FAILURE OF THE LOCAL-GLOBAL PRINCIPLE FOR ISOTROPY OF QUADRATIC FORMS OVER FUNCTION FIELDS

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Abstract. We prove the failure of the local-global principle, with respect to discrete valuations, for isotropy of quadratic forms in 2^n variables over function fields of transcendence degree $n \geq 2$ over an algebraically closed field of characteristic $\neq 2$. Our construction involves the generalized Kummer varieties considered by Borcea and by Cynk and Hulek as well as new results on the nontriviality of unramified cohomology of products of elliptic curves over discretely valued fields.

Introduction

The Hasse–Minkowski theorem states that if a quadratic form q over a number field is isotropic over every completion, then q is isotropic. This is the first, and most famous, instance of the *local-global principle* for isotropy of quadratic forms. Already for a function field of transcendence degree one over a number field, Witt [47] found examples of the failure of the local-global principle for isotropy of quadratic forms in 3 variables (and also 4). Lind [34] and Reichardt [38], and later Cassels [13], found examples of the failure of the local-global principle for isotropy of pairs of quadratic forms in 4 variables over $\mathbb{Q}$ (see [2] for a detailed account), giving examples of quadratic forms over the function field $\mathbb{Q}(t)$ by an application of the Amer–Brumer theorem [33], [23, Theorem 17.14]. Cassels, Ellison, and Pfister [14] found examples in 4 variables over the function field $\mathbb{R}(x, y)$.

Here, we are interested in the failure of the local-global principle for isotropy of quadratic forms over function fields of higher transcendence degree over algebraically closed fields. All our fields will be assumed to be of characteristic $\neq 2$ and all our quadratic forms nondegenerate. A quadratic form is called isotropic if it admits a nontrivial zero. If K is a field and v is a discrete valuation on K, we denote by K_v the fraction field of the completion (with respect to the v-adic topology) of the valuation ring of v. When we speak of the *local-global principle for isotropy of quadratic forms*, sometimes referred to as the strong Hasse principle, in a given dimension d over a given field K, we mean the following statement:

> If q is a quadratic form in d variables over K and q is isotropic over K_v for every discrete valuation v on K, then q is isotropic over K.

Our main result is the following.

Theorem 1. *The local-global principle for isotropy of quadratic forms fails to hold in dimension 2^n over any function field K of transcendence degree $n \geq 2$ over an algebraically closed field k of characteristic $\neq 2$ other than possibly the algebraic closure of a finite field.*

Previously, only the case of $n = 2$ was known, with the first explicit examples over $K = \mathbb{C}(x, y)$ appearing in [29], and later in [8] and [27]. For a construction,

using algebraic geometry, over any transcendence degree 2 function field over an algebraically closed field of characteristic 0, see [5], [6, §6]. In a previous version of this work, Theorem 1 was proved in the case of complex rational function fields, and left as a conjecture. Though we no longer need to make use of it, in §6, we also prove a "geometric presentation lemma" of general interest about the existence of double covers of varieties admitting nontrivial unramified cohomology in maximal degree, which was conjectured in an earlier version of this work and was shown to imply Theorem 1.

We recall that by Tsen–Lang theory [31, Theorem 6], such function fields are C_n-fields, hence have u-invariant 2^n, and thus all quadratic forms of dimension $> 2^n$ are already isotropic, thus we provide counterexamples to the local-global principle in the maximal dimension in which they could occur.

We mention that in the case of transcendence degree $n = 1$, where $K = k(X)$ for a smooth projective curve X over an algebraically closed field k, the local-global principle for isotropy of binary quadratic forms (the "global square theorem") holds when the genus of X is zero and fails when X has positive genus, see Remark 5.3.

Finally, when k is the algebraic closure of a finite field, our methods no longer work. Though one can use other techniques to handle the case of transcendence degree $n = 2$ (see Remark 5.4), proving the failure of the local-global principle for quadratic forms over function fields K of transcendence degree $n \geq 3$ over $\overline{\mathbb{F}}_p$ remains an open problem. Our method relies on proving the nontriviality of certain unramified cohomology classes in top degree, see §6. Already for $n = 3$, the existence of threefolds over $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$ admitting nontrivial unramified cohomology in degree 3 is an open problem related to the integral Tate conjecture, see [18, Question 5.4].

Our result relies on two new ingredients and one very useful trick. The trick, due to Bogomolov [9] and outlined in §1, is a kind of refinement of the existence of transcendence bases, and allows us to reduce the construction of counterexamples to the local-global principle over general function fields to the case of rational function fields. Next, our construction over rational function fields makes use of so-called generalized Kummer varieties, first considered by Borcea [12] and developed by Cynk and Hulek [20], which are constructed as quotients of products of elliptic curves and are birationally double covers of rational varieties. Finally, we prove a new result (Theorem 3.3) on the nontriviality of unramified cohomology on products of elliptic curves, which provides an arithmetic generalization of a result of Gabber [16, Appendice], see also Colliot-Thélène [17].

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1. Bogomolov's trick

Let K/k be a finitely generated field extension. Recall that K/k admits a finite *transcendence basis*, i.e., a set of elements $x_1, \dots, x_n \in K$ that are algebraically independent over k and such that $K/k(x_1, \dots, x_n)$ is a finite extension. The cardinality of any transcendence basis is equal to the transcendence degree of K/k.

A *projective model* of K/k is an integral projective k-variety X whose field of rational functions is k-isomorphic to K. By the classical Chow's lemma, every finitely generated field extension admits a projective model, where the dimension of the model coincides with the transcendence degree of the extension.

The following statement, a refinement of the existence of transcendence bases, can be traced back to Bogomolov, in the course of the proof of [9, Theorem 1.1], cf. [11, Proposition 20].

Lemma 1.1 (Bogomolov's trick)**.** *Let K/k be a finitely generated extension of transcendence degree n. Assume that k is infinite and that a projective model of K/k admits a smooth k-point. Then for any prime number p, there exists a transcendence basis $x_1, \dots, x_n \in K$ such that $K/k(x_1, \dots, x_n)$ is finite of degree prime to p.*

We remark that by the Lang–Nishimura theorem, see [32], [35] and also [39, Proposition A.6], the existence of a smooth k-point on a projective model of K/k implies that any other projective model admits a k-point. The condition that a projective model admits a smooth k-point also implies that any model is geometrically integral and generically smooth, see [45, Lemma 0CDW] and [45, Lemma 056V]. In particular, if k is algebraically closed, then any projective model of K/k admits a smooth k-point.

