

We now assume $c_1(E/N)^2 > 0$. If $\gamma(E/N) = 0$, we are in case (i).

If $\gamma(E/N) = 1$ and E/N is locally free, we are in case (iii). If E/N is not locally free, then [35, Proposition 2.4] shows that $(E/N)^{\vee\vee}$ has Clifford index 0, and we are in case (ii).

If $\gamma(E/N) = 2$ and E/N is locally free, we are in case (iv). If (E/N) is not locally free, then [35, Proposition 2.4] again shows that $(E/N)^{\vee\vee}$ has Clifford index 0 or 1, and we are in case (ii) or (iii), respectively. □

Remark 3.12. Furthermore, in case (iv) above, fixing the rank of A narrows the possibilities for the classification of E/N. For example, when A has rank 3 and E/N has Clifford index 2, then $E/N = E_{D,g^2_6}$ for a g^2_6 on a smooth curve D in the linear system of $\det(E/N)$.

Likewise, restricting the Clifford index of a LM bundle E similarly restricts to which linear system E corresponds. For example, if E is a LM bundle and $\gamma(E) = 1$ or $\gamma(E) = 2$, then a smooth irreducible curve $D \in |\det(E)|$ has $\gamma(D) \leq 2$ and is thus either hyperelliptic (when $\gamma(D) = 0$), trigonal or a plane quintic (when $\gamma(D) = 1$), or a plane sextic (when $\gamma(D) = 2$ and $\mathrm{rk}(E) = 3$).

One could similarly characterize gLM bundles of type (II) of higher Clifford index, using [35, Proposition 2.4] repeatedly as in Proposition 3.11, and then fixing the rank as above.

We recall a few lemmas which show when a linear series on a curve $C \in |H|$ is the restriction of a line bundle L on S.

Lemma 3.13. *Let (S,H) be a polarized K3 surface of genus $g \geq 2$, $C \in |H|$ be a smooth irreducible curve, and L a globally generated line bundle on S such that $L|_C$ is a g^r_d with $c_1(L).C = d < 2g-2$. Then if $h^1(S,L) = 0$, we have $L^2 = 2r - 2 - 2h^1(S, L(-C))$.*

Proof. Since H is basepoint free and $c_1(L(-C)).C = d - (2g-2) < 0$, we have $h^0(S, L(-C)) = 0$, as in the proof of [29, Proposition 2.1]. We now consider the short exact sequence for a divisor $C \subset S$ tensored with L,

$$0 \longrightarrow L(-C) \longrightarrow L \longrightarrow L|_C \longrightarrow 0.$$

By Riemann-Roch on C we have $h^1(S, L|_C) = h^1(C, L|_C) = r-d+g$, and as $h^1(S,L) = h^2(S,L) = 0$, the long exact sequence in cohomology and Serre duality give $h^2(S, L(-C)) = h^0(S, L(-C)^\vee) = r - d + g$. By Riemann-Roch on S, we have

$$h^0(S, L(-C)^\vee) - h^1(S, L(-C)) = 2 + \frac{c_1(L(-C))^2}{2} = 2 + \frac{c_1(L)^2 - 2d + 2g - 2}{2} = 1 - d + g + \frac{c_1(L)^2}{2}$$

thus $c_1(L)^2 = 2r - 2 - 2h^1(S, L(-C))$. □

Corollary 3.14. *Let (S,H) be a polarized K3 surface of genus $g \geq 2$, A a complete g^r_d on a smooth $C \in |H|$ with $d < 2g-2$. Let $N \in \mathrm{Pic}(S)$ be a line bundle with $h^0(S,N) \geq 2$ and $h^1(S,N) = 0$.*

Assume $H \otimes N^\vee$ is globally generated, satisfies $h^1(S, H \otimes N^\vee) = 0$, and is a lift of A. Then $c_1(H \otimes N^\vee)^2 = 2r - 2$.

Proof. We have $h^1(S, N) = 0$. Hence as $N^\vee = H \otimes N^\vee \otimes H^\vee$, Serre duality gives $0 = h^1(S, N^\vee) = h^1(S, H \otimes N^\vee(-C))$. Thus Lemma 3.13 shows that $(H - N)^2 = 2r - 2$. □

Remark 3.15 ([35] Remark 6)**.** The proof [35, Lemma 4.1] shows that as soon as we have a nontrivial $N \in \mathrm{Pic}(S)$ with $h^0(S, N) \neq 0$ and an injection $N \hookrightarrow E_{C,A}$, we have $h^0(S, \iota_*(A) \otimes (H \otimes N^\vee) \otimes \mathcal{O}_C) = h^0(C, A^\vee \otimes (H \otimes N^\vee)|_C) \neq 0$, i.e., the linear series $|A|$ is contained in $|(H \otimes N^\vee)|_C|$. We also note that if $h^1(S, N) = 0$, then

$$H^0(C, (H \otimes N^\vee) \otimes \mathcal{O}_C) = H^0(S, H \otimes N^\vee)|_C.$$

Lemma 3.16. *Let N be a line bundle and $0 \to N \to E \to E/N \to 0$ be a short exact sequence of coherent sheaves on a polarized K3 surface (S, H), where E/N is stable,* $\mathrm{rk}(E) = r+1$*, $c_1(E) = H$, $c_1(E)^2 = 2g - 2 \geq 0$. If $h^0(S, N) < 2$, then $c_2(E) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$.*

Proof. Since $\mu(N) \geq \mu(E) \geq 0$, we have $h^2(S, N) = 0$. Therefore if $h^0(S, N) < 2$ we have $c_1(N)^2 \leq -2$. Hence

$$c_1(E/N)^2 + 2c_1(N).c_1(E/N) = c_1(E)^2 - c_1(N)^2 \geq 2g - 2 + 2 = 2g$$

and

$$c_1(E/N).c_1(N) = c_1(N).(c_1(E) - c_1(N)) \geq \frac{2g-2}{r+1} + 2,$$

where the last inequality comes from the fact that $\mu(N) \geq \mu(E)$. Thus $\frac{c_1(E/N)^2}{2} \geq g - c_1(N).c_1(E/N)$.

Furthermore, since E/N is stable of rank r, the dimension of the moduli space of stable sheaves with Mukai vector $\nu(E/N)$, $M^s_{\nu(E/N)}$, has non-negative dimension. Thus $2rc_2(E/N) - (r-1)c_1(E/N)^2 - 2(r^2-1) \geq 0$, and we have $c_2(E/N) \geq r - \frac{1}{r} + \left(\frac{r-1}{2r}\right) c_1(E/N)^2$.

We now calculate $c_2(E) = c_1(E/N).c_1(N) + c_2(E/N) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$, as desired. □

We present a version of [34, Proposition 7.4] which motivates our proof strategy below.

Proposition 3.17. *Let (S, H) be a polarized K3 surface and A be a complete basepoint free g^r_d on a smooth irreducible curve $C \in |H|$ with $r \geq 2$ and let $E = E_{C,A}$. Suppose that E sits in a short exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

for some line bundle N and $c_2(E) = d < \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$. If E/N is stable, or E/N is semistable and there are no elliptic curves on S, then $|A|$ is contained in the restriction to C of the linear system $|H \otimes N^\vee|$ on S. Moreover, $H \otimes N^\vee$ is adapted to $|H|$ and $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq d - r - 3$.

Proof. By the previous lemma, $h^0(S, N) \geq 2$. We also have $h^0(S, \det E/N) \geq 2$ from [34, Lemma 3.3]. We note that $(E/N)^{\vee\vee}$ is globally generated off a finite set and

$$h^i(S, (E/N)^{\vee\vee}) = h^i(S, E/N) = 0 \text{ for } i = 1, 2.$$

Since $\det E/N = \det(E/N)^{\vee\vee}$ is basepoint free and nontrivial, $\det(E/N)$ is nef, thus $c_1(E/N)^2 \geq 0$. If $h^1(S, \det(E/N)) \neq 0$, then $c_1(E/N)^2 = 0$ by Saint-Donat. By [18, Proposition 1.1], there is a smooth elliptic curve $\Sigma \subset S$ such that $(E/N)^{\vee\vee} = \mathcal{O}(\Sigma)^{\oplus 3}$. This contradicts the stability of E/N (or the non-existence of elliptic curves on S), thus we must have $c_1(E/N)^2 \geq 2$ (hence $c_2(E/N) \geq r + 1 - \frac{2}{r}$) and $h^1(S, \det(E/N)) = 0$. This ensures that $h^0(C, \det(E/N) \otimes \mathcal{O}_C) = h^0(C, H \otimes N^\vee \otimes \mathcal{O}_C)$

does not depend on the curve $C \in |H|_s$. Hence $\det(E/N) = H \otimes N^\vee$ is adapted to $|H|$. We calculate

$$\begin{aligned}\gamma(\det(E/N) \otimes \mathcal{O}_C) &= c_1(E/N).c_1(E) - 2h^0(C, \det(E/N) \otimes \mathcal{O}_C) + 2 \\ &= c_1(E/N)^2 + c_1(N).c_1(E/N) - 2h^0(C, \det(E/N) \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(E/N)^2 - 2h^0(S, \det(E/N)) + c_1(N).c_1(E/N) + 2 \\ &= -2h^1(S, \det(E/N)) - 4 + c_1(N).c_1(E/N) + 2 \\ &= d - c_2(E/N) - 2 \leq d - r - 3.\end{aligned}$$

The claim that $|A|$ is contained in $|H \otimes N^\vee \otimes \mathcal{O}_C|$ is proved in the same way as in [35, Lemma 4.1]. □

Remark 3.18. In the above proposition, if A is of type g^3_d, then $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq d-r-3 = \gamma(A)$. However, as soon as $r \geq 4$, then $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C)$ may be bigger than $\gamma(A)$. However, Lelli-Chiesa proves in [35, Propositioon 5.1] that $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq \gamma(A)$ whenever $N \subset E$ is a saturated subsheaf and $h^1(S, N) = 0$.

4. Filtrations of Lazarsfeld–Mukai Bundles of Rank 4

Throughout this section, (S, H) is a polarized K3 surface of genus g, $C \in |H|$ is a smooth irreducible curve, A is a line bundle of type g^3_d on C with $d \leq g-1$, and $E = E_{C,A}$ is the LM bundle corresponding to A. Given E, we can take its JH filtration or take its HN filtration, further take JH filtrations of the properly semistable factors, lift the JH factors and expand the HN filtration of E to arrive at a *terminal filtration* such that all quotients are stable sheaves. We enumerate all the possibilities listing a filtration by the ranks of the terms, i.e., a filtration of type $1 \subset 4$ is a filtration $0 \subset N \subset E$ where $\mathrm{rk}(N) = 1$.

The terminal filtrations correspond to flags of E where each quotient is stable, hence the terminal filtrations are

$$\begin{aligned}&1 \subset 4, \quad 2 \subset 4, \quad 3 \subset 4, \\ &1 \subset 2 \subset 4, \quad 1 \subset 3 \subset 4, \quad 2 \subset 3 \subset 4, \\ &1 \subset 2 \subset 3 \subset 4.\end{aligned}$$

In order to apply Proposition 3.17, we want to show that given the g^3_d, E must have a terminal filtration of type $1 \subset 4$. In all other cases, we want to find a lower bound on $d = c_2(E)$. To this end, we find a bound for $c_2(E)$ in terms of the intersections of the Chern roots of the LM bundle E. We begin by noting a few general bounds, and then deal with each filtration.

