



Local law and rigidity for unitary Brownian motion

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Received: 1 March 2022 / Revised: 16 February 2023 / Accepted: 16 August 2023 /

Published online: 25 September 2023

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Abstract

We establish high probability estimates on the eigenvalue locations of Brownian motion on the N -dimensional unitary group, as well as estimates on the number of eigenvalues lying in any interval on the unit circle. These estimates are optimal up to arbitrarily small polynomial factors in N . Our results hold at the spectral edges (showing that the extremal eigenvalues are within $O(N^{-2/3+})$ of the edges of the limiting spectral measure), in the spectral bulk, as well as for times near 4 at which point the limiting spectral measure forms a cusp. Our methods are dynamical and are based on analyzing the evolution of the Cauchy transform of the empirical spectral measure along the characteristics of the PDE satisfied by the limiting spectral measure, that of the free unitary Brownian motion.

Mathematics Subject Classification 60H05 · 22C05

1 Introduction and main results

In this work we study Brownian motion on the unitary group $\mathbb{U}(N)$ of dimension N . One can define Brownian motion on $\mathbb{U}(N)$ by considering the left-invariant Riemannian metric induced by the inner product $\langle A, B \rangle = N \operatorname{tr}(AB^*)$ on the Lie algebra of skew-Hermitian matrices. Unitary Brownian motion is then the Markov diffusion process starting from the identity matrix with generator given by the Laplacian on $\mathbb{U}(N)$ associated to this metric.

The research of A.A. is supported by NSF Grant DMS-2102842.
The work of B.L. is partially supported by NSERC.

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It will be more convenient for us to consider the following equivalent definition of U_t as the solution of the Itô stochastic differential equation,

$$dU_t = iU_t dW_t - \frac{1}{2}U_t dt, \quad U_0 = \mathbf{1} \quad (1.1)$$

where W_t is a standard complex Hermitian Brownian motion. That is, if X_t and X'_t are $N \times N$ matrices of independent standard Brownian motions, then,

$$W_t = \frac{1}{\sqrt{4N}} \left(X_t + X_t^T + i(X'_t - (X'_t)^T) \right). \quad (1.2)$$

The process (1.1) admits strong solutions by standard results (see, e.g., Theorem 8.3 of [35]).

Unitary Brownian motion is well-studied in random matrix theory as well as in the context of free probability due to its connection with an object called free unitary Brownian motion. Of particular interest is the empirical spectral measure of U_t , which is a random, time-dependent measure on the unit circle defined by,

$$d\nu_{N,t}(x) := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(t)}(x) dx. \quad (1.3)$$

In the work [10], Biane showed that for fixed t , the measure $\nu_{N,t}$ converges almost surely to a measure on the unit circle which we will denote by ν_t . Identifying the unit circle with the angular coordinates $\theta \in (-\pi, \pi]$, the measure ν_t has a density $\rho_t(\theta)$ for any $t > 0$. For $t < 4$ the support of ρ_t is given by,

$$I_t := [-\Theta_t, \Theta_t], \quad \Theta_t := \frac{1}{2}\sqrt{(4-t)t} + 2 \arcsin \left(\sqrt{\frac{t}{4}} \right), \quad (1.4)$$

whereas for $t \geq 4$, the support is the entire unit circle. Moreover, for $t \geq 4$ the density is everywhere non-zero unless $t = 4$ in which case ρ_t vanishes only at π . In fact, ρ_t can be described as the spectral measure of free unitary Brownian motion, an object appearing in free probability. The limit of the $\nu_{N,t}$ was also derived independently by Rains in [43].

Since Biane's paper, there have been many works studying the convergence of $\nu_{N,t}$ to ρ_t . Concentration estimates and convergence for the empirical averages $\int f d\nu_{N,t}$ were established by Kemp [31] for various classes of f of low regularity. Meckes and Melcher [42] established explicit convergence rates in terms of the L^1 -Wasserstein metric. For $0 < t < 4$, the convergence of the spectral edge of U_t to $\pm\Theta_t$ was established by Collins, Dahlqvist and Kemp [17]. This work also established a multi-time, multi-matrix version of this result. The asymptotic Gaussian fluctuations of the empirical averages of $\int f d\nu_{N,t}$ were established by Lévy and Maida [38]. Multivariate fluctuations for trace polynomials of a two parameter family of diffusion processes

(including unitary Brownian motion as a special case) were studied by Cébron and Kemp [13].

The main contribution of the present work is to establish almost-optimal rates (i.e., up to polynomial N^ε factors) of convergence of $\nu_{N,t}$ to the limiting distribution ρ_t on the almost-shortest possible scales, as well as almost-optimal estimates on the eigenvalue locations. In the random matrix literature, these estimates are known as *local laws* and *rigidity* estimates, respectively. Our local laws are stated in Theorem 1.2 and in Corollary 1.3 below, showing that the number of eigenvalues in any sub-interval I of the unit circle is given by $N\rho_t(I) + \mathcal{O}(N^\varepsilon)$ for any $\varepsilon > 0$.

For times $t < 4$ we establish almost-optimal rates of convergence of the spectral edge of U_t to $\pm\Theta_t$. That is, the extremal eigenvalues are within distance $\mathcal{O}(N^{-2/3+\varepsilon})$ of $\pm\Theta_t$ (what is usually termed *edge rigidity* in the literature). Given the square-root behavior of the spectral measure at the edges, this is expected to be optimal up to the N^ε factor. We also derive almost-optimal edge rigidity results up to $t = 4 - N^{-1/2+\varepsilon}$ at which point the measure ρ_t forms a cusp (i.e., vanishes like a cube root) near $\theta = \pm\pi$. This is expected to be optimal as at later times, the natural inter-particle distance at the spectral edges exceeds the distance between $+\Theta_t$ and $-\Theta_t$. Our rigidity estimates are formulated in Corollaries 1.5 and 1.7 below. The scaling behaviors in the various parameter regimes will be given as the results are introduced.

To our knowledge, our results are the strongest available estimates on eigenvalue locations for unitary Brownian motion. Our methods are completely different from prior works on rates of convergence to ρ_t , relying on the method of characteristics from PDEs. Previous works were based on moment calculations and/or concentration estimates for heat kernels on Lie groups.

1.1 Discussion of methodology

There are many approaches in the literature for proving local laws and rigidity in random matrix theory. For Wigner matrices and related mean-field random matrices W , a multi-scale approach to analyzing the resolvent $(W - z)^{-1}$ giving estimates down to the optimal scale was developed by Erdős, Schlein and Yau [21–23]. For pedagogical overviews of this strategy see [9, 26]. Earlier local laws for random matrices on short scales appeared in [8, 28].

While powerful, the resolvent method makes heavy use of the matrix structure and independence between different entries; while U_t is associated with a matrix process, the correlation structure between entries is complicated and renders this approach intractable. Instead, the process U_t somewhat resembles what is known as (Hermitian) Dyson Brownian motion, whose definition we now recall. In the work [19], Dyson showed that the eigenvalues of $V + W_t$ for any Hermitian V (and W_t as in (1.2)) obey the system of SDEs,

$$d\mu_i = \sqrt{\frac{2}{N\beta}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{1}{\mu_i - \mu_j} dt - \frac{1}{2} \mu_i dt, \quad (1.5)$$

where $\beta = 2$. Moreover, at the level of formal calculation, Dyson showed that the eigenvalues $\lambda_i(t)$ of the matrix U_t satisfy the closed system of SDEs,

$$d\lambda_i = \frac{1}{\sqrt{N}} i\lambda_i dB_i - \frac{1}{N} \sum_{j \neq i} \frac{\lambda_j \lambda_i}{\lambda_i - \lambda_j} dt - \frac{1}{2} \lambda_i dt, \quad (1.6)$$

where the $\{B_i(t)\}_{i=1}^N$ are a family of independent standard Brownian motions. We will not make use of this SDE, but mention that well-definedness of this process was studied in [14].

Given the similar form of these two processes, it is therefore useful to recall how rigidity has been established in the study of DBM (1.5) for general β . Given Dyson's original derivation of (1.5) we see that for the special values $\beta = 1, 2, 4$, this eigenvalue process in fact comes from a matrix-valued Brownian motion that can be thought of as an additively deformed Gaussian Orthogonal/Unitary/Symplectic ensemble. In the case that the initial data is the 0 matrix, this is just a scaled Gaussian matrix, and so the local laws and rigidity follow from those for Wigner matrices. For general initial data, local laws and rigidity were developed by Lee and Schnelli [36, 37] as well as the second author with Yau [33, 34] still using resolvent methods, as the additive structure inherent in DBM allows this approach to work. However, for general β , the process (1.5) is no longer naturally associated with any matrix process and so no resolvent methods can work.

Nonetheless, the second author with Huang [30] showed that local laws and rigidity in fact hold for DBM with general β as well as a general potential (the equation (1.5) being associated with quadratic $V'(\mu) = \frac{1}{2}\mu$). This was further developed by the first author with Huang to study the spectral edges [1]. The method in these works is our starting point and so we review it here.

The works [1, 30] were based on a PDE-style approach to studying the Stieltjes transform,

$$m(z, t) := \frac{1}{N} \sum_{i=1}^N \frac{1}{\mu_i(t) - z}, \quad (1.7)$$

of the empirical eigenvalue measure associated to the Hermitian DBM (1.5), on short scales $\text{Im}[z] \sim N^{-1}$. For $\beta = 2$, this quantity satisfies,

$$dm(z, t) = \left(m(z, t) + \frac{z}{2}\right) \partial_z m(z, t) dt + \frac{1}{2} m(z, t) dt + dN_t, \quad (1.8)$$

for some Martingale N_t which turns out to be lower-order. This leads to the limiting complex Burger's equation,

$$\partial_t \tilde{m}(z, t) = \left(\tilde{m}(z, t) + \frac{z}{2}\right) \partial_z \tilde{m}(z, t) + \frac{1}{2} \tilde{m}(z, t). \quad (1.9)$$

These equations were considered by Rogers and Shi [45] and general potential analogues by Li, Li and Xie [39, 40]. The complex Burger's equation may be solved by

the elementary method of characteristics. The works [1, 30] were based on tracking the difference between $\tilde{m}(z_t, t)$ and $m(z_t, t)$ along characteristics z_t . Note that for any fixed z , it is relatively straightforward to see that the limiting $m(z, t)$ must satisfy (1.9). The main challenge in [1, 30] is to extend this to short scales and obtain optimal error estimates.

The natural analogue of the Stieltjes transform $m(z, t)$ above for measures supported on the unit circle is the following transform,

$$f(z, t) := \frac{1}{N} \operatorname{tr} \left(\frac{U_t + z}{U_t - z} \right) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu_{N,t}(\theta). \quad (1.10)$$

We will refer to this as the “Cauchy transform” although the Cauchy transform is usually defined slightly differently [15] (the usual definition differs from ours only by an affine transformation). Our choice of f is to match the work of Biane [10, 11] discussed in more detail below.

An application of Itô’s formula starting from (1.1) shows that $f(z, t)$ obeys the SDE,

$$df(z, t) = -\frac{zf(z, t)}{2} \partial_z f(z, t) dt + dM_t(z) \quad (1.11)$$

where $M_t(z)$ is a complex-valued Martingale defined via,

$$dM_t(z) = -\frac{2iz}{N} \operatorname{tr} \left(\frac{U_t}{(U_t - z)^2} dW_t \right) = -\frac{2iz}{N} \sum_{i,j=1}^N \left[\frac{U_t}{(U_t - z)^2} \right]_{ij} d(W_t)_{ij}, \quad (1.12)$$

where the second equality explicitly writes out the trace in terms of the matrix elements of $U_t(U_t - z)^{-2}$ and the matrix of stochastic differentials of the Hermitian Brownian motion W_t . The covariation process of M_t is easily calculated,

$$\begin{aligned} \langle dM, dM \rangle &= -\frac{4z^2}{N^3} \operatorname{tr} \left(\frac{U_t^2}{(U_t - z)^4} \right) dt \\ \langle dM, d\bar{M} \rangle &= \frac{4|z|^2}{N^3} \operatorname{tr} \left(\frac{|U_t|^2}{|U_t - z|^4} \right) dt. \end{aligned} \quad (1.13)$$

Based on this, one sees that the limiting Cauchy transform should be the solution to,

$$\partial_t \tilde{f}(z, t) = -\frac{z\tilde{f}(z, t)}{2} \partial_z \tilde{f}(z, t), \quad \tilde{f}(z, 0) = \frac{1+z}{1-z}, \quad (1.14)$$

the analogue of the complex Burger’s equation (1.9). Indeed, these equations were found by Biane [10, 11], and the Cauchy transform $\tilde{f}(z, t)$ characterizes the limiting spectral measure ν_t . We collect properties of this measure and its density $\rho_t(\theta)$ in Appendix A.

In the present work, we will analyze the equation (1.11) through the characteristics associated to (1.14). That is, if $z(t)$ is a time-dependent curve in $\mathbb{C}$ satisfying,

$$\frac{d}{dt}z(t) = \frac{z(t)\tilde{f}(z(t), t)}{2}, \quad (1.15)$$

then,

$$\frac{d}{dt}\tilde{f}(z(t), t) = 0. \quad (1.16)$$

In fact, Biane's work [11] shows that for any t the map $z \rightarrow z(t)$ is a conformal map of some domain onto the complement of the unit circle in $\mathbb{C}$, which allows one to construct $\tilde{f}(z, t)$ from the characteristics.

At a superficial level, we are translating the methods of [1, 30] from the real line to the unit circle. However, the translation is not at all straightforward and there are serious obstacles to be overcome once one moves to our new setting. We briefly mention a few now; they will be presented in more detail below as we discuss our results. First, it is not a-priori clear that this method could even work in the first place. In particular, it is crucial that the Martingale term $M_t(z)$ above can be controlled by the empirical Cauchy transform $f(z, t)$ itself. The precise form of the quadratic variation of M_t is therefore important. Similar considerations hold for controlling the term $\partial_z f(z, t)$ by $f(z, t)$. In fact, getting precise constants here is crucial due to the use of Gronwall's inequality in our proof.

One of the new novelties of our work is to deal with times close to $t \approx 4$ where the spectral measure $\rho_t(\theta)$ forms a cusp singularity near $\theta = \pm\pi$. Local laws near a cusp are in general delicate (see e.g., [6, 16, 20]) and in the random matrix setting have not been dealt with via the characteristics approach before our work (but see the work [29] which studies non-intersecting random walks via characteristics in a different context). Secondly, the works [1, 30] dealt only with short times $t = o(1)$ whereas we are interested in times of order 1. Coupled with the curvature of the unit circle, this requires a more detailed understanding of the behavior of the characteristics, especially near the spectral edges and cusps than was required before. Here, we partially rely on the semi-explicit form of the spectral measure and Cauchy transform $\tilde{f}(z, t)$. Finally, the results of [1] make somewhat strong assumptions on the initial data. This was primarily due to the treatment of general potentials in that work, which allowed for an analysis of the movement of the spectral edge. Here, our initial data is a delta function, falling outside the assumptions of [1]. In particular, our short-time analysis is more involved.

The characteristic approach has appeared in a few other works on the short-scale behavior of eigenvalues. Bourgade used characteristics to analyze a “stochastic advection equation” derived from a certain coupling between DBMs and obtained fine estimates on the local eigenvalue behavior of general Wigner matrices, including universality of the extreme gaps [12]. Von Soosten and Warzel used random characteristics to prove local laws for Wigner matrices [49] and to study delocalization in the Rosenzweig–Porter model [48].

1.2 Statement of main results

Cauchy transforms of measures μ on the unit circle $f_\mu(z)$ are traditionally studied for z in the open unit disc, $\{|z| < 1\}$. Due to the identity,

$$f_\mu(re^{i\theta}) = -\overline{f_\mu}(r^{-1}e^{i\theta}), \quad (1.17)$$

this is equivalent to studying the behavior for $|z| > 1$, and it is somewhat conceptually simpler for our techniques to treat $|z| > 1$. Based on this, our first main result is the following, identifying the rate of convergence of the empirical spectral measure of unitary Brownian motion down to the optimal scale (up to polynomial factors). In order to state our results we introduce the notion of *overwhelming probability*.

Definition 1.1 If $\mathcal{A}_i$ are events indexed by some set $i \in \mathcal{I}$ (and may depend on N) then we say that the family of events $\mathcal{A}_i$ hold with overwhelming probability if for all $D > 0$ there is a $C > 0$ so that,

$$\sup_{i \in \mathcal{I}} \mathbb{P}[\mathcal{A}_i^c] \leq N^{-D}, \quad (1.18)$$

for all $N \geq C$.

Theorem 1.2 Let $T > 0$ and $\varepsilon, \delta, \varsigma > 0$. For any $0 < t < T$, we define the domain,

$$\mathcal{B}_t := \left\{ z \in \mathbb{C} : 5 > \log |z| > \frac{N^\delta}{N|\operatorname{Re}[\tilde{f}(z, t)]]} \vee N^{-\varsigma} \right\}. \quad (1.19)$$

Then, with overwhelming probability we have uniformly for all $0 < t < T$ and $z \in \mathcal{B}_t$ that,

$$|f(z, t) - \tilde{f}(z, t)| \leq \frac{N^\varepsilon}{N \log |z|}. \quad (1.20)$$

Let us now explain the connection between the Cauchy transform and the spectral measure. For a general probability measure μ on the unit circle, one may recover μ via the weak limits,

$$\begin{aligned} (2\pi)d\mu(\theta) &= \lim_{r \uparrow 1} \left(\operatorname{Re} \left[\int \frac{e^{i\theta'} + re^{i\theta}}{e^{i\theta'} - re^{i\theta}} d\theta' \right] \right) d\theta \\ &= - \lim_{r \downarrow 1} \left(\operatorname{Re} \left[\int \frac{e^{i\theta'} + re^{i\theta}}{e^{i\theta'} - re^{i\theta}} d\theta' \right] \right) d\theta. \end{aligned} \quad (1.21)$$

However, this is not useful in order to obtain effective estimates on, e.g., the number of eigenvalues in an interval. The Helffer–Sjöstrand formula [18] for measures on $\mathbb{R}$ allows one to relate empirical averages of test functions to integrals of the Stieltjes transform over $\mathbb{C}$, turning estimates on the Stieltjes transform into effective estimates

on the eigenvalues [9]. In Sect. 6.1 we quickly develop a version of the Helffer–Sjöstrand formula for measures on the unit circle (like the usual HS formula, it is a consequence of Green’s theorem). Similar formulas have appeared before in the literature [41], but the form given here is well-adapted to our purposes. Using this and the above theorem as input, we obtain the following.

Corollary 1.3 *Let $T > 0$ and $\varepsilon > 0$. With overwhelming probability, the following holds uniformly over all intervals $I \subset (-\pi, \pi]$, and all $0 < t < T$,*

$$\left| \left| \{ \lambda_i(t) = e^{i\theta_i(t)} : \theta_i \in I \} \right| - N \int_I \rho_t(\theta) d\theta \right| \leq N^\varepsilon + t^{-1/2} N^{-1/\varepsilon}, \quad (1.22)$$

for N large enough.

The above corollary shows that as long as $t \geq N^{-C}$ for some $C > 0$ then the number of eigenvalues in any interval is given by the limiting spectral measure up to an arbitrarily small polynomial error, with very high probability. This is the optimal scaling up to perhaps replacing the N^ε error by some sort of logarithmic error.

Theorem 1.2 is proven in Sect. 2, and Corollary 1.3 is derived in Sect. 6.2. This is the most straightforward of our results as it is the most literal translation of the methods of [1, 30] to the unitary setting. Nonetheless, estimates as sharp as those of Corollary 1.3 were not known before this work.

Our method relies on an application of Gronwall’s inequality to the the function $t \rightarrow f(z(t), t) - \tilde{f}(z(t), t)$ where $z(t)$ is a characteristic as above. The most delicate estimates are associated with estimating the martingale term $M_t(z(t))$ defined above, as well as the argument around (2.23) and (2.24). The integral form of Gronwall’s inequality we apply involves an exponential term which this latter argument estimates. Here, we cannot lose any constants or else the error term would be far too large to close our argument.

Recall that for $t < 4$ the support of ρ_t is not yet the entire unit circle and is instead the interval $I_t := [-\Theta_t, \Theta_t]$ where Θ_t is as in (1.4). The estimates of Theorem 1.2 are insufficient to address the natural question of whether or not there are eigenvalues outside of I_t , or their typical distance from the edges of I_t .

As mentioned above, Collins, Dahlqvist and Kemp [17] showed that with high probability there are no eigenvalues outside any open set containing I_t . However, such a statement does not yield the correct order of fluctuations of the extremal eigenvalues of U_t .

Before stating our results regarding the extremal eigenvalues, let us first ascertain what we expect for the order of magnitude of the fluctuations. For $\delta < t < 4 - \delta$, we show in Appendix A that for $E > 0$ sufficiently small,

$$\rho_t(\Theta_t - E) = c_t E^{1/2} (1 + \mathcal{O}(E)). \quad (1.23)$$

That is, the spectral measure vanishes like a square-root at the edges of its spectrum. The square-root behavior near spectral edges is generic in random matrix theory and is associated with limiting Tracy–Widom fluctuations on the order of $\mathcal{O}(N^{-2/3})$. While not explicitly formulated, we expect that the estimates of [17] in fact show that the

extremal eigenvalues of U_t are no more than $\mathcal{O}(N^{-c})$ from the edges of I_t for some small, explicit $c > 0$. However, the interparticle distance associated with the square-root behavior is $\mathcal{O}(N^{-2/3})$ and so such an estimate nonetheless falls short.

Our next result shows that with overwhelming probability, the extremal eigenvalues are in fact within $\mathcal{O}(N^{-2/3+\varepsilon})$ of the edges of the support I_t . These are analogues of the well-known rigidity results in random matrix theory at the edge (see, e.g., [27] for the first such estimates for generalized Wigner matrices). This is also the analogue of the results of [1] on the Hermitian DBM in the unitary setting.

