Spectral gap estimates for mixed *p*-spin models at high temperature

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Abstract
We consider general mixed *p*-spin mean field spin glass models and provide a method
to prove that the spectral gap of the Dirichlet form associated with the Gibbs measure
is of order one at sufficiently high temperature. Our proof is based on an iteration
scheme relating the spectral gap of the N-spin system to that of suitably conditioned
subsystems.

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1 Introduction

In this paper we consider general mixed *p*-spin mean field models with energies
described by the Hamiltonian $H_N : \Sigma_N := \{-1, 1\}^N \to \mathbb{R}$ of the form

$$H_N(\sigma) := \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1 i_2 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + \sum_{1 \leq i \leq N} \eta_i \sigma_i,$$

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for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Sigma_N$. Here, the $g_{i_1 i_2 \ldots i_p}$ are i.i.d. standard Gaussian random variables for all tuples $(i_1, i_2 \ldots, i_p) \in \mathbb{N}^p$ with $i_k \neq i_l$ for each $1 \leq k \neq l \leq p$ and $p \in \mathbb{N}$. If two indices $i_k = i_l$ coincide, we set $g_{i_1 \ldots i_k \ldots i_l \ldots i_p} \equiv 0$. We remark here that if one appropriately changes the values of the inverse temperature couplings $\beta_2, \ldots, \beta_p, \ldots$; this is equivalent to the multi-linear model in which $g_{i_1, \ldots, i_p}$ is a standard normal variable for all choices of indices $i_1, \ldots, i_p$. Moreover, we assume that the temperature coefficients $\beta_p \geq 0$ are summable and the external field strengths are denoted by $\eta_i \in \mathbb{R}$. The Gibbs measure $\mu_N : \mathcal{P}(\Sigma_N) \rightarrow (0, 1)$ that corresponds to H_N is defined by

$$\mu_N(\sigma) := \frac{1}{Z_N} e^{H_N(\sigma)}, \quad Z_N := \sum_{\sigma \in \Sigma_N} e^{H_N(\sigma)},$$

where Z_N denotes the partition function of the system. In the following, we write $\langle \cdot \rangle$ and $\langle \cdot ; \cdot \rangle$ for the expectation and covariance, respectively, relative to μ_N, that is

$$\langle f \rangle := \sum_{\sigma \in \Sigma_N} f(\sigma) \mu_N(\sigma), \quad \langle f; g \rangle := \sum_{\sigma \in \Sigma_N} (f(\sigma) - \langle f(\sigma) \rangle)(g(\sigma) - \langle g(\sigma) \rangle) \mu_N(\sigma).$$

We are interested in the spectral gap of the Dirichlet form w.r.t. μ_N. More precisely, denote the discrete partial derivative in direction σ_i, $i \in \{1, \ldots, N\}$, by

$$(\partial_i f)(\sigma) := \frac{1}{2} \big(f(\sigma) - f(\hat{\sigma}_i) \big), \quad \text{where } \hat{\sigma}_i := (\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_N).$$

Then we can write $H_N(\sigma) = B_j(\sigma) \sigma_j + H_N^{(j)}(\sigma)$ s.t. the j-th cavity field $B_j : \Sigma_N \rightarrow \mathbb{R}$, defined by

$$B_j(\sigma) = \sigma_j \partial_j H_N(\sigma),$$

and $H_N^{(j)} = H_N - B_j$ do not depend on σ_j (notice that $\partial_i \sigma_j = \delta_{ij} \sigma_j$). Heuristically, one may think of B_j as the effective magnetic field acting on σ_j that is caused by the remaining spins σ_i for $i \neq j$.

With these definitions, we define the Dirichlet form weighted by the cavity fields B_j through

$$D(f) := \sum_{j=1}^{N} \left\langle \cosh^{-2}(B_j)(\partial_j f)^2 \right\rangle \tag{1.1}$$

for every $f : \Sigma_N \rightarrow \mathbb{R}$ and the (inverse of its) spectral gap by

$$a_{H_N} := \sup_{\substack{f: \Sigma_N \rightarrow \mathbb{R}, \\ f \neq const.}} \frac{\langle f; f \rangle}{D(f)}.$$

We remark here that the Dirichlet form in (1.1) is the same as the one in [8, Eq. (4)] for the Glauber Dynamics, namely

$$D(f) = \frac{1}{N} \sum_{i=1}^{N} \langle \mathrm{Var}(f(\sigma)|\sigma_{-i}) \rangle,$$

where conditioning on σ_{-i} means that we fix all spins except the ith one. Our main result is the following theorem.

Theorem 1.1 *For every $\epsilon > 0$, if $\beta := \sum_{p \geq 2} \sqrt{p^3 \log p}\, \beta_p$ is sufficiently small (depending on ϵ), then there exist constants $c, C > 0$, not depending on $N \in \mathbb{N}$, s.t.*

$$\mathbb{P}\big(\{a_{H_N} > 1 + \epsilon\}\big) \leq C \exp(-c\, N).$$

Remarks 1. We clearly have $D(f) = 0$ if and only if f is constant, i.e. $f \equiv \langle f \rangle$. Projecting onto the orthogonal complement of such functions in $L^2(d\mu_N)$, observe that a_{H_N} is indeed the inverse of the spectral gap of D (extended in the obvious way to a non-negative quadratic form in $L^2(d\mu_N)$) above its zero ground state energy. Theorem 1.1 shows that, at sufficiently high temperature, the spectral gap is of order one.
2. Notice that, due to the weight factors $\cosh^{-2}(B_j) \leq 1$, Theorem 1.1 implies also that the Dirichlet form of [4] in $L^2(d\mu_N)$ has a spectral gap of order one, at sufficiently high temperature.
3. Choosing linear functions $f = \sum_{j=1}^{N} c_j \sigma_j$ for $\|c\|_2 \neq 0$, Theorem 1.1 implies in particular that the correlation matrix $M = (\langle \sigma_i; \sigma_j \rangle)_{1 \leq i, j \leq N}$ has operator norm bounded by $1 + \epsilon$ with high probability, if β is small enough.
4. In principle, our arguments can be used to determine an explicit smallness threshold on β up to which our proof is valid. We have not optimized our constants, but rough estimates suggest that the largest β is of order $O(10^{-4})$. Since the smallness condition is due to technical limitations of our method, but without physical significance, we refrain from providing further details on this.

Spectral gap inequalities appear most prominently in the study of Markov chains. The difference between the largest and second largest eigenvalues of the Markov chain essentially dictates how fast the chain can equilibrate; with a large spectral gap, one can expect exponentially fast equilibriation. Markov chain Monte Carlo is one of the most efficient tools used to model physical systems. Thus, establishing a spectral gap inequality for a particular Markov chain model of interest would be critical to show numerical modelers that their results would result in accurate predictions of the behavior of these statistical models here. One classical Markov chain algorithm is the Glauber dynamics; the spectral gap we prove shows that the Glauber dynamics on the hypercube for the *p*-spin spin glass will equilibriate very quickly.

Due to its importance in the numerical sampling of statistical models, spectral gap inequalities have been of central interest to researchers. In the special case $\beta_2 > 0$ and $\beta_p = 0$ for $p > 2$, our model reduces to the well-known Sherrington-Kirkpatrick

model [13]. In this case, a logarithmic Sobolev inequality (LSI) for a slightly different Dirichlet form follows for $\beta < 1/4$ from the main result of [4], whose proof is based on a single-step renormalisation and Bakry-Émery theory [3]. However, the proof applied in this case is not readily generalizable to general mixed p-spin models. Indeed, the quadratic nature of the Sherrington-Kirkpatrick model allowed the authors of [4] to 'complete the square' on the Sherrington-Kirkpatrick Hamiltonian (in fact, the results of [4] apply to general quadratic models) and represent it as the convolution of a continuous model with a coupled Bernoulli variable. At this point, one can prove an LSI separately for each part by classical methods and then prove an LSI for the convolution through a short computation. The fact that the interaction in H_N, defined above, is not quadratic means that this method fails at the first step and there seems to be no natural way to decompose the p-spin model into simpler models as in [4].

The papers [2, 7, 8], prove spectral gap inequalities or LSI via a method of stochastic localization. These methods allow the authors to interpolate the Sherrington-Kirkpatrick model to a mixture of 2-spin Ising models with a rank 1 quadratic interaction. This reduces the question of the spectral gap of the SK model to the spectral gap of a simpler model. Since the p-spin models have a more complicated spin interaction, it is unclear that current methods would allow one to easily decompose the interaction of the p-spin model to simpler parts. The paper [1] combines stochastic localization with approximate message passing in order to sample distributions from the Sherrington-Kirkpatrick models; this method is different from Markov chain Monte Carlo. The paper [9] treats the technically simpler spherical case, where one has access to Bakry-Emery type methods.

For spectral gap estimates in the physically very different low temperature regime, see, however, [5] and the references therein, where it is shown under rather general conditions that a spectral gap inequality as implied by Theorem 1.1 does not hold true.

Since these methods do not readily extend to p-spin case, we use a different approach inspired by ideas related to self-consistent relations and martingale arguments as introduced in [11, 12]. Theorem 1.1 is the first spectral gap bound for general mixed p-spin models at sufficiently high temperature on the hypercube.