We slightly generalize the proof of [34, Lemma 4.1].

Proposition 4.1. *Let E a LM bundle with $c_1(E) = H$ and $\mu(E) = \frac{g-1}{2} > 0$ sitting in an exact sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

where M and M_1 are coherent sheaves. Suppose that the general smooth curve $C \in |H|$ has (constant) Clifford index $\gamma = \gamma(C)$. Then one has $c_1(M).c_1(M_1) \geq \gamma + 2$.

Proof. We write $\mu(F) = \mu_H(F)$. Since M_1 is a quotient of E, it is globally generated off a finite set of points. Moreover, we have $h^2(S, M_1) = 0$, thus $h^0(S, \det M_1) \geq 2$ by [34, Lemma 3.3] as the vector bundle $M_1^{\vee\vee}$ is globally generated off a finite number of points and $\det(M_1) := \det(M_1^{\vee\vee})$. As in [34, Lemma 3.2], we see that $\det M_1$ is basepoint free and nontrivial, thus $\mu(\det M_1) > 0$, $\mu(M) > 0$. Hence as $\mu(\det M) \geq \mu(M) > 0$, [34, Proposition 3.1] shows that $h^2(S, \det M_1) = 0$, $h^2(S, \det M) = 0$, and that $\det M_1$ is nef whereby $c_1(M_1)^2 \geq 0$.

Furthermore, as

$$\mu(M)=\frac{c_1(M).c_1(E)}{\mathrm{rk}(M)}=\frac{c_1(M).(c_1(M)+c_1(M_1))}{\mathrm{rk}(M)}\geq\frac{g-1}{2},$$

we have $c_1(M).c_1(M_1)\geq \mathrm{rk}(M)\frac{g-1}{2}-c_1(M)^2$. Since $h^2(S,\det M)=0$, we note that

$$h^0(S,\det M)\geq h^0(S,\det M)-h^1(S,\det M)=\chi(\det M)=2+\frac{c_1(M)^2}{2}.$$

Therefore, if $2>h^0(S,\det M)$, then $c_1(M)^2\leq -2$, and thus

$$c_1(M).c_1(M_1)\geq \mathrm{rk}(M)\frac{g-1}{2}+2\geq \mathrm{rk}(M)\gamma+2\geq\gamma+2$$

as $\mathrm{rk}(M)\geq 1$.

Hence from now on we assume that $h^0(S,\det M)\geq 2$. Since $\omega_C\otimes(\det M_1)^\vee\otimes\mathcal{O}_C=\det M\otimes\mathcal{O}_C$, the line bundle $\det M_1\otimes\mathcal{O}_C$ contributes to $\gamma(C)$. Tensoring the short exact sequence for $\mathcal{O}_C$ with $\det M_1$ gives

$$0\longrightarrow \det M^\vee\longrightarrow \det M_1\longrightarrow \det M_1\otimes\mathcal{O}_C\longrightarrow 0\ ,$$

which gives $h^0(C,\det M_1\otimes\mathcal{O}_C)\geq h^0(S,\det M_1)$. It follows that

$$\begin{aligned}\gamma(\det M_1\otimes\mathcal{O}_C)&=c_1(M_1).(c_1(M)+c_1(M_1))-2h^0(C,\det M_1\otimes\mathcal{O}_C)+2\\&\leq c_1(M_1)^2+c_1(M).c_1(M_1)-2\chi(\det M_1)-2h^1(S,\det M_1)+2\\&=-2+c_1(M).c_1(M_1)-2h^1(S,\det M_1).\end{aligned}$$

By assumption, we have $\gamma(\det M_1\otimes\mathcal{O}_C)\geq\gamma$, thus $c_1(M).c_1(M_1)\geq\gamma+2+2h^1(S,\det M_1)\geq\gamma+2$, as desired. ☐

Remark 4.2. It follows from the second half of the proof that if M and M_1 are coherent sheaves such that $c_1(M)+c_1(M_1)=c_1(E)$, $\det M_1\otimes\mathcal{O}_C$ (hence also $\det M\otimes\mathcal{O}_C$) contributes to $\gamma(C)$, and $h^2(S,\det M_1)=0$ (or $h^2(S,\det M)=0$), then $c_1(M).c_1(M_1)\geq\gamma(C)+2+2h^1(S,\det M_1)\geq\gamma(C)+2$ (or $c_1(M).c_1(M_1)\geq\gamma(C)+2+2h^1(S,\det M)\geq\gamma(C)+2$).

Proposition 4.3. *Let (S,H) be a polarized K3 surface, $C\in|H|$ a smooth irreducible curve, A a basepoint free line bundle on A of type g^3_d, and $E=E_{C,A}$. Suppose E sits in an exact sequence*

$$0\longrightarrow M\longrightarrow E\longrightarrow E/M\longrightarrow 0\ ,$$

where M and E/M are coherent torsion free sheaves on S and $\mu(M)\geq\mu(E)\geq\mu(E/M)$. If $\mathrm{rk}(M)\geq\mathrm{rk}(E/M)$, then $c_1(M)^2\geq c_1(E/M)^2$. If $\mathrm{rk}(M)>\mathrm{rk}(E/M)$, then $c_1(M)^2>c_1(E/M)^2$. In particular, $\det(E/M)\otimes\mathcal{O}_C$ contributes to $\gamma(C)$.

Proof. As in Proposition 4.1, we see $h^0(S,\det E/M)\geq 2$, $\mu(E/M)>0$, $\det(E/M)$ is nef, and $h^2(S,\det M)=0$. Since $h^0(S,\det E/M)\geq 2$, it remains to show that $h^0(S,\det M)\geq 2$.

We observe that

$$c_1(M)^2+c_1(M).c_1(E/M)=\mathrm{rk}(M)\mu(M)\geq\mathrm{rk}(E/M)\mu(E/M)=c_1(E/M)^2+c_1(M).c_1(E.M)$$

whence $c_1(M)^2\geq c_1(E/M)^2\geq 0$ as $\det(E/M)$ is nef.

Since $h^2(S,\det M)=0$, we have $h^0(S,\det M)\geq\chi(\det M)=2+\frac{c_1(M)^2}{2}$. Thus as $c_1(M)^2\geq 0$, $\det(E/M)\otimes\mathcal{O}_C$ contributes to $\gamma(C)$. ☐

For each terminal filtration not of the form $0\subset 1\subset 4$, we find a lower bound for $d=c_2(E)$. That is whenever E does not have a maximal destabilizing sub-line bundle, we find that d must be large. In effect, $c_2(E)$ controls the complexity of its Harder–Narasimhan and Jordan–Hölder filtrations.

4.1. **Filtration** $2 \subset 4$**.** We assume E is unstable with terminal filtration $0 \subset M \subset E$ with M and $M_1 = E/M$ stable rank 2 torsion free sheaves. Thus E sits in an exact sequence of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0\ .$$

We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{1}$$

$$d = c_2(E) = c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) \tag{2}$$

Lemma 4.4. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{\gamma}{2} + 4 + \frac{g-1}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $c_1(M).c_1(M_1) \geq \gamma + 2$. Stability of M and M_1 give $-2 \leq \langle \nu(M_{(1)}), \nu(M_{(1)})\rangle = 4c_2(M_{(1)}) - c_1(M_{(1)})^2 - 8$, thus $c_2(M_{(1)}) \geq \frac{3}{2} + \frac{c_1(M_{(1)})^2}{4}$.
We have

$$\frac{c_1(M)^2 + c_1(M_1)^2}{4} + \frac{c_1(M).c_1(M_1)}{2} = \frac{\mu(M) + \mu(M_1)}{2} = \frac{(c_1(M) + c_1(M_1))^2}{4} = \mu(E) = \frac{g-1}{2}.$$

We now calculate

$$\begin{aligned} d &= c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) \\ &\geq c_1(M).c_1(M_1) + 3 + \frac{c_1(M)^2 + c_1(M_1)^2}{4} \\ &= c_1(M).c_1(M_1) + 3 + \frac{g-1}{2} - \frac{c_1(M).c_1(M_1)}{2} \\ &\geq \frac{\gamma+2}{2} + 3 + \frac{g-1}{2}. \end{aligned}$$

as claimed. □

4.2. **Filtration** $3 \subset 4$**.** We assume $E = E_{C,A}$ is unstable with terminal filtration $0 \subset M \subset E$ with M a stable rank 3 torsion free sheaf. Thus E sits in an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \otimes I_\xi \longrightarrow 0$$

where N is a line bundle and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l(\xi) = d - c_1(M).c_1(N)$. We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N) \tag{3}$$

$$c_1(H) = c_1(E) = c_1(M) + c_1(N) \tag{4}$$

$$d = c_2(E) = c_1(N).c_1(M) + c_2(M) + l(\xi) \tag{5}$$

Lemma 4.5. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{2}{3}(\gamma + 2) + \frac{g}{2} + \frac{13}{6}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $c_1(N).c_1(M) \geq \gamma + 2$.

As M is stable, we have $-2 \leq \langle \nu(M), \nu(M) \rangle = 6c_2(M) - 2c_1(M)^2 - 18$, thus $c_2(M) \geq \frac{8+c_1(M)^2}{3}$. Thus

$$\begin{aligned} d &= c_1(N).c_1(M) + c_2(M) + l(\xi) \\ &\geq c_1(N).c_1(M) + \frac{c_1(M)^2}{3} + \frac{8}{3} \\ &\geq c_1(N).c_1(M) + \frac{g-1}{2} - \frac{c_1(N).c_1(M)}{3} + \frac{8}{3} \\ &\geq \frac{2}{3}(\gamma + 2) + \frac{g}{2} + \frac{13}{6}, \end{aligned}$$

as desired. □

4.3. **Filtration $1 \subset 2 \subset 4$.** We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 2$, and $E/M = M_1$ a stable torsion free sheaf. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{6}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(M_1) \tag{7}$$

(8)
$$d = c_2(E) = c_2(M) + c_2(M_1) + c_1(M).c_1(M_1) = c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1) + c_2(M_1)$$

Moreover, as M_1 is stable, we have

$$-2 \leq \langle \nu(M_1), \nu(M_1) \rangle = c_1(M_1)^2 - 4\chi(M_1) + 8 = 4c_2(M_1) - c_1(M_1)^2 - 8$$

thus $c_2(M_1) \geq \frac{3}{2} + \frac{c_1(M_1)^2}{4}$. Therefore we have

$$d \geq \frac{3}{2} + \frac{c_1(M_1)^2}{4} + c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1). \tag{9}$$

Lemma 4.6. *Suppose E is as above. Then $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(M_1) + c_1(M/N)) \geq \frac{g-1}{2} + 2$ and either $(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$;*
(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $c_1(M/N).(c_1(N) + c_1(M_1)) \geq 2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}$;*
(d) *$c_1(N).c_1(M/N) \geq \frac{g+3}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, N) \geq 2$
(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, N) < 2$
(iii) $h^0(S, M/N) < 2$ and $h^0(S, N) \geq 2$
(iv) $h^0(S, M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det M_1 \otimes N) \geq 2$ and $h^0(S, H \otimes N^\vee) = h^0(S, \det M_1 \otimes M/N) \geq 2$ as $\det M_1$ has global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(M_1).c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g-1}{2} + 2, \end{aligned}$$

thus the first statement of case (b) is proved. We now observe that $c_1(N \otimes \det M_1)^2 > c_1(M/N)^2$ which follows from the computation $c_1(N \otimes \det M_1)^2 - c_1(M/N)^2 \geq 2\mu(M_1) > 0$.