Proof. As above, since a projective model of K/k is geometrically integral, it is geometrically reduced, and hence K/k is separably generated by a result of MacLane, see [22, Theorem A1.3]. Hence, as in [25, Proposition I.4.9], there exists a projective hypersurface model $X \subset \mathbb{P}^{n+1}$ of K/k. Let d be the degree of X. If $d = 1$, then $X = \mathbb{P}^n$ and there is nothing to prove, so we can assume that $d > 1$.

Projection from a k-point in the complement of X (using that k is infinite) yields a dominant rational map $X \dashrightarrow \mathbb{P}^n$ of degree d. Indeed, it is dominant since the fibers of the projection are the intersections of X with the lines through the point, and such intersections are always nonempty, cf. [25, Theorem I.7.2]. Moreover, it is generically finite of degree d since any line through the point cannot be contained in X, hence must intersect X in a zero-dimensional scheme, which has length d. Similarly, projection from a smooth k-point P of X yields a dominant rational map $X \dashrightarrow \mathbb{P}^n$ of degree $d-1$. Indeed, since P is a smooth point, the tangent space to X at P has codimension 1 in $\mathbb{P}^{n+1}$, hence (again using that k is infinite) the general line in $\mathbb{P}^{n+1}$ through P meets X transversally at P and thus intersects X in a nonempty zero-dimensional scheme of degree d containing P as an irreducible component. Then the general fiber of this projection, which is the complement of P in the intersection of X with a general line through P, is nonempty (using $d > 1$) and has length $d-1$, cf. [24, Example 18.16]. Since d and $d-1$ are relatively prime, no prime number p can divide both, hence the associated extension of function fields $K = k(X)/k(\mathbb{P}^n) = k(x_1, \dots, x_n)$ can be chosen of degree prime to p. □

We remark that the hypothesis on a projective model admitting a k-point is essential. For example, if K/k is the function field of a smooth plane conic X with

no k-point, then there is no presentation of K as an odd degree extension of a rational function field $k(x)$. Indeed, X cannot acquire rational points over rational function fields (see [23, Lemma 7.15]) or extensions of odd degree (by Springer's theorem), but does acquire a rational point over its own function field.

We have the following immediate corollary of Bogomolov's trick.

Corollary 1.2. *Let K be a finitely generated field of transcendence degree n over an algebraically closed field k. Then there exists a transcendence basis $x_1, \dots, x_n \in K$ such that $K/k(x_1, \dots, x_n)$ is of odd degree.*

With this in mind, we now explain how Springer's theorem allows us to reduce the construction of counterexamples to the local-global principle for isotropy of quadratic forms over general function fields to the case of rational function fields.

Proposition 1.3. *Let q be a nondegenerate quadratic form over a field K' and let K/K' be a finite extension of odd degree. If q is a counterexample to the local-global principle for isotropy over K', then q_K is such a counterexample over K.*

Proof. By Springer's theorem, since q is anisotropic over K' and K/K' has odd degree, then q_K is anisotropic over K. To show that q_K is locally isotropic over K, let v be a discrete valuation on K, which then lies over a discrete valuation v' on K'. Since the completion K_v is a finite extension of the completion $K'_{v'}$ and since q is isotropic over $K'_{v'}$, we see that q_K is isotropic over K_v. □

2. Unramified cohomology of function fields

We now recall the notion of unramified cohomology, introduced in [19], restricting ourselves to mod 2 coefficients. Readers should consult the excellent survey [15] for further details. Let k be a field of characteristic $\neq 2$ and K/k be a finitely generated extension. By a *discrete valuation* v on K/k we mean a rank 1 discrete valuation v on K that is trivial on k.

For each discrete valuation v on K/k with residue field $\kappa(v)$, recall the residue map in Galois cohomology

$$\partial_v : H^n(K, \mu_2^{\otimes n}) \to H^{n-1}(\kappa(v), \mu_2^{\otimes n-1})$$

which arises from the Gysin sequence associated to the closed point in the spectrum of the valuation ring R_v of v, see [15, §3.3]. The residue map is uniquely determined by the property that $\partial_v\big((u_1)\cdots(u_{n-1})\cdot(\pi_v)\big) = (\overline{u}_1)\cdots(\overline{u}_{n-1})$, where π_v is a uniformizer and $u_1, \dots, u_{n-1}$ are units of R_v, and $\overline{u}$ means the image of a unit in $\kappa(v)$. The degree n unramified cohomology of K/k is defined by

$$H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n}) = \bigcap_v \ker\big(\partial_v : H^n(K, \mu_2^{\otimes n}) \to H^{n-1}(\kappa(v), \mu_2^{\otimes n-1})\big)$$

where the intersection ranges over all discrete valuations v on K/k. We say that an element $\alpha \in H^n(K, \mu_2^{\otimes n})$ is *unramified* if it belongs to $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n})$.

We recall two results about discrete valuations on rational function fields that will be useful later.

Proposition 2.1.

a) Let k be a field and $K = k(x_1, \dots, x_n)$ a rational function field over k with $n \geq 1$. For each $1 \leq m \leq n$, there exists a discrete valuation v on K/k satisfying $v(x_i) = 1$ for all $1 \leq i \leq m$ and $v(x_i) = 0$ for all $m+1 \leq i \leq n$.

b) Let k_0 be a field with a discrete valuation v_0 and residue field κ_0. Then there exists a discrete valuation v on the rational function field $K_0 = k_0(x_1, \dots, x_n)$, extending v_0 on k_0, and with residue field $\kappa_0(x_1, \dots, x_n)$.

Proof. For (a), let A be the localization of $k[x_1, \cdots, x_n]$ at the prime ideal $(x_1, \cdots, x_m)$. Then $R = A[y_1, \cdots, y_{m-1}]/(x_m - x_1y_1, \cdots, x_m - x_{m-1}y_{m-1})$ is an integral domain with field of fractions isomorphic to K. Furthermore, the ideal $\mathfrak{p}$ of R generated by the images of $x_1, \dots, x_m$ is a prime ideal and $R_{\mathfrak{p}}$ is a discrete valuation ring. The valuation on K/k given by this discrete valuation ring has the required properties. Geometrically, this corresponds to blowing up the model $\mathbb{P}^n$ of K/k along the linear subspace defined by $x_1 = \cdots = x_m = 0$.