Theorem 1.4 *Let $\delta > 0$ and $\varepsilon > 0$ be sufficiently small. With overwhelming probability, the following holds uniformly for all t satisfying $\delta < t < 4 - \delta$,*

$$\left| \left\{ i : \lambda_i(t) = e^{i\theta}, \theta \in [-\pi, \pi] \setminus [-\Theta_t - N^{-2/3+\varepsilon}, \Theta_t + N^{-2/3+\varepsilon}] \right\} \right| = 0, \quad (1.24)$$

and for all $0 \leq t \leq \delta$,

$$\left| \left\{ i : \lambda_i(t) = e^{i\theta}, \theta \in [-\pi, \pi] \setminus [-\Theta_t - N^{-\varepsilon/6}, \Theta_t + N^{-\varepsilon/6}] \right\} \right| = 0. \quad (1.25)$$

For short times $t \ll 1$, the measure ρ_t looks like an approximate semicircle centered at $\theta = 0$ of width $\sqrt{t}$ and $\rho_t(0) \asymp t^{-1/2}$. One therefore expects a different scaling of the interparticle distance for short times t . The error we obtain is not optimal for short times t , but this regime is not the main focus of our work and so we do not try to optimize our approach here.

Together with Corollary 1.3, we can then deduce the following rigidity estimates. To introduce them, we require some further notation. Note that for all times $t > 0$ the joint law of the eigenvalues of U_t has a density with respect to Haar measure, and so for each fixed time $t_0 > 0$, the eigenvalues are almost surely distinct. On the other hand, it follows from [14] that for distinct initial data, the solution to (1.6) exists as a strong solution for all times $t > t_0$ and moreover the eigenvalues do not intersect. Taking $t_0 \rightarrow 0$ it follows that with probability 1, the eigenvalues are distinct for all times $t > 0$. Since the eigenvalue locations are continuous functions of time and all start at the location $z = 1$ at $t = 0$ it follows that there is a labelling $\{\lambda_i(t)\}_{i=1}^N$ so that we can write $\lambda_i = e^{i\theta_i(t)}$ for continuous $\theta_i(t)$ starting at 0 and for all $t > 0$ satisfying,

$$\theta_1(t) < \theta_2(t) < \dots < \theta_N(t) < \theta_1(t) + 2\pi. \quad (1.26)$$

Note that it is possible for $\lambda_i(t) = e^{i\theta_i(t)}$ to wrap many times around the unit circle. E.g., each $\theta_i(t)$ can take any value in $\mathbb{R}$ such that the above ordering is respected. However, since the statement of Theorem 1.4 holds on an event of overwhelming probability for all t simultaneously, we can rule out any eigenvalues passing through the point $\theta = \pi$ on this event (this could also be concluded by passing from estimates holding on a sufficiently finely-spaced grid of times to all t using the Hoffman-Wielandt inequality). This observation allows us to formulate the following rigidity estimates.

Denote by $\gamma_i(t)$ the quantiles of ρ_t ,

$$\frac{i}{N} = \int_{-\pi}^{\gamma_i(t)} \rho_t(\theta) d\theta, \quad (1.27)$$

with the convention that γ_N is either the right spectral edge or π for $t \geq 4$. We have the following.

Corollary 1.5 *Let $\delta > 0$ and $\varepsilon > 0$. The following holds with overwhelming probability uniformly for all t satisfying $\delta < t < 4 - \delta$ and all $1 \leq i \leq N$. We have,*

$$|\theta_i(t) - \gamma_i(t)| \leq N^\varepsilon \frac{1}{N^{2/3} \min\{i^{1/3}, (N+1-i)^{1/3}\}}. \quad (1.28)$$

Theorem 1.4 is proven in Sect. 3. Corollary 1.5 follows in a straightforward manner from Corollary 1.3 and Theorem 1.4 and so we omit the proof (see, e.g., Section 3.3 of [30]).

Compared to the work [1], we encounter several new difficulties in our unitary setting in establishing these edge rigidity results. These mostly have to do with the fact that establishing the above results heavily depends on detailed analysis of the behavior of characteristics near the spectral edge. This behavior depends especially on the distance along the unit circle of characteristic from the location $\pm\Theta_t$ (the angular coordinate) as well as the distance from the unit circle. However, these coordinates introduce curvature whereas in the real line setting these are flat Cartesian coordinates of the real and imaginary part of the characteristic. This is further complicated by the fact that for short times t , the spectral measure is very peaked and that for long times t , the characteristics will leave the small neighbourhoods of the spectral edges for which we can develop expansions of $\tilde{f}(z, t)$.

We overcome the short-time difficulties mainly by sacrificing obtaining optimal estimates for short times; i.e., for short times we consider only characteristics that are somewhat far from the spectral edge. The second fact we use to overcome the difficulties associated with curvature and long times is monotonicity of the radial coordinate of the characteristics in time. This second fact is lacking for the general potential processes considered in [1, 30], and is one of the sources of the short-time restrictions in those works (“shocks” can develop for these processes). Essentially, monotonicity of the characteristics allows us to split the paths into “short” and “long” time regimes. In the short-time regimes, we can use a combination of analytic approaches to square-root measures and the semi-explicit formulas for $\tilde{f}(z, t)$ to control the behavior of characteristics close to spectral edges. In the long time regime, things are far away from the spectrum and so relevant quantities can usually be bounded by constants.

We now turn our attention to times $t \sim 4$. This is a critical time for unitary Brownian motion, as at $t = 4$, the two spectral edges $\pm\Theta_t$ merge, and the support of the density of states becomes the entire unit circle for later times t . In fact, as we show in Appendix A, the spectral measure at $t = 4$ has a cusp singularity,

$$\rho_4(\pi + E) = c|E|^{1/3}(1 + \mathcal{O}(|E|^{1/3})), \quad (1.29)$$

for some constant $c > 0$. Moreover, for times t near 4, the spectral measure undergoes a transition where the two edges gradually form a “near-cusp”, become a cusp, and then form a small local minimum as t ranges from slightly less than 4 to slightly larger than 4.

This behavior is identical to that found in the theory of the so-called quadratic vector equation. The quadratic vector equation is a generic equation characterizing the density of states of certain classes of mean field self-adjoint random matrix models. In a series of works [3–5], Ajanki, Erdős and Krüger carried out a systematic study of the solutions to the quadratic vector equation. In particular, they found that the only possible singularities that may occur are cusps, where the density of states vanishes like a cube root, and square roots, occurring at either external or internal edges. Moreover, they characterized transitional regimes where intervals of the density of states merge or split. In such regimes, they showed that the leading order of the density of states is always given by universal shape functions arising from Cardano’s formula for the roots of third-degree polynomials. There are two such functions; the first, Ψ_e corresponds to the case when two separate intervals merge and describe a transition from square-root to cubic behavior. The second, Ψ_m describes what occurs after the cusp forms, when the density of states has a small minimum.

In fact, in Appendix A we show that for times $t < 4$ that the density of states ρ_t of unitary Brownian motion is described by Ψ_e and for times $t > 4$ by Ψ_m , the universal shape functions of [3]. On the one hand, this is remarkable as there is no quadratic equation describing the density of states of unitary Brownian motion and moreover that the eigenvalues are on the unit circle instead of the real line. On the other hand, the universal shape functions arise from expansions of the Cauchy or Stieltjes transform near critical points (i.e., the spectral edges or minima) as soon as one is guaranteed that either the coefficients of the quadratic or cubic terms is non-degenerate and so this behavior is somewhat expected.

Theorems 1.2 and 1.4 above do not capture the behavior of the extremal eigenvalues in the case of the near-cusp, when times t are very close to 4. Using the formula for Θ_t above, we have for $t < 4$ that the gap between the two spectral edges scales like,

$$\Delta_t := 2(\pi - \Theta_t) = \frac{1}{3}(4 - t)^{3/2}(1 + \mathcal{O}(4 - t)). \quad (1.30)$$

By our calculations of ρ_t and the asymptotics of the shape function Ψ_e we have that in a vicinity of the edge, the density of states behaves like a re-scaled square-root,¹

$$\rho_t(\Theta_t - E) \asymp \frac{E^{1/2}}{(4 - t)^{1/4}} \asymp \frac{E^{1/2}}{\Delta_t^{1/6}}, \quad 0 \leq E \leq \Delta_t. \quad (1.31)$$

From the behavior of ρ_t it follows that the natural fluctuation scale of the extremal eigenvalues is $\Delta_t^{1/9} N^{-2/3}$. This is of the same order of magnitude as Δ_t when $4 - t = N^{-1/2}$. It follows that for $t \ll 4 - N^{-1/2}$ one expects that the extremal eigenvalues are

¹ The notation $\asymp$ means that for two positive possibly N -dependent quantities, $a_N \asymp b_N$ implies that there is a constant $C > 0$ so that $C^{-1}a_N \leq b_N \leq Ca_N$.

still located near their respective edges. For larger t , the fluctuations of the extremal eigenvalues is larger than Δ_t and so no such rigidity estimate is expected. The first statement is the content of the following theorem.

Theorem 1.6 *Let $\delta > 0$ and let $\varepsilon > 0$. With overwhelming probability the following holds. Uniformly for all t satisfying $2 < t < 4 - N^{-1/2+\delta}$ we have that,*

$$\left| \left\{ i : \lambda_i(t) = e^{i\theta}, \theta \in [-\pi, \pi] \setminus [-\Theta_t - N^{-2/3+\varepsilon} \Delta_t^{1/9}, \Theta_t + \Delta_t^{1/9} N^{-2/3+\varepsilon}] \right\} \right| = 0 \quad (1.32)$$

The above theorem is proven in Sect. 4. In principle, its proof could be absorbed into the proof of Theorem 1.4. However, handling the three separate scaling regimes, when t is small, of intermediate size, and close to 4, would require significant additional notation and burden the reader by overly complicating the proofs. We have chosen instead to treat the “short” and “long” time regimes separately; in fact we will use the result of Theorem 1.4 in our proof of Theorem 1.6, initializing the dynamics at an intermediate time $0 \ll t \ll 4$, conditional on the results of Theorem 1.4 holding.

Local laws and rigidity for random matrices exhibiting cusps were established in [20] using resolvent methods. The work [16] also establishes rigidity results for certain interpolating ensembles using a dynamical, PDE-based approach not related to our approach, although both works study the formation of cusps under eigenvalue dynamics.

The main obstacle in proving the above theorem is to understand how the cusp scaling affects the behavior of the characteristics. In particular, we must understand how the angular and radial coordinates of the characteristics are affected by this new scaling. Luckily, in the regime where we expect to prove edge rigidity, there is still a small interval where the density of states behaves like a square root, albeit rescaled by a factor involving Δ_t . The characteristics relevant to edge rigidity start close to the spectral edge, and some of our calculations of square-root behavior in the earlier short-time regime of Theorem 1.4 are applicable here, after finding appropriate re-scalings by Δ_t of the angular and radial characteristic coordinates. Nonetheless, we still need to handle the behavior of the characteristics for times of order 1, and so more analysis of the characteristics is required. This is further complicated by the curvature of our coordinate system as well as the fact that the scaling factor Δ_t is itself time-dependent and will in general differ by several orders of magnitude over the time intervals we consider.

We can use Theorem 1.6 together with Corollary 1.3 to deduce the following rigidity estimates, in a similar manner to Corollary 1.5. We omit the proof.

Corollary 1.7 *Let $\delta > 0$ and $\varepsilon > 0$. For $N^{-1/2+\delta} \leq 4 - t \leq 10^{-1}$ we have that the following estimates hold with overwhelming probability. Uniformly for all i satisfying $1 \leq i \leq N(4 - t)^2$ we have,*

$$|\theta_i(t) - \gamma_i(t)| \leq N^\varepsilon (4 - t)^{1/6} \frac{1}{N^{2/3} i^{1/3}} \quad (1.33)$$

and for $N(4-t)^2 \leq i \leq N/2$ we have,

$$|\theta_i(t) - \gamma_i(t)| \leq N^\varepsilon \frac{1}{N^{3/4} i^{1/4}}. \quad (1.34)$$

Analogous estimates hold for indices i near N .

Remark The reason for the two regimes of indices less than or greater than $N(4-t)^2$ is due to the behavior of the limiting spectral measure near $-1 = e^{i\pi}$ for times t close to 4. With $s = 4 - t$, we have that $\rho_t(\Theta_t - E) \asymp E^{1/2}/s^{1/4}$ for $E \leq s^{3/2}$ and $\rho_t(\Theta_t - E) \asymp E^{1/3}$ for $0.1 \geq E \geq s^{3/2}$ (see Proposition A.5 and (A.28)). The form of the RHS of the estimates (1.33) and (1.34) reflect the different interparticle distances in each of these regimes. $\square$

For larger times $t \gtrsim 4 - N^{-1/2}$, the optimal estimates for the Cauchy transform are in fact included in Theorem 1.2; compare with, e.g., the local laws of [20]. Note that Theorem 1.2 alone is insufficient to conclude rigidity estimates similar to Corollaries 1.5 or 1.7. In particular, the above results cannot rule out the case that after time $t \sim 4$, that all of the eigenvalues wrap around the unit circle many times.

However, due to the fact that $\theta_N(t)$ and $\theta_1(t)$ cannot cross, the “winding number” (we use this term loosely) can be determined from the behavior of the center of mass,

$$\bar{\theta}(t) := \frac{1}{N} \sum_{i=1}^N \theta_i(t). \quad (1.35)$$

From either (1.1) or (1.6) one can check that *formally* $d\bar{\theta} = N^{-1}dB$ for a Brownian motion B . In Sect. 5 we justify this using a careful application of the analytic functional calculus.

Proposition 1.8 *For any $t > 0$ we have almost surely that,*

$$\bar{\theta}(t) = \frac{1}{N} \text{tr}(W_t) \quad (1.36)$$

where W_t is the standard complex Hermitian Brownian motion in (1.1).

We expect that a version of the above statement could also be deduced from Lemma 5 of [42] but we provide a direct proof, aspects of which may be useful in other settings.

This allows us to deduce the following. Let us denote the extended quantiles $\tilde{\gamma}_i(t)$ of $\rho_i(t)$ as follows. For $1 \leq i \leq N$ we let $\tilde{\gamma}_i(t) = \gamma_i(t)$. For $i > N$ we

$$\tilde{\gamma}_i(t) = 2\pi + \gamma_{i-N}(t) \quad (1.37)$$

and for $i < 1$ we let

$$\tilde{\gamma}_i(t) = -2\pi + \gamma_{i+N}(t). \quad (1.38)$$

Corollary 1.9 *Let $\varepsilon > 0$ and $\delta > 0$. With overwhelming probability we have uniformly for all t satisfying $\delta < t < \delta^{-1}$ that,*

$$\tilde{\gamma}_{i-N^\varepsilon}(t) \leq \theta_i(t) \leq \tilde{\gamma}_{i+N^\varepsilon}(t). \quad (1.39)$$

Note that this is weaker for the edge eigenvalues for times $t < 4 - N^{-1/2+\varepsilon}$ than Corollaries 1.5 and 1.7, and is only useful in the regime where we can no longer rule out the existence of eigenvalues in the gap between the spectral edges, or there is no longer a gap. Corollary 1.9 is proven in Sect. 6.3.

1.3 Further discussion and motivation

In addition to being an intrinsic question about the nature of the process U_t , these local law and rigidity estimates have been well studied in the context of Hermitian random matrix theory. There, a primary motivation is the study of the universality of the local eigenvalue statistics: that is, whether or not the limiting local eigenvalue statistics coincide with those of the Gaussian ensembles, which admit exact formulas.

The short scale behaviors of the repulsive interaction terms of the eigenvalue process of the Hermitian and Unitary Brownian motions (1.5) and (1.6) are similar. Based on this and the general belief in the universality of large correlated systems, it is natural to conjecture that the local eigenvalue statistics of (1.6) should be given by the same statistics as the GUE in the limit $N \rightarrow \infty$.

Indeed, there have been many developments in the universality theory of the local eigenvalue statistics both within the larger context of random matrix theory, as well as that of the Hermitian Dyson Brownian motion started from general initial data. Note that if the initial data of (1.5) is not the 0 matrix, then the joint eigenvalue distribution of X_t is no longer that of a re-scaled GUE. Nonetheless, local scaling limits of DBM with general initial data has been obtained in [25, 32–34].

Universality has been established for wide classes of Hermitian random matrices. We refer the interested reader to, e.g., the book [26] for an overview of these developments as well as to the seminal papers of Tao and Vu [46, 47], and Erdős, Schlein and Yau [24].

Given these advances in the theory of Hermitian random matrices, it is natural to turn to the question of universality of unitary Brownian motion. An important tool in many of the proofs of Hermitian universality are the aforementioned rigidity and local law estimates.

The main contribution of the present work is to establish these results. It is then a subject of current investigation to use these estimates to prove the local universality of unitary Brownian motion: that the local eigenvalues statistics of U_t are given by the Tracy-Widom, and Pearcey and Sine kernels in the various scaling regimes of interest.

1.4 Notation

The notion of overwhelming probability was defined above in Definition 1.1. We let $c > 0$ and $C > 0$ denote small and large constants respectively. In general, we allow

them to increase or decrease from line to line. For two positive N -dependent quantities (or quantities depending on some auxiliary parameters, usually time t) the notation $a_N \asymp b_N$ means that there is a constant $C > 0$ so that $C^{-1}a_N \leq b_N \leq Ca_N$. The notation $a_N \ll b_N$ means $a_N/b_N \rightarrow 0$ as $N \rightarrow \infty$. We will use this notation sparingly, but somewhat informally. When used we always have an explicit estimate, e.g., $a_N \leq b_N/\log(N)$. For complex c_N and d_N , the notation $c_N = \mathcal{O}(d_N)$ means $|c_N| \leq C|d_N|$ for some $C > 0$.

1.5 Organization of paper

Sections 2, 3 and 4 are meant to be read in a relatively linear fashion. These sections prove Theorems 1.2, 1.4 and 1.4, respectively. The analysis in each section directly builds off that of the previous section. Section 2 treats the “bulk” local law (i.e., a local law with an error that is optimal only in the bulk) and the treatment of the characteristics is relatively straightforward. Section 3 treats the cases $t < 4 - \delta$ where ρ_t has a regular square-root edge. Here, the treatment of characteristics (and the resulting estimates of the quantities such as the Martingale term in evolution equation of $f(z, t)$) is more complicated. Finally, Sect. 4 deals with the formation of the cusp.

In the short Sect. 5 we prove Proposition 1.8, that the centre of mass, or averaged winding number, is described by a Brownian motion. In Sect. 6 we establish the analog of the Helffer–Sjöstrand formula and use it to deduce Corollary 1.3. The latter is very similar to what has appeared in [30] and so not all details are provided.

In Appendix A we establish various properties of the limiting spectral measure ρ_t . We use as input its characterization in terms of conformal maps of Biane [11], as well as arguments of Ajanki, Erdős and Krüger [3] involving solutions of Cauchy/Stieltjes transforms of approximate cubic equations, Cardano’s formula and the universal shape functions Ψ_m and Ψ_e . Finally, Appendix B collects various calculus-type inequalities used in the proof.

This is a shortened version prepared for publication in PTRF of the original manuscript. A longer version, containing all of the proofs omitted here, appears on the arXiv as [arXiv:2202.06714v3](https://arxiv.org/abs/2202.06714v3), i.e., as the third version v3, and is referenced in the current work as [2]. Whenever we omit a proof in the present manuscript, we precisely reference its location in [2].

2 Bulk estimates

In this section we prove Theorem 1.2. Fix a final time $T > 0$. This time may depend on N , but stays bounded above. We introduce the characteristic maps via

$$\mathcal{C}_t(z) = z \exp \left[-\frac{(T-t)\tilde{f}(z, T)}{2} \right]. \quad (2.1)$$

That is, the function $t \rightarrow \mathcal{C}_t(z)$ satisfies the characteristic equation (1.15) and has final condition $\mathcal{C}_T(z) = z$.

From the above, it is clear that the real parts of $f(z, t)$ and $\tilde{f}(z, t)$ will play important roles. We record here the identity,

$$\operatorname{Re}[f(z, t)] = \frac{1}{N} \sum_{i=1}^N \frac{1 - |z|^2}{|\lambda_i - z|^2}. \quad (2.2)$$

We will also have use for,

$$\partial_z f(z, t) = \frac{2}{N} \sum_{i=1}^N \frac{\lambda_i(t)}{(\lambda_i(t) - z)^2}, \quad (2.3)$$

and the inequality

$$|\partial_z f(z, t)| \leq \frac{2}{1 - |z|^2} \operatorname{Re}[f(z, t)]. \quad (2.4)$$

In the remainder of the section we will also make use of the spectral domains $\mathcal{B}_t$ that were defined above in (1.19). We collect some elementary properties of the characteristics.

Lemma 2.1 *For any $|z| > 1$, the map,*

$$t \rightarrow \log |\mathcal{C}_t(z)| \quad (2.5)$$

is decreasing in time. Secondly, there is a constant $C > 0$ so that for any $z \in \mathcal{B}_T$ we have,

$$|\mathcal{C}_t(z)| \leq C e^{CT}. \quad (2.6)$$

Proof The first claim follows from the fact that $\operatorname{Re}[\tilde{f}(z, t)] < 0$ for $|z| > 1$. For the second, let $z_t = \mathcal{C}_t(z)$. Note that $\tilde{f}(z_t, t) = \tilde{f}(z, T)$ for all t . If at any time t we have $|z_t| > 2$, then $|\tilde{f}(z_t, t)| \leq 4$ and so $|\mathcal{C}_t(z)| \leq |z|e^{2T}$. So either $|z_t| < 2$ for all t or $|z_t| \leq |z|e^{2T}$ for all t . This yields the claim. $\square$

We fix a final point $z \in \mathcal{B}_T$. We will first prove that Theorem 1.2 holds at a single z by tracking the evolution of $f(z, t)$ along the characteristic,

$$z_t := \mathcal{C}_t(z). \quad (2.7)$$

The extension to all z and all $0 < t < T$ will be detailed later. We now introduce the stopping time τ via,

$$\tau := \inf \left\{ s \in [0, T] : |f(z_s, s) - \tilde{f}(z_s, s)| > \frac{N^\varepsilon}{N \log |z_s|} \right\} \wedge T \quad (2.8)$$

where we choose $\varepsilon < \delta/10$, with δ as in the definition of $\mathcal{B}_T$. We first note that,

$$d\left(f(z_t, t) - \tilde{f}(z_t, t)\right) = -\frac{z_t \partial_z f(z_t, t)}{2} \left(f(z_t, t) - \tilde{f}(z_t, t)\right) + dM_t(z_t), \quad (2.9)$$

with the Martingale term defined above in (1.12). Hence,

$$f(z_\tau, \tau) - \tilde{f}(z_\tau, \tau) = \mathcal{E}_1(\tau) + \mathcal{E}_2(\tau) \quad (2.10)$$

where

$$\mathcal{E}_1(t) := -\int_0^t \frac{z_s \partial_z f(z_s, s)}{2} (f(z_s, s) - \tilde{f}(z_s, s)) ds \quad (2.11)$$

and

$$\mathcal{E}_2(t) := \int_0^t dM_s(z_s). \quad (2.12)$$

We first prove the following estimate on the martingale term.