If $c_1(N \otimes \det M_1)^2 < 0$, then also $c_1(M/N)^2 < 0$, and we calculate

$$\begin{aligned} 2g-2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(M_1))^2 \\ &= c_1(N \otimes \det M_1)^2 + 2c_1(N \otimes \det M_1).c_1(M/N) + c_1(M/N)^2 \\ &< 2(c_1(N) + c_1(M_1)).c_1(M/N), \end{aligned}$$

thus $c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$. Else $c_1(N \otimes \det M_1)^2 \geq 0$ and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, N \otimes \det M_1) \geq 2$ and so M/N contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M_1 \otimes M/N$, we have $h^0(S, \det M_1 \otimes M/N) \geq 2$. Thus as $h^0(S, N) \geq 2$, we see that $N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Therefore, as $h^0(S, M/N) < 2$, we have $c_1(M/N)^2 \leq -2$.

In cases (iii) and (iv), we have $c_1(M/N)^2 \leq -2$. We now calculate

$$\begin{aligned} 2g-2 = c_1(E)^2 &= c_1(M/N)^2 + c_1(N)^2 + c_1(M_1)^2 + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) \\ &\leq c_1(N)^2 + c_1(M_1)^2 + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) - 2 \\ &\leq c_1(N)^2 + g - 3 + 2c_1(M/N).c_1(N), \end{aligned}$$

thus

$$c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2}. \tag{10}$$

In case (iii), we observe that since

$$c_1(M/N).(c_1(N) + c_1(M_1)) + c_1(M/N)^2 = \mu(M/N) \geq \mu(E/N) = \frac{(c_1(E/N)).(c_1(E))}{3},$$

we have

$$\begin{aligned} c_1(M/N).(c_1(N) + c_1(M_1)) \geq -c_1(M/N)^2 + \frac{c_1(M/N)^2}{3} + \frac{c_1(M/N).(c_1(N) + c_1(M_1))}{3} \\ + \frac{c_1(M).c_1(M_1)}{3} + \frac{c_1(M_1)^2}{3}. \end{aligned}$$

And subtracting $c_1(M/N).(c_1(N) + c_1(M_1))/3$ from both sides and multiplying by $3/2$ yields

$$c_1(M/N).(c_1(N) + c_1(M_1)) \geq -c_1(M/N)^2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}.$$

Noting that $c_1(M/N)^2 \leq -2$ shows we are in case (c).

In case (iv), as $h^0(S, N), h^0(S, M/N) < 2$, we have $c_1(N)^2, c_1(M/N)^2 \leq -2$, thus Equation (10) gives $c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2} \geq \frac{g+1}{2} + 1 = \frac{g+3}{2}$, and we are in case (d). □

Lemma 4.7. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, and $m = D^2$ is the minimum self-intersection of an effective classes $D \in \mathrm{Pic}(S)$ (i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), then we have $d \geq \frac{5}{4}\gamma + \frac{m}{2} + 5$ or $d \geq 5 + \frac{3}{2}\gamma$. Moreover, when A is primitive, then we can assume $m \geq 2$.*

Proof. We write $2d \geq 3 + \frac{c_1(M_1)^2}{2} + c_1(N).c_1(E/N) + c_1(M/N).(c_1(N) + c_1(M_1)) + c_1(M).c_1(M_1)$, and apply bounds to each of the terms. From Proposition 4.1, we see that $c_1(N).c_1(E/N) \geq \gamma + 2$, and $c_1(M).c_1(M_1) \geq \gamma + 2$. In cases (a), (b), we have $c_1(M/N).(c_1(N) + c_1(M_1)) \geq \gamma + 2$. In case (c), we have $d \geq \frac{5}{4}\gamma + \frac{m}{2} + 5$. Finally, in case (d), we have $d \geq 2 + c_1(N).c_1(M/N) + c_1(M).c_1(M_1) \geq 2 + \frac{g+13}{2} + \gamma + 2$. And in any case, we have the desired inequality.

When A is primitive, M_1 is a gLM of type (II), and as $c_1(M_1)^2 \geq 0$ we have $c_2(M_1) > 0$, thus we cannot have $c_1(M_1)^2 = 0$ by Remark 3.10. Therefore m can be taken to be at least 2. □

4.4. **Filtration $1 \subset 3 \subset 4$.** We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 3$, and M/N a stable torsion free sheaf, and we call $E/M = N_1$. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) \geq \mu(E/N) \geq \mu(N_1) \tag{11}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \tag{12}$$

$$d = c_2(E) = c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \tag{13}$$

Moreover, since M/N is stable, we have

$$-2 \leq \langle \nu(M/N), \nu(M/N)\rangle = c_1(M/N)^2 - 4\chi(M/N) + 8 = 4c_2(M/N) - c_1(M/N)^2 - 8$$

thus $c_2(M/N) \geq \frac{3}{2} + \frac{c_1(M/N)^2}{4}$.

Lemma 4.8. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $\det(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(N_1) + c_1(M/N)) \geq \frac{g+3}{2} \geq \gamma(C) + 2$ and either $\det(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$;*
(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(N_1) + c_1(M/N))$;*
(d) *$\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq g + 1$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S, \det M/N) = h^2(S, M/N) = h^2(S, N) = 0$.

We have the following four cases:

(i) $h^0(S, \det M/N), h^0(S, N) \geq 2$
(ii) $h^0(S, \det M/N) \geq 2$ and $h^0(S, N) < 2$
(iii) $h^0(S, \det M/N) < 2$ and $h^0(S, N) \geq 2$
(iv) $h^0(S, \det M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes N^\vee) = h^0(S, \det M/N \otimes N_1) \geq 2$, and $h^0(S, H \otimes \det M/N^\vee) = h^0(S, N \otimes N_1) \geq 2$ as $\det M/N$, N, and N_1 have global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, thus $c_1(N) \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g+3}{2}. \end{aligned}$$

- If $c_1(N \otimes N_1)^2, c_1(M/N)^2 \geq 0$, then $\det M/N$ contributes to $\gamma(C)$ as $h^0(S, N \otimes N_1) = h^0(S, H - \det M/N) \geq 2$.
- If $c_1(N \otimes N_1)^2 \geq 2$ and $c_1(M/N)^2 < 0$, then as above $\det M/N$ contributes to $\gamma(C)$.
- If $c_1(N \otimes N_1)^2 < 0$ and $c_1(M/N)^2 \geq 0$ then we cannot say if $\det M/N$ contributes to $\gamma(C)$ as above. However, we calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 &= (c_1(M/N) + c_1(N \otimes N_1))^2 \\ &= c_1(M/N)^2 + 2c_1(M/N).(c_1(N) + c_1(N_1)) + c_1(N \otimes N_1)^2 \\ &< c_1(M/N)^2 + 2c_1(M/N).(c_1(N) + c_1(N_1)), \end{aligned}$$

  thus $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$.
- If $c_1(N \otimes N_1)^2, c_1(M/N)^2 < 0$, then the same calculation as above yields $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$.

Thus we are in case (b) of the lemma.

In case (iii), since $\det E/N = N_1 \otimes \det M/N$, [34, Lemma 3.3] implies that $h^0(S, N_1 \otimes \det M/N) \geq 2$. Thus since $h^0(S,N) \geq 2$, we see that $N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Furthermore, as $c_1(M/N)^2 + c_1(M/N).c_1(N) \geq c_1(N_1)^2 + c_1(N_1).c_1(N)$ and $c_1(N_1)^2 \geq 0 > c_1(M/N)^2$, we have $c_1(M/N)^2 + c_1(M/N).c_1(N) \geq c_1(N_1).c_1(N)$. Thus

$$\begin{aligned} c_1(M/N)^2 + c_1(M/N).c_1(N) - \frac{1}{2}(c_1(N).(c_1(N_1) + c_1(M/N))) &\geq c_1(M/N)^2 + \frac{c_1(M/N).c_1(N)}{2} - \frac{c_1(N).c_1(N_1)}{2} \\ &\geq \frac{c_1(M/N)^2}{2}, \end{aligned}$$

thus

$$\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(N_1) + c_1(M/N)),$$

and we are in case (c).

In case (iv), we see that $c_1(N)^2, c_1(M/N)^2 \leq -2$. We calculate

$$\begin{aligned} 2g - 2 = & c_1(E)^2 = (c_1(N) + c_1(N_1) + c_1(M/N))^2 \\ & \leq c_1(N_1)^2 + c_1(M/N)^2 + 2c_1(N).c_1(N_1) + 2c_1(N).c_1(M/N) + 2c_1(N_1).c_1(M/N) - 2 \\ & \leq g - 1 + 2c_1(N).c_1(M/N) + c_1(M/N)^2 - 2, \end{aligned}$$

thus $\frac{c_1(M/N)^2}{2} + c_1(N).c_1(M/N) \geq g + 1$, and we are in case (d). □

Remark 4.9. From the second half of the proof of Proposition 4.1, we see that in the situation above, if $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$, and if $\det M/N$ contributes to $\gamma(C)$, then we have $c_1(M/N).(c_1(N) + c_1(N_1)) \geq \gamma + 2 + 2h^1(S, \det M/N)$.

Lemma 4.10. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq \frac{3}{2}\gamma + 5$.*

Proof. We first see that if $c_1(M/N)^2 \geq 0$, then we are in cases (a) or (b) of the above lemma. Furthermore, we have $c_2(M/N) \geq 2$. Thus in case (a), we have

$$\begin{aligned} 2d &\geq 2(c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N)) \\ &\geq 4 + 2c_1(M/N).c_1(N) + 2c_1(N).c_1(N_1) + 2c_1(N_1).c_1(M/N) \\ &= 4 + c_1(M/N).(c_1(N) + c_1(N_1)) + c_1(N).(c_1(N_1) + c_1(M/N)) + c_1(N_1).(c_1(M/N) + c_1(N)) \\ &\geq 4 + 3(\gamma + 2), \end{aligned}$$

where the last inequality comes from Proposition 4.1. Thus $d \geq \frac{3}{2}\gamma + 5$. In case (b), we calculate as in case (a) and get $d \geq \frac{3}{2}\gamma + 5$ or

$$\begin{aligned} 2d &\geq 2\left(c_1(N).c_1(N_1) + c_1(N).c_1(M/N) + c_1(N_1).c_1(M/N) + \frac{c_1(M/N)^2}{4}\right) \\ &\geq g + c_1(N).c_1(M/N) + 2c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \\ &\geq g + 2(\gamma + 2), \end{aligned}$$

hence $d \geq \gamma + 2 + \frac{g}{2} \geq \frac{3}{2}\gamma + 5$.

If $c_1(M/N)^2 < 0$, in case (d), we have

$$\begin{aligned} d &\geq \frac{3}{2} + \frac{g+1}{2} + \frac{c_1(N).c_1(M/N)}{2} + c_1(M/N).c_1(N_1) + c_1(N).c_1(N_1) \\ &\geq \frac{g+4}{2} + k + \frac{g+1}{2} - \frac{c_1(M/N)^2}{4} \\ &\geq \gamma + 2 + g + \frac{7}{2}. \end{aligned}$$

If $0 > c_1(M/N)^2 \geq -6$, then $c_2(M/N) \geq 0$, thus $\chi(\det M/N) \leq 1$. Therefore $h^1(S, \det M/N) + 1 \geq h^0(S, \det M/N)$. Calculating as above, we see that

- in case (a), we have $d \geq \frac{3}{2}\gamma + 5$;
- in case (b), we have $d \geq \frac{3}{2}\gamma + 5$ or $d \geq \gamma + \frac{7}{2} + \frac{g+2}{2}$; and,
- in case (c), we have $d \geq \frac{3}{2}\gamma + 5$.