For (b), letting $R_0 \subset k_0$ be the valuation ring of v_0 and π_0 a uniformizer, we take the discrete valuation v on K_0 associated to the prime ideal in $R_0[x_1, \dots, x_n] \subset K_0$ generated by π_0. By construction, the residue field of v on K_0 is $\kappa_0(x_1, \dots, x_n)$. Geometrically, this corresponds to the special fiber of the model $\mathbb{P}^n_{R_0}$. □

3. Generalized Kummer varieties

In this section, we review a construction, considered in the context of modular Calabi–Yau varieties [20, §2] and [21], of a generalized Kummer variety attached to a product of elliptic curves. This recovers, in dimension 2, the Kummer K3 surface associated to a decomposable abelian surface, and in dimension 3, a class of Calabi–Yau threefolds of CM type considered by Borcea [12, §3]. We also prove some results about the unramified cohomology groups in top degree of products of elliptic curves and their associated generalized Kummer varieties.

Let $E_1, \dots, E_n$ be elliptic curves over an algebraically closed field k of characteristic $\neq 2$ and let $Y = E_1 \times \cdots \times E_n$. Let σ_i denote the negation automorphism on E_i and $E_i \to \mathbb{P}^1$ the associated quotient branched double cover. We extend each σ_i to an automorphism of Y by acting trivially on each E_j for $j \neq i$; the subgroup $G \subset \mathrm{Aut}(Y)$ they generate is an elementary abelian 2-group. Consider the exact sequence of abelian groups

$$1 \to H \to G \xrightarrow{\Pi} \mathbb{Z}/2 \to 0,$$

where Π is defined by sending each σ_i to 1. Then the product of the double covers $Y \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is the quotient by G and we denote by $Y \to X$ the quotient by the subgroup H. The intermediate quotient $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a double cover, branched over a reducible divisor of type $(4, \dots, 4)$. For $n = 2$, this divisor is the union of 4 vertical fibers and 4 horizontal fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ meeting in 16 points.

We point out that X is a singular degeneration of smooth Calabi–Yau varieties that (geometrically) admits a smooth Calabi–Yau model, see [20, Corollary 2.3] and [21, Section 4]. For $n = 2$, the minimal resolution of X is indeed isomorphic to the Kummer K3 surface $\mathrm{Kum}(E_1 \times E_2)$.

Given nontrivial classes $\gamma_i \in H^1_{\text{ét}}(E_i, \mu_2)$, we consider the cup product

$$\gamma = \gamma_1 \cdots \gamma_n \in H^n_{\text{ét}}(Y, \mu_2^{\otimes n}) \tag{1}$$

and its image in $H^n_{\text{ur}}(k(Y)/k, \mu_2^{\otimes n})$ under restriction to the generic point. These classes have been studied in [16]. We remark that γ is in the image of the restriction map $H^n(k(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1), \mu_2^{\otimes n}) \to H^n(k(Y), \mu_2^{\otimes n})$ in Galois cohomology since each γ_i is in the image of the restriction map $H^1(k(\mathbb{P}^1), \mu_2) \to H^1(k(E_i), \mu_2)$.

We make this more explicit as follows. Corresponding to each double cover $E_i \to \mathbb{P}^1$, choose a Weierstrass equation in Legendre form

$$y_i^2 = x_i(x_i - 1)(x_i - \lambda_i) \tag{2}$$

where x_i is a coordinate on $\mathbb{P}^1$ and $\lambda_i \in k \smallsetminus \{0, 1\}$, see [43, III.1.7]. Then the branched double cover $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is birationally defined by the equation

$$y^2 = \prod_{i=1}^{n} x_i(x_i - 1)(x_i - \lambda_i) = f(x_1, \ldots, x_n) \tag{3}$$

where $y = y_1 \cdots y_n$ in $k(Y)$, see [21, §3]. Up to an automorphism, we can, and henceforth will, choose the Legendre forms so that the image of γ_i under the map $H^1_{\text{ét}}(E_i, \mu_2) \to H^1(k(E_i), \mu_2)$ coincides with the square class $(x_i) \in k(E_i)/k(E_i)^{\times 2} = H^1(k(E_i), \mu_2)$ of the rational function x_i, which is then visibly in the image of the restriction map $H^1(k(\mathbb{P}^1), \mu_2) \to H^1(k(E_i), \mu_2)$. Hence we see that the (ramified) cup product class $\xi = (x_1) \cdots (x_n) \in H^n(k(x_1, \cdots, x_n), \mu_2^{\otimes n})$, restricts to the unramified class $\gamma \in H^n_{\text{ur}}(k(Y)/k, \mu_2^{\otimes n})$.

The first main result of this section is that the class ξ already restricts to an unramified class over the quadratic extension $k(X)$. We prove a more general result that can be viewed as a higher dimensional generalization of [15, §1].

Proposition 3.1. *Let k be an algebraically closed field of characteristic $\neq 2$ and $K = k(x_1, \ldots, x_n)$ a rational function field over k. For $1 \leq i \leq n$, let $f_i(x_i) \in k[x_i]$ be polynomials of even degree satisfying $f_i(0) \neq 0$, and let $f = \prod_{i=1}^n x_i f_i(x_i)$. Then the restriction of the class $\xi = (x_1) \cdots (x_n) \in H^n(K, \mu_2^{\otimes n})$ to $H^n(K(\sqrt{f}), \mu_2^{\otimes n})$ is unramified with respect to all discrete valuations.*

Proof. Let $L = K(\sqrt{f})$ and v a discrete valuation on L with valuation ring R, maximal ideal $\mathfrak{m}$, and residue field κ. Write ξ_L for the restriction of ξ to $H^n(L, \mu_2^{\otimes n})$.

Suppose $v(x_i) < 0$ for some i. Let d_i be the degree of f_i and consider the reciprocal polynomial $f_i^*(x_i) = x_i^{d_i} f_i(\frac{1}{x_i})$, so that $x_i f_i(x_i) = x_i^{d_i+1} \cdot \frac{1}{x_i} f_i^*(\frac{1}{x_i})$. Since d_i is even, we have that the polynomials $x_i f_i(x_i)$ and $\frac{1}{x_i} f_i^*(\frac{1}{x_i})$ have the same class in $K^\times/K^{\times 2}$.

Thus, up to replacing, for all i with $v(x_i) < 0$, the polynomial f_i by f_i^* in the definition of f and replacing x_i by $\frac{1}{x_i}$, we can assume that $v(x_i) \geq 0$ for all i without changing the extension L/K. Hence $k[x_1, \ldots, x_n] \subset R_v$.

Consider $\mathfrak{p} = k[x_1, \ldots, x_n] \cap \mathfrak{m}$. Then $\mathfrak{p}$ is a prime ideal of $k[x_1, \ldots, x_n]$ whose residue field $\kappa(\mathfrak{p})$ is a subfield of κ. Let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$ and L_v the completion of L at v. Then $K_\mathfrak{p}$ is a subfield of L_v.