2 Outline of the proof of Theorem 1.1

A simple computation shows that the spectral gap of a system with only one spin is equal to one (see Sect. 5 below for the short argument). In order to estimate the spectral gap of the N-spin system, we proceed iteratively over the system size. To this end, let us first introduce some additional notation. For disjoint subsets $A, B \subset \{1, \ldots, N\}$, we define $H_N^{[A,B]} \equiv H_{N,\sigma_A}^{[A,B]} : \Sigma_{N-|A \cup B|} \to \mathbb{R}$ by

$$H_{N,\sigma_A}^{[A,B]}(\sigma_{(A \cup B)^c}) = \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p \in B^c} g_{i_1 i_2 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + \sum_{i \in B^c} \eta_i \sigma_i,$$

where $S^c = \{1, \ldots, N\} \setminus S$ for $S \subset \{1, \ldots, N\}$, σ_S denotes $\sigma_S := (\sigma_i)_{i \in S}$ (in particular, we identify $\sigma_i = \sigma_{\{i\}}$) and where the coordinates of σ_A are understood to

be fixed. Observe that $H_N^{[A,B]}$ plays the role of the energy of the subsystem consisting of the spins σ_i, $i \in (A \cup B)^c$, conditionally on the spins σ_j for $j \in A$ and with the particles σ_j for $j \in B$ removed from the system. We then denote by $\langle \cdot \rangle_{[A,B]} \equiv \langle \cdot \rangle_{[A,B]}(\sigma_A)$ the conditional Gibbs measure induced by the reduced Hamiltonian $H_N^{[A,B]}$ and, in analogy to (1.1), we set

$$D_{[A,B]}(f) := \sum_{j \notin A \cup B} \left\langle \cosh^{-2}(B_j^{[A,B]})(\partial_j f)^2 \right\rangle_{[A,B]},$$

where $B_j^{[A,B]} := \sigma_j \partial_j H_N^{[A,B]}$. Notice that $B_j^{[A,B]}$ is formally obtained by setting σ_B equal to zero and fixing σ_A in B_j. More explicitly, it is given by

$$B_j^{[A,B]}(\sigma) = \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \left(\sum_{i_2,\dots,i_p \in B^c} g_{j i_2 \dots i_p} \sigma_{i_2} \dots \sigma_{i_p} + \dots \right.$$
$$\left. \dots + \sum_{i_1,\dots,i_{p-1} \in B^c} g_{i_1 i_2 \dots i_{p-1} j} \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_{p-1}} \right) + \eta_j \tag{2.1}$$

for $\sigma = (\sigma_A, \sigma_{A^c \cap B^c}) \in \Sigma_{N-|B|}$, with $\sigma_A \in \Sigma_{|A|}$ understood as being fixed.

Next, we define the spectral gap of the subsystem related to A, B by

$$a_{[A,B]} \equiv a_{[A,B]}(\sigma_A) := \sup_{\substack{f: \Sigma_{N-|A \cup B|} \to \mathbb{R}, \\ f \neq const.}} \frac{\langle f; f \rangle_{[A,B]}}{D_{[A,B]}(f)}, \tag{2.2}$$

and the maximal spectral gap over all $N - k$ spin subsystems by

$$a_{N-k} := \max_{\substack{A,B \subset \{1,\dots,N\}: \\ A \cap B = \emptyset, |A \cup B| = k}} \max_{\sigma_A \in \Sigma_{|A|}} a_{[A,B]}(\sigma_A). \tag{2.3}$$

For $i, j \in (A \cup B)^c$, we finally set $m_i^{[A,B]} = \langle \sigma_i \rangle_{[A,B]}$ and $m_{ij}^{[A,B]} = \langle \sigma_i; \sigma_j \rangle_{[A,B]}$.

Let us point out that, throughout the rest of this work, we keep the dependence of quantities like $\langle \cdot \rangle^{[A,B]}$, $H_N^{[A,B]}$, etc. on $\sigma_A \in \Sigma_{|A|}$ implicit, to ease the notation.

Our starting point for the proof of Theorem 1.1 is the following lemma.

Lemma 2.1 (Conditioning Lemma). *Let* $f : \Sigma_{N-k} \to \mathbb{R}$ *and let* A, B *be disjoint with* $|A \cup B| = k$, *then*

$$\langle f; f \rangle_{[A,B]} \leq \left(1 - \frac{1}{N-k} \right) a_{N-k-1} D_{[A,B]}(f) + \frac{1}{N-k} \sum_{j \notin A \cup B} \frac{\langle f; \sigma_j \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2}. \tag{2.4}$$

This inequality links the spectral gap $a_{[A,B]}$ over the $N - k$ spin system related to A and B with the maximal spectral gap a_{N-k-1} over the $N - k - 1$ spin systems, and

our proof of Theorem 1.1 is based on employing this relation inductively. The key point is then to show that the second term on the r.h.s. in (2.4) is under good control during the iteration. Heuristically, note that if we write $f = \sum_{j \notin A \cup B} c_j \sigma_j + h$ such that $\langle h; \sigma_j \rangle = 0$ (that is, up to a constant, h is the projection of f onto the orthogonal complement of the functions $\sigma \mapsto \sigma_j - m_j^{[A,B]}$ in $L^2(d\langle \cdot \rangle_{[A,B]}))$, then

$$\sum_{j \notin A \cup B} \frac{\langle f; \sigma_j \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} = \|(\Lambda^{[A,B]})^{1/2} M^{[A,B]} \mathbf{c}\|_2^2$$

for $\Lambda^{[A,B]} = \mathrm{diag}\big((1-(m_i^{[A,B]})^2)^{-1}\big)$, $M^{[A,B]} = (m_{ij}^{[A,B]})_{i,j \notin A \cup B}$ and $\mathbf{c} = (c_i)_{i \notin A \cup B}$. In particular, motivated by well-known results on the correlation matrix like for example [10], if we could ignore the order one entries on the diagonal of the correlation matrix $M^{[A,B]}$, we might expect the right hand to be of size $\epsilon \langle f; f \rangle$ for small ϵ if the inverse temperature coefficients are small enough, and with such a bound one could easily iterate (2.4) to conclude Theorem 1.1.

Of course, we can not simply ignore the diagonal of $M^{[A,B]}$ and therefore, we need to proceed slightly differently. First, in Sect. 3, we give the rough bound

$$\sum_{j \notin A \cup B} \frac{\langle f; \sigma_j \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} \leq C\sqrt{N-k} \langle f; f \rangle_{[A,B]}$$

on a set of probability close to one, uniformly in the subsets A, B. This bound implies that the maximal spectral gap can not have a large jump after adding one additional spin into the system. In the second step, we then control the error term in (2.4) through an improved iteration bound which, loosely speaking, has the form

$$\sum_{j \notin A \cup B} \frac{\langle f; \sigma_j \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} \leq \big(1 + O(\epsilon)\big) D_{[A,B]}(f) + O(\epsilon) \langle f; f \rangle_{[A,B]}$$

for small ϵ if the temperature coefficients are small and if we have a some a priori control on a_{N-k-1}. To obtain this bound, it turns out that we can follow a similar heuristics as above for the correlation matrix, but with the correlation matrix replaced by another matrix whose norm indeed turns out to be small at high temperature. Equipped with this improved estimate and the continuity argument from the first step, we use (2.4) inductively and conclude Theorem 1.1 in Sect. 5.

Let us conclude this overview with the proof of (2.4).

Proof of Lemma 2.1 By the total variance formula, we have for any $j \notin A \cup B$

$$\langle f; f \rangle_{[A,B]} = \langle \langle f(\cdot, \sigma_j); f(\cdot, \sigma_j) \rangle_{[A \cup \{j\}, B]} \rangle_{[A,B]} \tag{2.5}$$
$$+ \langle \langle f(\cdot, \sigma_j) \rangle_{[A \cup \{j\}, B]}; \langle f(\cdot, \sigma_j) \rangle_{[A \cup \{j\}, B]} \rangle_{[A,B]}.$$

By definition (2.3), we can bound the integrand in the first term on the r.h.s. in (2.5) by

$$\langle\langle f(\cdot, \sigma_j); f(\cdot, \sigma_j)\rangle_{[A\cup\{j\},B]}\rangle_{[A,B]}$$

$$\leq a_{N-k-1} \sum_{l\notin A\cup\{j\}\cup B} \left\langle\langle\cosh^{-2}(B_l^{[A\cup\{j\},B]})(\partial_l f(\cdot, \sigma_j))^2\rangle_{[A\cup\{j\},B]}\right\rangle_{[A,B]}$$

$$= a_{N-k-1} \sum_{l\notin A\cup\{j\}\cup B} \left\langle\cosh^{-2}(B_l^{[A,B]})(\partial_l f)^2\right\rangle_{[A,B]},$$

so that averaging over $j \notin A \cup B$ implies

$$\langle f; f\rangle_{[A,B]} \leq \left(1 - \frac{1}{N-k}\right) a_{N-k-1} D_{[A,B]}(f)$$

$$+ \frac{1}{N-k} \sum_{j\notin A\cup B} \langle\langle f(\cdot, \sigma_j)\rangle_{[A\cup\{j\},B]}; \langle f(\cdot, \sigma_j)\rangle_{[A\cup\{j\},B]}\rangle_{[A,B]}.$$

Finally, every function $\Sigma_{N-k} \ni \sigma \mapsto g(\sigma) = g(\sigma_j)$ that only depends on σ_j is a linear function $g(\sigma_j) = c\sigma_j + b$ for certain coefficients $b, c \in \mathbb{R}$. Thus $\langle g(\sigma_j); g(\sigma_j)\rangle_{[A,B]} = c^2(1 - (m_j^{[A,B]})^2)$ and $\langle g(\sigma_j); \sigma_j\rangle_{[A,B]} = c(1 - (m_j^{[A,B]})^2)$. We can rewrite this relationship as follows:

$$\langle g(\sigma_j); g(\sigma_j)\rangle_{[A,B]} = \frac{\langle g(\sigma_j); \sigma_j\rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2}.$$