If $c_2(M/N) < 0$, then the stability of M/N implies that $c_1(M/N)^2 \leq -8$ and

$$-2 \leq \langle \nu(M/N), \nu(M/N) \rangle = c_1(M/N)^2 + 8 - 4\chi(M/N) \leq -4\chi(M/N),$$

whereby $\chi(M/N) \leq 0$. We now consider inequalities associated with various filtrations that lead to the terminal $1 \subset 3 \subset 4$ filtration of E.

If the JH filtration of E is $1 \subset 3 \subset 4$, then we have $p(E) = p(M/N)$, which gives an equality of normalized Euler characteristics

$$\frac{\chi(M/N)}{2} = \frac{\chi(E)}{4} = \frac{g - \gamma + 1}{4}.$$

Thus $0 \geq 2\chi(M/N) = g - d + 7$, and hence $d \geq g + 7$.

If the HN filtration of E is $0 \subset M \subset E$ with $\mathrm{rk}(M) = 3$ and M properly semistable, then the JH filtration of M is $0 \subset N \subset M$. Hence $\mu(M/N) = \mu(M)$ and $\mu(M) > \mu(E)$. Thus

$$\frac{c_1(M/N)^2}{2} + \frac{c_1(M/N).c_1(N \otimes N_1)}{2} = \mu(M/N) > \mu(E) = \frac{g-1}{2},$$

hence

$$\begin{aligned} d &\geq \frac{3}{2} + \frac{c_1(M/N)^2}{4} + c_1(M/N).c_1(N \otimes N_1) + c_1(N).c_1(N_1) \\ &\geq \frac{3}{2} + \frac{g-1}{2} - \frac{c_1(M/N)^2}{4} + \frac{c_1(M/N).(c_1(N) + c_1(N_1))}{2} \\ &\geq \frac{3}{2} + \frac{g-1}{2} + \frac{c_1(N).(c_1(N_1) + c_1(M/N))}{2} + \frac{c_1(N_1).(c_1(N) + c_1(M/N))}{2} \\ &\geq \frac{3}{2} + \frac{g-1}{2} + \gamma + 1 \end{aligned}$$

where the last inequality comes from the fact that N_1 contributes to $\gamma(C)$, and that in cases (a),(b), and (c) $c_1(N).(c_1(N_1) + c_1(M/N)) \geq \gamma + 2$.

If the HN filtration of E is $0 \subset N \subset E$ with E/N properly semistable and the JH filtration of E/N is $0 \subset \overline{M} \subset E/N$ with $\mathrm{rk}(\overline{M}) = 2$, then we have an equality of normalized Euler characteristics

$$\frac{\chi(E) - \chi(N)}{3} = \frac{\chi(E/N)}{3} = \frac{\chi(\overline{M})}{2} = \frac{\chi(M/N)}{2}.$$

Thus $\chi(E) = g - \gamma + 1 = \frac{3\chi(M/N)}{2} + \chi(N)$, where $\gamma = d - 6$ is the Clifford index of the g^3_d on C. From the short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0\ ,$$

we have $\chi(N) = h^0(S,E) - h^0(S,E/N) \leq g-\gamma-1$ as $h^0(S,E/N) \geq 2$. Therefore

$$g-\gamma+1 = \chi(E) \leq \frac{3\chi(M/N)}{2} + g - \gamma - 1,$$

and thus $2 \leq \frac{3}{2}\chi(M/N) \leq 0$, which is a contradiction. Thus this does not occur, and in all cases we have at least $d \geq \frac{3}{2}\gamma + 5$, as claimed. ☐

4.5. **Filtration** $2 \subset 3 \subset 4$. We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with N a stable torsion free sheaf of rank $\mathrm{rk}(N) = 2$, $\mathrm{rk}(M) = 3$, and $N_1 = E/M$ a line bundle. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N_1) \tag{14}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(N_1) \tag{15}$$

$$d = c_2(E) = c_2(N) + c_1(N).c_1(M/N) + c_1(N).c_1(N_1) + c_1(M/N).c_1(N_1) \tag{16}$$

Moreover, as N is stable, we have $c_2(N) \geq \frac{3}{2} + \frac{c_1(N)^2}{4}$.

Lemma 4.11. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(N_1)+c_1(M/N)) \geq g+1$and either $(M/N)\otimes\mathcal{O}_C$ contributes to $\gamma(C)$ or $c_1(M/N).(c_1(N)+c_1(N_1)) \geq g$;*
(c) *$(\det N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, we can assume $c_1(N)^2 \geq 0$ and $c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(M/N) + c_1(N_1))$;*
(d) *$c_1(N)^2 \leq -2$ and $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S, \det N) = h^2(S, \det M) = h^2(S, M/N) = h^2(S, \det E/N) = 0$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, \det N) \geq 2$
(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, \det N) < 2$
(iii) $h^0(S, M/N) < 2$ and $h^0(S, \det N) \geq 2$
(iv) $h^0(S, M/N), h^0(S, \det N) < 2$.

In case (i), as N_1 has global sections, and $H - c_1(M/N) = c_1(N) + c_1(N_1)$ and $H - c_1(N) = c_1(N_1) + c_1(M/N)$, we see that both $(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$, and we are in case (a).

In case (ii), we have $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= 2\mu(N) - c_1(N)^2 \geq g - 1 + 2 = g + 1 \end{aligned}$$

We now observe that $c_1(\det N \otimes N_1)^2 \geq c_1(M/N)^2$ which follows from the following calculation

$$\begin{aligned} c_1(\det N \otimes N_1)^2 - c_1(M/N)^2 &= c_1(N)^2 + 2c_1(N).c_1(N_1) + c_1(N_1)^2 - c_1(M/N)^2 \\ &= 2\mu(N) + \mu(N_1) - \mu(M/N) \geq \mu(N) + \mu(N_1) > 0. \end{aligned}$$

If $c_1(\det N \otimes N_1)^2 < 0$, then also $c_1(M/N) < 0$, and we calculate

$$\begin{aligned} 2g-2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(N_1))^2 \\ &= c_1(\det N \otimes N_1)^2 + 2c_1(\det N \otimes N_1).c_1(M/N) + c_1(M/N)^2 \\ &< 2(c_1(N) + c_1(N_1)).c_1(M/N), \end{aligned}$$

thus $c_1(M/N).(c_1(N)+c_1(N_1)) \geq g$. Else $c_1(\det N \otimes N_1)^2 \geq 0$, and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det N \otimes N_1) \geq 2$, whereby $M/N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M/N \otimes N_1$, we have $h^0(S, \det M/N \otimes N_1) \geq 2$ by [34, Lemma 3.3]. Thus as $h^0(S, \det N) \geq 2$, we have that $\det N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus proving the first statement of case (c).

In cases (iii) and (iv), as $h^0(S, M/N) < 2$ we have $c_1(M/N)^2 \leq -2$. We now calculate

$$\begin{aligned}
2g-2 = c_1(E)^2 &= (c_1(N)+c_1(M/N)+c_1(N_1))^2 \\
&= c_1(N)^2 + c_1(M/N)^2 + c_1(N_1)^2 + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) \\
&\leq c_1(N)^2 + c_1(N_1)^2 + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) - 2 \\
&\leq g - 1 + c_1(N)^2 + 2c_1(M/N).c_1(N) - 2,
\end{aligned}$$

thus $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$. If $c_1(N)^2 \leq -2$, we are in case (d).

From now on, we assume $c_1(N)^2 \geq 0$. From the inequality $\mu(M/N) \geq \mu(N_1)$, we see that $c_1(N).c_1(M/N) > c_1(M/N)^2 + c_1(N).c_1(M/N) \geq c_1(N_1)^2 + c_1(N_1).c_1(M/N) \geq c_1(N_1).c_1(M/N)$. Thus

$$c_1(M/N).c_1(N) - \frac{1}{2}c_1(N).(c_1(M/N)+c_1(N_1)) = \frac{1}{2}(c_1(N).c_1(M/N) - c_1(N).c_1(N_1)) > 0,$$

and we are in case (c).

□

Lemma 4.12. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq 5 + \frac{3}{2}\gamma$.*

Proof. The proof follows the same argument as Lemma 4.7. □

4.6. **Filtration $1 \subset 2 \subset 3 \subset 4$.** We suppose E has a terminal filtration of the form

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 = E,$$

where $\mathrm{rk}(E_i) = i$, and E_i/E_{i+1} are torsion free sheaves of rank 1. Furthermore, we have

$$\mu(E_1) \geq \mu(E_2) \geq \mu(E_3) \geq \mu(E) = \frac{g-1}{2} \geq \mu(E/E_3) \tag{17}$$

$$\mu(E_1) \geq \mu(E_2/E_1) \geq \mu(E_3/E_2) \geq \mu(E/E_3) \tag{18}$$

$$\mu(E_i/E_j) \geq \mu(E/E_3) \text{ for } 1 \leq j < i \leq 4 \tag{19}$$

$$\begin{aligned} d = {} & c_1(E/E_3).(c_1(E_1)+c_1(E_2/E_1)+c_1(E_3/E_2)) + c_1(E_1).c_1(E_3/E_2) \\ & + c_1(E_2/E_1).c_1(E_3/E_2) + c_1(E_1).c_1(E_2/E_1) \end{aligned} \tag{20}$$

Letting $e_i := c_1(E_i/(E_{i-1}))$, be the Chern roots of E, and writing $e_i + e_j := c_1(E_i/E_{i-1} \otimes E_j/E_{j-1})$, we have

$$\begin{aligned} 4d = {} & e_1(e_2+e_3+e_4) + (e_1+e_2).(e_3+e_4) + (e_1+e_2+e_3).e_4 \\ & + (e_1+e_4).(e_2+e_3) + (e_1+e_3).(e_2+e_4) + (e_1+e_3+e_4).e_2 + (e_1+e_2+e_4).e_3 \end{aligned}$$

Lemma 4.13. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$,*

$$m := \min\{D^2 | D \in \mathrm{Pic}(S), D^2 \geq 0, \ D \textit{ is effective}\}$$

(i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), and

$$\mu = \min\{\mu(D) | D \in \mathrm{Pic}(S), D^2 \geq 0, \mu(D) > 0\},$$

we have $d \geq \frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2}$.

Proof. From Proposition 4.1 and Proposition 4.3, we see that $\det(E/E_i)\otimes\mathcal{O}_C$ contributes to $\gamma(C)$, and so we have $e_1(e_2+e_3+e_4)\geq\gamma+2$, $(e_1+e_2).(e_3+e_4)\geq\gamma+2$, and $(e_1+e_2+e_3).e_4\geq\gamma+2$. We also have $h^2(S,F)=0$ for $F=\det(E_i/E_j)$ and $F=E/E_3,\det E_i$.

It remains to bound the other four terms.