If $v(x_i) = 0$ for all i, then ξ_L is unramified at v. So suppose that $v(x_i) \neq 0$ for some i. By reindexing $x_1, \ldots, x_n$, we assume that there exists $m \geq 1$ such that $v(x_i) > 0$ for $1 \leq i \leq m$ and $v(x_i) = 0$ for $m + 1 \leq i \leq n$, i.e., we have $x_1, \ldots, x_m \in \mathfrak{p}$ and $x_{m+1}, \ldots, x_n \notin \mathfrak{p}$. In particular, the transcendence degree of $\kappa(\mathfrak{p})$ over k is $\leq n - m$.

First, suppose $f_i(x_i) \in \mathfrak{p}$ for some $m + 1 \leq i \leq n$. Since $f_i(x_i)$ is a product of linear factors in $k[x_i]$, we have that $x_i - a_i \in \mathfrak{p}$ for some $a_i \in k$, with $a_i \neq 0$ since $f_i(0) \neq 0$. Thus the image of x_i in $\kappa(\mathfrak{p})$ is equal to a_i and hence is a square in $K_\mathfrak{p}$. In particular, x_i is a square in L_v, thus ξ_L is trivial (hence unramified) at v, cf. [19, Proposition 1.4].

Now, suppose that $f_i(x_i) \notin \mathfrak{p}$ for all $m + 1 \leq i \leq n$. Then for each $1 \leq i \leq m$, we see that since $x_i \in \mathfrak{p}$ and $f_i(0) \neq 0$, we have $f_i(x_i) \notin \mathfrak{p}$. Consequently, we can assume

that $f = x_1 \cdots x_m u$ for some $u \in k[x_1, \ldots, x_n] \smallsetminus \mathfrak{p}$. We remark that $f = x_1 \cdots x_m u$ is a square in L, so that $(x_1 \cdots x_m) = (u)$ in $H^1(L, \mu_2)$.

For $m = 1$, we see that $\xi_L = (u) \cdot (x_2) \cdots (x_n)$ is unramified at v since u and $x_2, \ldots, x_n$ are units at v.

For $m > 1$, a computation with symbols

$$(x_1) \cdots (x_m) = (x_1) \cdots (x_{m-1}) \cdot (x_1 \cdots x_m) = (x_1) \cdots (x_{m-1}) \cdot (u) \in H^m(L, \mu_2^{\otimes m})$$

shows that $\xi_L = (x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)$. Since u and $x_{m+1}, \ldots, x_n$ are units at v, computing with the Galois cohomology residue homomorphism $\partial_v : H^n(L, \mu_2^{\otimes n}) \to H^{n-1}(\kappa(v), \mu_2^{\otimes n-1})$ from §2 shows that

$$\partial_v\big((x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)\big) = \alpha \cdot (\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n)$$

for some $\alpha \in H^{m-2}(\kappa(v), \mu_2^{\otimes m-2})$, where for any $h \in k[x_1, \ldots, x_n]$, we write $\overline{h}$ for the image of h in $\kappa(\mathfrak{p}) \subset \kappa$. Since the transcendence degree of $\kappa(\mathfrak{p})$ over k is $\leq n - m$ and k is algebraically closed, we have that $\kappa(\mathfrak{p})$ has 2-cohomological dimension $\leq n - m$ by [42, II.4.2 Proposition 11], so that $H^{n-m+1}(\kappa(\mathfrak{p}), \mu_2^{\otimes n-m+1}) = 0$. Since $\overline{u}, \overline{x}_i \in \kappa(\mathfrak{p})$, we then have that $(\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n)$ is trivial. In particular, $\partial_v(\xi_L)$ is trivial, and hence ξ_L is unramified at v. Finally, we have shown that the restriction ξ_L is unramified at all discrete valuations on L. □

As an immediate consequence, we deduce the fact that the class ξ restricts to an unramified class over $k(X) = k(x_1, \ldots, x_n)(\sqrt{f})$, where f is as in (3).

Proposition 3.2. *Let $E_1, \ldots, E_n$ be elliptic curves over an algebraically closed field k of characteristic $\neq 2$, given in the Legendre form (2), with $K = k(x_1, \ldots, x_n)$. Then the restriction of the class $\xi = (x_1) \cdots (x_n)$ in $H^n(K, \mu_2^{\otimes n})$ to $H^n(k(X), \mu_2^{\otimes n})$ is unramified at all discrete valuations.*

This unramified class on $k(X)$ restricts to the class γ on $k(Y)$ in (1), so without loss of generality, we will also call it γ. Finally, we will need conditions ensuring that our class γ is nontrivial over $k(X)$. For this, we must choose the elliptic curves $E_1, \ldots, E_n$ more carefully, and we will then show that γ is nontrivial over $k(Y)$, hence is nontrivial over $k(X)$. We proceed as follows.

First, we choose a subfield $k_0 \subset k$ admitting a discrete valuation v_0. This is possible unless k is the algebraic closure of a finite field; this is why we must henceforth assume that k is not the algebraic closure of a finite field. Then we choose E_i defined over k_0 with Weierstrass equation (2) satisfying $v_0(\lambda_i) > 0$. Finally, we appeal to the following arithmetic version, which was inspired by Bogomolov [10, §7], of a result of Gabber [16, Appendice].

Theorem 3.3. *Let k_0 be a field with a discrete valuation v_0 whose residue field has characteristic $\neq 2$. Let $E_1, \ldots, E_n$ be elliptic curves over k_0 given in the Legendre form (2), with $v_0(\lambda_i) > 0$ for all $1 \leq i \leq n$. Let $Y = E_1 \times \cdots \times E_n$ and k/k_0 be an algebraically closed extension. Then the class $\gamma \in H^n(k(Y), \mu_2^{\otimes n})$ in (1) is nontrivial.*