Choosing $\sigma \mapsto g(\sigma) = \langle f(\cdot, \sigma_j)\rangle_{[A\cup\{j\},B]}$, we obtain

$$\langle g; g\rangle_{[A,B]} = \frac{\langle\langle f(\cdot, \sigma_j)\rangle_{[A\cup\{j\},B]}; \sigma_j\rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} = \frac{\langle f; \sigma_j\rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2}.$$

$\square$

3 Continuity argument

As mentioned in the previous section, our first goal is to show that, assuming a_{N-k-1} to be of order one, the rate with which a_{N-k} may increase is not too large, with high probability. To make this more precise, let us set from now on

$$\beta := \sum_{p\geq 2} \sqrt{p^3 \log p}\, \beta_p \tag{3.1}$$

and let us define the good event Ω by

$$\Omega := \left\{ \sup_{A \cap B = \emptyset} \sup_{\sigma_A \in \Sigma_{|A|}} \sup_{\sigma \in \Sigma_{N-|A \cup B|}} \left\| \left(\partial_i B_j^{[A,B]}(\sigma) \right)_{1 \leq i,j \leq N-|A \cup B|} \right\| \leq 5\beta \right\}, \quad (3.2)$$

where $\| \cdot \|$ denotes the standard matrix norm $\|A\| := \sup_{\mathbf{c} \in \mathbb{R}^d : \|\mathbf{c}\|_2 = 1} \|A\mathbf{c}\|_2$, for $A \in \mathbb{R}^{d \times d}$ (recall that for symmetric $A = A^T$, the norm coincides with $\|A\| = \sup_{\mathbf{c} \in \mathbb{R}^d : \|\mathbf{c}\|_2 = 1} \mathbf{c}^T A \mathbf{c}$). Our specific definition of β comes from appropriate tensor bounds on $\|(\partial_i B_j^{[A,B]})\|$ that appear in Sect. 6. The event Ω is similar to events found in the literature such as [6, Appendix B], where one needs to control the matrix norm of $\|\nabla^2 H_N\|$, though our estimates are slightly more involved. Then, we prove in Sect. 6 below that $\mathbb{P}(\Omega) \geq 1 - e^{-N}$ (see Lemma 6.1). In the following, let us also denote by C_β the constant

$$C_\beta := (10^3 \beta)^2 \exp(10^3 \beta), \quad (3.3)$$

whose specific form becomes clearer in Sect. 6. The main result of this section reads as follows.

Proposition 3.1 *Let β be as in (3.1), Ω as in (3.2), C_β as in (3.3) and let $\epsilon \in (0, 10^{-2})$ where ϵ does not depend on β. Assume that β is sufficiently small. Then, there exists a universal constant $C > 0$ such that*

$$\mathbb{P}\left(\Omega \cap \bigcup_{k=0}^{N-2} \left\{ a_{N-k-1} C_\beta < \epsilon \text{ and } a_{N-k} > 5 a_{N-k-1} \right\} \right)$$

$$\leq C \, e^{(C\epsilon/C_\beta + 2) \log N + N \log 4 - N C_\beta^2 (\epsilon^2 C \beta^2)^{-1}}.$$

Prop. 3.1 requires a couple of auxiliary results and follows by combining Lemma 3.1 and Corollary 3.4 below. We start with the following observation.

Lemma 3.1 *For any disjoint $A, B \subset \{1, \ldots, N\}$ with $|A \cup B| = k$, we have that*

$$\sum_{i \notin A \cup B} \frac{\langle \sigma_i ; f \rangle_{[A,B]}^2}{1 - (m_i^{[A,B]})^2} \leq \| (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \| \langle f ; f \rangle_{[A,B]},$$

where $\Lambda^{[A,B]} = \left(\delta_{ij} [1 - (m_i^{[A,B]})^2]^{-1} \right)_{i,j \notin A \cup B}$ and $M^{[A,B]} = (m_{ij}^{[A,B]})_{i,j \notin A \cup B}$. As a result, $a_{[A,B]}$ is bounded by

$$\left(1 - \frac{\| (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \|}{N - k} \right) a_{[A,B]} \leq a_{N-k-1}. \quad (3.4)$$

Proof We write $f = \sum_{i \notin A \cup B} c_i \sigma_i + h$ with $\langle h ; \sigma_i \rangle_{[A,B]} = 0$ for all $i \notin A \cup B$. Notice that, up to a constant, h is the projection of f onto the orthogonal complement of the

functions $\sigma \mapsto \sigma_i - m_i^{[A,B]}$ in $L^2(d\langle\cdot\rangle_{[A,B]})$. Then

$$\sum_{i\notin A\cup B} \frac{\langle\sigma_i; f\rangle_{[A,B]}^2}{1-(m_i^{[A,B]})^2} = \sum_{i\notin A\cup B} \frac{(\sum_j m_{ij}^{[A,B]}c_j)^2}{1-(m_i^{[A,B]})^2} = \|(\Lambda^{[A,B]})^{1/2}M^{[A,B]}\mathbf{c}\|_2^2$$
$$\leq \|(M^{[A,B]})^{1/2}\Lambda^{[A,B]}(M^{[A,B]})^{1/2}\|\langle\mathbf{c}, M^{[A,B]}\mathbf{c}\rangle_2 \qquad (3.5)$$
$$\leq \|(M^{[A,B]})^{1/2}\Lambda^{[A,B]}(M^{[A,B]})^{1/2}\|\langle f; f\rangle_{[A,B]},$$

where $\langle\cdot,\cdot\rangle_2$ denotes the standard Euclidean inner product, $\|\cdot\|_2$ is its induced norm and $\|\cdot\|$ denotes the matrix norm, i.e. the maximal eigenvalue in case of a symmetric, positive semi-definite matrix like $(M^{[A,B]})^{1/2}\Lambda^{[A,B]}(M^{[A,B]})^{1/2}$. Noting that $\|(M^{[A,B]})^{1/2}\Lambda^{[A,B]}(M^{[A,B]})^{1/2}\| = \|(\Lambda^{[A,B]})^{1/2}M^{[A,B]}(\Lambda^{[A,B]})^{1/2}\|$, we conclude the claim using Lemma 2.1 and the definition of $a_{[A,B]}$ in (2.2). □

To proceed further, we need some a priori information on the distribution of the cavity fields $B_j^{[A,B]}$. In essence, the next result allows us to control the Gibbs expectation of exponentials of $B_j^{[A,B]}$ by the exponentials evaluated at the Gibbs expectation of $B_j^{[A,B]}$. To focus on the main line of the argument for the proof of Theorem 1.1, we defer the proof of the following technical key lemma to Sect. 6.

Lemma 3.2 *Let β be as in (3.1), Ω as in (3.2), $\epsilon \in (0, 10^{-2})$, $A, B \subset \{1, \ldots, N\}$ disjoint with $|A \cup B| = k$ and let $j \notin A \cup B$. Assume that β is sufficiently small. If $a_{[A,B\cup\{j\}]}C_\beta < \epsilon$, then we have in Ω that*

$$\frac{1}{\langle\cosh^2(B_j^{[A,B]})\rangle_{[A,B\cup\{j\}]}} \leq \langle\cosh^{-2}(B_j^{[A,B]})\rangle_{[A,B\cup\{j\}]} \leq (1+4\epsilon)\left(1-(m_j^{[A,B]})^2\right),$$
$$(3.6)$$

$$\langle e^{-KB_j^{[A,B]}}\rangle_{[A,B\cup\{j\}]} \leq (1+4\epsilon)e^{-K\langle B_j^{[A,B]}\rangle_{[A,B\cup\{j\}]}}, \qquad (3.7)$$

$$1 \leq \frac{\langle\cosh(KB_j^{[A,B]})\rangle_{[A,B\cup\{j\}]}}{\cosh\left(K\langle B_j^{[A,B]}\rangle_{[A,B\cup\{j\}]}\right)} \leq 1+4\epsilon \qquad (3.8)$$

uniformly in $K \in [-20, 20]$.

We use Lemma 3.2 to prove the following result.

Lemma 3.3 *Let Ω be as in (3.2), C_β as in (3.3), $\epsilon \in (0, 10^{-2})$ and set*

$$\Omega_{N-k-1,\epsilon} = \{a_{N-k-1}C_\beta < \epsilon\}. \qquad (3.9)$$

Then, there exists a universal constant $C > 0$ such that in the event $\Omega \cap \Omega_{N-k-1,\epsilon}$, we have for every disjoint A, B with $|A \cup B| = k$, $j \notin A \cup B$, every $\sigma_A \in \Sigma_{|A|}$ and every function $g : \Sigma_{N-k} \to \mathbb{R}$ that does not depend on σ_j that

$$\langle\sigma_j; g\rangle_{[A,B]}^2 \leq \frac{C\epsilon}{C_\beta}\left(1-(m_j^{[A,B]})^2\right)^2 D_{[A,B\cup\{j\}]}(g).$$

Proof Assume in the following that $g : \Sigma_{N-k} \to \mathbb{R}$ is a function that does not depend on σ_j. Then, writing $H_N^{[A,B]} = H_N^{[A,B\cup\{j\}]} + \sigma_j B_j^{[A,B]}$, we have that

$$
\langle g \rangle_{[A,B]} = \frac{\left\langle \sum_{\sigma_j \in \{\pm 1\}} g e^{\sigma_j B_j^{[A,B]}} \right\rangle_{[A,B\cup\{j\}]}}{\left\langle 2\cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}} = \frac{\left\langle g \sinh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}
$$