To bound $(e_2+e_3).(e_1+e_4)$, we note that $\mu(e_2+e_3)\geq\mu+\mu(e_3)\geq\mu+\mu(E/E_2)$, and thus

$$(e_2+e_3)^2+(e_1+e_4).(e_2+e_3)\geq\mu+\frac{(e_1+e_2).(e_3+e_4)}{2}+\frac{(e_3+e_4)^2}{2}\geq\mu+\frac{\gamma+2}{2}+\frac{(e_3+e_4)^2}{2}.$$

Furthermore, we note that $\mu(e_1+e_4)=\mu(e_1)+\mu(e_4)\geq\frac{g-1}{2}+\mu$, whereby

$$(e_1+e_4)^2+(e_1+e_4).(e_2+e_3)\geq\gamma.$$

Now if $h^0(S,e_1+e_4)<2$ then by considering the Euler characteristic we have $(e_1+e_4)^2\leq-2$, and thus $(e_1+e_4).(e_2+e_3)\geq\gamma+2$. If $h^0(S,e_2+e_3)<2$ then $(e_2+e_3)^2\leq-2$, and we have $(e_1+e_4).(e_2+e_3)\geq 2+\mu+\frac{\gamma+2}{2}+\frac{(e_3+e_4)^2}{2}$. By assumption, $(e_3+e_4)^2\geq m$, hence $(e_1+e_4).(e_2+e_3)\geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$ as well. Finally, if $h^0(S,e_1+e_4),h^0(S,e_2+e_3)\geq 2$, and thus they contribute to the $\gamma(C)$, and hence by Proposition 4.1 $(e_1+e_4).(e_2+e_3)\geq\gamma+2$. Therefore in either case, we have $(e_1+e_4).(e_2+e_3)\geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_3).(e_2+e_4)$, we note that $\mu(e_1+e_3)\geq\frac{g-1}{2}$, and hence

$$(e_1+e_3)^2+(e_1+e_3).(e_2+e_4)\geq\frac{g-1}{2}.$$

We also note that $\mu(e_2+e_4)\geq\mu+\mu(E/E_1)\geq\mu+\mu(E/E_2)$, whereby

$$(e_2+e_4)^2+(e_1+e_3).(e_2+e_4)\geq 1+\frac{(e_1+e_2).(e_3+e_4)}{2}+\frac{(e_3+e_4)^2}{2}\geq 1+\frac{\gamma+2}{2}+\frac{(e_3+e_4)^2}{2}.$$

As above, we have $(e_1+e_3).(e_2+e_4)\geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_3+e_4).e_2$, we note that $\mu(e_1+e_3+e_4)\geq\mu(e_1)\geq\frac{g-1}{2}$ and $\mu(e_2)\geq\mu(E/E_1)\geq\mu(E/E_2)$. Following the same argument as above, we have $(e_1+e_3+e_4).e_2\geq 3+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_2+e_4).e_3$, we note that $\mu(e_1+e_2+e_4)\geq\mu(e_1)\geq\frac{g-1}{2}$ and $\mu(e_3)\geq\mu(E/E_2)$. Following the same argument as above, we have $(e_1+e_2+e_4).e_3\geq 3+\frac{\gamma}{2}+\frac{m}{2}$.

Finally, we have that three of the terms in the expression for $4d$ are bounded below by $\gamma+2$, two by $3+\frac{\gamma}{2}+\frac{m}{2}$, and two by $3+\mu+\frac{\gamma}{2}+\frac{m}{2}$. Thus $d\geq\frac{5}{4}\gamma+\frac{\mu}{2}+\frac{m}{2}+\frac{9}{2}$, as desired. □

Remark 4.14. We note that in the proof above, μ is always at least the minimum slope of the determinant of a quotient of E.

5. Lifting g^3_ds

As above, (S,H) is a polarized K3 surface of genus g, $C\in|H|$ is a smooth irreducible curve of general Clifford index $\gamma=\lfloor\frac{g-1}{2}\rfloor$, A is a complete basepoint free g^3_d with $\rho(A)<0$, and $E=E_{C,A}$ the unstable LM bundle. Having attained the needed bounds on $c_2(E)$, we can prove our lifting results.

Theorem 5.1. *Let (S,H) be a polarized K3 surface of genus $g\neq 2,3,4,8$ and $C\in|H|$ a smooth irreducible curve of Clifford index γ. Let*

$$m:=\min\{D^2\mid D\in\operatorname{Pic}(S),\ D^2\geq 0,\ D\text{ is effective}\}$$

(i.e. there are no curves of genus $g'<\frac{m+2}{2}$ on S), and

$$\mu=\min\{\mu(D)\mid D\in\operatorname{Pic}(S),\ D^2\geq 0,\ \mu(D)>0\}.$$

If

$$d<\min\left\{g-1,\ \frac{5}{4}\gamma+\frac{\mu}{2}+\frac{m}{2}+\frac{9}{2},\ \frac{5}{4}\gamma+\frac{m}{2}+5,\ \frac{3}{2}\gamma+5,\ \frac{\gamma}{2}+\frac{g-1}{2}+4\right\},$$

then there is a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A| \subseteq |M \otimes \mathcal{O}_C|$ and $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$. Moreover, one has $c_1(M).C \leq \frac{3g-3}{2}$.

Proof. The LM bundle E has $c_2(E) = d$. If $g \neq 2, 3, 4, 8$, then $d < \frac{5g+19}{6}$. As

$$d < \min\left\{g-1,\ \frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2},\ \frac{5}{4}\gamma + \frac{m}{2} + 5,\ \frac{3}{2}\gamma + 5,\ \frac{\gamma}{2} + \frac{g-1}{2} + 4\right\}$$

by assumption, the only terminal filtration of E is of type $1 \subset 4$. Thus by Proposition 3.17, the result follows. □

We remark that Theorem 2 is a special case of Theorem 5.1 since if S has no elliptic curves, then $m \geq 2$ so that $d < \frac{5}{4}\gamma + 6$. Considering the bounds obtained in Section 4, we have also proved the following proposition.

Proposition 5.2. *With A as above, the bundle $E_{C,A}$ only admits a terminal filtration of type $1 \subset 4$, $1 \subset 2 \subset 4$, or $1 \subset 2 \subset 3 \subset 4$.*

Proof. We simply solve $\rho(g, 3, d) < 0$ for d and compare it to the bounds obtained for each terminal filtration. □

6. Maximal Brill–Noether Loci in Genus ≤ 23

In this section, we identify the maximal Brill–Noether loci in genus 3–19, 22, and 23, proving Theorem 1. Our technique combines known results about non-containments of Brill–Noether loci, work by Lelli-Chiesa [34] on lifting of rank 2 linear systems and linear systems computing the Clifford index, together with our lifting results for rank 3 linear systems above.

6.1. Genus 3–6.

By Clifford's theorem, any Brill–Noether special curve of genus 3 or 4 is hyperelliptic, hence $\mathcal{M}^1_{g,2}$ is the only maximal (and expected maximal) Brill–Noether locus. Similarly, in genus 5, every Brill–Noether special curve has gonality ≤ 3, hence $\mathcal{M}^1_{5,3}$ is the only maximal (and expected maximal), Brill–Noether locus. Thus Conjecture 1 holds in genus 3–5. In genus 6, we verify the conjecture as well.

Proposition 6.1. *The maximal Brill–Noether loci in genus 6 are $\mathcal{M}^1_{6,3}$ and $\mathcal{M}^2_{6,5}$.*

Proof. $\mathcal{M}^1_{6,3}$ and $\mathcal{M}^2_{6,5}$ are the expected maximal Brill–Noether loci. It remains to show that they are distinct. Since $\rho(6, 1, 3) = -2$ and $\rho(6, 2, 5) = -3$, results on the codimension of Brill–Noether loci (e.g., [11, 13, 49]) imply that $\mathcal{M}^1_{6,3} \nsubseteq \mathcal{M}^2_{6,5}$. A smooth plane quintic curve C has genus 6. By a well-known result of Max Noether [22], C has gonality 4, hence has no g^1_3. Thus $\mathcal{M}^2_{6,5} \nsubseteq \mathcal{M}^1_{6,3}$. □

6.2. Unexpected containments in genus 7–9.

In each genus 7-9, there are two expected maximal Brill–Noether loci, and we give detailed constructions of the unexpected containments between them. In genus 7 and 9, we are indebted to Hannah Larson for pointing them out. These are $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$, $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, and $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$. Thus in these genera, there is a unique maximal Brill–Noether locus.

Proposition 6.2. *Every Brill–Noether special curve of genus 7 has a g^1_4.*

Proof. The expected maximal Brill–Noether loci in genus 7 are $\mathcal{M}^1_{7,4}$ and $\mathcal{M}^2_{7,6}$. We show that every smooth genus 7 curve with a g^2_6 has a g^1_4. Let $\phi : C \to \mathbb{P}^2$ be the map given by the g^2_6. If the g^2_6 is not very ample, then C has a g^1_4. Thus we can assume ϕ is a nondegenerate embedding, so that $\phi(C)$ is a plane curve of degree 2,3, or 6. If $\phi(C)$ has degree 2 or 3, then $\phi(C)$ has arithmetic genus 0 or 1, and thus C has a g^1_4. If $\phi(C)$ has degree 6, then it has arithmetic genus 10. Hence $\phi(C)$ must have a singular point of multiplicity ≥ 2. Projecting from this point gives a g^1_k for $k \leq 4$, hence a g^1_4. □

Proposition 6.3 (Mukai [41, Lemma 3.8])**.** *Every Brill–Noether special curve of genus* 8 *has a* g^2_7*.*

Proof. The maximal Brill–Noether loci in genus 8 are $\mathcal{M}^1_{8,4}$ and $\mathcal{M}^2_{8,7}$. We show that a curve C of genus 8 with a g^1_4 has a g^2_7. Let A be a line bundle of type g^1_4 on C. If C has a g^2_6 then it has a g^2_7, thus we may assume that C has no g^2_6, hence no g^3_8 (Serre adjoint to a g^2_6). Similarly, we can assume C has no g^1_3 (as twice a g^1_3 is a g^2_6), whence $|A|$ is basepoint free. Furthermore, the Serre adjoint A' of A is of type g^4_{10} and is very ample as there is no g^3_8. Hence $|A'|$ exhibits C as degree 10 curve in $\mathbb{P}^4$. This embedding of C has 8 trisecant lines by the Berzolari formula

$$\#\{\text{trisecant lines to } C\} = \frac{(d-2)(d-3)(d-4)}{6} - g(d-4),$$

where g is the genus of C and d is the degree of C in $\mathbb{P}^4$, see [33]. Projecting from one of the trisecant lines gives a g^2_7. □

Proposition 6.4. *Every Brill–Noether special curve of genus* 9 *has a* g^1_5*.*

Proof. The expected maximal Brill–Noether loci in genus 9 are $\mathcal{M}^1_{9,5}$ and $\mathcal{M}^2_{9,7}$. We will show, similarly to Proposition 6.2, that every smooth genus 9 curve with a g^2_7 has a g^1_5. Let $\phi : C \to \mathbb{P}^2$ be the map given by the g^2_7. If the g^2_7 is not very ample, then C has a g^1_5. Thus we can assume ϕ is a nondegenerate embedding, so that $\phi(C)$ is a plane curve of degree 7, so has arithmetic genus 15. Hence $\phi(C)$ must have a singular point of multiplicity ≥ 2. Projecting from this point gives a g^1_k for $k \leq 5$, hence a g^1_5. □

Remark 6.5. The constructions in genus 7–9 rely on projections from secant linear spaces. Given a very ample linear system of type g^r_d defining an embedding $C \to \mathbb{P}^r$ of degree d, if C admits a k-secant l-dimensional linear subspace of $\mathbb{P}^r$, then projection from that linear subspace results in a g^{r-l-1}_{d-k}. The expected dimension of the space of l-dimensional linear spaces of $\mathbb{P}^r$ that are k-secant to C is classically known to be $k-(k-l-1)(r-l)$, see [17]. Secant linear spaces for which this expected dimension is nonnegative (resp. negative) are called *expected* (resp. *unexpected*). When the expected dimension is 0, there are unwieldy enumerative formulas for the expected number of such secant linear spaces generalizing the Berzolari formula, see [2, VIII.4]. We have checked that the only cases when an expected maximal g^r_d (or its Serre adjoint) admits an expected k-secant l-dimensional linear space and such that the associated g^{r-l-1}_{d-k} is also Brill–Noether special (and not Serre adjoint to a g^r_d) are the three cases discussed above in genus 7–9. Thus no additional unexpected containments of expected maximal Brill–Noether loci can arise from expected secant linear spaces. Unexpected secant linear spaces could potentially give rise to other unexpected containments, but these should not exist if we believe various versions of the Donagi–Morrison conjecture for expected maximal Brill–Noether special linear systems, see [35, Theorem 1.4].