Proposition 2.2 *For all $\varepsilon_1 > 0$ we have,*

$$\mathbb{P}\left[\exists t \in [0, \tau] : |\mathcal{E}_2(t)| > \frac{N^{\varepsilon_1}}{N \log |z_t|}\right] \leq C \log(N) e^{-cN^{\varepsilon_1}}. \quad (2.13)$$

Proof We fix a sequence of intermediate times t_k with $k = 1, \dots, M$ in $[0, T]$ such that $\log |z_{t_k}| \leq 2 \log |z_{t_{k+1}}|$ and $t_M = T$. Then $M \leq C \log(N)$ for some $C > 0$ since $\log |z_0|$ is bounded by Lemma 2.1. Let $\tau_k = \tau \wedge t_k$. The quadratic variation of $\mathcal{E}_2(\tau_k)$ satisfies,

$$\begin{aligned} \langle \mathcal{E}_2(\tau_k), \bar{\mathcal{E}}_2(\tau_k) \rangle &= \frac{4}{N^2} \int_0^{\tau_k} |z_s|^2 \frac{1}{N} \sum_{i=1}^N \frac{1}{|\lambda_i(s) - z_s|^4} ds \\ &\leq \frac{4}{N^2} \int_0^{\tau_k} \frac{|z_s|^2}{(|z_s| - 1)^2} \frac{1}{N} \sum_{i=1}^N \frac{1}{|\lambda_i(s) - z_s|^2} ds \\ &\leq \frac{C}{N^2} \int_0^{\tau_k} \frac{|\operatorname{Re}[f(z_s, s)]|}{(\log |z_s|)^3} ds. \end{aligned} \quad (2.14)$$

In the first inequality we used the trivial estimates $|\lambda_i(s) - z_s| \geq |z_s| - 1$. In the last inequality we used Lemma 2.1 to bound $|z_s|$ as well as the representation (2.2). We also used that $\log(r) \leq r - 1$ for $r > 1$. For $s < \tau$ we have

$$|\operatorname{Re}[f(z_s, s)] - \operatorname{Re}[\tilde{f}(z_s, s)]| \leq \frac{N^\varepsilon}{N \log |z_s|} \leq \frac{N^\varepsilon}{N \log |z_T|}. \quad (2.15)$$

By definition of $\mathcal{B}_T$ we see that

$$\frac{N^\varepsilon}{N \log |z_T|} \leq N^{\varepsilon-\delta} |\operatorname{Re}[\tilde{f}(z_T, T)]| = N^{\varepsilon-\delta} |\operatorname{Re}[\tilde{f}(z_s, s)]| \quad (2.16)$$

where we used that $\tilde{f}$ is constant along characteristics. Since $\varepsilon < \delta$ we therefore see that,

$$|\operatorname{Re}[f(z_s, s)]| \leq (1 + N^{\varepsilon-\delta}) |\operatorname{Re}[\tilde{f}(z_s, s)]| \leq \left(1 + \frac{1}{\log(N)}\right) |\operatorname{Re}[\tilde{f}(z_s, s)]| \quad (2.17)$$

for $s < \tau$. Therefore,

$$\begin{aligned} \int_0^{\tau_k} \frac{|\operatorname{Re}[f(z_s, s)]|}{(\log |z_s|)^3} ds &\leq 2 \int_0^{\tau_k} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{(\log |z_s|)^3} ds \leq 2 \int_0^{\tau_k} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{(\log |z_s|)^3} ds \\ &\leq \frac{2}{(\log |z_{t_k}|)^2}. \end{aligned} \quad (2.18)$$

In the last inequality we used that $\partial_u \log |z_u| = -\frac{1}{2} |\operatorname{Re}[\tilde{f}(z_u, u)]|$. By the Burkholder-Davis-Gundy (BDG) inequality (see, e.g., [44]) and a union bound, we conclude that,

$$\mathbb{P} \left[\exists k : \sup_{0 \leq t \leq \tau_k} |\mathcal{E}_2(t)| > \frac{N^{\varepsilon_1}}{2N \log |z_{t_k}|} \right] \leq C \log(N) e^{-cN^{\varepsilon_1}}. \quad (2.19)$$

Let $0 < t < \tau$ and let k be such that $\tau_k \leq t < \tau_{k+1}$. On the complement of the event on the LHS of (2.19) we have,

$$|\mathcal{E}_2(t)| \leq \frac{N^{\varepsilon_1}}{2N \log |z_{t_k}|} \leq \frac{N^{\varepsilon_1}}{N \log |z_t|} \quad (2.20)$$

by the choice of the t_k 's. This completes the proof. $\square$

Proof of Theorem 1.2 Let $z \in \mathcal{B}_T$ with characteristic z_t as above. Let τ be the stopping time as defined above. Let Υ be the event of Proposition 2.2 and define,

$$g(s) := \frac{|z_s \partial_z f(z_s, s)|}{2}. \quad (2.21)$$

By Gronwall's inequality we have for all $0 < t < \tau$ on the event Υ ,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq \int_0^t g(s) \exp \left[\int_s^t g(u) du \right] \frac{N^{\varepsilon_1}}{N \log |z_s|} ds + \frac{N^{\varepsilon_1}}{N \log |z_t|}. \quad (2.22)$$

We now estimate the integral of the function g that appears above in the argument of the exponential function. Using first (2.4) we have,

$$\begin{aligned} \int_s^t g(u) du &= \int_s^t \frac{|z_u| |\partial_z f(z_u, u)|}{2} du \leq \int_s^t \frac{|z_u| |\operatorname{Re}[f(z_u, u)]|}{|z_u|^2 - 1} du \\ &\leq \int_s^t \frac{|\operatorname{Re}[f(z_u, u)]|}{2 \log |z_u|} du. \end{aligned} \quad (2.23)$$

In the last inequality we used the elementary inequality,

$$\frac{x}{x^2 - 1} \leq \frac{1}{2 \log(x)} \quad (2.24)$$

which holds for $x > 1$ (see Lemma B.1). Using now (2.17) we have,

$$\begin{aligned} \int_s^t \frac{|\operatorname{Re}[f(z_u, u)]|}{2 \log |z_u|} du &\leq \left(1 + \frac{1}{\log(N)}\right) \int_s^t \frac{|\operatorname{Re}[\tilde{f}(z_u, u)]|}{2 \log |z_u|} du \\ &= \left(1 + \frac{1}{\log(N)}\right) \log \left(\frac{\log |z_s|}{\log |z_t|} \right) \end{aligned} \quad (2.25)$$

where we used in the last line that $\partial_u \log |z_u| = -\frac{1}{2} |\operatorname{Re}[\tilde{f}(z_u, u)]|$. Therefore,

$$\exp \left[\int_s^t g(u) du \right] \leq C \frac{\log |z_s|}{\log |z_t|}. \quad (2.26)$$

We used the fact that the assumption $C \geq \log |z_u| \geq N^{-c}$ from the definition of $\mathcal{B}_T$ implies that

$$\left(\frac{\log |z_s|}{\log |z_t|} \right)^{1/\log(N)} \leq C. \quad (2.27)$$

Substituting (2.26) into (2.22) yields,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq \frac{CN^{\varepsilon_1}}{N \log |z_t|} \int_0^t g(s) ds + \frac{N^{\varepsilon_1}}{N \log |z_t|}. \quad (2.28)$$

From the definition of g and (2.4) and (2.17) we see that,

$$g(s) \leq C \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\log |z_s|} \quad (2.29)$$

for $s < \tau$. Therefore,

$$\int_0^t g(s) ds \leq C \log(N) \quad (2.30)$$

using that $\log |z_t| \geq N^{-\epsilon}$. We conclude that for all $0 < t < \tau$ that,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq C \frac{\log(N) N^{\epsilon_1}}{N \log |z_t|}. \quad (2.31)$$

Taking $\epsilon_1 < \epsilon/2$, where $\epsilon > 0$ is in the definition of the stopping time, we see that on the event Υ , we must have $\tau = T$ for all N large enough. This proves the estimate of Theorem 1.2 at the final time T and at the point z . The extension to all points $z \in \mathcal{B}_T$ may be done by first taking a union bound over $\mathcal{O}(N^C)$ points and then using

$$|f_\mu(z) - f_\mu(w)| \leq C|z - w| \left(\frac{1}{\log |z|} + \frac{1}{\log |w|} \right)^2 \quad (2.32)$$

valid for any Cauchy transform and $|z|, |w|$ in bounded regions of $\mathbb{C}$. The above estimate is elementary and can be proven directly from the definition

$$f_\mu(z) := \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)$$

and the fact that for $|z| > 1$ we have for the quantity in the denominator $|e^{i\theta} - z| \geq |z| - 1 \geq c \log |z|$.

The extension to all times $t < T$ is done in a similar manner; one first proves the estimate for all $z \in \mathcal{B}_t$ for all times t in a well-spaced grid of $[0, T]$ of at most size $\mathcal{O}(N^A)$, some $A > 0$ to be determined. Then one notes that the proof also gives that the estimate holds along the entire characteristic, and along each characteristic, we have, e.g., $|z_t - z_s| \leq CN^C |t - s|$, for some $C > 0$ depending on $c > 0$. We need only take A large enough depending on $C > 0$. This completes the proof of Theorem 1.2. $\square$

3 Edge estimates

Fix $T < 4$. We will need to consider characteristics ending at many times and so we introduce the characteristic map,

$$C_{s,t}(z) = z \exp \left[-\frac{(t-s)\tilde{f}(z,t)}{2} \right] \quad (3.1)$$

for $0 \leq s \leq t$. We will need to establish several properties about the behavior of characteristics near the edge Θ_t . The proof of the following lemma, an exercise in calculus, and can be found in Appendix B of [2].

Lemma 3.1 *Let ρ be a measure on $[-\pi, \pi]$ such that $\rho(\theta) = \rho(-\theta)$ that is supported in $[-E, E]$ for $0 < E < \pi$. Assume either that $\rho(\theta) \leq M$ or $E < \pi/8$. For any $\epsilon > 0$ there is a $c_\epsilon > 0$, depending only on E, M and $\epsilon > 0$ so that for $0 < r - 1 < c_\epsilon$ and*

$E < \theta < \pi - \varepsilon$ we have,

$$\operatorname{Im}[f(re^{i\theta})] < \operatorname{Im}[f(e^{i\theta})] < \operatorname{Im}[f(e^{iE})] \quad (3.2)$$

where $f(z)$ is the Cauchy transform of ρ . We also have for $E < \theta \leq \varphi \leq \pi$,

$$0 \leq \operatorname{Im}[f(e^{i\varphi})] \leq \operatorname{Im}[f(e^{i\theta})], \quad (3.3)$$

with equality occurring only in trivial cases.

We now prove the following.

Lemma 3.2 *Let $T < 4$ and $z = re^{i(\Theta_t + \kappa)}$ for $\kappa > 0$. There is a small $c > 0$ and $d > 0$ so that,*

$$\operatorname{Im}[\tilde{f}(z, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)] \leq -d\sqrt{\kappa} \quad (3.4)$$

for $0 < t < T$ and all $0 < r - 1 < c$ and $\kappa < \pi - \Theta_t$.

Proof By Lemma B.3, the desired estimate hold for all $\kappa < \varepsilon$ and $0 < r - 1 < \varepsilon$ for some $\varepsilon > 0$. In particular,

$$\operatorname{Im}[\tilde{f}(e^{i(\Theta_t + \varepsilon/2)}, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)] \leq -c_1 \quad (3.5)$$

for some $c_1 > 0$. Since for $t < T$ the measures are supported away from π we see that there is a $\delta > 0$ so that

$$|\operatorname{Im}[\tilde{f}(z, t)] - \operatorname{Im}[\tilde{f}(-1, t)]| \leq \frac{c_1}{2}, \quad (3.6)$$

for $|z + 1| < \delta$. It follows that for all $|z + 1| < \delta$,

$$\operatorname{Im}[\tilde{f}(z, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t})] \leq \frac{c_1}{2} + \operatorname{Im}[\tilde{f}(e^{i(\Theta_t + \varepsilon/2)}, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t})] \leq -\frac{c_1}{2} \quad (3.7)$$

where we used the second estimate of Lemma 3.1. This proves the desired estimate for $|z + 1| < \delta$ after possibly decreasing the value of $d > 0$.

On the other hand, by Lemma 3.1 we conclude that there is a $c_2 > 0$ so that for $0 < r - 1 < c_2$ and $\varepsilon/2 < \kappa < \pi - \Theta_t - \delta/2$ that,

$$\operatorname{Im}[\tilde{f}(re^{i(\Theta_t + \kappa)}, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)] \leq \operatorname{Im}[\tilde{f}(e^{i(\Theta_t + \varepsilon/2)}, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t})] \leq -c_1. \quad (3.8)$$

This concludes the proof. $\square$

The following contains the properties of the characteristics that we will need.

Proposition 3.3 *Let $T < 4$. There is a $\alpha > 0$ and $\mathfrak{b} > 0$ depending on T so that the following holds. Let $z_s = C_{s,t}(z)$ for any $t < T$, where $z = re^{i\theta}$ satisfies $0 < r - 1 < \alpha$ and $\Theta_t < \theta < \pi$. Denote $z_s = r_s e^{i(\Theta_s + \kappa_s)}$. Let s_* be,*

$$s_* = \inf\{s < t : (r_s - 1) < \alpha\}. \quad (3.9)$$

Then for $s_ < s < t$ we have,*

$$\sqrt{\kappa_s} \geq \sqrt{\kappa_t} + \mathfrak{b}(t - s) \quad (3.10)$$

and for $s \leq s_$ we have $r_s \geq 1 + \alpha$. Furthermore, let $D(s)$ be a function that obeys,*

$$\sqrt{D(s)} \leq \sqrt{D(t)} + \frac{\mathfrak{b}}{2}(t - s), \quad s < t. \quad (3.11)$$

If $\kappa_t \geq D(t)$ then for $s_ < s < t$ we have,*

$$\kappa_s \geq D(s) + \frac{\mathfrak{b}}{2}\sqrt{\kappa_t}(t - s). \quad (3.12)$$

Finally, characteristics do not cross the real axis in the complex plane.

Proof We take α so small so that the conclusion of Lemma 3.2 for $z = re^{i\theta}$ with $r - 1 < 10\alpha$. For $s < s_*$ we have $r_s > \alpha$ because the radial coordinate is decreasing in time. Let s_1 be,

$$s_1 = \inf\{s < t : \kappa_s > 0\}. \quad (3.13)$$

Note that $s_1 < t$ as we assume $\kappa_t > 0$. We claim that $s_1 \leq s_*$. For $t > s > s_1 \vee s_*$ we have, the following calculation,

$$\begin{aligned} \partial_s \kappa_s &= \frac{1}{2} \operatorname{Im}[\tilde{f}(z_s, s)] - \frac{1}{2} \operatorname{Im}[\tilde{f}(e^{i\Theta_s}, s)] \\ &= \frac{1}{2} \operatorname{Im}[\tilde{f}(z_t, t)] - \frac{1}{2} \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)] - \frac{1}{2} \left(\sqrt{\frac{4-s}{s}} - \sqrt{\frac{4-t}{t}} \right) \\ &\leq \frac{1}{2} [\operatorname{Im}[\tilde{f}(z_t, t)] - \frac{1}{2} \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)]] - c(t - s) \\ &\leq -d\sqrt{\kappa_t} - c(t - s) \end{aligned} \quad (3.14)$$

The second line is from the fact that $\tilde{f}(e^{i\Theta_t}, t) = i\sqrt{4t^{-1} - 1}$ for all t and that $\tilde{f}$ is constant along characteristics. The third line is straightforward. The last line follows from Lemma 3.2. In particular, we see that κ_s is a decreasing function for $s > s_* \vee s_1$. Since $\kappa_t > 0$ it follows that $s_1 \leq s_*$. Therefore, the final inequality in (3.14) holds for all $s_* < s < t$. Therefore via integration we obtain,

$$\kappa_s \geq \kappa_t + 2c_2(t - s)\sqrt{\kappa_t} + c_2(t - s)^2 \quad (3.15)$$

for some small $c_2 > 0$ for all $s_* < s < t$. This is equivalent to the desired inequality. For the last inequality of the proposition, we have that

$$\sqrt{\kappa_s} \geq \sqrt{\kappa_t} + \mathfrak{b}(t-s) \geq \sqrt{D(t)} + \mathfrak{b}(t-s) \geq \sqrt{D(s)} + \frac{\mathfrak{b}}{2}(t-s). \quad (3.16)$$

We can square this to get $\kappa_s \geq D(s) + \mathfrak{b}(t-s)\sqrt{D(s)}$ as well as $\kappa_s \geq \kappa_t + 2\mathfrak{b}(t-s)\sqrt{\kappa_t}$. If $D(s) \geq \kappa_t$, we can use the first inequality. Otherwise, we can use the second.

The final claim that characteristics not cross the real axis is due to the fact that the imaginary part of $\tilde{f}(z, t)$ vanishes for purely real z , due to the symmetry of ρ_t . $\square$

The real part of the Cauchy transform $f(z, t)$ can be used to detect the presence of outlying eigenvalues. In order to use it, we first need the following estimates on how $\tilde{f}(z, t)$ behaves. The proof is for the most part elementary and is deferred to Appendix B of the longer version [2].

Lemma 3.4 *Let $\delta > 0$. Let $z = (1 + \eta)e^{i(\Theta_t + \kappa)}$. Uniformly in the region,*

$$0 < \eta < 5, \quad 0 < \kappa < \pi - \Theta_t \quad (3.17)$$

we have for $\delta < t < 4 - \delta$ that,

$$c \frac{\eta}{\sqrt{\kappa + \eta}} \leq |\operatorname{Re}[\tilde{f}(z, t)]| \leq C \frac{\eta}{\sqrt{\kappa + \eta}} \quad (3.18)$$

*for some $c, C > 0$. For $0 < t < \frac{1}{2}$ we also have,*²

$$\eta c \leq |\operatorname{Re}[\tilde{f}(z, t)]| \leq C \frac{\eta}{\kappa^2}. \quad (3.19)$$

As advertised, the following lemma allows us to use estimates for the Cauchy transform outside the spectrum to rule out the presence of outlying eigenvalues.

Lemma 3.5 *Let $\delta > 0$ and assume $\delta < t < 4 - \delta$. Let $\varepsilon > 0$ and $0 \leq k \leq \frac{2}{3}$. Suppose that the estimate,*

$$\left| f(z, t) - \tilde{f}(z, t) \right| \leq \frac{N^\varepsilon}{N\sqrt{\eta + \kappa}\sqrt{\eta}} \quad (3.20)$$

holds for all $z = re^{i\theta}$ for any r and θ satisfying,

$$\Theta_t + N^{-2/3+k+5\varepsilon} \leq |\theta| \leq \pi, \quad N^{-2/3+k/4+\varepsilon} \leq r - 1 \leq c \quad (3.21)$$

where $c > 0$ is any positive constant. Then there are no eigenvalues $\lambda = e^{i\theta}$ for $\Theta_t + N^{-2/3+k+5\varepsilon} \leq |\theta| \leq \pi$. The same conclusion holds for $t < \delta$ if we take $k = 2/3 - 5\varepsilon - \varepsilon/6$.

² An anonymous referee pointed out that this could be improved to $|\operatorname{Re}[\tilde{f}(z, t)]| \asymp t^{-3/4} \frac{\eta}{\sqrt{\kappa + \eta}}$ if $|z - \Theta_t| \leq \sqrt{t}$ and $\asymp \eta|z|^{-2}$ if $|z - \Theta_t| > \sqrt{t}$ (at least for z in the upper-half plane and $\kappa > 0$) using the short-time behavior of the spectral measure proved in Proposition A.6.

Proof Let λ be an eigenvalue in $f(z, t)$. We will show that if z is of the form $re^{i\theta}$ with $r > 1$ and $|\theta - \arg \lambda| < r - 1$, then $\operatorname{Re}[f(z, t)] \geq \frac{1}{N(r-1)}$. By rotational invariance along the unit circle, it suffices to consider $\lambda = 1$.

We see by direct calculation that

$$\begin{aligned} \operatorname{Re} \left[\frac{z-1}{z+1} \right] &= \frac{r^2-1}{(r-1)^2+2r(1-\cos\theta)} \geq \frac{r^2-1}{(r-1)^2+r\theta^2} \geq \frac{r^2-1}{(r-1)^2(1+r)} \\ &\geq \frac{1}{r-1}. \end{aligned} \quad (3.22)$$

To get the second inequality, we used $1 - \cos \theta \leq \frac{\theta^2}{2}$. The third inequality comes from $\theta < r - 1$.

Thus, we see that if z of the form $re^{i\theta}$ with $r > 1$ and $|\theta - \arg \lambda| < r - 1$, we know that $\operatorname{Re}[f(z, t)] \geq \frac{1}{N(r-1)}$ provided that λ is an eigenvalue appearing in the empirical measure for $f(z, t)$.

Now, consider the point $z = (1 + N^{-2/3+k/4+\varepsilon})e^{i\theta}$, where θ is an angle in the region $|\theta| \in [\Theta_t + N^{-2/3+k+5\varepsilon}, \pi]$.