(3.10)

and furthermore

$$
\langle \sigma_j ; g \rangle_{[A,B]} = \frac{\left\langle \sum_{\sigma_j \in \{\pm 1\}} \sigma_j g e^{\sigma_j B_j^{[A,B]}} \right\rangle_{[A,B\cup\{j\}]}}{\left\langle 2\cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}
$$
$$
- \frac{\left\langle \sum_{\sigma_j \in \{\pm 1\}} \sigma_j e^{\sigma_j B_j^{[A,B]}} \right\rangle_{[A,B\cup\{j\}]}}{\left\langle 2\cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}} \frac{\left\langle \sum_{\sigma_j \in \{\pm 1\}} g e^{\sigma_j B_j^{[A,B]}} \right\rangle_{[A,B\cup\{j\}]}}{\left\langle 2\cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}.
$$

To simplify the notation, let us abbreviate for the rest of the proof

$$
\langle \cdot \rangle_* = \langle \cdot \rangle_{[A,B\cup\{j\}]}, \quad \xi = \frac{\sinh(B_j^{[A,B]})}{\left\langle \cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}, \quad \zeta = \frac{\cosh(B_j^{[A,B]})}{\left\langle \cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}},
$$

so that, by the independence of g of σ_j, we find

$$
\langle \sigma_j ; g \rangle_{[A,B]} = \langle \xi g \rangle_* - \langle \xi \rangle_* \langle g \zeta \rangle_* = \langle \xi ; g \rangle_* - \langle \xi \rangle_* \langle g ; \zeta \rangle_* = \left\langle (\xi - \langle \xi \rangle_* \zeta); g \right\rangle_*.
$$

Notice that in the second step, we used $\langle \zeta \rangle_* = 1$. Observing also that

$$
\langle \xi \rangle_* = \frac{\left\langle \sinh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh(B_j^{[A,B]}) \right\rangle_{[A,B\cup\{j\}]}} = \langle \sigma_j \rangle_{[A,B]} = m_j^{[A,B]},
$$

(3.11)

we apply Cauchy-Schwarz and obtain the upper bound

$$
\langle \sigma_j ; g \rangle_{[A,B]}^2 \le \langle g ; g \rangle_* \left\langle \xi - m_j^{[A,B]} \zeta ; \xi - m_j^{[A,B]} \zeta \right\rangle_*
$$
$$
\le a_{N-k-1} D_{[A,B\cup\{j\}]}(g) \left\langle (\xi - m_j^{[A,B]} \zeta)^2 \right\rangle_*,
$$

(3.12)

where in the last step we applied spectral gap inequality $\langle g ; g \rangle_* \le a_{N-k-1} D_{[A,B\cup\{j\}]}(g)$. In Ω_{N-k-1}, we know that $a_{N-k-1} C_\beta < \epsilon$, so the claim follows if we show that the

last factor on the r.h.s. of the previous estimate is bounded by some absolute constant $C > 0$ times $(1 - (m_j^{[A,B]})^2)^2$. To this end, we bound

$$
\begin{aligned}
&\left\langle (\xi - m_j^{[A,B]}\zeta)^2 \right\rangle_* \\
&= \left\langle \cosh(B_j^{[A,B]}) \right\rangle_*^{-2} \left\langle \cosh^2(B_j^{[A,B]}) \left(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \right)^2 \right\rangle_* \\
&\leq \left\langle \cosh(B_j^{[A,B]}) \right\rangle_*^{-2} \left\langle \cosh^4(B_j^{[A,B]}) \right\rangle_*^{1/2} \left\langle \left(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \right)^4 \right\rangle_*^{1/2}.
\end{aligned}
$$

In $\Omega \cap \Omega_{N-k-1,\epsilon}$ we can apply Lemma 3.2, because $a_{N-k-1}C_\beta < \epsilon$ implies in particular that $a_{[A,B\cup\{j\}]}C_\beta < \epsilon$. Expanding $\cosh(x) = (e^x + e^{-x})/2$, this yields

$$
\left\langle \cosh(B_j^{[A,B]}) \right\rangle_*^{-2} \left\langle \cosh^4(B_j^{[A,B]}) \right\rangle_*^{1/2} \leq C \tag{3.13}
$$

and we also claim that

$$
\left\langle \left(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \right)^4 \right\rangle_* \leq C \left(1 - (m_j^{[A,B]})^2 \right)^4. \tag{3.14}
$$

Assuming the validity of (3.14) for the moment, the lemma follows by combining the previous four estimates. Hence, let us focus on the proof of (3.14). As explained below, this bound follows by giving a uniform (in N) lower bound on $1 + m_j^{[A,B]}$ or $1 - m_j^{[A,B]}$, depending on whether $\langle B_j^{[A,B]} \rangle_* \geq 0$ or $\langle B_j^{[A,B]} \rangle_* \leq 0$. To see this, let us assume first that $\langle B_j^{[A,B]} \rangle_* \geq 0$. Then, Lemma 3.2 implies

$$
\begin{aligned}
\left\langle \mathbf{1}\{ B_j^{[A,B]} \leq -1/2 \} \right\rangle_* &\leq \exp(-K/2) \left\langle \exp\left(-K B_j^{[A,B]} \right) \right\rangle_* \\
&\leq (1 + \epsilon) \exp(-K/2) \exp\left(-K \langle B_j^{[A,B]} \rangle_* \right) \leq 2 \exp(-K/2).
\end{aligned}
$$

for every $0 \leq K \leq 20$. Choosing $0 \leq K \leq 20$ suitably, we get

$$
\begin{aligned}
1 + m_j^{[A,B]} &= 1 + \left\langle \tanh\left(B_j^{[A,B]} \right) \right\rangle_{[A,B]} \\
&\geq 1 + \tanh(-1/2) - \frac{\langle \sinh^2(B_j^{[A,B]}) \rangle_*^{1/2}}{\langle \cosh(B_j^{[A,B]}) \rangle_*} \left\langle \mathbf{1}\{ B_j^{[A,B]} \leq -1/2 \} \right\rangle_*^{1/2} \\
&\geq 1 - \tanh(1/2) - 2 \exp(-K/4) \geq 1/(1 + e) > 0.
\end{aligned}
$$

Using this lower bound, the elementary bounds $0 \leq 1 - \tanh(x) \leq 2e^{-2x}$ and $\cosh^2(x) \leq \cosh(2x)$ for all $x \in \mathbb{R}$ and once again Lemma 3.2, we find that

$$
\begin{aligned}
\big\langle \big(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \big)^4 \big\rangle_* &\leq C \big\langle \big(1 - \tanh(B_j^{[A,B]}) \big)^4 \big\rangle_* + C \big(1 - m_j^{[A,B]} \big)^4 \\
&\leq C \big\langle e^{-8B_j^{[A,B]}} \big\rangle_* + C \big(1 - (m_j^{[A,B]})^2 \big)^4 \\
&\leq C e^{-8 \langle B_j^{[A,B]} \rangle_*} + C \big(1 - (m_j^{[A,B]})^2 \big)^4
\end{aligned}
$$

as well as

$$
\begin{aligned}
e^{-2\langle B_j \rangle_*} &\leq \frac{1}{2} \cosh \big(2 \langle B_j^{[A,B]} \rangle_* \big)^{-1} \leq C \big\langle \cosh \big(2 B_j^{[A,B]} \big) \big\rangle_*^{-1} \\
&\leq C \big\langle \cosh^2 \big(B_j^{[A,B]} \big) \big\rangle_*^{-1} \leq C \big(1 - (m_j^{[A,B]})^2 \big),
\end{aligned}
$$

where we used (3.6) in the last step. Combining the previous two display yields (3.14) for $\langle B_j^{[A,B]} \rangle_* \geq 0$.

If $\langle B_j^{[A,B]} \rangle_* < 0$, we proceed similarly as above and the bound (3.14) follows from estimating $\langle \mathbf{1}\{B_j^{[A,B]} \geq 1/2\} \rangle_* \leq 2 \exp(-K/2)$ so that $1 - m_j^{[A,B]} \geq 1/(1+e)$ by choosing a suitable $0 \leq K \leq 20$, and combining this with

$$
\begin{aligned}
\big\langle \big(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \big)^4 \big\rangle_* &\leq C \big\langle \big(1 + \tanh(B_j^{[A,B]}) \big)^4 \big\rangle_* + C \big(1 + m_j^{[A,B]} \big)^4 \\
&\leq C e^{8 \langle B_j^{[A,B]} \rangle_*} + C \big(1 - (m_j^{[A,B]})^2 \big)^4 \\
&\leq C \cosh \big(2 \langle B_j^{[A,B]} \rangle_* \big)^{-4} + C \big(1 - (m_j^{[A,B]})^2 \big)^4 \\
&\leq C \big(1 - (m_j^{[A,B]})^2 \big)^4.
\end{aligned}
$$

$\square$

Corollary 3.4 *Let Ω be as in (3.2), C_β as in (3.3), $\epsilon \in (0, 10^{-2})$ and recall $\Omega_{N-k-1,\epsilon}$ defined in (3.9). Then, there exists a universal constant $C > 0$, s.t. in $\Omega \cap \Omega_{N-k-1,\epsilon}$, we have for every A, B be disjoint with $|A \cup B| = k < N$ and every $\sigma_A \in \Sigma_{|A|}$ that*

$$
\| (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \| \leq C \sqrt{\frac{\epsilon}{C_\beta}} \sqrt{N - k} + 1.
$$

Here, we recall that $\Lambda^{[A,B]} = \Lambda^{[A,B]}(\sigma_A) = \big(\delta_{ij} [1 - (m_i^{[A,B]})^2]^{-1} \big)_{i,j \notin A \cup B}$ as well as $M^{[A,B]} = M^{[A,B]}(\sigma_A) = (m_{ij}^{[A,B]})_{i,j \notin A \cup B}$.