6.3. **Genus** 10–13**.** We first establish a few useful lemmas which, in effect, say that if $\mathrm{Pic}(S) = \langle H, L\rangle$ looks like it is obtained by lifting a g^r_d on $C \in |H|$ to a line bundle L, then L is in fact a lift of a g^r_d. Moreover, for these lifts, we would like the line bundle to be basepoint free, which is true if the g^r_d is primitive. In particular, our next lemma shows that if a curve C on a K3 surface strictly contains a Brill–Noether special linear system, then it is primitive.

Lemma 6.6. *Let* (S, H) *be a polarized K3 surface of genus* g*,* $C \in |H|$ *a smooth connected curve, and* $A \in \mathrm{Pic}(C)$ *be a line bundle of type* g^r_d*. Suppose that* $\rho(g, r, d) < 0$ *and* C *has no Brill–Noether special linear series of Clifford index smaller than* A*. Then* A *is primitive.*

Proof. We note that $\gamma(\omega_C \otimes A^\vee) = \gamma(A)$, $\rho(A) = \rho(\omega_C \otimes A^\vee)$, $\gamma(A-P) < \gamma(A)$ when P is a basepoint of A, and $\rho(g, r, d-1) < \rho(g, r, d)$. Suppose A has a basepoint P. Then $A - P$ has strictly smaller Clifford index and is Brill–Noether special. By assumption, C cannot admit the linear series $|A - P|$. Thus A is basepoint free. Likewise, if $\omega_C \otimes A^\vee$ has a basepoint P, then $\omega_C \otimes A^\vee - P$ is Brill–Noether special and has smaller Clifford index, which cannot be the case. □

Parts of the following Lemma go back to Farkas in [15] and Rathmann's Theorem (see [28, 46]).

Lemma 6.7. *Let (S, H) be a polarized K3 surface of genus g in the Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$, i.e., with* $\mathrm{Pic}(S)$ *admitting a primitive embedding of the sublattice*

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

Let $C \in |H|$ be a smooth irreducible curve.

(i) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ and $2 \le r, d \le g-1$, then L is nef.
(ii) If L and $H - L$ are basepoint free, $r \ge 2$, and $0 < d \le g-1$, then $L \otimes \mathcal{O}_C$ is a g^r_d. (The assumption on basepoint free-ness is achieved if for example S has no (-2)-curves, or can be checked numerically.)
(iii) Suppose that $L \otimes \mathcal{O}_C$ is a g^r_d with $\gamma(r,d) > \lfloor \frac{g-1}{2} \rfloor$ and $\rho(g,r,d) < 0$ and that all lattices obtained by lifting special linear systems of general Clifford index or lower cannot be contained in $\mathrm{Pic}(S)$*. Then C has Clifford index $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$, maximal gonality $\lfloor \frac{g+3}{2} \rfloor$, and Clifford dimension* 1.
(iv) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ is associated to an expected maximal g^r_d, then the assumption on lattices in (iii) holds.
(v) Suppose that $\gamma(r,d) \le \lfloor \frac{g-1}{2} \rfloor$, $\rho(g,r,d) < 0$, and that all lattices obtained by lifting special linear systems A not of type g^r_d with $\gamma(A) \le \lfloor \frac{g-1}{2} \rfloor$ cannot be contained in $\mathrm{Pic}(S)$*. Then $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r,d)$.*

Proof. To prove (i) we show that for any (-2)-curve $\Gamma = aH + bL \in \Lambda^r_{g,d}$, we have $\Gamma.L \ge 0$. We note that as Γ is a (-2)-curve, a and b must have opposite sign. We prove (i) in three cases.

First suppose $a > 0$ and $b < 0$. Then as $\Gamma.H \ge 1$ and $a > 0$, we have $b\Gamma.L \le -2$, thus as $b < 0$, $\Gamma.L \ge 0$.

Second, suppose $a < -1$ and $b > 0$. Then since $\Gamma.H \ge 1$, we have $a\Gamma.H \le -2$. Thus $b.\Gamma.L \ge 0$, and since $b > 0$ we must have $\Gamma.L \ge 0$.

Lastly, suppose $a = -1$ and $b > 0$. We see that if $\Gamma.H \ge 2$, then we can follow the same argument as above to see that L is nef. Thus the only remaining case is when $a = -1$ and $\Gamma.H = 1$. We calculate $2g - 2 = (H + \Gamma)^2 = (bL)^2 = b^2(2r-2)$, hence $b^2 = \frac{g-1}{r-1} \in \mathbb{Z}$. From $\Gamma.H = 1$, we see $b = \frac{2g-1}{d}$, and plugging this in to $2g - 2 = b^2(2r-2)$ yields

$$d^2(g-1) = (2g-1)^2(r-1).$$

Looking modulo $g - 1$, we immediately see that $r - 1 \equiv 0 \mod g-1$, hence $\frac{r-1}{g-1} \in \mathbb{Z}$, and thus $r = g$, which is a contradiction. Thus L is always nef.

To prove (ii), we note that L is clearly a lift of a $g^{r'}_d$ on C for some $r' \ge 0$. Since $0 < d \le g-1$, we see that $L^2, (H-L)^2 > 0$. Furthermore, since $H.L, H.(H-L) > 0$, both these line bundles are nontrivial and intersect H positively, hence $h^0(S,L), h^0(S,H-L) \ge 2$. By assumption, L and $H - L$ are basepoint free, and thus globally generated. Therefore Corollary 3.14 applies. Thus, as $L^2 = 2r-2$, we see that $L \otimes \mathcal{O}_C$ must be a divisor of type g^r_d. Hence (ii) is proved.

To prove (iii), we note that a $g^1_{d'}$ with $\rho(g,1,d') < 0$ has Clifford index $\gamma(g^1_{d'}) < \lfloor \frac{g-1}{2} \rfloor$. Suppose for contradiction that C has lower than general Clifford index. Then by [27, Lemma 8.3] and [35, Theorem 4.2] we would be able to lift some special linear system computing $\gamma(C)$ to a divisor $L' \in \mathrm{Pic}(S)$, and by assumption $\langle H, L' \rangle$ cannot be contained in $\mathrm{Pic}(S)$. Thus C has general Clifford index. The same argument shows that C cannot have a special linear system computing its Clifford index. Thus C has a $g^1_{\lfloor \frac{g+3}{2} \rfloor}$ which computes the Clifford index. Hence C has maximal gonality and Clifford dimension 1.

To prove (iv), we note that if C had any Brill–Noether special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) \leq \frac{g-1}{2}$, then it has a g^r_d with $\gamma = \frac{g-1}{2}$ or a g^1_d with $\gamma(g^1_d) = \frac{g-1}{2} - 1$. Thus we only need to consider lattices $\Lambda^r_{g,d}$ associated to those g^r_d. The proof is now Proposition 1.8(i). Thus (iv) is proved.

To prove (v), we note again that $L \otimes \mathcal{O}_C$ is a $g^{r'}_d$. If $r' \neq r$, then $\gamma(C) \neq \gamma(r,d)$ and some line bundle A would compute $\gamma(C)$. Thus there would exists some lift of A to a line bundle L', but again the lattice $\langle H, L'\rangle \nsubseteq \mathrm{Pic}(S)$. Hence $r' = r$ and we see that $L \otimes \mathcal{O}_C$ is a g^r_d. Similarly, $\gamma(C) = \gamma(r,d)$. □

Remark 6.8. If $\Lambda^r_{g,d}$ has a (-2)-curve, there are still some ways to check that L and $H - L$ are basepoint free. Namely, if they are both nef, then we can check they are basepoint free by checking if there are any elliptic curves on S. Namely if $N \in \mathrm{Pic}(S)$ is nef and there are no elliptic curves, then N is basepoint free by a well-known result of Saint-Donat. To numerically check if $D \in \mathrm{Pic}(S)$ is nef, one can check whether $D.\Gamma \geq 0$ for any (-2)-curve Γ.

One can also check that $L \otimes \mathcal{O}_C$ is a g^r_d by enumerating all of the degree d $g^{r'}_d$ on C and using Lelli-Chiesa's lifting results to show that $\mathrm{Pic}(S)$ cannot have a lift of a $g^{r'}_d$ for $r' \neq r$.

Remark 6.9. As noted in [4, Corollary 4.11, Remark 4.12], it follows from Lemma 6.7(iii) that for $\mathcal{M}^r_{g,d}$ expected maximal, $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}$ for $g \geq 14$. Indeed, for (S,H) with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$, it follows from Lemma 6.7(iv) and [27, Lemma 8.3] that $\gamma(L \otimes \mathcal{O}_C) > \lfloor\frac{g-1}{2}\rfloor$. Now the Riemann–Roch computation of [4, Corollary 4.11] that $L \otimes \mathcal{O}_C$ is a g^s_d for some $s \geq r$. By the trivial containments of Brill–Noether loci, C has a g^r_d and maximal gonality.

We can now prove that the maximal Brill–Noether loci in genus 10–19, 22, and 23 are as predicted by Conjecture 1. The proof in genus 10–13 uses Brill–Noether theory for curves of fixed gonality and various results distinguishing lattices above. In genus 14–19, 22 and 23, the main strategy to distinguish the expected maximal Brill–Noether loci, for example to show that $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$, is to prove that for a very general K3 surface $(S,H) \in \mathcal{K}^r_{g,d}$, a curve $C \in |H|$ has a g^r_d but not a $g^{r'}_{d'}$. This is done by, first, applying Lemma 6.7 to deduce that C has a g^r_d, and second, assuming that C has a $g^{r'}_{d'}$ and then using various lifting results to produce a line bundle M on S that is numerically incompatible with $\mathrm{Pic}(S)$.

For the rest of the section, we summarize the various arguments, organized by genus.

In low genus, where there are no non-computing linear systems, we argue by the Clifford index of C and can assume that a g^r_d computes the Clifford index of $C \in |H|$. Then Lelli-Chiesa's lifting results [35] suffice to verify Conjecture 1 in genus 10–13.