Applying the triangle inequality and the estimates (3.20) and (3.18), we see that for large enough N ,

$$\begin{aligned} |\operatorname{Re}[f(z, t)]| &\leq |f(z, t) - \tilde{f}(z, t)| + |\operatorname{Re}[\tilde{f}(z, t)]| < C \frac{\eta}{\sqrt{\kappa} + \eta} + \frac{N^\varepsilon}{N\sqrt{\eta}\sqrt{\kappa} + \eta} \\ &< \frac{1}{N\eta}. \end{aligned} \quad (3.23)$$

Indeed, for the final inequality, we use the hypotheses on η and θ to estimate,

$$\begin{aligned} \frac{\eta}{\sqrt{\kappa} + \eta} &\leq N^{-1/3-k/4-3/2\varepsilon}, \quad \frac{N^\varepsilon}{N\sqrt{\eta}\sqrt{\kappa} + \eta} \leq N^{-1/3-5k/8-2\varepsilon}, \quad \frac{1}{N\eta} \\ &= N^{-1/3-k/4-\varepsilon} \end{aligned} \quad (3.24)$$

This shows $|\operatorname{Re}[f(z, t)]| < \frac{1}{N\eta}$ and therefore $e^{i\theta}$ cannot be an eigenvalue of the empirical measure associated to $f(z, t)$.

We now consider the case $0 < t < \delta$ and $k = 2/3 - 5\varepsilon - \varepsilon/6$. We can instead apply the trivial bound $|\operatorname{Re}[\tilde{f}(z, t)]| \leq C \frac{\eta}{\kappa^2}$ (3.19). We see that at the same choice of η and κ , we have

$$\frac{\eta}{\kappa^2} \leq N^{-1/2+\varepsilon/24}, \quad \frac{N^\varepsilon}{N\sqrt{\eta}\sqrt{\kappa} + \eta} \leq N^{-3/4+59/48\varepsilon}, \quad \frac{1}{N\eta} = N^{-1/2+7/24\varepsilon}. \quad (3.25)$$

Therefore, for $z = re^{i\theta}$ as in the statement of the lemma we see that $|\operatorname{Re}[f(z, t)]| < \frac{1}{N\eta}$. This completes the proof. $\square$

With the above results in hand, we can begin our proof of Theorem 1.4. Before doing so, we need to introduce further notation. Let $T < 4$. We consider times $t < T$. Let $\mathfrak{a}, \mathfrak{b} > 0$ be the constants of Proposition 3.3. Let $\delta > 0$ and $\varepsilon > 0$. Assume $\varepsilon < 10^{-6}$. These parameters will be fixed until the end of the proof of Theorem 1.4.

Introduce $D(t)$ to be the function

$$D(t) = N^{-\varepsilon/6}, 0 \leq t \leq \delta, \quad D(t) = \max \left\{ (N^{-\varepsilon/12} - \frac{\mathfrak{b}}{10}(t - \delta))^2, N^{-2/3+5\varepsilon} \right\}. \quad (3.26)$$

The function $D(t)$ satisfies the hypotheses of Proposition 3.3. In fact, with this choice of $D(t)$ we have for any choices of $s < t$ that

$$\sqrt{D(s)} \leq \sqrt{D(t)} + \frac{\mathfrak{b}}{10}(t - s). \quad (3.27)$$

Additionally, define $k(t)$ to be the solution of $N^{-2/3+k(t)+5\varepsilon} = D(t)$. Define the spectral domains $\mathcal{G}_t$ by,

$$\mathcal{G}_t := \{z = re^{i\theta} : \Theta_t + N^{-2/3+k(t)+5\varepsilon} \leq |\theta| \leq \pi, N^{-2/3+k(t)/4+\varepsilon} \leq r - 1 \leq \mathfrak{a}/2\}. \quad (3.28)$$

For any characteristic $z_t = re^{i\theta}$ with $r > 1$ define $\kappa(z_t) = |\theta| - \Theta_t$ and $\eta(z_t) = r - 1$. Consider the control parameter,

$$B(z_t) := \frac{1}{N\sqrt{\kappa(z_t) + \eta(z_t)}\sqrt{\eta(z_t)}} \mathbf{1}_{\kappa(z_t) > 0} + \frac{1}{N\eta(z_t)} \mathbf{1}_{\kappa(z_t) < 0}. \quad (3.29)$$

Lemma 3.6 *Let $0 < t < T$ and let $z_s = C_{s,t}(z)$ where $z \in \mathcal{G}_t$. Then, for all $0 < s < t$ we have,*

$$N^\varepsilon B(z_s) \leq \frac{1}{\log(N)} |\operatorname{Re}[\tilde{f}(z_s, s)]|. \quad (3.30)$$

Proof Let s_* be as in Proposition 3.3. For $s \leq s_*$ we have that

$$N^\varepsilon B(z_s) \leq CN^{\varepsilon-1}. \quad (3.31)$$

On the other hand, by Lemma 3.4, $|\operatorname{Re}[\tilde{f}(z_s, s)]| \geq c$ for some $c > 0$ for such z_s . We now consider $s_* < s < t$. By Proposition 3.3 it follows that,

$$\kappa(z_s) \geq \kappa(z_t) \geq N^{-2/3+k(t)+5\varepsilon} \quad (3.32)$$

and $\eta(z_s) \geq N^{-2/3+k(t)/4+\varepsilon}$. First consider $t \leq \delta$. Then, $k(t) = 2/3 - 5\varepsilon - \varepsilon/6$ and so

$$N^\varepsilon B(z_s) \leq N^{-3/4+10\varepsilon}. \quad (3.33)$$

Moreover, $\operatorname{Re}[\tilde{f}(z_s, s)] \geq c\eta(z_s) \geq N^{-1/2-5\varepsilon}$. This proves the estimate for such t . Consider now $t > \delta$. In this case, the desired inequality can be rewritten as

$$\log(N)N^\varepsilon B(z_s) \leq |\operatorname{Re}[\tilde{f}(z_s, s)]| = |\operatorname{Re}[\tilde{f}(z_t, t)]|, \quad (3.34)$$

using the fact that $\tilde{f}$ is constant along characteristics. Since $B(z_s) \leq B(z_t)$ for $s > s_*$, this reduces to whether,

$$\eta(z_t)^{3/2} \geq \log(N)N^{\varepsilon-1}. \quad (3.35)$$

But this holds since $\eta(z_s) \geq N^{-2/3+\varepsilon}$. $\square$

We introduce a grid of times $t_i = iTN^{-10}$ for $i = 1, \dots, N^{10}$. For each such t_i we let $\{u_j^i\}_{j=1}^{N^{10}}$ be a well-spaced grid of $\mathcal{G}_{t_i}$. We then define the characteristics,

$$z_j^i(s) := \mathcal{C}_{s, t_i}(u_j^i). \quad (3.36)$$

We define each characteristic $z_j^i(s)$ only for $0 < s < t_i$. We introduce the stopping times,

$$\tau_{ij} := \inf\{s \in (0, t_i) : |f(z_j^i(s), s) - \tilde{f}(z_j^i(s), s)| > N^{\varepsilon/2}B(z_j^i(s))\} \quad (3.37)$$

where the infimum of the empty set is $+\infty$. Then, we introduce

$$\tau := \min_{i,j} \tau_{ij} \wedge T. \quad (3.38)$$

We want to prove that $\tau = T$. First observe that since $|\tilde{f}(z, t)| \leq N$ for any $z \in \mathcal{G}_t$, we have $|z_{s_1} - z_{s_2}| \leq CN|s_1 - s_2|$ for any characteristic z_s ending in some $\mathcal{G}_t$. It follows that for each i and s satisfying $t_i - TN^{-10} < s \leq t_i$ that the points $\{z_j^i(s)\}_{j=1}^{N^{10}}$ are a well-spaced grid of $\mathcal{G}_s$ in the sense that every $z \in \mathcal{G}_s$ is no further than N^{-8} away from the closest of these points. Since $f(z, t)$ and $\tilde{f}(z, t)$ are Lipschitz functions with Lipschitz constant less than CN for $|z| > 1 + N^{-1}$ we see that for every $s < \tau$ that,

$$|f(z, s) - \tilde{f}(z, s)| \leq N^\varepsilon B(z) \quad (3.39)$$

for any $z \in \mathcal{G}_s$. From Lemma 3.5 we conclude that at any time $s < \tau$ there are no eigenvalues of the form $\lambda = e^{i\theta}$ for $\Theta_s + N^{-2/3+k(s)+5\varepsilon} < |\theta| \leq \pi$.

As in the proof of Theorem 1.2 we write, for any $t < t_i \wedge \tau$,

$$f(z_j^i(t)) - \tilde{f}(z_j^i(t)) = \mathcal{E}_1(t)_j^i + \mathcal{E}_2(t)_j^i \quad (3.40)$$

where

$$\mathcal{E}_1(t)_j^i = -\frac{1}{2} \int_0^s z_j^i(s)(\partial_z f)(z_j^i(s), s)(f(z_j^i(s), s) - \tilde{f}(z_j^i(s), s))ds \quad (3.41)$$

and

$$\mathcal{E}_2(t)_j^i = \int_0^t dM_s(z_j^i(s)). \quad (3.42)$$

We now prove the following estimate on the stochastic term.

Proposition 3.7 *Let $\varepsilon_1 > 0$. Then for all i, j as above,*

$$\mathbb{P} \left[\exists t \in (0, t_i \wedge \tau) : \left| \mathcal{E}_2(t)_j^i \right| > N^{\varepsilon_1 + \varepsilon/8} B(z_j^i(t)) \right] \leq C \log(N) e^{-cN^{\varepsilon_1}}. \quad (3.43)$$

Proof For simplicity of notation let us denote $z_t := z_j^i(t)$. We fix a sequence of intermediate times $0 < s_1, s_2, \dots, s_M = t_i$ such that,

$$\frac{1}{2} B(z_{s_k}) \leq B(z_{s_{k+1}}) \leq 2B(z_{s_k}), \quad (3.44)$$

for every i . We can take $M \leq C \log(N)$. Recall also the time $s_* < t_i$ defined in Proposition 3.3. Define also $\kappa_t = \kappa(z_t)$ and $\eta_t = \eta(z_t)$. We calculate the quadratic variation,

$$\begin{aligned} \langle \bar{\mathcal{E}}_2(s_k \wedge \tau) \mathcal{E}_2(s_k \wedge \tau) \rangle &\leq \frac{C}{N^2} \int_0^{s_k \wedge \tau} \frac{1}{N} \sum_{n=1}^N \frac{1}{|\lambda_n(t) - z_t|^4} dt \\ &\leq \frac{C}{N^2} \int_{s_* \wedge \tau}^{s_k \wedge \tau} \frac{1}{N} \sum_{n=1}^N \frac{1}{|\lambda_n - z_t|^4} dt + \frac{C}{N^2} \end{aligned} \quad (3.45)$$

where the integral in the last line is interpreted as 0 if $s_k \leq s_*$. When $s_k > s_*$ we continue to estimate the integral. For $s_* < t < \tau$ we have,

$$|\lambda_n(t) - z_t|^2 \geq c \left((\kappa_t - N^{-2/3+k(t)+5\varepsilon})^2 + \eta_t^2 \right) \quad (3.46)$$

Note that in particular, we also used that $\kappa_t \geq N^{-2/3+k(t)+5\varepsilon} = D(t)$ for $s_* < t < s_k$ due to (3.12) and the fact that $\kappa_{s_k} \geq D(s_k)$. Therefore,

$$\begin{aligned} \frac{1}{N^2} \int_{s_* \wedge \tau}^{s_k \wedge \tau} \frac{1}{N} \sum_{n=1}^N \frac{1}{|\lambda_n - z_t|^4} dt &\leq \frac{C}{N^2} \int_{s_* \wedge \tau}^{s_k \wedge \tau} \frac{|\operatorname{Re}[f(z_t, t)]|}{\eta_t \left((\kappa_t - N^{-2/3+k(t)+5\varepsilon})^2 + \eta_t^2 \right)} dt \\ &\leq \frac{C}{N^2} \int_{s_* \wedge \tau}^{s_k \wedge \tau} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \left((\kappa_t - N^{-2/3+k(t)+5\varepsilon})^2 + \eta_t^2 \right)} dt, \end{aligned} \quad (3.47)$$

where in the second line we used that Lemma 3.6 together with the definition of the stopping time τ implies that for $t < \tau$ we have,

$$|\operatorname{Re}[f(z_t, t)]| \leq |\operatorname{Re}[\tilde{f}(z_t, t)]| + N^{\varepsilon/2} B(z_t) \leq 2|\operatorname{Re}[\tilde{f}(z_t, t)]|.$$

We use now,

$$\begin{aligned}
 \kappa_t - N^{-2/3+k(t)+5\varepsilon} &= \kappa_t - D(t) \\
 &\geq \kappa_{s_k} + 2\mathfrak{b}\sqrt{\kappa_{s_k}}(s_k - t) + \mathfrak{b}^2(s_k - t)^2 - D(t) \\
 &\geq \kappa_{s_k} + 2\mathfrak{b}\sqrt{\kappa_{s_k}}(s_k - t) + \mathfrak{b}^2(s_k - t)^2 \\
 &\quad - D(s_k) - \mathfrak{b}\sqrt{D(s_k)}(s_k - t) - \frac{\mathfrak{b}^2}{4}(s_k - t)^2 \\
 &\geq \mathfrak{b}\sqrt{\kappa_{s_k}}(s_k - t).
 \end{aligned} \tag{3.48}$$

In the first inequality we used (3.10). In the second inequality we used the square of the inequality $\sqrt{D(t)} \leq \sqrt{D(s_k)} + \frac{\mathfrak{b}}{2}(s_k - t)$. In the last inequality we used $\kappa_{s_k} \geq D(s_k)$.

Therefore,

$$\frac{1}{N^2} \int_{s_* \wedge \tau}^{s_k \wedge \tau} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \left((\kappa_t - N^{-2/3+k(t)+5\varepsilon})^2 + \eta_t^2 \right)} dt \leq \frac{C}{N^2} \int_{s_*}^{s_k} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \left((\kappa_{s_k} (t - s_k)^2 + \eta_t^2) \right)} dt. \tag{3.49}$$

We need to consider two cases. First, consider $\eta_{s_k} \leq \kappa_{s_k}$. From the fact that $\kappa(s_k) \geq D(s_k)$ and that $D(s_k) = N^{-\varepsilon/6}$ if $s_k < \delta$ we see from (3.18) and (3.19) that,

$$|\operatorname{Re}[\tilde{f}(z_{s_k}, s_k)]| \leq C \frac{\eta_{s_k}}{\sqrt{\kappa_{s_k}}} N^{\varepsilon/4}. \tag{3.50}$$

Then using this, as well as that η_t is decreasing and $\tilde{f}(z_t, t)$ is constant along characteristics we have,

$$\begin{aligned}
 \frac{1}{N^2} \int_{s_*}^{s_k} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \left((\kappa_{s_k} (t - s_k) + \eta_t^2) \right)} dt &\leq \frac{C N^{\varepsilon/4}}{N^2} \frac{1}{\sqrt{\kappa_{s_k} + \eta_{s_k}}} \int_{s_*}^{s_k} \frac{1}{\kappa_{s_k} (t - s_k)^2 + \eta_{s_k}^2} dt \\
 &\leq \frac{C N^{\varepsilon/4}}{N^2} \frac{1}{\sqrt{\kappa_{s_k} + \eta_{s_k}}} \frac{1}{\sqrt{\kappa_{s_k} \eta_{s_k}}} \\
 &\leq \frac{C N^{\varepsilon/4}}{N^2} \frac{1}{(\kappa_{s_k} + \eta_{s_k})(\eta_{s_k})}.
 \end{aligned} \tag{3.51}$$

The second estimate follows via direct integration and in the last inequality we used the assumption $\kappa_{s_k} \geq \eta_{s_k}$. We now consider the case $\eta_{s_k} \geq \kappa_{s_k}$. In this case we proceed similarly to Proposition 2.2,

$$\begin{aligned}
 \frac{1}{N^2} \int_{s_*}^{s_k} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \left((\kappa_{s_k} (t - s_k) + \eta_t^2) \right)} dt &\leq \frac{1}{N^2} \int_{s_*}^{s_k} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t^3} dt \leq \frac{C}{N^2 \eta_{s_k}^2} \\
 &\leq \frac{C N^{\varepsilon/4}}{N^2 \eta_{s_k} (\kappa_{s_k} + \eta_{s_k})}.
 \end{aligned} \tag{3.52}$$

By the BDG inequality,

$$\mathbb{P} \left[\sup_{s \in (0, s_k)} |\mathcal{E}_2(s \wedge \tau)| > N^{\varepsilon_1 + \varepsilon/8} B(z_{s_k}) \right] \leq C e^{-cN^{\varepsilon_1}}. \quad (3.53)$$

Taking a union bound over the $O(\log(N))$ choices of s_k and using (3.44) we conclude the proof, similarly as in the proof of Proposition 2.2. $\square$

Proof of Theorem 1.4 With Proposition 3.7 in hand, the proof of Theorem 1.4 is similar to the proof of Theorem 1.2. Let Υ be the event of Proposition 3.7, after taking an intersection over all choices of $i, j \leq N^{10}$. Let $z_t := z(t)_j^i$ for notational simplicity. On the event of Υ we have the inequality for $0 < t < \tau$,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq \int_0^t g(s) |f(z_s, s) - \tilde{f}(z_s, s)| ds + N^{\varepsilon_1 + \varepsilon/8} B(z_t), \quad (3.54)$$

where we denoted,

$$g(s) = \frac{|z_s(\partial_z f)(z_s, s)|}{2}. \quad (3.55)$$

Hence, via Gronwall's inequality we obtain for $0 < t < \tau$,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq \int_0^t g(s) \exp \left[\int_s^t g(u) du \right] N^{\varepsilon_1 + \varepsilon/8} B(z_s) ds + N^{\varepsilon_1 + \varepsilon/8} B(z_t). \quad (3.56)$$

Similar to the proof of Theorem 1.2, now using Lemma 3.6, we have

$$\exp \left[\int_s^t g(u) du \right] \leq C \frac{\log |z_s|}{\log |z_t|} \quad (3.57)$$

as well as

$$g(s) \leq C \frac{\operatorname{Re}[\tilde{f}(z_s, s)]}{\eta_s}. \quad (3.58)$$

Therefore,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq C \frac{N^{\varepsilon_1 + \varepsilon/8}}{\eta_t} \int_0^t |\operatorname{Re}[\tilde{f}(z_s, s)]| B(z_s) ds + N^{\varepsilon_1 + \varepsilon/8} B(z_t). \quad (3.59)$$

We must consider a few different cases. First, let us consider the case that $t < s_*$. Then, we see that

$$\frac{N^{\varepsilon_1+\varepsilon/8}}{\eta_t} \int_0^t |\operatorname{Re}[\tilde{f}(z_s, s)]| B(z_s) ds + N^{\varepsilon_1+\varepsilon/8} B(z_t) \leq C \frac{N^{\varepsilon_1+\varepsilon/8}}{N}. \quad (3.60)$$

So in the remainder of the proof we consider the case $t \geq s_*$. We now consider a few different cases depending on whether or not t_i (the end-time of the characteristic z_t) is small or large. First, assume that $t_i < \delta$. Then, using (3.19),

$$\frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_t} = \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t} \leq C \left(|\kappa_t|^{-2} \right) \leq C N^{\varepsilon/3} \quad (3.61)$$

we have,

$$\frac{N^{\varepsilon_1+\varepsilon/8}}{\eta_t} \int_0^t |\operatorname{Re}[\tilde{f}(z_s, s)]| B(z_s) ds \leq N^{11\varepsilon/24+\varepsilon_1} B(z_t) \quad (3.62)$$

because $B(z_t)$ is effectively decreasing along characteristics (it may not be decreasing for $t < s_*$ but for such t it is $\mathcal{O}(N^{-1})$). We take $\varepsilon_1 < \varepsilon/100$.

Now, we assume that $t_i > \delta$. Then for $s < \delta/2$ we have that either $\eta_s \geq c$ or $\kappa_s \geq c$ by Proposition 3.3 depending on whether s is smaller or larger than s_* . Then for such s we have $B(z_s) \leq C N^{-1} \eta_t^{-1/2}$ and so,

$$\begin{aligned} \frac{N^{\varepsilon_1+\varepsilon/8}}{\eta_t} \int_0^{t \wedge \delta/2} |\operatorname{Re}[\tilde{f}(z_s, s)]| B(z_s) ds &\leq C N^{\varepsilon_1+\varepsilon/8} \frac{1}{N \eta_t^{1/2}} \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t} \\ &\leq C N^{\varepsilon_1+\varepsilon/8} B(z_t). \end{aligned} \quad (3.63)$$

In the final inequality we used $\operatorname{Re}[\tilde{f}(z_t, t)] \eta_t^{-1} \leq C(\kappa_t + \eta_t)^{-1/2}$ (recall that $t \geq s_*$) if $t > \delta/2$ and (3.18). If $t < \delta/2$ then since $t \geq s_*$ we have $\kappa_t \geq c$ since $t < t_i - \delta/2$ and so $|\operatorname{Re}[\tilde{f}(z_t, t)]| \eta_t^{-1}$ is bounded.