Proof For $\mathbf{c} = (c_i)_{i \notin A \cup B} \in \mathbb{R}^{N-k}$, we have that

$$
\begin{aligned}
&\mathbf{c}^{\mathsf{T}} (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \mathbf{c} \\
&= \sum_{j \notin A \cup B} \frac{c_j}{\sqrt{1 - (m_j^{[A,B]})^2}} \bigg\langle \sigma_j; \sum_{i \notin A \cup B} \frac{c_i \sigma_i}{\sqrt{1 - (m_i^{[A,B]})^2}} \bigg\rangle_{[A,B]}.
\end{aligned}
$$

Fixing $j \notin A \cup B$ and setting $g_j(\sigma) := \sum_{i \notin A \cup B \cup \{j\}} \left(1 - (m_i^{[A,B]})^2\right)^{-1/2} c_i \sigma_i$, it is clear that g_j does not depend on σ_j and Lemma 3.3 implies that

$$0 \le \mathbf{c}^{\mathsf{T}} (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \mathbf{c}$$

$$\le C \sqrt{\frac{\epsilon}{C_\beta}} \sum_{j \notin A \cup B} \sqrt{1 - (m_j^{[A,B]})^2} |c_j| \sqrt{\sum_{i \notin A \cup B \cup \{j\}} \frac{\langle \cosh^{-2}(B_i^{[A,B]}) \rangle_{[A, B \cup \{j\}]} c_i^2}{1 - (m_i^{[A,B]})^2} + \|\mathbf{c}\|_2^2}$$

Using (3.6), we obtain

$$\mathbf{c}^{\mathsf{T}} (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \mathbf{c} \le C \sqrt{\frac{\epsilon}{C_\beta}} \sqrt{N - k} \|\mathbf{c}\|_2^2 + \|\mathbf{c}\|_2^2.$$

Since the constant $C > 0$ on the right hand side in the last bound depend neither on A, B nor on the spin configuration σ_A (implicitly contained in the expectations w.r.t. the conditional Gibbs measure $\langle \cdot \rangle_{[A, B]}$), this proves the lemma. $\qquad\square$

The previous corollary is not, yet, enough to conclude Prop. 3.1. To this end, we need another consequence of Lemma 3.2.

Lemma 3.5 *Let β be as in (3.1), Ω as in (3.2), C_β as in (3.3) and let $\epsilon \in (0, 10^{-2})$. Assume that β is sufficiently small and recall $\Omega_{N-k-1,\epsilon}$ defined in (3.9). Moreover, let $T \in \mathbb{N}$ be fixed (independently of N), let $N - T \le k \le N - 1$ and define for $A, B \subset \{1, \ldots, N\}$ disjoint with $|A \cup B| = k$ and $\sigma_A \in \Sigma_{|A|}$ the event*

$$\Omega_{A, B, \sigma_A} := \left\{ \|(\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2}\| > \frac{11}{10} \sqrt{N - k} \right\}.$$

Then, there is a universal constant $C > 0$ such that

$$\mathbb{P}\left(\bigcup_{\substack{A \cap B = \emptyset, \ \sigma_A \in \Sigma_{|A|} \\ |A \cup B| = k}} \left(\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_{A, B, \sigma_A}\right) \right)$$

$$\le C \, e^{(T+2)\log N + N \log 4 - N C_\beta (C T \epsilon \beta^2)^{-1}}.$$

Proof Let $T \in \mathbb{N}$ be fixed and $A, B \subset \{1, \ldots, N\}$ be disjoint with $N - T \le k = |A \cup B| \le N - 1$. Now, consider the auxiliary field $B_j^{[A, A^c]}$ which, formally, is obtained from $B_j^{[A,B]}$ by setting $\sigma_{A^c} = 0$ (note that $B \subset A^c$). Then, $B_j^{[A, A^c]}$ is obviously a

function of σ_A alone and by (2.1), we have that

$$
(B_j^{[A,B]} - B_j^{[A,A^c]})(\sigma)
$$

$$
= \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \Bigg(\sum_{\substack{i_2,\ldots,i_p \in B^c, \\ \exists\, 2 \leq l \leq p:\, i_l \in A^c \cap B^c}} g_{j i_2 \ldots i_p} \sigma_{i_2} \ldots \sigma_{i_p} + \ldots
$$

$$
\ldots + \sum_{\substack{i_1,\ldots,i_{p-1} \in B^c, \\ \exists\, 1 \leq l \leq p-1:\, i_l \in A^c \cap B^c}} g_{i_1 i_2 \ldots i_{p-1} j} \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_{p-1}} \Bigg)
$$

for every $\sigma = (\sigma_A, \sigma_{A^c \cap B^c}) \in \Sigma_{N-|B|}$.

Moreover, given $\sigma \in \Sigma_{N-|B|}$, it is straightforward to see that $\left(B_j^{[A,B]} - B_j^{[A,A^c]}\right)(\sigma)$ is a centered Gaussian random variable with variance bounded by

$$
\mathbb{E}\big(B_j^{[A,B]} - B_j^{[A,A^c]}\big)^2(\sigma) \leq C \sum_{p \geq 2} \frac{N^{p-2}}{N^{p-1}} p \beta_p^2 T \leq \frac{C T \beta^2}{N}
$$

for some universal constant $C > 0$. Setting

$$
\Omega_\delta := \left\{ \sup_{\substack{A,B \subset \{1,\ldots,N\}: \\ A \cap B = \emptyset,\, |A \cup B| \geq N-T,}} \sup_{j \notin A \cup B} \sup_{\sigma \in \Sigma_{N-|B|}} \left| (B_j^{[A,B]} - B_j^{[A,A^c]})(\sigma) \right| > \delta \right\}
$$

for $\delta \in (0, 1)$, we conclude that

$$
\begin{aligned}
\mathbb{P}(\Omega_\delta) &\leq \binom{N}{T} T^2 (N-T) 2^{N-T} 2^{N-|B|} e^{-N\delta^2/(2CT\beta^2)} \\
&\leq \exp\left((T+2)\log N + N \log 4 - \frac{N\delta^2}{CT\beta^2} \right).
\end{aligned} \tag{3.15}
$$

Now consider the event $\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_\delta^c$. If A, B are disjoint s.t. $|A \cup B| = k$, $\sigma_A \in \Sigma_{|A|}$ and $g : \Sigma_{N-k} \to \mathbb{R}$ is a function that does not depend on σ_j, for some $j \notin A \cup B$, we can proceed as in (3.12) to (3.13) of Lemma 3.3 to conclude that

$$
\langle \sigma_j; g \rangle^2_{[A,B]} \leq C a_{N-k-1} D_{[A,B\cup\{j\}]}(g) \left\langle \left(\tanh(B_j^{[A,B]}) - m_j^{[A,B]} \right)^4 \right\rangle^{1/2}_{[A,B\cup\{j\}]}
$$

for some universal $C > 0$. By definition of Ω_δ, we have

$$
\begin{aligned}
C &\left\langle \left(\tanh\left(B_j^{[A,B]} \right) - m_j^{[A,B]} \right)^4 \right\rangle_* \\
&= C \left\langle \left(\tanh\left(B_j^{[A,B]} \right) - \tanh\left(B_j^{[A,A^c]} \right) \right. \right. \\
&\qquad\qquad \left. \left. + \left\langle \tanh\left(B_j^{[A,A^c]} \right) - \tanh\left(B_j^{[A,B]} \right) \right\rangle_{[A,B]} \right)^4 \right\rangle_* \leq C\delta^4
\end{aligned}
$$

in $\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_\delta^c$, where we used that $B_j^{[A,A^c]}$ is constant, conditionally on σ_A, and that $x \mapsto \tanh(x)$ in $\mathbb{R}$ is globally Lipschitz (with Lipschitz constant bounded by one). That is, in $\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_\delta^c$, we have that

$$\langle \sigma_j; g \rangle_{[A,B]}^2 \le \frac{C\epsilon}{C_\beta} \delta^2 \big(1 - (m_j^{[A,B]})^2\big)^2 D_{[A,B\cup\{j\}]}(g)$$

uniformly in A, B, σ_A and g. Therefore, arguing as in the proof of Corollary 3.4, we find that in $\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_\delta^c$, we have

$$\sup_{\|\mathbf{c}\|_2=1} \mathbf{c}^\mathsf{T}(\Lambda^{[A,B]})^{1/2} M^{[A,B]}(\Lambda^{[A,B]})^{1/2}\mathbf{c} \le C\delta\sqrt{\frac{\epsilon}{C_\beta}}\sqrt{N-k}+1 \le \frac{11}{10}\sqrt{N-k}$$

for all $\delta > 0$ small enough so that $\delta \le \frac{1}{10}C^{-1}\big(\frac{C_\beta}{\epsilon}\big)^{1/2}$. Since this bound is true uniformly in the sets A, B with $|A \cup B| = k$ (with $N - T \le k \le N - 1$) and $\sigma_A \in \Sigma_{|A|}$, we conclude that

$$\bigcup_{\substack{A \cap B=\emptyset, \\ |A\cup B|=k}} \bigcup_{\sigma_A \in \Sigma_{|A|}} \big(\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_{A,B,\sigma_A}\big) \subset \Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_{\delta_0} \subset \Omega_{\delta_0}$$

for $\delta_0 = C^{-1}\big(\frac{C_\beta}{\epsilon}\big)^{1/2}$. Hence, the bound (3.15) (for $\delta = \delta_0$) concludes the lemma. $\square$

We can now combine the previous lemmas to prove the main result of this section.