Proposition 6.10. *For any $10 \leq g \leq 13$ and any positive integers r, d, r', d' such that*

- $r' \geq 2$,
- $\rho(g,r,d), \rho(g,r',d') < 0$,
- $\Delta(g,r,d), \Delta(g,r',d') < 0$, *and*
- $2 < \gamma(r',d') \leq \gamma(r,d) \leq \lfloor\frac{g-1}{2}\rfloor$,

there is a polarized K3 surface $(S,H) \in \mathcal{K}^r_{g,d}$ such that a curve $C \in |H|$ admits a g^r_d but not a $g^{r'}_{d'}$. Thus $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$.

Proof. First assume that $r' \geq 2$. We let $(S,H) \in \mathcal{K}^r_{g,d}$ be a very general and $C \in |H|$ a smooth irreducible curve of genus g. As in Proposition 1.8 (i), no lattices obtained by lifting special linear systems on C can be contained in $\mathrm{Pic}(S)$. By Lemma 6.7 (v) we see that $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r,d)$. We suppose for contradiction that C admits a $g^{r'}_{d'}$. We cannot have $\gamma(r',d') < \gamma(r,d)$, as then the g^r_d does not compute the Clifford index of C. Hence $\gamma(r',d') = \gamma(r,d)$. But now [35, Theorem 4.2] shows that we have a Donagi–Morrison lift $M \in \mathrm{Pic}(S)$ of the $g^{r'}_{d'}$, and by Proposition 1.8 (i) again, we see that $\langle H, M\rangle \nsubseteq \mathrm{Pic}(S)$ unless the Donagi–Morrison lift of the

$g^{r'}_{d'}$ is of type g^r_d, which only occurs when $r, r' \geq 2$. In this case, the Lazarsfeld–Mukai bundle $E_{C,g^{r'}_{d'}}$ has a quotient E with $\gamma(E) = 0$, and one checks that none of the cases of Lemma 3.7 can occur (for a detailed computation see Proposition 6.15). Thus C cannot admit a $g^{r'}_{d'}$. □

Remark 6.11. When $r' = 1$, case (a) of Lemma 3.7 can occur, hence we assume $r' \geq 2$. In fact, in genus 11, $\mathcal{M}^1_{11,5} \subseteq \mathcal{M}^2_{11,9}$. However, for expected maximal loci, the codimensions of the expected maximal $\mathcal{M}^1_{g,k}$ and $\mathcal{M}^2_{g,d}$ loci rule out similar containments.

Corollary 6.12. *In genus* 10–13, *Conjecture 1 holds. The maximal Brill–Noether loci*

- *in genus* 10 *are* $\mathcal{M}^1_{10,5}$ *and* $\mathcal{M}^2_{10,8}$*;*
- *in genus* 11 *are* $\mathcal{M}^1_{11,6}$ *and* $\mathcal{M}^2_{11,9}$*;*
- *in genus* 12 *are* $\mathcal{M}^1_{12,6}$, $\mathcal{M}^2_{12,9}$, *and* $\mathcal{M}^3_{12,11}$*;*
- *in genus* 13 *are* $\mathcal{M}^1_{13,7}$, $\mathcal{M}^2_{13,10}$, *and* $\mathcal{M}^3_{13,12}$.

Proof. Propositions 6.10 and 1.6 suffice to verify the conjecture in genus 10–13. □

6.4. **Genus** 14–15**.** The arguments in genus 14 and 15 only require the lifting results of Lelli-Chiesa [35] and the preliminary results above.

Proposition 6.13. *In genus* 14, *the maximal Brill–Noether loci are* $\mathcal{M}^1_{14,7}$, $\mathcal{M}^2_{14,11}$, *and* $\mathcal{M}^3_{14,13}$.

Proof. These loci are the expected maximal Brill–Noether loci in genus 14, thus it remains to show that there are no containments among them. By Proposition 1.6, $\mathcal{M}^1_{14,7} \nsubseteq \mathcal{M}^2_{14,11}$ and $\mathcal{M}^1_{14,7} \nsubseteq \mathcal{M}^3_{14,13}$. By Lemma 6.7 (*iii*), we see that there are curves which admit a g^2_{11} or a g^3_{13} and have maximal gonality $\lfloor \frac{14+3}{2} \rfloor = 8$, whereby $\mathcal{M}^2_{14,11} \nsubseteq \mathcal{M}^1_{14,7}$ and $\mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^1_{14,7}$. Since $\rho(14, 2, 11) = -1$ and $\rho(14, 3, 13) = -2$, and noting that therefore $\mathcal{M}^2_{14,11}$ has codimension 1 and $\mathcal{M}^3_{14,13}$ has codimension at least 2 in $\mathcal{M}_{14}$, we see that $\mathcal{M}^2_{14,11} \nsubseteq \mathcal{M}^3_{14,13}$. Finally, Lelli-Chiesa's lifting of rank 2 linear systems [34] shows that $\mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^2_{14,11}$. □

The proof in genus 15 follows the same argument as genus 14 above.

Proposition 6.14. *In genus* 15, *the maximal Brill–Noether loci are* $\mathcal{M}^1_{15,8}$, $\mathcal{M}^2_{15,11}$, *and* $\mathcal{M}^3_{15,14}$.

6.5. **Genus** 16–17**.** In genus 16 and 17, the proofs are slightly complicated by the fact that one cannot expect to always lift a linear system $A \in \mathrm{Pic}(C)$ to a line bundle on S, but under the Donagi–Morrison conjecture, we can at least find a Donagi–Morrison lift, i.e., a line bundle $N \in \mathrm{Pic}(S)$ such that $|A| \subseteq |N \otimes \mathcal{O}_C|$ with $\gamma(N \otimes \mathcal{O}_C) \leq \gamma(A)$, see Definition 1.12.

Proposition 6.15. *The maximal Brill–Noether loci in genus* 16 *are* $\mathcal{M}^1_{16,8}$, $\mathcal{M}^2_{16,12}$, *and* $\mathcal{M}^3_{16,14}$.

Proof. As above, it remains to show that there are no containments among these loci. One can check, as in Remark 6.8, that for L in $\Lambda^3_{16,14}$, $L \otimes \mathcal{O}_C$ is in fact a g^3_{14}. We note that there are no (-2)-curves in $\Lambda^2_{15,12}$. Hence Lemma 6.7 applies for $\mathrm{Pic}(S)$ either $\Lambda^2_{16,12}$ or $\Lambda^3_{16,14}$. Thus $\mathcal{M}^2_{16,12} \nsubseteq \mathcal{M}^1_{16,8}$ and $\mathcal{M}^3_{16,14} \nsubseteq \mathcal{M}^1_{16,8}$. Furthermore, we have $\mathcal{M}^1_{16,8} \nsubseteq \mathcal{M}^2_{16,12}$ and $\mathcal{M}^1_{16,8} \nsubseteq \mathcal{M}^3_{16,14}$ from Proposition 1.6. Since $\rho(16, 2, 12) = -2$ and $\rho(16, 3, 14) = -4$, we see that $\mathcal{M}^2_{16,12} \nsubseteq \mathcal{M}^3_{16,14}$. It remains to show that there are curves with a g^3_{14} and no g^2_{12}.

Suppose that $\mathrm{Pic}(S) = \Lambda^3_{16,14}$, and suppose C has a line bundle A of type g^2_{12}. Then by [34, Theorem 1], there is a Donagi–Morrison lift of A. It can easily be checked that if the Donagi–Morrison lift M is not of type g^3_{14}, then M can not be contained in $\mathrm{Pic}(S)$. Thus we can assume that M is of type g^3_{14} and $M^2 = 4$. However, by Lemma 3.8, we see that $\gamma(E_{C,A}/N) = 0$, and each of the cases in Lemma 3.7 cannot hold. In case (c), one appeals to [48, Theorem 5.2] which shows that a curve is hyperelliptic only if there is an irreducible curve $B \subset S$ of genus 1 or 2. However, this would yield $B^2 = 0$ or $B^2 = 2$, both of which are too small. Thus there can be no such M, and thus C cannot admit a g^2_{12}. Thus $\mathcal{M}^3_{16,14} \nsubseteq \mathcal{M}^2_{16,12}$. □

The proof in genus 17 follows the same argument as genus 16 above.

Proposition 6.16. *The maximal Brill–Noether loci in genus* 17 *are* $\mathcal{M}^1_{17,9}$*,* $\mathcal{M}^2_{17,13}$*, and* $\mathcal{M}^3_{17,15}$*.*

6.6. **Genus** 18**.** The proof in genus 18 is slightly complicated by the fact that in showing the non-containment $\mathcal{M}^2_{18,13} \not\subseteq \mathcal{M}^3_{18,16}$, the bound in Theorem 5.1 does not rule out the possibility of a $1 \subset 2 \subset 4$ terminal filtration. The other non-contaimnents are similar to the proofs above. We give a proof of this non-trivial non-containment.

Proposition 6.17. *The maximal Brill–Noether loci in genus* 18 *are* $\mathcal{M}^1_{18,9}$*,* $\mathcal{M}^2_{18,13}$*, and* $\mathcal{M}^3_{18,16}$*.*

Proof. The only non-containment requiring additional analysis is $\mathcal{M}^2_{18,13} \not\subseteq \mathcal{M}^3_{18,16}$. The other non-containments follow the arguments above.

In Theorem 5.1, the bound on d to ensure that a Donagi–Morrison lift exists for a g^3_{16} on a general $(S,H) \in \mathcal{K}^2_{18,13}$ is 16, and hence we are not guaranteed to have a Donagi–Morrison lift by using Proposition 3.17. However, the bound is sufficient to show that the LM bundle $E_{C,A}$ associated to the g^3_{16} can only have a terminal filtration of type $1 \subset 4$ or $1 \subset 2 \subset 4$. We argue that the terminal filtration of type $1 \subset 2 \subset 4$ cannot exist.

Suppose $\mathrm{Pic}(S) = \Lambda^2_{18,13}$, and that C has g^3_{16}. Lemma 6.7 shows that C has $\gamma(C) = 8$. Suppose also that $E = E_{C,g^3_{16}}$ has a $1 \subset 2 \subset 4$ terminal filtration, which is $0 \subset N \subset M \subset E$ where N is a line bundle and M has rank 2. We show that this leads to a contradiction. We have $c_1(N).c_1(E/N) \geq \gamma(C) + 2$ by Proposition 4.3. Furthermore, C has general Clifford index by Lemma 6.7. Up to replacing N with its saturation, we can assume E/N is a gLM bundle of type (II), and a computation gives $\gamma(E) = c_1(N).c_1(E/N) + \gamma(E/N) - 2$, thus $\gamma(E/N) \leq 2$.

One can easily check that S has no elliptic curves, hence one of the four cases in Proposition 3.11 occur. In case (i) and (ii), one checks the cases in Lemma 3.7, and finds that none can occur. Thus for a smooth irreducible $D \in |\det(E/N)|$, D is either trigonal, a plane quintic, or a plane sextic, see Remark 3.12. If C is hyperelliptic or trigonal, one finds a Donagi–Morrison lift of the g^1_2 or the g^1_3, which cannot be contained in $\mathrm{Pic}(S)$. Thus we may assume $\gamma(D) = 2$. As the condition $(*)$ from [35, Theorem 4.2] applies, we obtain a Donagi–Morrison lift of the g^2_6, which again cannot be contained in $\mathrm{Pic}(S)$. Thus E cannot have a $1 \subset 2 \subset 4$ filtration.