Now we have still to estimate the integral over $[t \wedge \delta/2, t]$ in the case that $t_i > \delta$. Since this integral is 0 if $t < \delta/2$ we may assume that $t > \delta/2$. Then, using freely (3.18), we have,

$$\begin{aligned} \frac{N^{\varepsilon_1+\varepsilon/8}}{\eta_t} \int_{\delta/2 \vee s_*}^t |\operatorname{Re}[\tilde{f}(z_s, s)]| B(z_s) ds &\leq \frac{C}{N} \frac{N^{\varepsilon_1+\varepsilon/8}}{\sqrt{\kappa_t + \eta_t}} \int_{\delta/2 \vee s_*}^t \frac{\eta_s}{\sqrt{\kappa_s + \eta_s} \eta_s^{3/2}} ds \\ &\leq \frac{C}{N} \frac{N^{\varepsilon_1+\varepsilon/8}}{\sqrt{\kappa_t + \eta_t}} \int_{\delta/2 \vee s_*}^t \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_s^{3/2}} ds \leq \frac{C N^{\varepsilon_1+\varepsilon/8}}{N \sqrt{\kappa_t + \eta_t} \sqrt{\eta_t}} \end{aligned} \quad (3.64)$$

The integral over $[\delta/2, \delta/2 \vee s_*]$ contributes $N^{\varepsilon_1+\varepsilon/8-1}(\kappa_t + \eta_t)^{-1/2}$ because $\eta_s \geq c$ here. Therefore, taking $\varepsilon_1 < \varepsilon/100$ we see that we have proven that,

$$|f(z_t, t) - \tilde{f}(z_t, t)| \leq C N^{23\varepsilon/48} B(z_t) \ll N^{\varepsilon/2} B(z_t). \quad (3.65)$$

on the event Υ . It follows that $\tau = T$. $\square$

4 Cusp estimates

In this section we prove Theorem 1.6. We will use the following reverse time parameterization. We will denote by uppercase T , and S the usual forward time parameterization, so that we will consider S and T close to 4. We will introduce,

$$T = 4 - t, \quad S = 4 - s \quad (4.1)$$

so that t and s will usually obey,

$$N^{-1/2+\delta} \leq s, t \leq \frac{1}{10}. \quad (4.2)$$

We will often substitute in T or t into functions of time. When we write t it is understood that we are evaluating the function at $4 - t$. For example, the gap between edges is,

$$\Delta_t = \Delta_T = 2(\pi - \Theta_T) = \frac{1}{3}t^{3/2}(1 + \mathcal{O}(t)). \quad (4.3)$$

The proof of Theorem 1.6 is similar in structure to the proof of Theorem 1.4. We first establish analogues of the estimates proven there before proceeding to the main body of the proof.

Lemma 4.1 *There is a constant $C > 0$ so that the following holds. For $0 < t < \frac{1}{10}$ and $z = (1 + \eta)e^{i(\Theta_t + \kappa)}$ with η and κ satisfying,*

$$0 < \eta < \Delta_t, \quad 0 < \kappa < \pi - \Theta_t \quad (4.4)$$

we have that

$$\frac{1}{C} \frac{\eta}{\sqrt{\eta + \kappa}} \leq \Delta_t^{1/6} |\operatorname{Re}[\tilde{f}(z, t)]| \leq C \frac{\eta}{\sqrt{\eta + \kappa}} \quad (4.5)$$

Proof Denoting $\theta = \kappa + \Theta_t$ we have,

$$-\operatorname{Re}[\tilde{f}(z, t)] \asymp \eta \int \frac{\rho_t(x)}{\eta^2 + \sin^2(\frac{\theta-x}{2})} dx. \quad (4.6)$$

Due to symmetry of the density,

$$\int \frac{\rho_t(x)}{\eta^2 + \sin^2(\frac{\theta-x}{2})} dx \asymp \int_0^\pi \frac{\rho_t(x)}{\eta^2 + \sin^2(\frac{\theta-x}{2})} dx. \quad (4.7)$$

Changing coordinates and using the fact that $\sin^2(x) \asymp x^2$ for $|x| \leq \pi/2$ we have,

$$\int_0^\pi \frac{\rho_t(x)}{\eta^2 + \sin^2(\frac{\theta-x}{2})} dx \asymp \int_0^{\Theta_t} \frac{\rho_t(\Theta_t - x)}{\eta^2 + (\kappa + x)^2} dx. \quad (4.8)$$

For an upper bound we split the integral into the regions $[0, \Delta_t]$ and $[\Delta_t, \Theta_t]$. For the latter region we can bound $\rho_t(\Theta_t - x) \leq Cx^{1/3}$,

$$\int_{\Delta_t}^{\Theta_t} \frac{\rho_t(\Theta_t - x)}{\eta^2 + (\kappa + x)^2} dx \leq C \int_{\Delta_t}^{\Theta_t} (\Delta_t + x)^{-5/3} dx \leq C \Delta_t^{-2/3} \leq C \Delta_t^{-1/6} \frac{1}{\sqrt{\kappa + \eta}}, \quad (4.9)$$

where in the last inequality we used the assumption $\kappa + \eta \leq 2\Delta_t$. In the region $[0, \Delta_t]$ we use $\rho_t(\Theta_t - x) \Delta_t^{1/6} \leq Cx^{1/2}$,

$$\begin{aligned} \int_0^{\Delta_t} \frac{\rho_t(\Theta_t - x)}{\eta^2 + (\kappa + x)^2} dx &\leq C \Delta_t^{-1/6} \int_0^{\Delta_t} \frac{x^{1/2}}{\eta^2 + (\kappa + x)^2} dx \\ &\leq C \Delta_t^{-1/6} \int_0^{\Delta_t} (x + \kappa + \eta)^{-3/2} dx \leq C \Delta_t^{-1/6} \frac{1}{\sqrt{\kappa + \eta}}. \end{aligned} \quad (4.10)$$

This completes the proof of the upper bound. For the lower bound,

$$\begin{aligned} \int_0^{\Theta_t} \frac{\rho_t(\Theta_t - x)}{\eta^2 + (\kappa + x)^2} dx &\geq \int_0^{2\Delta_t} \frac{\rho_t(\Theta_t - x)}{\eta^2 + (\kappa + x)^2} dx \geq c \Delta_t^{-1/6} \int_0^{2\Delta_t} \frac{\sqrt{x}}{\eta^2 + (\kappa + x)^2} dx \\ &\geq c \Delta_t^{-1/6} \int_{\frac{\kappa+\eta}{2}}^{\kappa+\eta} \frac{\sqrt{x}}{\eta^2 + (\kappa + x)^2} dx \geq c \Delta_t^{-1/6} \int_{\frac{\kappa+\eta}{2}}^{\kappa+\eta} (\kappa + \eta)^{-3/2} dx \geq c \Delta_t^{-1/6} \frac{1}{\sqrt{\kappa + \eta}}. \end{aligned} \quad (4.11)$$

We used above the assumption that $\kappa + \eta \leq 2\Delta_t$. This completes the proof. $\square$

Lemma 4.2 Recall the convention $T = 4 - t$. There are $\mathfrak{a}, \mathfrak{b} > 0$ so that the following holds. Let $z_s = \mathcal{C}_{S,T}(z)$ denote a characteristic ending at a point $z = (1 + \eta)e^{i(\Theta_t + \kappa)}$ where $0 < \eta < \mathfrak{a}$ and $0 < \kappa < \Delta_T/2$. Let S_* be defined as,

$$S_* = \sup\{S : |z_S| \geq 1 + \mathfrak{a}\Delta_S\}. \quad (4.12)$$

Then for $S_* \vee (4 - 10^{-1}) < S < T$ we have,

$$\Delta_S^{1/6} \sqrt{\kappa_S} > \Delta_T^{1/6} \sqrt{\kappa_T} + \mathfrak{b}(T - S), \quad (4.13)$$

and also that κ_S is decreasing. For $S < S_*$ we have $\eta_S \geq \eta_{S_*} \geq \mathfrak{a}\Delta_{S_*} \geq \mathfrak{a}\Delta_T$.

Proof We let $100\mathfrak{a}$ be the constant from Proposition B.4. For $S \in (S_*, T)$ we can apply the estimate from this proposition to obtain,

$$\frac{d}{dS} \kappa_S = \frac{1}{2} \left(\operatorname{Im}[\tilde{f}(z_S, S)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_S}, S)] \right) \leq -c \Delta_S^{-1/6} \sqrt{\kappa_S}, \quad (4.14)$$

as long as $\kappa_S \geq 0$. We see that κ_S is decreasing so if $\kappa_T > 0$ then $\kappa_S > 0$ for $S > S_* \vee (4 - 10^{-1})$. Moreover, since Δ_S is decreasing, we see that

$$\frac{d}{dS} \left(\Delta_S^{1/3} \kappa_S \right) \leq -c \left(\Delta_S^{1/3} \kappa_S \right)^{1/2}. \quad (4.15)$$

The differential inequality $\partial_t g \leq -c g^{1/2}$ is solved by considering $\partial_t g^{1/2} \leq -c/2$ and so we see that for $S \in (S_* \vee (4 - 10^{-1}), T)$ that

$$\Delta_S^{1/6} \sqrt{\kappa_S} \geq \Delta_T^{1/6} \sqrt{\kappa_T} + c(T - S) \quad (4.16)$$

for some $c > 0$ as desired. This completes the proof. $\square$

The following establishes estimates on the behavior of the characteristics near the spectral edge.

Proposition 4.3 *Let $z_s = \mathcal{C}_{s,t}(z)$ be a characteristic ending at a point z at final time $t < 10^{-1}$ (recall the reversed time convention $T = 4 - t$). Assume at the final time t the estimates,*

$$\kappa_t > N^{-2/3+5\varepsilon} \Delta_t^{1/9}, \quad \Delta_t \geq N^{-3/4+9\varepsilon} \quad (4.17)$$

hold. Moreover assume $0 < \eta_t < \mathfrak{a} \Delta_t$. Let,

$$s_* = \inf \{s : \eta_s > \mathfrak{a} \Delta_s\}. \quad (4.18)$$

Note $S_ = 4 - s_*$ where S_* is as in Lemma 4.2. For $t < s < s_* \wedge 10^{-1}$ we have,*

$$\partial_s (\kappa_s - N^{-2/3+5\varepsilon} \Delta_s) \geq 0. \quad (4.19)$$

For the next two estimates, continue to assume $t < s < s_ \wedge 10^{-1}$. There is furthermore a constant $\mathfrak{d} > 0$ so that the following holds. If $s - t \leq \Delta_t^{1/6} \sqrt{\kappa_t}$ then,*

$$(\kappa_s - N^{-2/3+5\varepsilon} \Delta_s^{1/9})^2 \geq \mathfrak{d}(s - t)^2 \Delta_t^{-1/3} \kappa_t. \quad (4.20)$$

If $s - t \geq \Delta_t^{1/6} \sqrt{\kappa_t}$ then,

$$(\kappa_s - N^{-2/3+5\varepsilon} \Delta_s^{1/9})^2 \geq \mathfrak{d} \kappa_t^2. \quad (4.21)$$

Proof By direct calculation, one can see that $|\partial_t \Theta_t| \leq C t^{1/2} \leq C \Delta_t^{1/3}$. Therefore, proceeding as in the proof of Lemma 4.2 we have for $t < s < s_*$,

$$\frac{d}{ds} (\kappa_s - N^{-2/3+5\varepsilon} \Delta_s^{1/9}) \geq c \Delta_s^{-1/6} \sqrt{\kappa_s} - C N^{-2/3+5\varepsilon} \Delta_s^{-5/9}, \quad (4.22)$$

for some $c, C > 0$. Consider the set

$$\mathcal{A} := \{s : t \leq s < s_* : \kappa_s < N^{-2/3+5\varepsilon} \Delta_s^{1/9}\}. \quad (4.23)$$

Note that by assumption there is a small interval around t not contained in $\mathcal{A}$. Assume that $\mathcal{A}$ is not empty and let $F = \inf \mathcal{A}$, so that $s_* > F > t$. At F we clearly must have that

$$\kappa_F = N^{-2/3+5\varepsilon} \Delta_F^{1/9}. \quad (4.24)$$

We now show that the RHS of (4.22) is strictly positive at $s = F$. Indeed, we evaluate

$$\Delta_F^{10/9-1/3} \kappa_F = N^{-2/3+5\varepsilon} \Delta_F^{8/9} \geq N^{-2/3+5\varepsilon} \Delta_t^{8/9} \geq N^{3\varepsilon} (N^{-2/3+5\varepsilon})^2 \quad (4.25)$$

which shows that the RHS of (4.22) is strictly positive for N large enough. In particular, the derivative on the LHS of (4.22) is strictly positive at $s = F$, showing that $\kappa_s \geq N^{-2/3+5\varepsilon} \Delta_s^{1/9}$ in an open interval containing F . This contradicts the assumption that F was an infimum, proving that $\mathcal{A}$ is empty.

Substituting the lower bound $\kappa_s \geq N^{-2/3+5\varepsilon} \Delta_s^{1/9}$ into the RHS of (4.22) we then find that,

$$\frac{d}{ds} (\kappa_s - N^{-2/3+5\varepsilon} \Delta_s^{1/9}) \geq c \Delta_s^{-1/9} N^{-1/3+5\varepsilon/2} - C N^{-2/3+5\varepsilon} \Delta_s^{-5/9}. \quad (4.26)$$

Positivity of the RHS is equivalent to

$$\Delta_s^{4/9} \geq \frac{C}{c} N^{-1/3+5\varepsilon/2}. \quad (4.27)$$

However, by assumption $\Delta_t^{4/9} \geq N^{-1/3+4\varepsilon}$. We conclude the proof of (4.19).

We turn now to the proof of the remaining inequalities. From (4.13) we have

$$\begin{aligned} \Delta^{1/3}(s) \kappa(s) - N^{-2/3+5\varepsilon} \Delta^{4/9}(s) &\geq [\Delta^{1/3}(t) \kappa(t) - N^{-2/3+5\varepsilon} \Delta^{4/9}(t)] \\ &\quad + [N^{-2/3+5\varepsilon} \Delta^{4/9}(t) - N^{-2/3+5\varepsilon} \Delta^{4/9}(s)] \\ &\quad + 2c(s-t) \Delta^{1/6}(t) \sqrt{\kappa(t)}, \end{aligned} \quad (4.28)$$

for some $c > 0$. Note that the first term on the RHS is positive. First, assume that $s - t \leq \Delta_t^{1/6} \sqrt{\kappa_t}$. From the fact that $\kappa_t \leq \Delta_t$ and $\Delta_t \leq Ct^{3/2}$ we see that $s \leq Ct$ for some $C > 0$. It follows from the mean value theorem that,

$$\begin{aligned} N^{-2/3+5\varepsilon} |\Delta_t^{4/9} - \Delta_s^{4/9}| &\leq C N^{-2/3+5\varepsilon} (s-t) \Delta_t^{-2/9} \\ &\leq C(s-t) \Delta_t^{1/6} \sqrt{\kappa_t} N^{-3\varepsilon/2}. \end{aligned} \quad (4.29)$$

The estimate (4.20) easily follows. Consider now the case $s - t \geq \Delta_t^{1/6} \sqrt{\kappa_t}$. Let t_* be the time such that

$$t_* = t + \Delta_t^{1/6} \sqrt{\kappa_t}. \quad (4.30)$$

Applying (4.19) we see that

$$\kappa_s - N^{-2/3+5\varepsilon} \Delta_s \geq \kappa_{t_*} - N^{-2/3+5\varepsilon} \Delta_{t_*}. \quad (4.31)$$

Since the estimate (4.20) holds at $s = t_*$ we see that

$$(\kappa_{t_*} - N^{-2/3+5\varepsilon} \Delta_{t_*})^2 \geq c \kappa_t^2 \quad (4.32)$$

for some $c > 0$. This completes the proof of the proposition. $\square$

The following is similar to Lemma 3.5 and the proof is deferred to Appendix B of [2].

Lemma 4.4 *Let $\varepsilon > 0$. Consider the domain,*

$$\mathcal{D} := \{z = (1 + \eta)e^{i\theta} : \Delta_t^{1/9} N^{-2/3+\varepsilon} < \eta < 2\Delta_t^{1/9} N^{-2/3+\varepsilon}, \Theta_t + N^{-2/3+5\varepsilon} \Delta_t^{1/9} < |\theta| \leq \pi\}. \quad (4.33)$$

Assume for all $z \in \mathcal{D}$ we have the estimate,

$$|\tilde{f}(z, t) - f(z, t)| \leq \frac{N^\varepsilon}{N\sqrt{\eta}\sqrt{\kappa + \eta}}. \quad (4.34)$$

Then there are no eigenvalues of the form $\lambda = e^{i\theta}$ with $\Theta_t + N^{-2/3+5\varepsilon} \Delta_t < |\theta| \leq \pi$.

Lemma 4.5 *Let $\varepsilon > 0$ and let $t < N^{-1/10}$ with $\Delta_t \geq N^{-3/4+9\varepsilon}$. Let z_s be a characteristic that ends at time t in the region,*

$$\{z = (1 + \eta)e^{i\theta} : N^{-2/3+\varepsilon} \Delta_t^{1/9} \leq \eta \leq 2N^{-2/3+\varepsilon} \Delta_t^{1/9}, \Theta_t + N^{-2/3+5\varepsilon} \Delta_t^{1/9} \leq |\theta| \leq \pi\}. \quad (4.35)$$

Then for $t < s < 10^{-1}$ we have,

$$\eta_s \leq C N^{-\varepsilon/2} \Delta_s. \quad (4.36)$$

Proof Define the functions

$$h_1(u) = \log |z_t| + u N^{\varepsilon/2} \frac{\eta_t}{\Delta_t^{1/6} \sqrt{\kappa_t}} \quad (4.37)$$

and

$$h_2(u) = N^{-\varepsilon/2} \Delta_t + u t^{1/2} N^{-\varepsilon}. \quad (4.38)$$

By the definition of the characteristics,

$$\log |z_s| = \log |z_t| + (s - t) \frac{|\operatorname{Re}[\tilde{f}(z, t)]|}{2}, \quad (4.39)$$

and so for $0 \leq u < 10^{-1}$ we have,

$$\eta_{t+u} \leq Ch_1(u) \quad (4.40)$$

for some large C , using Lemma 4.1. Since $\partial_u \Delta_u \geq cu^{1/2}$ we have that

$$h_2(u) \leq N^{-\varepsilon/2} \Delta_{t+u}, \quad (4.41)$$

for N large enough. Note that $h_2(0) > h_1(0)$. We claim for $u < 10^{-1}$ that

$$h'_2(u) \geq h'_1(u) \quad (4.42)$$

which will yield the claim. This is equivalent to,

$$N^{3\varepsilon/2} \eta_t \leq t^{1/2} \Delta_t^{1/6} \sqrt{\kappa_t}. \quad (4.43)$$

The LHS is less than $2N^{5\varepsilon/2-2/3} \Delta_t^{1/9}$. The RHS is larger than $cN^{5\varepsilon/2-1/3} \Delta_t^{1/2} \Delta_t^{1/18}$. But,

$$\Delta_t^{1/2-1/9+1/18} = \Delta_t^{4/9} \geq N^{-1/3+4\varepsilon} \quad (4.44)$$

by assumption. $\square$

Lemma 4.6 Fix a time t . Let $z_s = C_{s,t}(z)$ be a characteristic terminating at a point $z = (1 + \eta)e^{i(\Theta_t + \kappa)}$ where

$$N^{-2/3+\varepsilon} \Delta_t^{1/9} < \eta < 2N^{-2/3+\varepsilon} \Delta_t^{1/9}, \quad N^{-2/3+5\varepsilon} \Delta_t^{1/9} \leq \kappa < \pi - \Theta_t. \quad (4.45)$$

Let t_1 and t_2 be two times $t \leq t_1 < t_2 < 10^{-1}$ and assume that for $t_1 < s < t_2$ the estimate

$$|\tilde{f}(z_s, s) - f(z_s, s)| \leq 2|\operatorname{Re}[\tilde{f}(z_s, s)]| \quad (4.46)$$

holds and that at each time s there are no eigenvalues of the form $\lambda = e^{i\theta}$ for $\Theta_s + N^{-2/3+5\varepsilon} \Delta_s^{1/9} < |\theta| \leq \pi$. Then,

$$\int_{t_1}^{t_2} \frac{1}{N} \sum_{i=1}^N \frac{1}{|\lambda_i(s) - z_s|^4} ds \leq C \log(N) B(z_{t_1})^2 \quad (4.47)$$

Proof First we consider the case that $\alpha\kappa_{t_1} \leq 10\eta_{t_1}$. In this case, it suffices to estimate the integrand by $C \log(N)\eta_{t_1}^{-2}$. This argument is similar to that appearing in the proof of Proposition 2.2 and so we omit it.

For the remainder of the proof we therefore assume that $\kappa_{t_1}\alpha \geq 10\eta_{t_1}$. In particular, this implies that $\eta_{t_1} \leq \alpha\Delta_{t_1}/10$.