Proof of Prop. 3.1 By Eq. (3.4) from Lemma 3.1 and by Corollary 3.4, there exists a universal constant $C > 0$ s.t. for all A, B disjoint with $|A \cup B| = k < N$, we have in $\Omega \cap \Omega_{N-k-1,\epsilon}$ that

$$a_{[A,B]} = a_{[A,B]}(\sigma_A) \le \Big(1 - C\sqrt{\epsilon}/\sqrt{C_\beta(N-k)}\Big)^{-1} a_{N-k-1}.$$

Here, we set $\Omega_{N-k-1,\epsilon} = \{a_{N-k-1}C_\beta < \epsilon\}$ as before. Taking the sup over the spin configurations $\sigma_A \in \Sigma_{|A|}$ and over all subsystems described by disjoint sets A, $B \subset \{1, \ldots, N\}$ with $|A \cup B| = k < N$, this implies that in $\Omega \cap \Omega_{N-k-1,\epsilon}$

$$a_{N-k} \le 5a_{N-k-1}$$

for all $k < N - T$, where $T := (5C/4)^2(\epsilon/C_\beta)$. In other words, for such k, we have

$$\Omega \cap \Omega_{N-k-1,\epsilon} \cap \{a_{N-k} > 5a_{N-k-1}\} = \emptyset.$$

For the remaining $N - T \le k \le N - 2$, on the other hand, Lemma 3.1 and the fact that $\frac{11}{10\sqrt{2}} \le \frac{4}{5}$ imply that

$$\Omega \cap \Omega_{N-k-1,\epsilon} \cap \{a_{N-k} > 5a_{N-k-1}\} \subset \bigcup_{\substack{A \cap B = \emptyset, \\ |A \cup B| = k}} \bigcup_{\sigma_A \in \Sigma_{|A|}} \left(\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_{A,B,\sigma_A}\right),$$

where

$$\Omega_{A,B,\sigma_A} := \left\{ \left\| (\Lambda^{[A,B]})^{1/2} M^{[A,B]} (\Lambda^{[A,B]})^{1/2} \right\| (\sigma_A) > \frac{11}{10} \sqrt{N - |A \cup B|} + 1 \right\}.$$

By the preceding Lemma 3.5, the probability of this event is bounded by

$$\mathbb{P}\left(\bigcup_{\substack{A \cap B = \emptyset, \\ |A \cup B| = k}} \bigcup_{\sigma_A \in \Sigma_{|A|}} \left(\Omega \cap \Omega_{N-k-1,\epsilon} \cap \Omega_{A,B,\sigma_A}\right) \right)$$

$$\leq C \, e^{(T+2)\log N + N \log 4 - N C_\beta (CT\epsilon\beta^2)^{-1}}$$

$$\leq C \, e^{(C\epsilon/C_\beta + 2)\log N + N \log 4 - N C_\beta^2 (C\epsilon^2\beta^2)^{-1}}$$

for a universal constant $C > 0$. Finally, taking a union bound over these remaining k with $N - T \leq k \leq N - 2$, we conclude the claim. $\qquad\square$

4 Improved iteration estimate

As pointed out in the introduction, the main difficulty in deriving an upper bound on the spectral gap a_{H_N} lies in the fact that a priori it is not simple how to control the error term on the r.h.s. in (2.4). While in the previous section, we have used the two point correlation matrix to control the error term, this is not, yet, enough to iterate (2.4) and obtain a meaningful upper bound on a_{H_N}. However, controlling the error term both through the Dirichlet form and the norm of a related matrix, we get sufficiently strong control on the error that allows us to iterate (2.4) and to give, in combination with Prop. 3.1, an inductive proof of Theorem 1.1. The main result of this section reads as follows.

Proposition 4.1 *Let β be as in (3.1), Ω as in (3.2) and C_β as in (3.3). For $\epsilon \in (0, 10^{-2})$, set*

$$\Omega_{N-k-1,\epsilon} = \left\{ a_{N-k-1} \, C_\beta < \epsilon \right\}.$$

Then, for β sufficiently small, we have in the event $\Omega \cap \Omega_{N-k-1,\epsilon}$ that

$$\left(1 - \frac{C\beta^2 e^{C\beta} \max(1, a_{N-k})^2}{\epsilon \, (N-k)} \right) a_{N-k} \leq \left(1 - \frac{1}{N-k} \right) a_{N-k-1} + \frac{(1+4\epsilon)^5}{N-k}$$

for some universal constant $C > 0$.

To derive the above result, our starting point is once again the upper bound (2.4). However, in this section we relate the error term (the second term on the r.h.s. in (2.4)) to the Dirichlet form, up to another error that is indeed small at sufficiently high temperature. To be more precise, let $f : \Sigma_{N-|A\cup B|} \to \mathbb{R}$, then

$$
\langle f; \sigma_j \rangle_{[A,B]} - \langle \partial_j f; \sigma_j \rangle_{[A,B]}
$$

$$
= \frac{1}{2} \Big\langle (f(\sigma) + f(\hat{\sigma}_j)) \sum_{\sigma_j \in \{\pm 1\}} \frac{(\sigma_j - m_j^{[A,B]}) e^{\sigma_j B_j^{[A,B]}}}{2 \langle \cosh (B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}} \Big\rangle_{[A,B\cup\{j\}]}
$$

$$
= \frac{1}{2} \Big\langle f(\sigma) \Big((\sigma_j - m_j^{[A,B]}) - e^{-2\sigma_j B_j^{[A,B]}} (\sigma_j + m_j^{[A,B]}) \Big) \Big\rangle_{[A,B]} =: \frac{1}{2} \langle f h_j^{[A,B]} \rangle_{[A,B]},
$$

where we set

$$
h_j^{[A,B]} = (\sigma_j - m_j^{[A,B]}) - e^{-2\sigma_j B_j^{[A,B]}} (\sigma_j + m_j^{[A,B]}).
$$

By choosing $f \equiv 1$, we observe that $\langle h_j^{[A,B]} \rangle_{[A,B]} = 0$, so that in fact

$$
\langle f; \sigma_j \rangle_{[A,B]} = \langle \partial_j f; \sigma_j \rangle_{[A,B]} + \frac{1}{2} \langle f; h_j^{[A,B]} \rangle_{[A,B]} \tag{4.1}
$$

for general $f : \Sigma_{N-|A\cup B|} \to \mathbb{R}$. The last identity shows that the error term in (2.4) can be controlled by $D_{[A,B]}(f)$, through the first term on the r.h.s. in (4.1), and, as explained below, by the norm of $S^{[A,B]} = (S_{ij}^{[A,B]})_{1 \le i,j \le N-|A\cup B|}$, defined by

$$
S_{ij}^{[A,B]} = \frac{\langle h_i^{[A,B]}; h_j^{[A,B]} \rangle_{[A,B]}}{\sqrt{1 - (m_i^{[A,B]})^2} \sqrt{1 - (m_j^{[A,B]})^2}}. \tag{4.2}
$$

The crucial observation is that the operator norm of $S^{[A,B]}$ is small if β is sufficiently small, assuming some rough a priori information on a_{N-k-1}, like in the previous section. We make this more precise in the following auxiliary lemmas and conclude Prop. 4.1 at the end of this section by combining Lemma 4.1 and Lemma 4.3 below.

Lemma 4.1 *Let $\epsilon \in (0, 10^{-2})$, Ω be as in (3.2), C_β as in (3.3) and recall $\Omega_{N-k-1,\epsilon}$ defined in (3.9). Assume that β is sufficiently small. Then, for disjoint $A, B \subset \{1, \ldots, N\}$ s.t. $|A \cup B| = k$, we have in $\Omega \cap \Omega_{N-k-1,\epsilon}$ that*

$$
\sum_{j \notin A \cup B} \frac{\langle \sigma_j; f \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} \le (1 + 4\epsilon)^5 D_{[A,B]}(f) + \frac{1}{2\epsilon} \| S^{[A,B]} \| \langle f; f \rangle_{[A,B]}.
$$

As a consequence, we have in $\Omega \cap \Omega_{N-k-1,\epsilon}$ that

$$\left(1 - \frac{1}{2\epsilon} \frac{\|S^{[A,B]}\|}{N-k}\right) a_{[A,B]} \le \left(1 - \frac{1}{N-k}\right) a_{N-k-1} + \frac{(1+4\epsilon)^5}{N-k}.$$

Proof Let $j \notin A \cup B$. From (4.1), we obtain that

$$\langle f; \sigma_j \rangle^2_{[A,B]} \le (1+\epsilon)\langle \partial_j f; \sigma_j \rangle^2_{[A,B]} + \left(\frac{1}{4} + \frac{1}{4\epsilon}\right)\langle f; h_j^{[A,B]} \rangle^2_{[A,B]}.$$

Also, since $\sigma_j \partial_j f$ does not depend on σ_j, we get

$$\begin{aligned}
\langle \partial_j f; \sigma_j \rangle_{[A,B]} &= \langle (1 - \sigma_j m_j^{[A,B]})\sigma_j \partial_j f \rangle_{[A,B]} \\
&= \langle (1 - m_j^{[A,B]} \tanh(B_j^{[A,B]}))\sigma_j \partial_j f \rangle_{[A,B]},
\end{aligned}$$

so that by Cauchy-Schwarz

$$\begin{aligned}
\langle \partial_j f; \sigma_j \rangle^2_{[A,B]} &\le \langle \cosh^{-2}(B_j^{[A,B]})(\partial_j f)^2 \rangle_{[A,B]} \\
&\quad \times \langle (\cosh(B_j^{[A,B]}) - m_j^{[A,B]} \sinh(B_j^{[A,B]})^2 \rangle_{[A,B]}.
\end{aligned}$$