Proof. Let $K_0 = k_0(x_1, \ldots, x_n)$ and let γ_0 be the restriction of the class $\xi_0 = (x_1) \cdots (x_n) \in H^n(K_0, \mu_2^{\otimes n})$ to $H^n(k_0(Y), \mu_2^{\otimes n})$. Letting κ_0 be the residue field of v_0, by Proposition 2.1*b)* we can extend v_0 to a discrete valuation on K_0 with residue field $\kappa_0(x_1, \ldots, x_n)$. We remark that each $x_i \in K_0$ is a unit with respect

to this valuation. Since $k_0(Y)/K_0$ is a finite separable extension, we can further extend this valuation to a discrete valuation $\tilde{v}$ on $k_0(Y)$. Writing

$$k_0(Y) = k_0(x_1, \cdots, x_n)(\sqrt{x_1(x_1-1)(x_1-\lambda_1)}, \cdots, \sqrt{x_n(x_n-1)(x_n-\lambda_n)})$$

then since $\tilde{v}(\lambda_i) > 0$ and $\tilde{v}(x_i) = 0$ for all i, we have that the residue field of $\tilde{v}$ is

$$\tilde{\kappa} = \kappa_0(x_1, \cdots, x_n)(\sqrt{x_1 - 1}, \cdots, \sqrt{x_n - 1}).$$

Since each x_i is a unit at $\tilde{v}$, the class γ_0 is unramified at $\tilde{v}$, and has specialization $\tilde{\xi}_0 = (x_1) \cdots (x_n) \in H^n(\tilde{\kappa}, \mu_2^{\otimes n})$.

We now argue that $\tilde{\xi}_0$ is nontrivial, hence that γ_0 is nontrivial. To this end, by Proposition 2.1*a)* there is a valuation v_n on $\kappa_0(x_1, \cdots, x_n)$ such that $v_n(x_i) = 0$ for $1 \leq i \leq n-1$ and $v_n(x_n) = 1$, and we denote by $\tilde{v}_n$ an extension to $\tilde{\kappa}$, which is separable over $\kappa_0(x_1, \ldots, x_n)$ and unramified at $\tilde{v}_n$. Thus $\tilde{v}_n$ is trivial on the subfield

$$\tilde{\kappa}_n = \kappa_0(x_1, \cdots, x_{n-1})(\sqrt{x_1 - 1}, \cdots, \sqrt{x_{n-1} - 1})$$

and satisfies $\tilde{v}_n(x_n) = 1$. Then the residue field of $\tilde{v}_n$ is $\tilde{\kappa}_n(\sqrt{-1})$ and the residue of the class $\tilde{\xi}_0$ at $\tilde{v}_n$ is simply $(x_1) \cdots (x_{n-1})$. Repeatedly taking residues using this process, we arrive at the class $(x_1) \in H^1(\kappa_0(x_1)(\sqrt{-1}, \sqrt{x_1 - 1}), \mu_2)$, which is nontrivial, hence $\tilde{\xi}_0$ is nontrivial. Thus $\gamma_0 \in H^n(k_0(Y), \mu_2^{\otimes n})$ is nontrivial.

Now let k/k_0 be any algebraically closed field extension and let $\bar{k}_0$ be the algebraic closure of k_0 in k. First, we show that the restriction of γ_0 to $H^n(\bar{k}_0(Y), \mu_2^{\otimes n})$ is nontrivial. This is equivalent to the restriction of γ_0 to $H^n(l_0(Y), \mu_2^{\otimes n})$ being nontrivial for every finite algebraic extension l_0/k_0. Letting w_0 be an extension of v_0 to l_0, we still have that $w_0(\lambda_i) > 0$ for all i, so we can apply what we have already proved. Second, since γ_0 is unramified, its restriction to $H^n(\bar{k}_0(Y), \mu_2^{\otimes n})$ and further to $H^n(k(Y), \mu_2^{\otimes n})$, remains unramified and coincides with the class γ. Then we can appeal to the rigidity property for unramified cohomology, which implies that the restriction map $H^n_{\mathrm{ur}}(\bar{k}_0(Y)/\bar{k}, \mu_2^{\otimes n}) \to H^n_{\mathrm{ur}}(k(Y)/k, \mu_2^{\otimes n})$ is an isomorphism, see [15, §4.4], showing that γ is nontrivial. □

Additional aspects and applications of the argument in the proof of Theorem 3.3 will be the subject of forthcoming work [7]. In particular, $\mu_2^{\otimes n}$ coefficients can be replaced by $\mu_\ell^{\otimes n}$ coefficients for any positive integer ℓ prime to the residue characteristic of k_0. We content ourselves with giving one application here, which is a new proof of (a generalization of) Gabber's result [16, Appendice].

Corollary 3.4. *Let k be a field of characteristic $\neq 2$ and K/k an algebraically closed extension. Let $E_1, \ldots, E_n$ be elliptic curves over K whose j-invariants are algebraically independent over k. Let $Y = E_1 \times \cdots \times E_n$. Then the class $\gamma \in H^n(K(Y), \mu_2^{\otimes n})$ in (1) is nontrivial.*

Proof. Since K is algebraically closed, each elliptic curve E_i can be put into Legendre form (2). Hence Y is defined over the field $k_0 = k(\lambda_1, \ldots, \lambda_n)$. Since the j-invariant of E_i is a rational function in λ_i, the algebraic independence of $j(E_1), \ldots, j(E_n)$ over k implies the algebraic independence of $\lambda_1, \ldots, \lambda_n$ over k. By Proposition 2.1*a)*, there exists a discrete valuation v_0 on k_0 such that $v_0(\lambda_i) > 0$ for all i, and then we can apply Theorem 3.3. □

4. Hyperbolicity over a quadratic extension

Let K be a field of characteristic $\neq 2$. We will need the following result about isotropy of quadratic forms, generalizing a well-known result in the dimension four case, see [41, Ch. 2, Lemma 14.2].

Proposition 4.1. *Let q be a quadratic form over K of dimension divisible by 4 and discriminant d, and let $L = K(\sqrt{d})$. If q is hyperbolic over L then q is isotropic over K.*

Proof. If $d \in K^{\times 2}$, then $K = L$ and there is nothing to prove, so suppose $d \notin K^{\times 2}$. To get a contradiction, we will assume q is anisotropic. Since q_L is hyperbolic, we then have $q \simeq <1, -d> \otimes q_1$ for some quadratic form q_1 over K, see [41, Ch. 2, Theorem 5.2]. Since the dimension of q is divisible by four, the dimension of q_1 is divisible by two, and a computation of the discriminant shows that $d \in K^{\times 2}$, which is a contradiction. □

For $n \geq 1$ and $a_1, \ldots, a_n \in K^\times$, recall the n-fold Pfister form

$$\ll a_1, \ldots, a_n \gg = <1, -a_1> \otimes \cdots \otimes <1, -a_n>$$

and the associated symbol $(a_1) \cdots (a_n)$ in the Galois cohomology group $H^n(K, \mu_2^{\otimes n})$. Then $\ll a_1, \ldots, a_n \gg$ is hyperbolic if and only if $\ll a_1, \ldots, a_n \gg$ is isotropic if and only if $(a_1) \cdots (a_n)$ is trivial. For the fact that isotropic Pfister forms are hyperbolic, see [41, Ch. 4, Corollary 1.5]. The fact that the triviality of $(a_1) \cdots (a_n)$ implies the hyperbolicity of $\ll a_1, \ldots, a_n \gg$ is a consequence of the Milnor conjectures for the Witt group, as proved by Voevodsky [46] and Orlov, Vishik, Voevodsky [36].