Let s_* be as in Lemma 4.2, that is

$$s_* := \inf\{s > t : \eta_s > \alpha\Delta_s\}. \quad (4.48)$$

Note that by Lemma 4.5 we have $s_* > t_2$. For $t_1 < s < t_2$ we know from (4.19) that $\kappa_s \geq N^{-2/3+5\varepsilon}\Delta_s^{1/9}$ because this is satisfied at $s = t$. In particular, define

$$\tilde{t} = t_1 + \Delta_{t_1}^{1/6} \sqrt{\kappa_{t_1}}. \quad (4.49)$$

We assume $t_1 < \tilde{t} < t_2$. The other cases are easy to deal with, as one has to treat only one of the regions of integration described below. Then for $t_1 < s < \tilde{t}$ we have,

$$(\kappa_s - N^{-2/3+5\varepsilon}\Delta_s^{1/9})^2 \geq \mathfrak{d}(s - t_1)^2 \Delta_{t_1}^{-1/3} \kappa_{t_1} \quad (4.50)$$

and for $\tilde{t} < s < t_2$,

$$(\kappa_s - N^{-2/3+5\varepsilon}\Delta_s^{1/9})^2 \geq \mathfrak{d}\kappa_{t_1}^2, \quad (4.51)$$

by applying Proposition 4.3. We will use the estimate,

$$\int_{t_1}^{t_2} \frac{1}{N} \sum_{i=1}^N \frac{1}{|\lambda_i(s) - z_s|^4} ds \leq \int_{t_1}^{t_2} \frac{1}{(\kappa_s - N^{-2/3+5\varepsilon}\Delta_s^{1/9})^2 + \eta_s^2} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_s} ds. \quad (4.52)$$

We start with the region $s > \tilde{t}$. For such s we can apply (4.51) and estimate,

$$\begin{aligned} \int_{\tilde{t}}^{t_2} \frac{1}{(\kappa_s - N^{-2/3+5\varepsilon}\Delta_s^{1/9})^2 + \eta_s^2} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_s} ds &\leq \frac{C}{\kappa_{t_1}^2} \int_{\tilde{t}}^{t_2} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_s} ds \\ &\leq \frac{C}{\kappa_{t_1}^2} \log(N) \\ &\leq \frac{C}{\eta_{t_1}(\kappa_{t_1} + \eta_{t_1})} \log(N) \end{aligned} \quad (4.53)$$

We consider now the region $s < \tilde{t}$. By applying (4.50) and Lemma 4.1 (as well as the constancy of $\tilde{f}$ along characteristics, and the fact that η_s is increasing in s) we obtain,

$$\int_{t_1}^{\tilde{t}} \frac{1}{(\kappa_s - N^{-2/3+5\varepsilon}\Delta_s^{1/9})^2 + \eta_s^2} \frac{|\operatorname{Re}[\tilde{f}(z_s, s)]|}{\eta_s} ds$$

$$\begin{aligned}
&\leq \frac{C}{\Delta_{t_1}^{1/6} \sqrt{\kappa_{t_1} + \eta_{t_1}}} \int_{t_1}^{\tilde{t}} \frac{1}{(s-t)^2 \Delta_{t_1}^{-1/3} \kappa_{t_1} + \eta_{t_1}^2} ds \\
&\leq \frac{C}{\eta_{t_1} \sqrt{\kappa_{t_1} \sqrt{\kappa_{t_1} + \eta_{t_1}}}}.
\end{aligned} \tag{4.54}$$

This completes the proof. $\square$

Define the domain,

$$\begin{aligned}
\mathfrak{B}_s := \{z = (1 + \eta)e^{i\theta} : N^{-2/3+\varepsilon} \Delta_s^{1/9} < \eta < 2N^{-2/3+\varepsilon} \Delta_s^{1/9}, N^{-2/3+5\varepsilon} \Delta_s^{1/9} \\
+ \Theta_s < |\theta| \leq \pi\}.
\end{aligned} \tag{4.55}$$

We will assume $s > N^{-1/2+9\varepsilon}$ so that $\Delta_s > cN^{-3/4+10\varepsilon}$. We fix $S_0 = 4 - s_0$ with $s_0 = 10^{-1}$, and the final time,

$$S_f := 4 - s_f, \quad s_f = N^{-1/2+9\varepsilon} \tag{4.56}$$

Similar to the proof of Theorem 1.4, we introduce a polynomial number of characteristics as follows. We introduce times T_i by

$$T_0 = S_0, \quad T_i = T_{i-1} + \frac{S_f - S_0}{N^{10}}, \tag{4.57}$$

for $i = 1, \dots, N^{10}$. At each T_i we introduce a well-spaced mesh of $\mathcal{B}_{T_i}$ of size N^{10} denoted by $\{u_j^i\}_{j=1}^{10}$. We introduce the characteristics,

$$z_j^i(S) := \mathcal{C}_{S, T_i}(u_j^i). \tag{4.58}$$

Note that each characteristic is defined only for $0 \leq s \leq T_i$. For $i \geq 1$ we introduce the stopping times,

$$\tau_{ij} = \inf\{S \in (S_0, T_i) : |f(z_j^i(S), S) - \tilde{f}(z_j^i(S), S)| > N^{\varepsilon/2} B(z_j^i(S))\}, \tag{4.59}$$

with the infimum of the empty set being $+\infty$. Let $\mathcal{A}_0$ be the event,

$$\mathcal{A}_0 = \{\exists(i, j) : |f(z_j^i(S_0), S_0) - \tilde{f}(z_j^i(S_0), S_0)| > N^{\varepsilon_1} \inf_{S_0 < S < T_i} B(z_j^i(S))\}^c \tag{4.60}$$

Lemma 4.7 *The event $\mathcal{A}_0$ holds with overwhelming probability.*

Proof We will show that the desired estimates hold on the event of Proposition B.5, with ε, δ in that proposition statement chosen sufficiently small.

Fix a single characteristic $z_S := z_i^j(S)$ and let $S_0 < S < T_i$ be fixed. By (4.19) and Lemma 4.5 it follows that $\kappa_S > 0$ for all S and that $\kappa_{S_0} \geq cN^{-2/3+5\varepsilon}$. From (B.11) we see that

$$|f(z_{S_0}, S_0) - \tilde{f}(z_{S_0}, S_0)| \leq N^{\varepsilon_1} B(z_{S_0}). \quad (4.61)$$

Since $\kappa_S > 0$ and κ_S and η_S are both decreasing in S that,

$$B(z_S) \geq B(z_{S_0}) \quad (4.62)$$

for $S > S_0$. This yields the claim. $\square$

We also let τ_0 be the stopping time that equals $+\infty$ on $\mathcal{A}_0$ and is S_0 on $\mathcal{A}_0^c$. We now introduce the stopping time,

$$\tau = \left(\min_{i,j} \tau_{ij} \right) \wedge \tau_0 \wedge S_f. \quad (4.63)$$

Using the Lipschitz continuity of $\tilde{f}(z, S)$ and $f(z, S)$ on the domains $\mathcal{B}_S$ we have from Lemma 4.4 that for any $S_0 < S < \tau$ that there are no eigenvalues of the form $\lambda = e^{i\theta}$ for $\Theta_S + N^{-2/3+5\varepsilon} \Delta_S^{1/9} < |\theta| \leq \pi$.

Lemma 4.8 *Let z_S be a characteristic terminating at time T_i in the domain,*

$$\{z = (1 + \eta)e^{i\theta} : N^{-2/3+\varepsilon} \Delta_{T_i}^{1/9} \leq \eta \leq \alpha \Delta_{T_i}/10, \Theta_{T_i} + N^{-2/3+5\varepsilon} \Delta_{T_i}^{1/9} \leq |\theta| \leq \pi\}. \quad (4.64)$$

Then for $S_0 < S < T_i$ we have,

$$N^\varepsilon B(z_S) \leq \frac{1}{\log(N)} |\operatorname{Re}[\tilde{f}(z_S, S)]|. \quad (4.65)$$

Proof Note that $\tilde{f}(z_S, S)$ is constant. At $S = T_i$ we have,

$$\begin{aligned} N^\varepsilon B(z_{T_i}) &= N^\varepsilon \frac{1}{N\sqrt{\eta_{T_i}(\eta_{T_i} + \kappa_{T_i})}} \leq C \frac{N^\varepsilon \Delta_{T_i}^{1/6}}{N\eta_{T_i}^{3/2}} |\operatorname{Re}[\tilde{f}(z_{T_i}, T_i)]| \\ &\leq CN^{-\varepsilon/2} |\operatorname{Re}[\tilde{f}(z_{T_i}, T_i)]| \end{aligned} \quad (4.66)$$

where we used $\eta_{T_i} \geq \Delta_{T_i}^{1/9} N^{-2/3+\varepsilon}$ which holds by assumption. Let S_* be as in Lemma 4.2. By Lemma 4.2 we have

$$B(z_S) \leq B(z_{T_i}) \quad (4.67)$$

for all $S > S_*$. If $S_* > S_0$ then for $S < S_*$ we have $\eta_S \geq \eta_{S_*} \geq c\Delta_{S_*} \geq c\Delta_{T_i} \geq c\kappa_{T_i}$ and so

$$B(z_S) \leq (N\eta_S)^{-1} \leq CB(z_{T_i}). \quad (4.68)$$

This completes the proof. $\square$

With similar notation as in the other sections we have for any $S_0 \leq S \leq T_i \wedge \tau$,

$$f(z_j^i(S), S) - \tilde{f}(z_j^i(S), S) = f(z_j^i(S_0), S_0) - \tilde{f}(z_j^i(S_0), S_0) = \mathcal{E}_1(S)_j^i + \mathcal{E}_2(S)_j^i, \quad (4.69)$$

where

$$\mathcal{E}_1(S)_j^i = -\frac{1}{2} \int_{S_0}^S z_j^i(U) (\partial_z f)(z_j^i(U), U) (f(z_j^i(U), U) - \tilde{f}(z_j^i(U), U)) dU \quad (4.70)$$

and

$$\mathcal{E}_2(S)_j^i = \int_{S_0}^S dM_U(z_j^i(U)). \quad (4.71)$$

For the martingale term we have the following.

Proposition 4.9 *For any i, j and $\varepsilon_1 > 0$ we have,*

$$\mathbb{P} \left[\exists S \in (S_0, T_i \wedge \tau) : |\mathcal{E}_2(S)_j^i| > N^{\varepsilon_1} B(z_j^i(S)) \right] \leq C \log(N) e^{-cN^{\varepsilon_1}}. \quad (4.72)$$

Proof This is proven in an almost identical manner to Proposition 3.7. The quadratic variation is bounded using Lemma 4.6. Note that the condition (4.46) is a consequence of Lemma 4.8. $\square$

Proof of Theorem 1.6 Let Υ be the intersection of the event of Proposition 4.9 and $\mathcal{A}_0$ so that Υ holds with overwhelming probability. Let $z_S = z_j^i(S)$ be a characteristic. On the event Υ we have for $S_0 < S < T_i \wedge \tau$ that,

$$|f(z_S, S) - \tilde{f}(z_S, S)| \leq \int_0^S g(U) |f(z_U, U) - \tilde{f}(z_U, U)| dU + 2N^{\varepsilon_1} B(z_S) \quad (4.73)$$

by the definition of $\mathcal{A}_0$ and (4.69), where

$$g(U) = \frac{|z_U (\partial_z f)(z_U, U)|}{2}. \quad (4.74)$$

Hence, via Gronwall's inequality we obtain,

$$|f(z_S, S) - \tilde{f}(z_S, S)| \leq 2 \int_0^S g(u) \exp \left[\int_U^S g(w) dw \right] N^{\varepsilon_1} B(z_U) du + 2N^{\varepsilon_1} B(z_S). \quad (4.75)$$

As in the proofs of Theorems 1.2 and 1.4 we have, using Lemma 4.8,

$$\exp \left[\int_U^S g(w) dw \right] \leq C \frac{\eta_U}{\eta_S}, \quad (4.76)$$

as well as

$$g(U) \leq C \frac{|\operatorname{Re}[\tilde{f}(z_U, U)]|}{\eta_U}. \quad (4.77)$$

Hence,

$$|f(z_S, S) - \tilde{f}(z_S, S)| \leq C \frac{N^{\varepsilon_1}}{\eta_S} \int_0^S |\operatorname{Re}[\tilde{f}(z_U, U)]| B(z_U) dU + 2N^{\varepsilon_1} B(z_S). \quad (4.78)$$

We now split into a few different cases. Suppose first that $\eta_S \geq \kappa_S$. Then,

$$\int_0^S |\operatorname{Re}[\tilde{f}(z_U, U)]| B(z_U) dU \leq \frac{1}{N} \int_0^S |\operatorname{Re}[\tilde{f}(z_U, U)]| \eta_U^{-1} dU \leq CN^{-1} \log(N) \quad (4.79)$$

and so,

$$|f(z_S, S) - \tilde{f}(z_S, S)| \leq CN^{\varepsilon_1} B(z_S) + C \log(N) N^{\varepsilon_1} \eta_S \leq C \log(N) N^{\varepsilon_1} B(z_S) \quad (4.80)$$

where we used the assumption that $\eta_S \geq \kappa_S$ in the last inequality. Now assume that $\eta_S \leq \kappa_S$. Similar to the convention of Lemma 4.2 we define,

$$S_* = \sup\{S_0 < U < S : |z_U| \geq 1 + \alpha \Delta_U\}. \quad (4.81)$$

with the convention that the supremum of the empty set is $-\infty$. By Lemma 4.5 it follows that $S_* = -\infty$. Therefore, for all $S_0 < U < S$ we have,

$$|\operatorname{Re}[\tilde{f}(z_U, U)]| \asymp \frac{1}{\Delta_U^{1/6}} \frac{\eta_U}{\sqrt{\kappa_U + \eta_U}}, \quad (4.82)$$

as $\kappa_U > 0$ for all $S_0 < U < S$.

Let us define $S_1 = 4 - 2s$. Let us assume that $S_0 < S_1$. The other case is easier, requiring consideration of only one of the regions of integration below. We consider first the region $U \in (S_1, S)$. Then, for such U we have $\Delta_U \leq C\Delta_S$, and so

$$\begin{aligned} \frac{1}{\eta_S} \int_{S_1}^S |\operatorname{Re}[\tilde{f}(z_U, U)]| N B(z_U) dU &\leq \frac{1}{\sqrt{\kappa_S + \eta_S} \Delta_S^{1/6}} \int_{S_1}^S \frac{1}{\sqrt{\kappa_U + \eta_U} \sqrt{\eta_U}} dU \\ &\leq C \frac{1}{\sqrt{\kappa_S + \eta_S} \Delta_S^{1/6}} \int_{S_1}^S \Delta_U^{1/6} |\operatorname{Re}[\tilde{f}(z_U, U)]| \eta_U^{-3/2} dU \\ &\leq C \frac{1}{\sqrt{\kappa_S + \eta_S}} \int_{S_1}^S |\operatorname{Re}[\tilde{f}(z_U, U)]| \eta_U^{-3/2} dU \\ &\leq C \frac{1}{\sqrt{\kappa_S + \eta_S} \eta_S^{1/2}}. \end{aligned} \quad (4.83)$$

Now we consider $U \in (S_0, S_1)$. For such U , let $U = 4 - u$. Note $u \geq 2s$. In particular, from (4.13) we conclude that

$$\Delta_U^{1/6} \sqrt{\kappa_U} \geq cu \quad (4.84)$$

which implies that $C\Delta_U \geq \kappa_U \geq c\Delta_U$. From Lemma 4.1 we see that for $U \in (S_0, S_1)$,

$$c \frac{\eta_U}{\Delta_U^{2/3}} \leq |\operatorname{Re}[\tilde{f}(z_U, U)]| \leq C \frac{\eta_U}{\Delta_U^{2/3}}. \quad (4.85)$$

Since $\tilde{f}(z_U, U)$ is constant along characteristics we deduce from this that

$$\Delta_U^{1/6} \leq C \frac{\eta_U^{1/4} \Delta_{S_1}^{1/6}}{\eta_{S_1}^{1/4}} \leq C \frac{\eta_U^{1/4} \Delta_S^{1/6}}{\eta_S^{1/4}} \quad (4.86)$$

The second inequality uses that $\eta_{S_1} \geq \eta_S$ and that $\Delta_{S_1} \asymp \Delta_S$ by the choice of S_1 . Hence,

$$\begin{aligned} \frac{1}{\eta_S} \int_{S_0}^{S_1} |\operatorname{Re}[\tilde{f}(z_U, U)]| N B(z_U) dU &\leq \frac{C}{\sqrt{\kappa_S + \eta_S} \Delta_S^{1/6}} \int_{S_0}^{S_1} \frac{1}{\sqrt{\kappa_U + \eta_U} \sqrt{\eta_U}} dU \\ &\leq \frac{C}{\sqrt{\kappa_S + \eta_S} \Delta_S^{1/6}} \int_{S_0}^{S_1} \Delta_U^{1/6} |\operatorname{Re}[\tilde{f}(z_U, U)]| \eta_U^{-3/2} dU \\ &\leq \frac{C}{\sqrt{\kappa_S + \eta_S} \eta_S^{1/4}} \int_{S_0}^S |\operatorname{Re}[\tilde{f}(z_U, U)]| \eta_U^{-5/4} dU \\ &\leq C \frac{1}{\sqrt{\kappa_S + \eta_S} \eta_S^{1/2}}. \end{aligned} \quad (4.87)$$

In the first inequality we used constancy of $\tilde{f}$ along characteristics and Lemma 4.1. In the second inequality we used Lemma 4.1 again. In the third inequality we used (4.86).

Summarizing, we see that on the event Υ we have for any $S_0 < S < \tau \wedge T_i$ that

$$|f(z_S, S) - \tilde{f}(z_S, S)| \leq C \log(N) N^{\varepsilon_1} B(z_S). \quad (4.88)$$

Taking $\varepsilon_1 < \varepsilon/100$ we see that we must have $\tau > T_i$ for every i . Therefore, $\tau = T$ and we conclude the theorem. $\square$

5 Center of mass evolution

In this section we prove Proposition 1.8. We will not track dependence of constants on the parameter N , as Proposition 1.8 is a statement in soft analysis. The reader should note that we will therefore change notation and consider unitary matrices of size n instead of N evolving according to (1.1). We do this in order to emphasize the ineffectiveness of the estimates in the dimension of the system.

Fix some time $t_0 > 0$ and assume that there are no eigenvalues in the set $\{z = e^{i\theta} : \theta \in I_0\}$ for $I_0 = [\theta_0 - L/2, \theta_0 + L/2]$ at time t_0 and $\theta_0 \in [0, 2\pi)$. Let $\tau > t_0$ be the first time an eigenvalue enters this set. Let Γ be the contour,

$$\begin{aligned} \Gamma := & \{z = re^{i\theta} : 1/2 < r < 3/2, \theta = \theta_0 \pm L/4\} \\ & \cup \{z = re^{i\theta} : r = 1/2, 3/2, \theta \notin [\theta_0 - L/4, \theta_0 + L/4]\}. \end{aligned} \quad (5.1)$$

That is, it encloses all of the eigenvalues on the unit circle but avoids the ray $\{re^{i\theta_0} : r \geq 0\}$. A schematic diagram of the contour Γ is given in Fig. 1.

We use this contour as we will take a logarithm with branch cut being this ray. Then for any time $t_0 < t_1 < \tau$ we have,

$$i\theta_i(t_1) - i\theta_i(t_0) = \frac{1}{2\pi i} \int_{\Gamma} g_{\theta_0}(z) \left(\frac{1}{z - \lambda_i(t_1)} - \frac{1}{z - \lambda_i(t_0)} \right) dz \quad (5.2)$$

where $g_{\theta_0}(z)$ is the branch of the logarithm holomorphic in $\mathbb{C} \setminus \{z = re^{i\theta_0}, r \geq 0\}$. This formula holds due to the fact that no eigenvalue crosses the angle θ_0 in this time interval, and so for this entire time interval, $\theta_i(t) = 2\pi k + \tilde{\theta}_i(t)$ where k is a constant integer and $\theta_0 < \tilde{\theta}_i(t) < \theta_0 + 2\pi$.

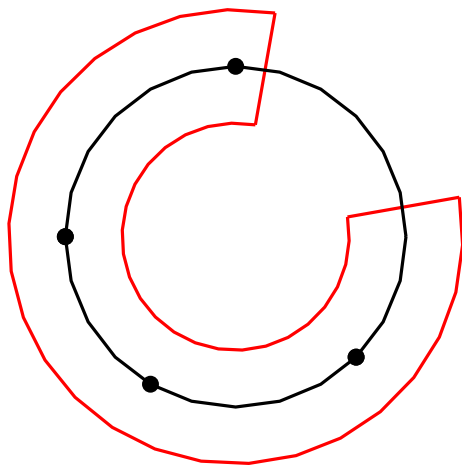
Introducing,

$$m(z, t) = \frac{1}{n} \sum_{i=1}^n \frac{1}{\lambda_i(t) - z} = \frac{1}{n} \text{tr} \frac{1}{U_t - z} \quad (5.3)$$

we therefore have,

$$\bar{\theta}(t_1 \wedge \tau) - \bar{\theta}(t_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{-g_{\theta_0}(z)}{i} (m(z, t_1 \wedge \tau) - m(z, t_0)) dz. \quad (5.4)$$

Fig. 1 Schematic diagram of contour Γ defined in (5.1). Contour in red; unit circle in black. Eigenvalues are the thick black dots enclosed by the red contour (color figure online)



For $t_0 < t < \tau$, $m(z, t)$ is well-defined for $z \in \Gamma$ and so we may apply the Itô formula. Since,

$$f(z, t) = 1 + 2zm(z, t) \quad (5.5)$$

we have from (1.11) that (abbreviating $g = g_{\theta_0}$ and $m = m(z, t)$),

$$\begin{aligned} d \frac{1}{2\pi i} \int_{\Gamma} ig(z)m(z, t)dz &= \frac{1}{2\pi i} \int_{\Gamma} \frac{-ig}{2}(1 + 2zm)(m + zm')dzdt \\ &+ \frac{1}{2\pi i} \int_{\Gamma} g(z) \frac{1}{n} \operatorname{tr} \left(\frac{U_t}{(U_t - z)^2} dW \right) dz. \end{aligned} \quad (5.6)$$

Observe that by the holomorphic functional calculus,

$$\frac{1}{2\pi i} \int_{\Gamma} g(z) \frac{1}{(U_t - z)^2} dz = U_t^{-1}, \quad (5.7)$$

and so

$$\frac{1}{2\pi i} \int_{\Gamma} g(z) \frac{1}{n} \operatorname{tr} \left(\frac{U_t}{(U_t - z)^2} dW \right) dz = \frac{1}{n} \operatorname{tr} (dW). \quad (5.8)$$

The first line of (5.6) turns out to vanish identically. To see this, we will evaluate all of the integrals explicitly using the Cauchy integral formula as well as the variants

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{(z - a)(z - b)} dz = \frac{F(b) - F(a)}{b - a} \quad (5.9)$$

and

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{(z-a)^2(z-b)} dz = -\frac{F'(a)}{b-a} + \frac{F(b)-F(a)}{(b-a)^2} \quad (5.10)$$

valid for F analytic in the appropriate domain and Γ encircling a, b . We have,

$$\begin{aligned} \frac{1}{2\pi i} \int g(z)(1+2zm)(m+zm')dz &= \frac{1}{2\pi i} \int g(z)(m+2zm^2+zm'+2z^2mm')dz \\ &= \frac{1}{2\pi i} \int dz g(z) \left\{ \frac{1}{n} \sum_i \frac{1}{\lambda_i - z} + 2 \frac{z}{n^2} \sum_{i \neq j} \frac{1}{(\lambda_i - z)(\lambda_j - z)} \right. \\ &\quad + z(2n^{-1} + 1) \frac{1}{n} \sum_i \frac{1}{(\lambda_i - z)^2} \\ &\quad \left. + 2 \frac{z^2}{n^2} \sum_{i \neq j} \frac{1}{(\lambda_i - z)(\lambda_j - z)^2} + 2 \frac{z^2}{n^2} \sum_i \frac{1}{(\lambda_i - z)^3} \right\} \\ &= -\frac{1}{n} \sum_i g(\lambda_i) + \frac{2}{n^2} \sum_{i \neq j} \frac{\lambda_i g(\lambda_i) - \lambda_j g(\lambda_j)}{\lambda_i - \lambda_j} + \frac{2n^{-1} + 1}{n} \sum_i (g(\lambda_i) + 1) \\ &\quad + \frac{1}{n^2} \sum_{i \neq j} \frac{4\lambda_j g(\lambda_j) + 2\lambda_j}{\lambda_i - \lambda_j} - \frac{2}{n^2} \sum_{i \neq j} \frac{\lambda_i^2 g(\lambda_i) - \lambda_j^2 g(\lambda_j)}{(\lambda_i - \lambda_j)^2} - \frac{1}{n^2} \sum_i (2g(\lambda_i) + 3). \end{aligned} \quad (5.11)$$

Now, note that

$$\frac{2}{n^2} \sum_{i \neq j} \frac{\lambda_i^2 g(\lambda_i) - \lambda_j^2 g(\lambda_j)}{(\lambda_i - \lambda_j)^2} = 0 \quad (5.12)$$

by symmetry as well as,

$$-\frac{1}{n} \sum_i g(\lambda_i) + \frac{2n^{-1} + 1}{n} \sum_i (g(\lambda_i) + 1) - \frac{1}{n^2} \sum_i (2g(\lambda_i) + 3) = 1 - \frac{1}{n} \quad (5.13)$$

Finally,

$$\begin{aligned} \frac{2}{n^2} \sum_{i \neq j} \frac{\lambda_i g(\lambda_i) - \lambda_j g(\lambda_j)}{\lambda_i - \lambda_j} + \frac{1}{n^2} \sum_{i \neq j} \frac{4\lambda_j g(\lambda_j) + 2\lambda_j}{\lambda_i - \lambda_j} &= \frac{2}{n^2} \sum_{i \neq j} \frac{\lambda_j}{\lambda_i - \lambda_j} \\ &= -\frac{n(n-1)}{n^2} \end{aligned} \quad (5.14)$$

and so indeed the term on the first line of (5.6) vanishes. It follows that for any $t_0 < t_1 \leq \tau$ we have

$$\bar{\theta}(t_1) - \bar{\theta}(t_0) = \frac{1}{n} (\operatorname{tr} W_{t_1} - \operatorname{tr} W_{t_0}). \quad (5.15)$$

Fix a final time $T > 1$ and a large integer $m > 0$, and intermediate times $t_i = iT/m$. Let θ_k , $k = 1, 2, \dots, 4n$, be $4n$ equally spaced points along the unit circle. For any $\ell > 0$ define the arcs,

$$Z_{k,\ell} = \{z = e^{i\theta} : |\theta - \theta_k| < \ell/(100n)\}. \quad (5.16)$$

Throughout the remainder of the proof we will make use of the arcs $\{Z_{k,1}\}_k$ and $\{Z_{k,1/2}\}_k$, i.e., the choices $\ell = 1, 1/2$.