Now, splitting $\cosh(x) - y\sinh(x) = \frac{1}{2}(1-y)e^x + \frac{1}{2}(1+y)e^{-x}$ for $x, y \in \mathbb{R}$ and recalling (3.10) and that $m_j^{[A,B]} = \langle \sinh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]} / \langle \cosh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}$ as in (3.11), we bound

$$\begin{aligned}
&\frac{1}{4} \frac{\langle \cosh(B_j^{[A,B]}) - \sinh(B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}} \frac{\langle \cosh(B_j^{[A,B]}) \exp(2B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}} \\
&\le \frac{1}{4} \frac{\langle \exp(-B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}}{\cosh^2(\langle B_j^{[A,B]} \rangle_{[A,B\cup\{j\}]})} \frac{\langle \cosh^2(B_j^{[A,B]}) \rangle^{1/2}_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}} \langle \exp(4B_j^{[A,B]}) \rangle^{1/2}_{[A,B\cup\{j\}]} \quad (4.3) \\
&\le \frac{1}{4}(1+4\epsilon)^4 \langle \cosh^2(B_j^{[A,B]}) \rangle^{-1}_{[A,B\cup\{j\}]}
\end{aligned}$$

as well as

$$\begin{aligned}
&\frac{1}{4} \frac{\langle \cosh(B_j^{[A,B]}) + \sinh(B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}} \frac{\langle \cosh(B_j^{[A,B]}) \exp(-2B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}} \\
&\le \frac{1}{4} \frac{\langle \exp(B_j^{[A,B]}) \rangle^2_{[A,B\cup\{j\}]}}{\cosh^2(\langle B_j^{[A,B]} \rangle_{[A,B\cup\{j\}]})} \frac{\langle \cosh^2(B_j^{[A,B]}) \rangle^{1/2}_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]}} \langle \exp(-2B_j^{[A,B]}) \rangle_{[A,B\cup\{j\}]} \quad (4.4) \\
&\le \frac{1}{4}(1+4\epsilon)^4 \langle \cosh^2(B_j^{[A,B]}) \rangle^{-1}_{[A,B\cup\{j\}]},
\end{aligned}$$

where we used repeatedly the bound (3.7) from Lemma 3.2. Using also the first bound (3.6) from Lemma 3.2, the previous bounds yield altogether that

$$\frac{\langle \partial_j f; \sigma_j \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2} \leq (1 + 4\epsilon)^4 \langle \cosh^{-2}(B_j^{[A,B]})(\partial_j f)^2 \rangle_{[A,B]}.$$

Combining this bound with Lemma 2.1, we obtain

$$\langle f; f \rangle_{[A,B]} \leq a_{N-k-1}\left(1 - \frac{1}{N-k}\right) D_{[A,B]}(f) + \frac{(1+\epsilon)(1+4\epsilon)^4}{N-k} D_{[A,B]}(f)$$
$$+ \frac{1}{2\epsilon} \frac{1}{N-k} \sum_{j \notin A \cup B} \frac{\langle f; h_j^{[A,B]} \rangle_{[A,B]}^2}{1 - (m_j^{[A,B]})^2}.$$

To conclude the lemma, the last term on the r.h.s. of the last equation can be estimated similarly as in (3.5). □

To obtain the improved iteration bound of Proposition 4.1, the previous lemma suggests to study the norm of $S^{[A,B]}$, introduced in (4.2). We do this in two main steps and, similarly as in the previous section, we need a technical preparation whose proof is explained in Sect. 6.

Lemma 4.2 *Let β be as in (3.1), Ω as in (3.2), $A, B \subset \{1, \ldots, N\}$ be disjoint with $|A \cup B| = k$ and for $q \in \mathbb{N}$, let*

$$I_q^{[A,B]} := \max\left[\max_{i \notin A \cup B} \sum_{j \notin A \cup B} |\partial_i B_j^{[A,B]}|^q, \max_{j \notin A \cup B} \sum_{i \notin A \cup B} |\partial_i B_j^{[A,B]}|^q\right].$$

Then, for every $q \geq 2$, we have in Ω that $I_q^{[A,B]} \leq (5\beta)^q$.

Moreover, defining $X_q^{[A,B]}$ as the matrix with entries

$$X_{q,ij}^{[A,B]} := (\partial_i B_j^{[A,B]})^q \quad \text{for } 1 \leq i, j \leq N - k,$$

the matrix norm of $X_q^{[A,B]}$ is bounded in Ω by $\|X_q^{[A,B]}\| \leq (5\beta)^q$, for every $q \in \mathbb{N}$.

With the input of Lemma 4.2, we can prove the following result.

Lemma 4.3 *Let β be as in (3.1), Ω as in (3.2), C_β as in (3.3), $\epsilon \in (0, 10^{-2})$, $A, B \subset \{1, \ldots, N\}$ disjoint with $|A \cup B| = k$ and let $S^{[A,B]}$ be as in (4.2). Assume that β is sufficiently small and set*

$$\Omega_{N-k-1,\epsilon} = \{a_{N-k-1} C_\beta < \epsilon\}.$$

Then, there exists some universal $C > 0$ such that in $\Omega \cap \Omega_{N-k-1,\epsilon}$, we have

$$\|S^{[A,B]}\| \leq C\beta^2 \exp(C\beta) \max(1, a_{[A,B]})^2.$$

Proof We split the bound into two main steps, based on

$$
\begin{aligned}
0 \leq \sum_{i,j \notin A \cup B} c_i S_{ij}^{[A,B]} c_j &= \left\langle \sum_{i \notin A \cup B} c_i \frac{h_i^{[A,B]}}{\sqrt{1 - (m_i^{[A,B]})^2}} ; \sum_{j \notin A \cup B} c_j \frac{h_j^{[A,B]}}{\sqrt{1 - (m_j^{[A,B]})^2}} \right\rangle_{[A,B]} \\
&\leq a_{[A,B]} \sum_{i \notin A \cup B} \left\langle \cosh^{-2}(B_i^{[A,B]}) \left(\sum_{j \notin A \cup B} \frac{\partial_i h_j^{[A,B]}}{\sqrt{1 - (m_j^{[A,B]})^2}} c_j \right)^2 \right\rangle_{[A,B]} \\
&\leq 2a_{[A,B]} \sum_{i \notin A \cup B} \left\langle \cosh^{-2}(B_i^{[A,B]}) \left(\sum_{j \notin A \cup B \cup \{i\}} \frac{\partial_i h_j^{[A,B]}}{\sqrt{1 - (m_j^{[A,B]})^2}} c_j \right)^2 \right\rangle_{[A,B]} \\
&\quad + 2a_{[A,B]} \sum_{i \notin A \cup B} \frac{c_i^2}{1 - (m_i^{[A,B]})^2} \left\langle \cosh^{-2}(B_i^{[A,B]}) (\partial_i h_i^{[A,B]})^2 \right\rangle_{[A,B]} \\
&=: T_1 + T_2
\end{aligned}
\tag{4.5}
$$

for $\mathbf{c} \in \mathbb{R}^{N-k}$. We bound the contributions T_1 and T_2, defined on the r.h.s. in (4.5), separately. To this end, a straightforward computation yields first of all that

$$
\partial_i h_j^{[A,B]}(\sigma) = \begin{cases} -\frac{1}{2} e^{-2\sigma_j B_j^{[A,B]}} (m_j^{[A,B]} + \sigma_j) X_{ij}^{[A,B]}, & i \neq j, \\ -\sigma_j \sinh\left(2B_j^{[A,B]}\right) \left(\tanh\left(B_j^{[A,B]}\right) - m_j^{[A,B]}\right), & i = j, \end{cases}
$$

where $X^{[A,B]} = \left(X_{ij}^{[A,B]}\right)_{1 \leq i,j \leq N-k}$ is defined by

$$
X_{ij}^{[A,B]} = 1 - \exp\left(2\sigma_j \partial_i B_j^{[A,B]}\right) = \sum_{q=1}^{\infty} \frac{2^q}{q!} (\sigma_j \partial_i B_j^{[A,B]})^q.
$$

In particular, Lemma 4.2 implies that in Ω, we have that

$$
\left\| X^{[A,B]} \right\| \leq \sum_{q=1}^{\infty} \frac{2^q}{q!} \left\| X_q^{[A,B]} \right\| \leq C\beta \exp(C\beta)
$$

for some universal $C > 0$ and thus

$$
T_1 \leq C\beta^2 \exp(C\beta) a_{[A,B]} \sum_{j \notin A \cup B} \left(1 - \left(m_j^{[A,B]}\right)^2\right)^{-1} \left\langle e^{-4\sigma_j B_j} (m_j^{[A,B]} + \sigma_j)^2 \right\rangle_{[A,B]} c_j^2.
$$

To control the r.h.s. further, we compute

$$
\begin{aligned}
&\left\langle e^{-4\sigma_j B_j^{[A,B]}} (m_j^{[A,B]} + \sigma_j)^2 \right\rangle_{[A,B]} \\
&= \frac{\left(1 + \left(m_j^{[A,B]}\right)^2\right)\left\langle \cosh\left(3B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]} - 2m_j^{[A,B]}\left\langle \sinh\left(3B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh\left(B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}} \\
&= \frac{1}{2} \frac{\left(1 - m_j^{[A,B]}\right)^2\left\langle e^{3B_j^{[A,B]}}\right\rangle_{[A,B\cup\{j\}]} + \left(1 + m_j^{[A,B]}\right)^2\left\langle e^{-3B_j^{[A,B]}}\right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh\left(B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}}.
\end{aligned}
$$