For $d \in K^\times$ and $n \geq 2$, we will consider quadratic forms of discriminant d related to n-fold Pfister forms, as follows. Write $\ll a_1, \ldots, a_n \gg$ as $q_0 \perp <(-1)^n a_1 \ldots a_n>$, then define $\ll a_1, \ldots, a_n; d \gg = q_0 \perp <(-1)^n a_1 \ldots a_n d>$. For example:

$$\begin{aligned} \ll a; d \gg &= <1, -ad> \\ \ll a, b; d \gg &= <1, -a, -b, abd> \\ \ll a, b, c; d \gg &= <1, -a, -b, -c, ab, ac, bc, -abcd> \end{aligned}$$

for $n = 1, 2, 3$, respectively. We remark that every quadratic form of dimension 4 is similar to one of this type. We also remark that $\ll a_1, \ldots, a_n; d \gg$ becomes isomorphic to $\ll a_1, \ldots, a_n \gg$ over $K(\sqrt{d})$. In general, these quadratic forms are examples of *twisted Pfister forms* in the sense of Hoffmann [26].

Proposition 4.2. *Assume $n \geq 2$. If $q = \ll a_1, \ldots, a_n; d \gg$ and $L = K(\sqrt{d})$ then q is isotropic if and only if q_L is isotropic if and only if $(a_1) \cdots (a_n) \in H^n(L, \mu_2^{\otimes n})$ is trivial.*

Proof. If q is isotropic then q_L is isotropic. If q_L is isotropic, then as mentioned above, it is hyperbolic as it is a Pfister form, hence by Proposition 4.1 (since q has dimension 2^n and $n \geq 2$), q is isotropic over K. As previously mentioned above (and consequence of the Milnor conjectures), $(a_1) \cdots (a_n) \in H^n(L, \mu_2^{\otimes n})$ is trivial if and only the Pfister form q_L is isotropic. □

This generalizes a well-known result about quadratic forms of dimension 4, see [41, Ch. 2, Lemma 14.2].

5. Failure of the local global principle

In this section, we prove our main Theorem 1 by providing a construction of quadratic forms over function fields that are locally isotropic yet globally anisotropic. First we prove a general result about the generalized Pfister forms in Section 4.

Proposition 5.1. *Let k be an algebraically closed field of characteristic $\neq 2$ and K/k a finitely generated extension of transcendence degree $n \geq 2$. Let $a_1, \ldots, a_n, d \in K^\times$ be such that the symbol $(a_1)\cdots(a_n)$ in $H^n(K, \mu_2^{\otimes n})$ becomes unramified over $L = K(\sqrt{d})$. Then the quadratic form $q = \ll a_1, \ldots, a_n; d \gg$ is locally isotropic over K.*

Proof. Let v be a discrete valuation on K and w an extension to L, with completions K_v and L_w and residue fields $\kappa(v)$ and $\kappa(w)$, respectively. By assumption, the restriction of the symbol $(a_1)\cdots(a_n)$ to $H^n(L, \mu_2^{\otimes n})$ is unramified at w. By cohomological purity for discrete valuation rings (cf. [15, §3.3]) we have a surjective map $H^n_{\text{ét}}(R_w, \mu_2^{\otimes n}) \to H^n_{\text{ur}}(L_w/k, \mu_2^{\otimes n})$ where $R_w \subset L_w$ is the valuation ring. By proper base change (cf. [4, XII.5.5], see also a general result of Gabber [45, Tag 09ZI]), we have an isomorphism $H^n_{\text{ét}}(R_w, \mu_2^{\otimes n}) \cong H^n(\kappa(w), \mu_2^{\otimes n})$. Since $\kappa(w)/k$ has transcendence degree $< n$ by Abhyankar's inequality [1, Corollary 1(1)] and k is algebraically closed, we have that $\kappa(w)$ has 2-cohomological dimension $< n$ by [42, II.4.2 Proposition 11]. From all this, we deduce that $H^n_{\text{ur}}(L_w/k, \mu_2^{\otimes n}) = 0$. In particular, the symbol $(a_1)\cdots(a_n)$ has trivial restriction to $H^n(L_w, \mu_2^{\otimes n})$. Thus by Proposition 4.2, we have that q_{K_v} is isotropic. Finally, as this holds for every discrete valuation v on K, the quadratic form q is locally isotropic over K. □

Now, we will utilize our constructions in Section 3. Let k be an algebraically closed field of characteristic $\neq 2$ that is not the algebraic closure of a finite field. Let $k_0 \subset k$ be a subfield with a discrete valuation v_0 whose residue field has characteristic $\neq 2$. Let $E_1, \ldots, E_n$ be elliptic curves over k_0 given in the Legendre form (2), with $v_0(\lambda_i) > 0$ for all $1 \leq i \leq n$. Let $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be the double cover defined by $y^2 = \prod_{i=1}^n x_i(x_i - 1)(x_i - \lambda_i) = f(x_1, \ldots, x_n)$ in (3), and consider the quadratic form

$$q = \ll x_1, \ldots, x_n; f \gg \tag{4}$$

over the rational function field $k(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = k(x_1, \ldots, x_n)$, as in Section 4.

Our main result is that for $n \geq 2$, the quadratic form q shows the failure of the local-global principle for isotropy, with respect to all discrete valuations, for quadratic forms of dimension 2^n over $k(x_1, \ldots, x_n)$.

Theorem 5.2. *Let k be an algebraically closed field of characteristic $\neq 2$ that is not the algebraic closure of a finite field and assume $n \geq 2$. The quadratic form $q = \ll x_1, \ldots, x_n; f \gg$ as in (4) is anisotropic over $k(x_1, \ldots, x_n)$ yet is isotropic over the completion at every discrete valuation.*

Proof. Write $K = k(x_1, \ldots, x_n)$ and $L = K(\sqrt{f})$. By Proposition 3.2, the restriction of the symbol $(x_1)\cdots(x_n) \in H^n(K, \mu_2^{\otimes n})$ to L is unramified. Hence Proposition 5.1 implies that q is locally isotropic at every discrete valuation on K.