For each $0 \leq i \leq m$ and $1 \leq k \leq 4n$ define the stopping time $\tau_{i,k}$ as follows. If at time t_i there is an eigenvalue in $Z_{k,1}$ let $\tau_{i,k} = t_i$. Otherwise, $\tau_{i,k}$ be the first time $t_i < t \leq t_{i+1}$ that an eigenvalue hits the set $Z_{k,1/2}$. If this does not occur, let $\tau_{i,k} = \infty$. Then, let $\tau_i = \min_k \tau_{i,k}$. By the pigeonhole principle we have $\tau_i > t_i$ for all i . Finally, let $\tau^{(m)} = \min_i \tau_i$.

From the above discussion we have on the event $\tau^{(m)} = \infty$ and by telescoping that for any $0 < t < T$ we have

$$\bar{\theta}(t) = \frac{1}{n} \operatorname{tr} (W_t). \quad (5.17)$$

The proof of Proposition 1.8 will be complete once we prove that

$$\lim_{m \rightarrow \infty} \mathbb{P} [\tau^{(m)} = \infty] = 1. \quad (5.18)$$

The remainder of this section is devoted to this. Consider the points,

$$z_{\pm,k} = (1 + \eta) \exp \left[i \left(\theta_k \pm \frac{3}{4} \frac{1}{100n} \right) \right]. \quad (5.19)$$

Note that for any k that if $Z_{k,1}$ contains no eigenvalues at time t_i , then

$$|\operatorname{Re}[f(z_{\pm,k}, t_i)]| \leq C_1 n^2 \eta \quad (5.20)$$

for some $C_1 > 0$. On the other hand, if for this k we have that $\tau_{i,k} < \infty$ then there is some $t \in [t_i, t_{i+1}]$ such that,

$$|\operatorname{Re}[f(z_{+,k}, t)]| + |\operatorname{Re}[f(z_{-,k}, t)]| > \frac{c_1}{n\eta}. \quad (5.21)$$

Choosing η small enough so that

$$\frac{c_1}{n\eta} > 10C_1 n^2 \eta \quad (5.22)$$

we have that

$$\{\tau_i < \infty\} \subseteq \bigcup_{k=1}^{4n} \left\{ \exists t \in [t_i, t_{i+1}] : |f(z_{+,k}, t) - f(z_{+,k}, t_i)| + |f(z_{-,k}, t) - f(z_{-,k}, t_i)| > \frac{c_2}{10n\eta} \right\}. \quad (5.23)$$

The parameter $\eta > 0$ is fixed for the remainder of the proof. We see by (1.11) that (abbreviating $z = z_{\pm,k}$)

$$|f(z, t) - f(z, t_i)| \leq CT \frac{1}{m\eta^3} + (M_t - M_{t_i}). \quad (5.24)$$

By the BDG inequality,

$$\mathbb{P} \left[\sup_{t_i < t < t_{i+1}} |M_{t_i} - M_t| > s \right] \leq C \exp \left[-csm^{1/2}n^{-2}T^{-1} \right]. \quad (5.25)$$

So taking $s = c_2/(100n\eta)$ we see that for all m large enough, there are constants depending on n and η such that

$$\mathbb{P}[\tau_i < \infty] \leq Ce^{-m^{1/2}c}. \quad (5.26)$$

This completes the proof of Proposition 1.8. $\square$

6 Rigidity

6.1 Helffer–Sjöstrand formula

This section establishes an analog of the Helffer–Sjöstrand formula for measures on the unit circle. Recall,

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \quad (6.1)$$

as well as Green's theorem,

$$F(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{(\partial_{\bar{z}} F)(x, y)}{\lambda - (x + iy)} dx dy \quad (6.2)$$

for any $F \in C^2$ of compact support. In polar coordinates we recall,

$$\partial_{\bar{z}} F(r, \theta) = \frac{e^{i\theta}}{2} \left(\partial_r F + \frac{i}{r} \partial_\theta F \right). \quad (6.3)$$

For any F supported in an annulus,

$$F(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} F}{2z} \frac{\lambda + z}{\lambda - z} dx dy. \quad (6.4)$$

Let $\varphi : [0, 2\pi] \rightarrow \mathbb{R}$ be a function on the unit circle that extends to a smooth 2π -periodic function on $\mathbb{R}$. We define the quasi-analytic extension of φ by,

$$\tilde{\varphi}(r, \theta) := (\varphi(\theta) - i \log(r) \varphi'(\theta)) \chi(r) \quad (6.5)$$

where $\chi(r)$ is a function that is 1 on $[3/4, 4/3]$ and 0 outside of $[1/2, 2]$. We may also assume that,

$$\chi(r) = \chi(r^{-1}). \quad (6.6)$$

If μ is any measure on the unit circle with Cauchy transform $f_\mu(z)$ we see that,

$$\begin{aligned} \int \varphi(\theta) d\mu(\theta) &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} \tilde{\varphi}}{2z} f_\mu(z) dx dy \\ &= \frac{1}{4\pi} \int (\varphi(\theta) - i \log(r) \varphi'(\theta)) \chi'(r) f_\mu(z) dr d\theta \\ &\quad + \frac{1}{4\pi} \int \varphi''(\theta) \log(r) \chi(r) f_\mu(z) r^{-1} dr d\theta. \end{aligned} \quad (6.7)$$

This is the Helffer–Sjöstrand formula we require to establish our eigenvalue estimates and will be used in the next subsections.

6.2 Proof of Corollary 1.3

We follow Section 3.3 of [30] very closely, as the estimates of Theorem 1.2 and the formula (6.7) combine in the exactly same manner there as here to prove Corollary 1.3. Introduce,

$$\eta(\theta) = \inf_{\eta > 1} \{(\eta - 1) |\operatorname{Re}[\tilde{f}(\eta e^{i\theta})]| \geq N^{\varepsilon-1}\}. \quad (6.8)$$

Note that from Lemma B.2,

$$\eta \rightarrow (\eta - 1) |\operatorname{Re}[\tilde{f}(\eta e^{i\theta})]| \quad (6.9)$$

is an increasing function for $\eta > 1$. We first assume $t < 2$. We consider $I = [\theta_0, \pi]$ for some $0 < \theta_0 < \pi$. The case $-\pi < \theta_0 < 0$ requires only notational changes. Define,

$$\tilde{\eta} := \inf_{\eta: \log \eta \geq N^{1-\varepsilon}} \{\eta : \max_{\theta_0 \leq x \leq \theta_0 + \eta} \eta(x) \leq \eta\}. \quad (6.10)$$

Let $\tilde{\theta}$ be

$$\tilde{\theta} := \operatorname{argmax}_{\theta_0 - \tilde{\eta} \leq x \leq \theta_0} \eta(x) \quad (6.11)$$

so that

$$\eta(\tilde{\theta}) = \tilde{\eta}. \quad (6.12)$$

We let φ be a function that is 1 on $[\theta_0, \pi]$ and 0 outside of $[\theta_0 - \log \tilde{\eta} \wedge \frac{1}{10}, \pi + N^{2\varepsilon-1}]$, with $|\varphi^{(k)}(x)| \leq C_k(N^{1-2\varepsilon})^k$ for x near π and $|\varphi^{(k)}(x)| \leq C_k(\log \tilde{\eta})^{-k}$ for x near θ_0 .

By (6.7) we have,

$$\begin{aligned} \left| \frac{1}{N} \sum_i \varphi(\lambda_i(t)) - \int \varphi(\theta) \rho_t(\theta) d\theta \right| &\leq C \int (|\varphi(\theta)| + |\varphi'(\theta)|) |\chi'(r)| |S(z)| dr d\theta \\ &\quad + \left| \int \varphi''(\theta) \log(r) \chi(r) S(z) r^{-1} dr d\theta \right|, \end{aligned} \quad (6.13)$$

where

$$S(z) = f(z, t) - \tilde{f}(z, t). \quad (6.14)$$

On the event that the estimates of Theorem 1.2 hold we see that,

$$\int (|\varphi(\theta)| + |\varphi'(\theta)|) |\chi'(r)| |S(z)| dr d\theta \leq C \frac{N^\varepsilon}{N}. \quad (6.15)$$

For the second term, note that the measure $r^{-1} dr$ is invariant under the transformation $r \rightarrow r^{-1}$ so that,

$$\begin{aligned} &\left| \int \varphi''(\theta) \log(r) \chi(r) S(z) r^{-1} dr d\theta \right| \\ &= 2 \left| \int_0^{2\pi} \int_{r>1} \varphi''(\theta) \log(r) |\operatorname{Re}[S(z)]| \chi(r) r^{-1} dr d\theta \right| \end{aligned} \quad (6.16)$$

Recall that $\varphi''(x) = 0$ unless $x \in [\pi, \pi + N^{2\varepsilon-1}]$ or $x \in [\theta_0 - \log \tilde{\eta} \wedge \frac{1}{10}, \theta_0]$. Note that for $r \geq \eta_c$ where $\log \eta_c := N^{-\epsilon}$ we have,

$$|\operatorname{Re}[S(z)]| \leq C \frac{N^\varepsilon}{N \log |z|} \quad (6.17)$$

using Lemma B.2 (see Remark 3.3 of [30] for a similar argument). Therefore,

$$\int_{\theta \in [\pi, \pi + N^{2\varepsilon-1}]} \int_{r>1} |\varphi''(\theta)| \log(r) |\operatorname{Re}[S(z)]| \chi(r) r^{-1} dr d\theta$$

$$\begin{aligned}
&\leq \int_{\theta \in [\pi, \pi + N^{2\varepsilon-1}]} \int_{r > \eta_\epsilon} |\varphi''(\theta)| \log(r) |\operatorname{Re}[S(z)]| \chi(r) r^{-1} dr d\theta \\
&\quad + C \int_{\theta \in [\pi, \pi + N^{2\varepsilon-1}]} \int_{1 < r < \eta_\epsilon} |\varphi''(\theta)| \chi(r) dr d\theta \leq C \frac{N^{3\varepsilon}}{N} + \frac{C}{N^\epsilon} \leq C \frac{N^{3\varepsilon}}{N}.
\end{aligned} \tag{6.18}$$

We used (6.17) in the first integral and the estimate $|(r-1)\operatorname{Re}[S(re^{i\theta})]| \leq C$ for the second. For the region where $\theta \in [\theta_0 - \log \tilde{\eta} \wedge \frac{1}{10}, \theta_0] =: J$ we first bound,

$$\begin{aligned}
&\int_{\theta \in J} \int_{1 < r < \tilde{\eta}} |\log(r)| |\operatorname{Re}[S(z)] \varphi''(\theta) \chi(r)| dr d\theta \\
&\leq \int_{\theta \in J} \int_{1 < r < \eta_\epsilon} |\log(r)| |\operatorname{Re}[S(z)] \varphi''(\theta) \chi(r)| dr d\theta \\
&\quad + \int_{\theta \in J} \int_{\eta_\epsilon < r < \tilde{\eta}} |\log(r)| |\operatorname{Re}[S(z)] \varphi''(\theta) \chi(r)| dr d\theta \\
&\leq C \frac{\eta_\epsilon}{\tilde{\eta}} + \frac{CN^\varepsilon}{N} \leq CN^{\varepsilon-1}.
\end{aligned} \tag{6.19}$$

For the first integral we used $|(r-1)\operatorname{Re}[S]| \leq 2$ and that $\tilde{\eta} \geq N^{1-\epsilon}$. For the second region we used again (6.17). For the contribution of $r > \tilde{\eta}$ we have, by partial integration

$$\begin{aligned}
&\int_{\theta \in J} \int_{r > \tilde{\eta}} \log(r) \varphi''(\theta) \chi(r) \operatorname{Re}[S] r^{-1} dr d\theta \\
&= \int_{\theta \in J} \theta'(\theta) \log(\tilde{\eta}) \operatorname{Im}[S] \tilde{\eta}^{-1} d\theta \\
&\quad - \int_{\theta \in J} \int_{r > \tilde{\eta}} \varphi'(\theta) \partial_r (\log(r) \chi(r) r^{-1}) \operatorname{Im}[S] dr d\theta.
\end{aligned} \tag{6.20}$$

By definition of $\tilde{\eta}$, all z appearing in the above integration lie in $\mathcal{B}_t$. Therefore, we may apply the estimate on S of Theorem 1.2 and obtain that both of these integrals are bounded above by $C \log(N) N^{\varepsilon-1}$.

When these estimates hold we therefore conclude that,

$$|\{i : \theta_i(t) \in I\}| \leq N \int \varphi(\theta) \rho_t(\theta) d\theta + CN^{3\varepsilon}. \tag{6.21}$$

For $t < 2$, ρ_t has no support for $|\theta - \pi| < \delta$ some $\delta > 0$. Therefore,

$$N \int \varphi(\theta) \rho_t(\theta) d\theta \leq N \int_I \rho_t(\theta) d\theta + N \int_{\theta_0 - \log \tilde{\eta} \wedge \frac{1}{10}}^{\theta_0} \rho_t(\theta) d\theta. \tag{6.22}$$

There are two cases. If $\tilde{\eta} = N^{-c}$ then the second integral is bounded by $\tilde{\eta}/\sqrt{t} = N^{-c}t^{-1/2}$. Otherwise,

$$\int_{\theta_0 - \log \tilde{\eta} \wedge \frac{1}{10}}^{\theta_0} \rho_t(\theta) d\theta \leq C(\tilde{\eta} - 1) |\operatorname{Re}[\tilde{\eta} e^{i\tilde{\theta}}]| \leq C \frac{N^{3\varepsilon}}{N}. \quad (6.23)$$

The lower bound for the number of $\theta_i \in I$ follows similarly. For $t > 2$, then the densities ρ_t all satisfy that $\inf_{|\theta| < c} \rho_t(\theta) > c$ for some $c > 0$. In this case, one uses intervals with one end-point at $\theta = 0$. In (6.22) there is then a second term on the RHS,

$$\int_{-N^{2\varepsilon-1}}^0 \rho_t(\theta) d\theta \leq C N^{2\varepsilon-1}. \quad (6.24)$$

Everything else is identical. This proves the corollary. $\square$

6.3 Proof of Corollary 1.9

It follows from Proposition 1.8 that with overwhelming probability,

$$\sup_{0 \leq t \leq T} |\bar{\theta}(t)| \leq \frac{N^\varepsilon}{N} \quad (6.25)$$

for any $\varepsilon > 0$. Next, for each i , we may write,

$$\theta_i(t) = 2\pi n_i(t) + \varphi_i(t) \quad (6.26)$$

for $n_i(t)$ an integer and $-\pi \leq \varphi_i(t) \leq \pi$. For any fixed $t > 0$ it follows from the fact that the eigenvalues never cross that the set $\{n_1(t), n_2(t), \dots, n_N(t)\}$ contains at most two consecutive integers whose absolute values we will denote by $m(t), m(t) + 1$. Let $N_1(t)$ be the number of eigenvalues s.t. $|n_i(t)| = m(t)$ and $N_2(t)$ be the number of eigenvalues s.t. $|n_i(t)| = m(t) + 1$. Let μ be a probability measure on $[-\pi, \pi]$ with a density. We have,

$$\int_0^\pi \theta d\mu(\theta) = \int_0^\pi \mu([t, \pi]) dt, \quad (6.27)$$

and since $\int_{-\pi}^\pi \theta \rho_t(\theta) d\theta = 0$ we have,

$$\left| \frac{1}{N} \sum_i \varphi_i(t) \right| \leq \frac{N^\varepsilon}{N} \quad (6.28)$$

by Corollary 1.3. From this it follows that,

$$(1 + m(t))N_2(t) + m(t)N_1(t) \leq N^\varepsilon \quad (6.29)$$

with overwhelming probability. In particular, $m(t) = 0$ and the number of eigenvalues s.t. $|\theta_i(t)| > \pi$ is at most N^ε with overwhelming probability. Using this, the remainder of Corollary 1.9 follows from Corollary 1.3 in a straightforward manner similar to the proof of Corollary 3.2 of [30]. $\square$

A Density of states calculations

Recall the limiting spectral measure $\rho_t(\theta)$ and its Cauchy transform,

$$\tilde{f}(z, t) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \rho_t(\theta) d\theta. \quad (\text{A.1})$$

Note that,

$$\tilde{f}(z, 0) = \frac{1+z}{1-z}, \quad (\text{A.2})$$

and that $\tilde{f}$ is constant along characteristics,

$$t \rightarrow z \exp \left[\frac{t}{2} \tilde{f}(z, 0) \right] = z \exp \left[\frac{t}{2} \frac{1+z}{1-z} \right]. \quad (\text{A.3})$$

Moreover, define the region,

$$\Gamma_t := \left\{ z \in \mathbb{C} : \operatorname{Re}[z] > 0 \text{ and } \left| \frac{z-1}{z+1} e^{\frac{t}{2}z} \right| < 1 \right\}. \quad (\text{A.4})$$

Then, the function $\tilde{f}(z, t)$ is a conformal map of the open unit disc into Γ_t [11]. The domain Γ_t has the following form. We let $x_+(t) > 1$ be the smallest solution larger than 1 of

$$\frac{1-x}{x+1} e^{\frac{t}{2}x} = 1, \quad (\text{A.5})$$

and if $t > 4$ we let, $x_-(t)$ be the largest solution less than 1 of

$$\frac{x-1}{x+1} e^{\frac{t}{2}x} = 1. \quad (\text{A.6})$$

If $t \leq 4$ set $x_-(t) = 0$. For $x \in (x_-(t), x_+(t))$ set

$$k_t(x) := \sqrt{\frac{(x+1)^2 - (x-1)^2 e^{tx}}{e^{tx} - 1}}. \quad (\text{A.7})$$

If $t < 4$ the boundary of Γ_t is the union of the vertical line on the imaginary axis,

$$\{z = x + iy : x = 0, |y| \leq \sqrt{4t^{-1} - 1}\} \quad (\text{A.8})$$

and the curves

$$\{z = x \pm ik_t(x) : 0 < x \leq x_+(t)\}. \quad (\text{A.9})$$

When $t \geq 4$ the boundary is the union of the curves,

$$\{z = x \pm ik_t(x) : x_-(t) \leq x \leq x_+(t)\}. \quad (\text{A.10})$$

Moreover, $\tilde{f}(z, t)$ extends to a bijection of the closed unit disc to the closure of Γ_t . It is also a conformal map of the complement of the closed unit disc in $\mathbb{C}$ to $-\Gamma_t$.