Therefore E can only have a terminal filtration of type $1 \subset 4$, and Conjecture 2 holds for the g^3_{16}. The rest of the argument is now similar to the arguments above. ☐

6.7. **Genus** 19**.**

Proposition 6.18. *The maximal Brill–Noether loci in genus* 19 *are* $\mathcal{M}^1_{19,10}$*,* $\mathcal{M}^2_{19,14}$*, and* $\mathcal{M}^3_{19,17}$*.*

Proof. To apply Theorem 5.1, it suffices to note that when $\mathrm{Pic}(S) = \Lambda^2_{19,14}$, then we have $\mu \geq 2$ and hence the Donagi–Morrison conjecture holds for a g^3_{17} on a smooth $C \in |H|$, otherwise the argument is similar to Proposition 6.15. ☐

Remark 6.19. In [35, Appendix A, Remark 12], Knutsen and Lelli-Chiesa construct examples of K3 surfaces S of Picard rank 2 such that a smooth irreducible curve $C \subset S$ has a Brill–Noether special linear system A of rank 3 with $\rho(A) = -1$ whose Lazarseld–Mukai bundle $E_{C,A}$ admits no effective sub-line bundle. That is, Proposition 3.17 cannot be used to find a Donagi–Morrison lift of A. Here, we give an explicit example and explain how it relates to our results.

We first recall Knutsen and Lelli-Chiesa's construction. For even integers $a, b \geq 4$ and $d = a + b$, let S be a K3 surface with $\mathrm{Pic}(S) = \Lambda^b_{a,d}$. Suppose that $\mathrm{Pic}(S)$ has no classes of self-intersection -2 or 0. There are infinitely many choices of a and b that satisfy these hypotheses, and such that every element of the linear systems $|H|$ and $|L|$ are reduced and irreducible; these are examples of the so-called *Knutsen K3 surfaces* in [1]. Thus general curves $C_1 \in |H|$ and $C_2 \in |L|$ are smooth of genus a and b, and by Lazarsfeld's theorem [32], are Brill–Noether general, in particular, have generic gonality $k_1 = (a+2)/2$ and $k_2 = (b+2)/2$, respectively. Let E_1 and E_2 be the LM bundles associated to gonality pencils $g^1_{k_1}$ on C_1 and a $g^1_{k_2}$ on C_2. As these pencils are Brill–Noether general, the LM bundles E_1 and E_2 are simple, hence admit no injective map from an effective line bundle N. A calculation using Remark 3.4 shows that the vector bundle $E = E_1 \oplus E_2$ is a LM bundle associated to a linear system A of type $g^3_{k_1+k_2+d}$ on a smooth irreducible curve $C \in |H + L|$. We note that $g(C) = 2d - 1$, and that $\rho(A) = -1$. However, since E admits no injective map $N \hookrightarrow E$, the linear system A admits no Donagi–Morrison lift, and so Conjecture 2 fails for (C, A).

By construction, E has a $2 \subset 4$ terminal filtration. Checking the bound from Lemma 4.4, one finds that $\gamma(C) \leq d - 2$, thus C does not have general Clifford index. In fact, one can verify using Lemma 6.7 that $L|_C$ is a line bundle of type g^b_{d+2b-2}, which has $\gamma(L|_C) = d - 2$. We note that A is non-computing, and does not compute the special Clifford index $\widetilde{\gamma}(C)$. However, the linear system $L|_C$ does compute $\widetilde{\gamma}(C)$, and has a (Donagi–Morrison) lift by construction. Hence, while this is a counterexample to the Donagi–Morrison conjecture, it does not give a negative answer to Question 2.

The first case where such an example shows the failure of Conjecture 2 for (C, A) is genus 19, with $a = 6$ and $b = 4$. The corresponding polarized K3 surface $(S, H+L)$ of genus 19 has $\mathrm{Pic}(S) = \Lambda^4_{19,16}$ with basis $H + L, L$. In the proof of Proposition 6.18, we needed the Donagi–Morrison Conjecture (Conjecture 2) for linear systems on curves on a different lattice polarized K3 surface, showing that our bounded version (Theorem 5.1) is in some sense tight (at least in genus 19).

6.8. **Genus** $20-21$**.** We briefly list what is known and summarize the needed non-containments to verify Conjecture 1 in genus 20 and 21 stemming from these techniques. We note that Conjecture 1 is known to hold in genus 20 and 21, see [3, 4, 5].

The expected maximal Brill–Noether loci in genus 20 are $\mathcal{M}^1_{20,10}$, $\mathcal{M}^2_{20,15}$, and $\mathcal{M}^3_{20,17}$, and $\mathcal{M}^4_{20,19}$. We state the following propositions without proof, as they follow the arguments above.

Proposition 6.20. *In genus* 20*, the loci* $\mathcal{M}^1_{20,10}$*,* $\mathcal{M}^2_{20,17}$*, and* $\mathcal{M}^4_{20,19}$ *are maximal. There are also non-containments*

- $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^1_{20,10}$ *and*
- $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^2_{20,17}$.

In fact, to verify Conjecture 1 in genus 20 following this reduction, it remains to show the non-containment $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^4_{20,19}$. Current lifting methods do not suffice to prove the last non-containment, as there are no known general lifting results for linear systems of rank 4. If Conjecture 2 holds in rank 4, then this would suffice. Another approach to verifying Conjecture 1 in genus 20 is to show that the codimension of $\mathcal{M}^3_{20,17}$ is at least the expected value of 4 and the codimension of $\mathcal{M}^4_{20,19}$ is at least the expected value of 5.

Similarly, the expected maximal Brill–Noether loci in genus 21 are $\mathcal{M}^1_{21,11}$, $\mathcal{M}^2_{21,15}$, and $\mathcal{M}^3_{21,18}$, and $\mathcal{M}^4_{21,20}$. And these techniques suffice to prove that some expected maximal loci are indeed maximal.

Proposition 6.21. *In genus* 21*, the loci* $\mathcal{M}^1_{21,11}$ *and* $\mathcal{M}^4_{21,20}$ *are maximal. There are also non-containments*

- $\mathcal{M}^2_{21,15} \not\subseteq \mathcal{M}^1_{21,11}$,
- $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^1_{21,11}$,
- $\mathcal{M}^2_{21,15} \not\subseteq \mathcal{M}^3_{21,18}$, *and*
- $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^2_{21,15}$.

Our results herein thus reduce the verification of Conjecture 1 in genus 21 to the verification of the non-containments $\mathcal{M}^2_{21,15} \not\subseteq \mathcal{M}^4_{21,20}$ and $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^4_{21,20}$. Again, Conjecture 2 in rank 4 would suffice. Another approach is by verifying that the codimension of $\mathcal{M}^4_{21,20}$ is the expected value of 4, since $\rho(21, 2, 15) = \rho(21, 3, 18) = -3$ and thus the corresponding loci have codimension 3 in $\mathcal{M}_{21}$.

6.9. **Genus** 22.

Proposition 6.22. *The maximal Brill–Noether loci in genus* 22 *are* $\mathcal{M}^1_{22,11}$*,* $\mathcal{M}^2_{22,16}$*, and* $\mathcal{M}^3_{22,19}$*, and* $\mathcal{M}^4_{22,21}$*.*

Proof. In genus 22, [6, Corollary 3.5] shows that the loci $\mathcal{M}^2_{22,16}$ and $\mathcal{M}^3_{22,19}$ are distinct. The argument then follows Proposition 6.15. □

6.10. **Genus** 23. Finally, we provide a proof in genus 23. We note that Farkas proved in [14] that the Brill–Noether divisors $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, and $\mathcal{M}^3_{23,20}$ are mutually distinct. Our results, and those of Lelli-Chiesa [34], provide a different proof for these non-containments. However, the full proof of Conjecture 1 in genus 23 requires our improved lifting results.

Proposition 6.23. *The maximal Brill–Noether loci in genus* 23 *are* $\mathcal{M}^1_{23,12}$*,* $\mathcal{M}^2_{23,17}$*,* $\mathcal{M}^3_{23,20}$*, and* $\mathcal{M}^4_{23,22}$*.*

Proof. Since $\rho(23,1,12) = \rho(23,2,17) = \rho(23,3,20) = -1$ and $\rho(23,4,22) = -2$, Eisenbud and Harris [13] show that the corresponding loci are irreducible of codimension 1 in $\mathcal{M}_g$ and that $\mathcal{M}^4_{23,22}$ has codimension ≥ 2, hence the other loci cannot be contained in $\mathcal{M}^4_{23,22}$. Since there are no (-2)-curves in the Picard lattices of a general K3 surface in $\mathcal{K}^2_{23,17}$, $\mathcal{K}^3_{23,20}$, and $\mathcal{K}^4_{23,22}$, we see by Lemma 6.7 that none of the loci are contained in $\mathcal{M}^1_{23,12}$. One can check that for a very general K3 surface in $\mathcal{K}^4_{23,22}$, the minimal positive self-intersection is 4. Hence by Theorem 5.1, if $C \in |H|$ had a g^3_{20} then by considering the Donagi–Morrison lifts, one finds that L is the only possible Donagi–Morrison lift of the g^3_{20}. Therefore $\gamma(E/N) = 0$, and one then argues as in the proof of Proposition 6.15. Thus $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^3_{23,20}$. The lifting results in [34] similarly show that $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$ and $\mathcal{M}^3_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$. Since the latter two are codimension 1 and irreducible, they are distinct. Thus all of the Brill–Noether loci are distinct. □

6.11. **Codimension 1 and 2 loci.** Here we prove Conjecture 1 for two infinite sequences of genera. Recall that Brill–Noether loci with $\rho = -1, -2$ are irreducible and, as shown in [6, 8], loci with the same $\rho \in \{-1, -2\}$ are not contained in each other. Thus, if all of the expected maximal Brill–Noether loci have the same $\rho \in \{-1, -2\}$, then Conjecture 1 holds.

Proposition 6.24. *If* $g + a = \operatorname{lcm}(1, 2, \ldots, n)$ *for some* $n \in \mathbb{N}_{\geq 3}$*, then every expected maximal Brill–Noether locus has* $\rho = -a$*. In particular, Conjecture 1 holds for genus* g *such that*

$$g + 1 \text{ or } g + 2 \in \{\operatorname{lcm}(1, 2, \ldots, n) \text{ for some } n \in \mathbb{N}_{\geq 3}\}.$$

Proof. We simply observe that the condition $-a = \rho(g, r, d) = g - (r + 1)(g - d + r)$ is equivalent to $g + a = (r + 1)(g - d + r)$ factoring non-trivially. In particular, for $g + a = \operatorname{lcm}(1, 2, \ldots, n)$, there is such a non-trivial factorization for all $r \leq \lfloor \sqrt{g} - \frac{1}{2} \rfloor$, which covers all expected maximal Brill–Noether loci.

Finally, we note as before that two Brill–Noether loci with $\rho = -1$ (resp. $\rho = -2$) are not contained in each other. Thus for $g + 1 \in \{\operatorname{lcm}(1, 2, \ldots, n)$ for some $n \in \mathbb{N}_{\geq 3}\}$, (resp. $g + 2$), Conjecture 1 holds. □

We note that Conjecture 1 implies that all Brill–Noether loci with $\rho \in \{-1, -2\}$ (not necessarily equal) are indeed distinct.

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