The restriction of the symbol $(x_1)\cdots(x_n) \in H^n(K, \mu_2^{\otimes n})$ to L is nontrivial since its further restriction to $k(E_1 \times \cdots \times E_n)$ is nontrivial by Theorem 3.3. Hence Proposition 4.2 implies that q is anisotropic over K. □

Proof of Theorem 1. By Bogomolov's trick (Corollary 1.2), we find $x_1, \dots, x_n \in K$ such that $K/k(x_1, \dots, x_n)$ has odd degree. If q is as in (4), then by Theorem 5.2, q is anisotropic yet locally isotropic over $k(x_1, \dots, x_n)$. Finally, by Proposition 1.3, q is a counterexample to the local-global principle for isotropy over K. □

To give an explicit example, let $a, b, c \in \overline{\mathbb{Q}} \smallsetminus \{0, 1\}$ be any algebraic integers all divisible by a common odd prime ideal in a number field containing them. For example, take $a = b = c = 3$. Then over the function field $K = \mathbb{C}(x, y, z)$, the quadratic form

$$q = <1, x, y, z, xy, xz, yz, (x-1)(y-1)(z-1)(x-a)(y-b)(z-c)>$$

is isotropic over every completion K_v associated to a discrete valuation v on K, and yet q is anisotropic over K.

Remark 5.3. Let k be any algebraically closed field of characteristic $\neq 2$. When K/k is a finitely generated field of transcendence degree 1, then $K = k(X)$ for a smooth projective curve X over k. Any binary quadratic form q over K is similar to $\ll a \gg = <1, -a>$ for some $a \in K^\times$, and q is isotropic if and only if a is a square. For any discrete valuation v on K, we have that q is isotropic over K_v if and only if a is a square in K_v, equivalently (since k is algebraically closed and characteristic $\neq 2$), $v(a)$ is even. Thus if q is locally isotropic at all discrete valuations on K then the divisor of the rational function a on X can be written as $2D$ for a divisor D on X. The divisor class of D is 2-torsion in $\mathrm{Pic}(X)$ and it is trivial if and only if a is a square in K. Conversely, if X admits a nontrivial 2-torsion element of $\mathrm{Pic}(X)$, then twice this element is the divisor of a rational function $a \in K$ and the local-global principle fails for $\ll a \gg$. Thus the local-global principle for isotropy fails for K if and only if the Picard group of X admits a nontrivial 2-torsion element, equivalently (again, since k is algebraically closed and characteristic $\neq 2$), the genus of X is positive. Equivalently, the local-global principle for isotropy holds for quadratic forms over K if and only if K/k is purely transcendental.

In fact, we see that Proposition 4.2 (and hence Proposition 5.1) is false for $n = 1$ by considering the trivial class in $H^1(K, \mu_2)$ and $d \in K^\times$ any nonsquare.

Remark 5.4. When k is the algebraic closure of a finite field of characteristic $\neq 2$, Theorem 5.2 still holds for $n = 2$ assuming that the elliptic curves E_1 and E_2 are not isogenous. Indeed, by Proposition 3.2, the restriction of the symbol $(x_1) \cdot (x_2) \in H^2(K, \mu_2^{\otimes 2})$ to L is still unramified, and the only thing left to verify is that it is nontrivial. We can check this by further restriction to $k(E_1 \times E_2)$, where the symbol is the restriction to the generic point of a class in $H^1_{\text{ét}}(E_1, \mu_2) \otimes H^1_{\text{ét}}(E_2, \mu_2)$ by §3. However, standard computations of the Brauer group of $E_1 \times E_2$, cf. [44, §3], show that if E_1 is not isogenous to E_2, then in fact $\mathrm{Br}(E_1 \times E_2) \cong H^1_{\text{ét}}(E_1, \mu_2) \otimes H^1_{\text{ét}}(E_2, \mu_2)$, so that each such cup product class is indeed nontrivial in the Brauer group. Then, as before, Proposition 5.1 implies that the local-global principle for isotropy fails for q as in (4) over K, hence Theorem 1 also holds in this case.

6. A geometric presentation lemma

The method for producing locally isotropic but globally anisotropic quadratic forms of dimension 2^n over function fields of transcendence degree n presented in this work is different from the one employed in [6, §6] for $n = 2$. There, we first proved a kind of geometric presentation lemma about the existence of nontrivial

unramified cohomology (in degree 2) over quadratic extensions. Specifically, using Hodge theory, we proved [6, Proposition 6.4] that given any smooth projective surface S over an algebraically closed field of characteristic zero, there exists a double cover $T \to S$ with T smooth and $H^2_{\mathrm{ur}}(k(T)/k, \mu_2^{\otimes 2}) = \mathrm{Br}(T)[2] \neq 0$. It has been an open question ever since whether such a geometric presentation lemma holds for unramified cohomology in higher degree.

Conjecture 6.1. *Let K be a finitely generated field of transcendence degree n over an algebraically closed field k of characteristic $\neq 2$. Then either $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n}) \neq 0$ or there exists a separable quadratic extension L/K such that $H^n_{\mathrm{ur}}(L/k, \mu_2^{\otimes n}) \neq 0$.*

Assuming this conjecture, we can give a more direct proof of the existence of quadratic forms representing a failure of the local-global principle for isotropy without using the construction involving generalized Kummer varieties in §3.

Proposition 6.2. *Let K be a finitely generated field of transcendence degree n over an algebraically closed field k of characteristic $\neq 2$. If Conjecture 6.1 holds for K, then the local-global principle for isotropy of quadratic forms fails to hold in dimension 2^n over K.*

Before proceeding with the proof of Proposition 6.2, we recall a standard application of the Milnor conjectures for the Witt group. Since we could not find a suitable reference, we also provide a proof.

Lemma 6.3. *Let K be a field of characteristic $\neq 2$. If K is a C_n-field then every element in $H^n(K, \mu_2^{\otimes n})$ is a symbol.*

Proof. By the Milnor conjectures for the Witt group, as proved by Voevodsky [46] and Orlov, Vishik, Voevodsky [36], there exists a surjective homomorphism $e_n : I^n(K) \to H^n(K, \mu_2^{\otimes n})$ taking n-fold Pfister forms to symbols, where $I^n(K)$ is the nth power of the fundamental ideal of the Witt group of K. Thus it suffices to prove that every element in $I^n(K)$ is represented by a Pfister form. Let q be an anisotropic quadratic form representing a class in $I^n(K)$. By the Arason–Pfister Hauptsatz (see [41, Ch. 4, Theorem 5.6]), q has dimension $\geq 2^n$, but since we are assuming that K is a C_n-field, every quadratic form of dimension $> 2^n$ is isotropic, hence q has dimension 2^n.