For the moment we consider the case $t < 4$. We have that $\tilde{f}(z, t)$ sends the arc

$$\{z = e^{i\theta} : \pi \geq |\theta| > \Theta_t\} \quad (\text{A.11})$$

to the boundary of Γ_t that intersects the imaginary axis. Here,

$$\Theta_t := \frac{1}{2}\sqrt{(4-t)t} + \arccos\left(1 - \frac{t}{2}\right) = \frac{1}{2}\sqrt{(4-t)t} + 2\arcsin\left(\sqrt{\frac{t}{4}}\right) \quad (\text{A.12})$$

Using the expansion,

$$\arcsin(1-x) = \frac{\pi}{2} - \sqrt{2}x^{1/2} - \frac{\sqrt{2}}{12}x^{3/2} + \mathcal{O}(x^{5/2}) \quad (\text{A.13})$$

one can check that the gap satisfies,

$$\Delta_t = 2(\pi - \Theta_t) = \frac{s^{3/2}}{3} + \mathcal{O}(s^{5/2}) \quad (\text{A.14})$$

where $t = 4 - s$. We require the following a-priori bound. First, note that for any Cauchy transform of a measure on the unit circle and $z = re^{i\varphi}$,

$$\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \rho(\theta) d\theta = (1 - |z|^2) \int_0^{2\pi} \frac{1}{|e^{i\theta} - z|^2} \rho(\theta) d\theta + i2r \int_0^{2\pi} \frac{\sin(\varphi - \theta)}{|e^{i\theta} - z|^2} \rho(\theta) d\theta. \quad (\text{A.15})$$

The proof of the following appears in Appendix A of [2].

Proposition A.1 *The following estimate holds for all $|z| < 1$ and $t > 0$,*

$$|\tilde{f}(z, t)| \leq 3 \left(1 + t^{-1/2}\right). \quad (\text{A.16})$$

The extension of $\tilde{f}$ to the closed unit disc satisfies,

$$\frac{\tilde{f}(z, t) - 1}{\tilde{f}(z, t) + 1} e^{\frac{t}{2} \tilde{f}(z, t)} = z. \quad (\text{A.17})$$

Let,

$$w_0 = i\sqrt{\frac{4}{t} - 1}, \quad z_0 = e^{i\Theta_t}. \quad (\text{A.18})$$

Note that this is the value of $\tilde{f}(z_0, t) = w_0$. Our goal is now to derive an approximate cubic equation via Taylor expansion for $\tilde{f}(z, t)$ where z are points on the boundary of the unit disc close to z_0 . This will allow for conclusions about the behavior of ρ_t near the edge later. In order to facilitate this expansion, we introduce the coordinates $E > 0$ and $q \in \mathbb{C}$ via,

$$z = e^{i(\Theta_t - E)}, \quad \tilde{f}(z, t) = w_0 - iq \quad (\text{A.19})$$

so that $\text{Im}[q] \geq 0$, as $\text{Re}[\tilde{f}(z, t)] > 0$ on the open unit disc, with strict inequality extending to points on the boundary inside the support of the spectral measure.

With these coordinates,

$$\rho_t(E) = \frac{1}{2\pi} \text{Im}[q((1^-)e^{iE})] \quad (\text{A.20})$$

in the sense of boundary values. In the longer version [2] we implement a Taylor expansion of the self-consistent equation for $\tilde{f}(z, t)$ to obtain the following.

Proposition A.2 *Let $0 < t < 4$. Let $t = 4 - s$. Let $\tilde{f}(z + z_0, t) = w_0 - iq$. The following holds for $|q| \leq t^{-1/2}/10 + \mathbf{1}_{\{t>1\}}100$. First,*

$$z = iq^2 e^{i\Theta_t} \frac{t^{3/2}s^{1/2}}{8} + iq^3 e^{i\Theta_t} \frac{(3-t)t^2}{24} + \mathcal{O}(t^{5/2}|q|^4) \quad (\text{A.21})$$

Second, if $z = e^{i(\Theta_t - E)}$ then $\tilde{f}(z, t) = w_0 - iq$ satisfies,

$$0 = E - i\frac{E^2}{2} + q^2 \frac{t^{3/2}s^{1/2}}{8} + q^3 \frac{(3-t)t^2}{24} + \mathcal{O}(t^{5/2}|q|^4 + |E|^3) \quad (\text{A.22})$$

Remark Due to Proposition A.1, the assumption on q holds for all z for $1 < t < 4$.

Using the above expansion, we can derive the square-root behavior for times away from 0 and 4. A proof appears in Appendix A of [2].

Proposition A.3 *Let $\delta > 0$ and assume $\delta < t < 4 - \delta$. Then, for $|E| \leq c_\delta$ we have,*

$$\rho_t(\Theta_t - E) = \frac{E^{1/2}}{\pi} \sqrt{\frac{2}{t^{3/2}(4-t)^{1/2}}} (1 + \mathcal{O}(E)). \quad (\text{A.23})$$

At $t = 4$, the following shows that we have an exact cusp. The proof can be found in Appendix A of [2] and also serves as a useful warm-up for the near-cusp case.

Proposition A.4 *Let $t = 4$. Then,*

$$\rho_t(\pi - E) = E^{1/3} \left(\frac{3}{2} \right)^{1/3} \frac{\sqrt{3}}{4\pi} \left(1 + \mathcal{O}(|E|^{1/3}) \right) \quad (\text{A.24})$$

We will now consider the behavior of ρ_t for times t close to 4. For this, we will follow relatively closely the arguments of Section 9 of the work [3] of Ajanki, Erdős and Krüger. There, they showed how one may use Cardano's formula for roots of cubics to find expressions for spectral measures. As the coefficients of our equation are explicit, we are able to short-cut some of the arguments of [3] in adapting them to our setting.

Following [3], we introduce the universal edge shape function,

$$\Psi_e(\lambda) = \frac{\sqrt{(1+\lambda)\lambda}}{(1+2\lambda+2\sqrt{(1+\lambda)\lambda})^{2/3} + (1+2\lambda-2\sqrt{(1+\lambda)\lambda})^{2/3} + 1} \quad (\text{A.25})$$

As advertised, we characterize the spectral measure in the case of two nearby edges in terms of this shape function. A proof of the below Proposition appears in Appendix A of [2].

Proposition A.5 *Let $3.5 < t < 4$. Then,*

$$\rho_t(\Theta_t - E) = \Delta_t^{1/3} \left(\frac{\Delta_t^{1/6} (t-3)t^{5/4}}{2\sqrt{2}s^{1/4}\pi} \right) \Psi_e \left(\frac{E}{\Delta_t} \right) + \mathcal{O} \left(\min\{E^{3/2}s^{-5/4}, E^{2/3}\} \right), \quad (\text{A.26})$$

or equivalently,

$$\rho_t(\Theta_t - E) = s^{1/2} \left(\frac{t^{5/4}(t-3)}{\pi 2\sqrt{6}} \right) \Psi_e \left(\frac{3E}{s^{3/2}} \right) + \mathcal{O} \left(\min\{E^{3/2}s^{-5/4}, E^{2/3}\} \right) \quad (\text{A.27})$$

The asymptotics of the shape function are given in (9.63) of [3],

$$\Delta_t^{1/3} \Psi_e \left(\frac{E}{\Delta_t} \right) \asymp \begin{cases} \frac{E^{1/2}}{\Delta_t^{1/6}}, & E \leq \Delta_t \\ E^{1/3}, & E \geq \Delta_t \end{cases}. \quad (\text{A.28})$$

The short time regime is as follows. A proof appears in Appendix A of [2].

Proposition A.6 *There is a $\delta > 0$ so that for all $t < \frac{1}{10}$ we have,*

$$\rho_t(\Theta_t - E) = \frac{\sqrt{2}}{\pi s^{1/4} t^{1/2}} \sqrt{\frac{E}{t^{1/2}}} \left(1 + \mathcal{O}(Et^{-1/2})\right) \quad (\text{A.29})$$

for $|E| \leq t^{1/2}\delta$.

We consider now $t > 4$. Let $w_0 = x_-(t) := x$. This is the value of $\tilde{f}(z, t)$ at $z = -1$. Note that it is purely real. The equation defining w_0 is

$$e^{tx/2} = \frac{2}{1-x} - 1. \quad (\text{A.30})$$

Introducing $t = 4 + s$ we can expand this to find,

$$x^2(4 - t^3/24) - x2s(1 + s/8) - s = \mathcal{O}(x^3). \quad (\text{A.31})$$

We see that $|x| \leq Cs^{1/2}$, and then that,

$$x = s^{1/2}(4 - t^3/24)^{-1/2}(1 + \mathcal{O}(s^{1/2})) = \sqrt{3}s^{1/2}/2 + \mathcal{O}(s). \quad (\text{A.32})$$

Similar to the regime $t < 4$ we derive a self-consistent equation for the Cauchy transform, but now in the regime $t > 4$. The proof appears in Appendix A of [2].

Proposition A.7 *There is a $\delta > 0$ so that for $4 < t < 4 + \delta$ the following holds. Let $z = e^{i(\pi-E)}$ and $\tilde{f}(z, t) = w_0 - iq$. Then,*

$$E + Aq + Biq^2 + Cq^3 = \mathcal{O}(|E|^2 + |q|^4) \quad (\text{A.33})$$

where

$$A = \frac{4w_0^2 + sw_0^2 - s}{2(1 + w_0)(1 - w_0)} \quad (\text{A.34})$$

and

$$B = \left(w_0(2t - t^2/8) - (t - 4)^2/8 + w_0^2(t + t^2/8) + w_0^3t^2/8\right) \frac{1}{(1 + w_0)^2(w_0 - 1)} \quad (\text{A.35})$$

and

$$C = \frac{1}{(1 + w_0)(w_0 - 1)} \left(\frac{t^3}{48}(w_0^2 - 1) + \frac{t^2}{4} - \frac{t}{1 + w_0} + \frac{2}{(1 + w_0)^2}\right) \quad (\text{A.36})$$

We have $A > 0$ and $C < 0$ and $B < 0$, and $|A| \asymp s$ and $|B| \asymp s^{1/2}$ and $|C| \asymp 1$.

We now find an expression for ρ_t near $\theta = \pi$ for t just larger than 4. The proof of the following may be found in Appendix A of [2], and closely follows Section 9.1 of [3]. We define,

$$\Psi_m(\lambda) := \frac{\sqrt{1+\lambda^2}}{(\sqrt{1+\lambda^2} + \lambda)^{2/3} + (\sqrt{1+\lambda^2} - \lambda)^{2/3} - 1} - 1. \quad (\text{A.37})$$

Proposition A.8 *There is a $\delta > 0$ so that for $4 < t < 4 + \delta$ we have for $|E| \leq \delta$ that,*

$$\rho_t(\pi - E) - \rho_t(\pi) = \frac{s^{1/2}}{4\pi} (1 + J) \Psi_m \left(\frac{6E}{s^{3/2}} \right) + \mathcal{O} \left(\min\{|E|s^{-1/2}, |E|^{2/3}\} \right) \quad (\text{A.38})$$

and

$$\rho_t(\pi) = \frac{\sqrt{3}s^{1/2}}{4\pi} (1 + \mathcal{O}(s^{1/2})), \quad (\text{A.39})$$

and above

$$J = 6\gamma_1 + 3\gamma_1^2 - 2\gamma_2 = \mathcal{O}(s^{1/2}) \quad (\text{A.40})$$

where γ_i are introduced in the proof below.

The asymptotics of Ψ_m are,

$$s^{1/2} \Psi_m \left(\frac{6E}{s^{3/2}} \right) \asymp \begin{cases} \frac{|E|^2}{s^{3/2}}, & |E| \leq s^{3/2} \\ |E|^{2/3}, & |E| \geq s^{3/2} \end{cases}. \quad (\text{A.41})$$

From the characterization of Γ_t we see that the function $\rho_t(\theta)$ is monotonic in $[0, \pi]$. Hence, we conclude the following.

Lemma A.9 *Let $\delta > 0$. Then there is a $c > 0$ so that if either $t > 4 + \delta$ and $\theta \in [0, \pi]$ or $t > \delta$ and $0 \leq \theta \leq \Theta_t - \delta$ (where we set $\Theta_t = \pi$ for $t \geq 4$) we have,*

$$\rho_t(\theta) \geq c. \quad (\text{A.42})$$

If $t < \delta$ and $0 \leq \theta < \Theta_t - \sqrt{t}\delta_1$ some $\delta_1 > 0$ then there is a c_1 depending on δ and δ_1 such that,

$$\rho_t(\theta) \geq c_1 t^{-1/2}. \quad (\text{A.43})$$

B Auxilliary Proofs

This appendix collects a few results from calculus and other routine calculations. Proofs are all deferred to Appendix B of [2].

Lemma B.1 For $x > 1$ we have,

$$x \log(x) \leq \frac{x^2 - 1}{2} \quad (\text{B.1})$$

Lemma B.2 Let $f_\mu(z)$ be a Cauchy transform of a probability measure μ on the unit circle. Then the function,

$$r \rightarrow (r - 1)|\operatorname{Re}[f_\mu(re^{i\theta})]| \quad (\text{B.2})$$

is increasing for $r > 1$.

Lemma B.3 Let $\rho_t(E)$ be as above and $T < 4$. There is an $\varepsilon > 0$ so that with $z = re^{i(\Theta_t + \kappa)}$ with,

$$0 \leq r - 1 < \varepsilon, \quad 0 < \kappa < \varepsilon \quad (\text{B.3})$$

we have,

$$\operatorname{Im}[\tilde{f}(z, t)] - \operatorname{Im}[\tilde{f}(e^{i\Theta_t}, t)] \leq -d\sqrt{\kappa}. \quad (\text{B.4})$$

for some $d > 0$, uniformly in $0 < t < T$.

B.1 Cusp calculation

In this section we consider general densities on the unit circle $\rho(x)dx$ with support $[-\pi + \Delta/2, \pi - \Delta/2]$ where $0 < \Delta < \frac{1}{100}$. We assume that ρ is symmetric, $\rho(x) = \rho(-x)$. Let $\Theta = \pi - \Delta/2$ denote the edge. We moreover assume that there is a $C_1 > 0$ so that

$$\frac{1}{C_1}\sqrt{x} \leq \Delta^{1/6}\rho(\Theta - x) \leq C_1\sqrt{x}, \quad 0 < x < \Delta \quad (\text{B.5})$$

and

$$\frac{1}{C_1}x^{1/3} \leq \rho(\Theta - x) \leq C_1x^{1/3}, \quad \Delta < x < \Theta. \quad (\text{B.6})$$

A proof of the following appears in Appendix B.1 of [2].

Proposition B.4 Under the above assumptions there is a $\frac{1}{2} > c_1 > 0$ and a $d > 0$ depending only on $C_1 > 0$ so that the following holds. For $z = (1 + \eta)e^{i(\Theta + \kappa)}$ with $0 < \eta < c_1\Delta$ and $0 < \kappa < \Delta/2$,

$$\operatorname{Im}[f(z)] - \operatorname{Im}[f(e^{i\Theta})] \leq -d\Delta^{-1/6}\sqrt{\kappa}. \quad (\text{B.7})$$

B.2 Extended spectral domain

The following proposition is used in the Proof of Theorem 1.6. It uses only as inputs Theorems 1.2 and 1.4, and its proof appears in Appendix B.2 of [2].

Proposition B.5 *Let $c < t < 4 - c$ for some $c > 0$. Let $\varepsilon, \delta > 0$. In the domain*

$$\mathcal{D}_1 := \{z = (1 + \eta)e^{i\theta} : N^{\delta-1} < \eta < \delta^{-1}\} \quad (\text{B.8})$$

the estimate

$$|\tilde{f}(z, t) - f(z, t)| \leq \frac{N^\varepsilon}{N\eta} \quad (\text{B.9})$$

holds with overwhelming probability. In the domain,

$$\mathcal{D}_2 := \{z = (1 + \eta)e^{i\theta} : N^{\delta-1} < \eta < \delta^{-1}, \Theta_t + N^{-2/3+\delta} < |\theta| \leq \pi\} \quad (\text{B.10})$$

the estimate

$$|\tilde{f}(z, t) - f(z, t)| \leq \frac{N^\varepsilon}{N(\eta + \kappa)} \quad (\text{B.11})$$

holds with overwhelming probability.

C General β

In this section we discuss extensions of our result to general $\beta > 0$. In this case, we consider the N particles all on $\mathbb{R}$,

$$d\theta_i(t) = \sqrt{\frac{2}{N\beta}} dW_i(t) + \frac{1}{2N} \sum_{j \neq i} \cot\left(\frac{\theta_i(t) - \theta_j(t)}{2}\right) dt \quad (\text{C.1})$$

where $\theta_i(0) = 0$ for all i and $\{W_i\}_i$ are a family of independent standard Brownian motions. We leave aside discussion of the existence of solutions to the above system for now and proceed at a formal level. Setting, $\lambda_i(t) = e^{i\theta_i(t)}$, we find

$$d\lambda_i = i\lambda_i \sqrt{\frac{2}{N\beta}} dW_i - \frac{1}{N} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} dt - \frac{1}{2} \lambda_i dt + \frac{\lambda_i(\beta - 2)}{2\beta N} dt \quad (\text{C.2})$$

recovering the dynamics (1.6) when $\beta = 2$. Letting now,

$$f(z, t) := \frac{1}{N} \sum_{i=1}^N \frac{\lambda_i + z}{\lambda_i - z} \quad (\text{C.3})$$

as before we find,

$$df(z, t) = -\frac{zf(z, t)}{2}\partial_z f(z, t) + dM_t^\beta(z) + \frac{\beta - 2}{2\beta N} \left(z\partial_z f(z, t) + z^2\partial_z^2 f(z, t) \right). \quad (\text{C.4})$$

where

$$dM_t^\beta(z) = -\sqrt{\frac{2}{N\beta}} \sum_{i=1}^N \frac{2iz\lambda_i(t)}{(\lambda_i(t) - z)^2} dW_i(t) \quad (\text{C.5})$$

In order to extend our main results, Theorems 1.2, 1.4 and 1.6 to the case of general β we must examine how the equation (C.4) changes when $\beta \neq 2$. First, the martingale $M_t^\beta(z)$ can be handled the exactly the same as $\beta = 2$ as the quadratic variations are the same up to constant factors.

What is at first non-obvious is whether or not the last term on the RHS (C.4) drastically changes the behavior of f or not. Recall that our proof strategy is to introduce a stopping time τ which is the first time an inequality of the form $|f(z_t, t) - \tilde{f}(z_t, t)| \leq N^\varepsilon \Phi(z_t)$ is violated along some collection of characteristics, where $\Phi(z_t)$ is some control parameter. In the proof of Theorem 1.2 we take $\Phi(z) = 1/N\eta$ (where we recall the notation $\eta = |z| - 1$) and in the other cases $\Phi(z) = B(z)$ with $B(z)$ defined in (3.29).

In order for our proofs to go through it suffices to check that for $t < \tau$ along any characteristic z_t we consider that,

$$\frac{1}{N} \int_0^t \left(|\partial_z f(z_s, s)| + |\partial_z^2 f(z_s, s)| \right) ds \leq N^{\varepsilon/2} \Phi(z_t) \ll N^\varepsilon \Phi(z_t). \quad (\text{C.6})$$

In all three cases (bulk, edge and cusp) we have that along the characteristics we consider we have $N^\varepsilon \Phi(z_t) \leq |\operatorname{Re}[\tilde{f}(z_t, t)]|$ (see Lemma 3.6 and Lemma 4.8 for the edge and cusp cases; the bulk case is obvious, see e.g., (2.17)) and so we may estimate the integrand on the LHS by

$$\frac{1}{N} \left(|\partial_z f(z_t, t)| + |\partial_z^2 f(z_t, t)| \right) \leq C \frac{|\operatorname{Re}[\tilde{f}(z_t, t)]|}{\eta_t \operatorname{dist}(z, \lambda)} \quad (\text{C.7})$$

where λ denotes the set of eigenvalues. For the bulk case, Theorem 1.2, we can use simply $\operatorname{dist}(z, \lambda) \geq \eta_t$ and find that the integral of the RHS of (C.7) is bounded above by $(N\eta_t)^{-1}$, which is sufficient.

For the edge regime, Theorem 1.4, one can modify appropriately the part of the proof of Proposition 3.7 that begins with (3.47) and ends with (3.52) to find an estimate of $CN^{\varepsilon/4} \log(N)(N(\kappa_t + \eta_t))^{-1}$ for the integral of the RHS of (C.7) over the time interval $[0, t]$. Similarly, in the cusp regime one can modify the part of the proof of Lemma 4.6 that starts with (4.52) and ends with (4.54) to arrive at the same estimate as in the cusp case.

The conclusion is then that formally the arguments of Sects. 2–4 extend to the more general (C.4) with only minor changes. However, all of this rests on the definition of notion of solution of (C.1) which we now briefly discuss.

In the work [14], Cépa and Lépingle have developed a solution theory for the equation (C.1) and various generalizations. In particular they show that for all choices of $\beta > 0$ and all choices of initial data, the equation (C.1) has a unique strong solution. The caveat is that when $\beta < 1$, the particles may collide. In such cases, the work [14] shows that the local time of collisions is sufficiently small so that, for example, the cotangent on the RHS (C.1) is a.s. locally integrable. When $\beta \geq 1$, even if the initial data has coincident particles, there are no collisions for any $t > 0$ almost surely. We expect the methods outlined above apply to the processes constructed in [14].

Alternatively, at least in the case $\beta \geq 1$, one can completely avoid discussion of collisions as follows. First, note that if the particles are initially distinct then one can construct a unique strong solution in an elementary way following, e.g., Chapter 4.3 of [7], and substituting in the Lyapunov function used in the proof of Theorem 3.1 of [14] to rule out collisions.

Suppose now that we consider (C.1) with initial data $\theta_i^\varepsilon = i\varepsilon$ where eventually we take $\varepsilon \rightarrow 0$. As long as, say, $\varepsilon \leq e^{-N}$, it should not be much trouble to recover all of the results of this paper for the θ^ε process, uniformly in $e^{-N} \geq \varepsilon > 0$.

Convergence of the sample paths $\theta_i^\varepsilon(t)$ to some limiting set of sample paths may then be proven using the maximum principle argument of Lemma 2.3 of [30] which establishes $\sup_{i, 0 < t < T} |\theta_i^\varepsilon(t) - \theta_i^\delta(t)| \leq C_T \sup_i |\theta_i^\varepsilon(0) - \theta_i^\delta(0)|$. The argument of Lemma 2.3 of [30] goes through for the process (C.1) after usage of the partial fraction expansion,

$$\pi \cot(\pi x) = \lim_{n \rightarrow \infty} \sum_{|i| \leq n} \frac{1}{x + i}. \quad (\text{C.8})$$

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