Now we recall that every anisotropic form q of dimension 2^n in $I^n(K)$ is similar to a Pfister form over (any field) K, cf. [28, Corollaire 4.3.7]. Indeed, let $K(q)$ be the function field of the projective quadric defined by q. Then $q_{K(q)} \in I^n(K(q))$. Since q is isotropic over $K(q)$, the anisotropic part of $q_{K(q)}$ over $K(q)$ has dimension smaller than 2^n, hence by the Arason–Pfister Hauptsatz must be zero, thus q is hyperbolic over $K(q)$. Being anisotropic over K and hyperbolic over $K(q)$, the quadratic form q is thus similar to a Pfister form over K, see [41, Ch. 4, Theorem 5.4(i)].

Since K is assumed to be a C_n-field, and $I^{n+1}(K)$ is additively generated by $(n+1)$-fold Pfister forms by [41, Ch. 4, Lemma 5.5], which are hyperbolic as soon as they are isotropic by [41, Ch. 4, Corollary 1.5], we conclude that $I^{n+1}(K) = 0$. We now argue that any quadratic form in $I^n(K)$ that is similar to a Pfister form is actually a Pfister form. Indeed, if ψ is any Pfister form in $I^n(K)$ and $a \in K^\times$, then $\ll a \gg \otimes \psi = \psi \perp -a\psi$ is in $I^{n+1}(K) = 0$, hence $\psi \cong a\psi$. Thus our anisotropic quadratic form q in $I^n(K)$ is a Pfister form, proving the desired statement. □

We do not know, in the spirit of [42, II.4.5 Remark 3] and [30], whether the statement of Lemma 6.3 holds for Galois cohomology modulo ℓ for primes $\ell \neq 2$.

Proof of Proposition 6.2. First, by Lemma 6.3, every element in $H^n(K, \mu_2^{\otimes n})$ is a symbol since K is a C_n-field by Tsen–Lang theory [31]. Proposition 5.1 (applied with $d = 1$) implies that for any symbol $(a_1)\cdots(a_n)$ in $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n})$, the n-fold Pfister form $\ll a_1, \ldots, a_n \gg$ is locally isotropic. If we assume that $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n}) \neq 0$, then taking a nontrivial unramified symbol $(a_1)\cdots(a_n)$, the n-fold Pfister form $\ll a_1, \ldots, a_n \gg$ is locally isotropic but is anisotropic by Proposition 4.2, giving a counterexample to the local-global principle for isotropy over K.

Now assume that $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n}) = 0$ and that $H^n_{\mathrm{ur}}(L/k, \mu_2^{\otimes n}) \neq 0$ for some separable quadratic extension $L = K(\sqrt{d})$ of K. By Tsen–Lang theory (e.g., [42, II.4.5]), L is also a C_n-field, hence by Lemma 6.3 every element in $H^n(L, \mu_2^{\otimes n})$ is a symbol. Thus we can choose a nontrivial unramified symbol $(a_1)\cdots(a_n) \in H^n_{\mathrm{ur}}(L/k, \mu_2^{\otimes n})$. Since the corestriction map $H^n(L, \mu_2^{\otimes n}) \to H^n(K, \mu_2^{\otimes n})$ preserves unramified cohomology, and we have assumed that $H^n_{\mathrm{ur}}(K/k, \mu_2^{\otimes n}) = 0$, we see that the corestriction of $(a_1)\cdots(a_n)$ is trivial. By the restriction-corestriction exact sequence for Galois cohomology, see [3, Satz 4.5] or [42, I§2 Exercise 2], we have that $(a_1)\cdots(a_n)$ is in the image of the restriction map $H^n(K, \mu_2^{\otimes n}) \to H^n(L, \mu_2^{\otimes n})$, and thus we can take $a_1, \ldots, a_n \in K^\times$. Then by Proposition 5.1, the twisted Pfister form $\ll a_1, \ldots, a_n; d \gg$ is locally isotropic over K but globally anisotropic. □

However, under the hypothesis in which we prove Theorem 1, namely, that k is not the algebraic closure of a finite field, our method allows us to prove Conjecture 6.1.

Theorem 6.4. *Let k be an algebraically closed field of characteristic $\neq 2$. If k is not the algebraic closure of a finite field then Conjecture 6.1 holds for any finitely generated field K of transcendence degree n over k.*

Proof. By Bogomolov's trick (Corollary 1.2) we consider K as an extension K/K_0 of odd degree over a rational function field $K_0 = k(x_1, \ldots, x_n)$. By Theorem 3.3, the symbol $(x_1)\cdots(x_n) \in H^n(K_0, \mu_2^{\otimes n})$ is nontrivial over the (separable) quadratic extension $L_0 = K_0(\sqrt{f})$ for $f \in K$ defined by (3). Since K/K_0 and L/K_0 have relatively prime degree, $L = K \otimes_{K_0} L_0$ is a quadratic extension of K and L/L_0 has odd degree. Thus by a standard restriction-corestriction argument, the symbol $(x_1)\cdots(x_n)$ remains nontrivial when restricted from L_0 to L. By Proposition 3.2, it is unramified over L_0, hence it remains unramified over L. □

Remark 6.5. When k is the algebraic closure of a finite field of characteristic $\neq 2$, then Conjecture 6.1 holds for $n = 2$. Indeed, following the proof of Theorem 6.4, we only need to show that $(x_1)\cdot(x_2) \in H^2(K_0, \mu_2^{\otimes n})$ is nontrivial over the quadratic extension $L_0 = K_0(\sqrt{f})$, which follows from Remark 5.4.

Thus we have reduced Conjecture 6.1 to k the algebraic closure of a finite field. However, the construction of nontrivial higher degree unramified cohomology on varieties over a finite field (or the algebraic closure of a finite field) is an open problem. In degree 3, this is related to the integral Tate conjecture. Currently, there are no known smooth projective threefolds over a finite field with nontrivial unramified cohomology in degree 3; investigating this is a favorite problem of Colliot-Thélène, see [18, Question 5.4]. The smallest known dimensions in which such varieties exist is 5 (see [37]), and recently, 4 (see [40]). Of course, one wonders whether the cup product class on a product of three elliptic curves, as in §3, is nontrivial over a finite field. One might also investigate the same class on the associated generalized Kummer variety over a finite field.

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