Now, recalling once more that $m_j^{[A,B]} = \frac{\langle \sinh(B_j^{[A,B]})\rangle_{[A,B\cup\{j\}]}}{\langle \cosh(B_j^{[A,B]})\rangle_{[A,B\cup\{j\}]}}$ as in (3.11), a similar computation as in (4.3) and (4.4) using Lemma 3.2 shows that in $\Omega \cap \{a_{N-k-1,\epsilon}C_\beta < \epsilon\}$

$$
1 \pm m_j^{[A,B]} = \frac{2\left\langle e^{\pm B_j^{[A,B]}}\right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh\left(B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}} \leq C \frac{e^{\pm\left\langle B_j^{[A,B]}\right\rangle_{[A,B\cup\{j\}]}}}{\left\langle \cosh\left(B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}}
$$

and

$$
\left\langle e^{-4\sigma_j B_j^{[A,B]}} (m_j^{[A,B]} + \sigma_j)^2 \right\rangle_{[A,B]} \leq C \left\langle \cosh^2\left(B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}^{-1} \leq C\left(1 - \left(m_j^{[A,B]}\right)^2\right)
$$

for some absolute constant $C > 0$. Consequently, it holds true that $T_1 \leq C\beta^2 \exp(C\beta) a_{[A,B]} \|\mathbf{c}\|_2^2$.

Next, we consider the contribution T_2, defined in (4.5). In this case, in order to extract a factor β, we need to apply the spectral gap inequality again. Setting

$$
u_i := \tanh\left(B_i^{[A,B]}\right) - m_i^{[A,B]},
$$

we apply Cauchy-Schwarz and find

$$
\begin{aligned}
&\left\langle \cosh^{-2}(B_i^{[A,B]})\left(\partial_i h_i^{[A,B]}\right)^2 \right\rangle_{[A,B]} \\
&= \frac{\left\langle 4\sinh^2\left(B_i^{[A,B]}\right)\cosh\left(B_i^{[A,B]}\right)\left(\tanh\left(B_i^{[A,B]}\right) - m_i^{[A,B]}\right)^2\right\rangle_{[A,B\cup\{i\}]}}{\left\langle \cosh\left(B_i^{[A,B]}\right)\right\rangle_{[A,B\cup\{i\}]}} \\
&\leq C \frac{\left\langle \cosh^6\left(B_i^{[A,B]}\right)\right\rangle_{[A,B\cup\{i\}]}^{1/2}}{\left\langle \cosh\left(B_i^{[A,B]}\right)\right\rangle_{[A,B\cup\{i\}]}}\left\langle u_i^4\right\rangle_{[A,B]}^{1/2}.
\end{aligned}
$$

Then, since $\langle u_i \rangle_{[A,B]} = 0$, we can bound $\langle u_i^4 \rangle_{[A,B]} \leq \langle u_i^2; u_i^2 \rangle_{[A,B]} + \langle u_i; u_i \rangle_{[A,B]}^2$ and by the spectral gap inequality, we get

$$\langle u_i; u_i \rangle_{[A,B]} \leq a_{[A,B]} \sum_{k \notin A \cup B} \left\langle (\partial_k u_i)^2 \right\rangle_{[A,B]},$$

$$\left\langle u_i^2; u_i^2 \right\rangle_{[A,B]} \leq 4 a_{[A,B]} \sum_{k \notin A \cup B} \left\langle (\partial_k u_i)^2 (u_i - \partial_k u_i)^2 \right\rangle_{[A,B]}.$$

Using the elementary bound $|\tanh(x) - \tanh(y)| \leq 2 \cosh^{-2}(x)|x - y|e^{|x-y|}$ for $x, y \in \mathbb{R}$, we then obtain

$$\left| \partial_k u_i^{[A,B]} \right| \leq 2 \cosh^{-2} \left(B_i^{[A,B]} \right) \left| \partial_k B_i^{[A,B]} \right| \exp \left(\left| \partial_k B_i^{[A,B]} \right| \right)$$

so that Lemma 4.2 implies for β small enough (so that $5\beta < 1$) that

$$\langle u_i; u_i \rangle_{[A,B]} \leq 4 a_{[A,B]} \sum_{k \notin A \cup B} \left\langle \cosh^{-4} \left(B_i^{[A,B]} \right) \left| \partial_k B_i^{[A,B]} \right|^2 \exp \left(2 \left| \partial_k B_i^{[A,B]} \right| \right) \right\rangle_{[A,B]}$$

$$\leq C\beta^2 \exp(C\beta) a_{[A,B]} \left\langle \cosh^{-4} \left(B_i^{[A,B]} \right) \right\rangle_{[A,B]}$$

$$= C\beta^2 \exp(C\beta) a_{[A,B]} \frac{\left\langle \cosh^{-3} \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}}{\left\langle \cosh \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}}$$

Analogously, we obtain that

$$\left\langle u_i^2; u_i^2 \right\rangle_{[A,B]} \leq C\beta^2 \exp(C\beta) a_{[A,B]} \frac{\left\langle \cosh^{-3} \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}}{\left\langle \cosh \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}}$$

and combining this with the previous estimates, we conclude that

$$T_2 \leq C a_{[A,B]}^{3/2} \sum_{i \notin A \cup B} \frac{\beta^2 \exp(C\beta) c_i^2}{1 - (m_i^{[A,B]})^2} \frac{\left\langle \cosh^6 \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}^{1/2}}{\left\langle \cosh \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}^2} \left\langle \cosh^{-3} \left(B_i^{[A,B]} \right) \right\rangle_{[A,B\cup\{i\}]}$$

$$\leq C\beta^2 \exp(C\beta) \max(1, a_{[A,B]})^2 \|\mathbf{c}\|_2^2$$

Here, the second step follows from Lemma 3.2, arguing as before. In conclusion, we have shown that for all $\mathbf{c} \in \mathbb{R}^{N-k}$, we have that

$$0 \leq \langle \mathbf{c}, S^{[A,B]} \mathbf{c} \rangle_2 \leq C\beta^2 \exp(C\beta) \max(1, a_{[A,B]})^2 \|\mathbf{c}\|_2.$$

for some universal $C > 0$, i.e. $\|S^{[A,B]}\| \leq C\beta^2 \exp(C\beta) \max(1, a_{[A,B]})^2$. $\qquad\square$

Proof of Prop. 4.1 We combine Lemmas 4.1 and 4.3, which implies directly the improved iteration bound in Proposition 4.1. $\qquad\square$

5 Proof of Theorem 1.1

In this section, we combine Prop. 3.1 and Prop. 4.1 to prove Theorem 1.1.

Proof of Theorem 1.1 W.l.o.g., we can assume that ϵ is sufficiently small. In particular, we assume in the following that $\epsilon \in (0, 10^{-2})$ so that we can apply the main results of the previous sections.

We proceed inductively and before giving the details, let us first of all show that $a_1 = 1$. To see this, let $A, B \subset \{1, \ldots, N\}$ be disjoint such that $|A \cup B| = N - 1$ and let $i \notin A \cup B$. Then, the Gibbs measure $\langle \cdot \rangle_{[A,B]}$ is simply the coin tossing measure associated to the external field $B_i^{[A,B]}$. In other words, it is determined by the probabilities $\langle \mathbf{1}_{\sigma_i = \pm 1} \rangle_{[A,B]} = \frac{1}{2} \pm \frac{1}{2} \tanh \left(B_i^{[A,B]} \right)$. Set $p_i := \frac{1}{2} + \frac{1}{2} \tanh \left(B_i^{[A,B]} \right)$ and let $f : \{\pm 1\} \to \mathbb{R}$ be a function s.t. $f(1) \neq f(-1)$. Moreover, assume without loss of generality that $\langle f \rangle_{[A,B]} = 0$, i.e. that

$$f(-1) = -\frac{p_i}{1 - p_i} f(1).$$

Then, we find that

$$\langle f^2 \rangle_{[A,B]} = p_i f(1)^2 + (1 - p_i) f(-1)^2 = \frac{p_i}{1 - p_i} f(1)^2$$
$$= p_i (1 - p_i) \langle (f(1) - f(-1))^2 \rangle_{[A,B]} = D(f)$$

and thus $a_{[A,B]} = 1$. Since $A, B \subset \{1, \ldots, N\}$ were arbitrary, this means that $a_1 = 1$.

Now, let $\epsilon > 0$ be sufficiently small. We choose the inverse temperature $\beta = \epsilon^{3/4} \ll 1$ so that

$$\frac{\beta}{\epsilon} = \epsilon^{-1/4} \gg 1 \quad \text{and} \quad \frac{\beta}{\epsilon^{1/2}} = \epsilon^{1/4} \ll 1.$$

In particular, we can assume in the following w.l.o.g. that ϵ is so small that

(i) $\quad \dfrac{C_\beta^2}{C_1 \epsilon^2 \beta^2} = \mathcal{O}(1)\epsilon^{-1/2} > 2 \log 4,$

(ii) $\quad (1 + 40\epsilon^{1/4}) C_\beta = \mathcal{O}(1) \, \epsilon^{3/2} < \epsilon \quad$ and

(iii) $\quad 25(1 + 40\epsilon^{1/4})^2 \dfrac{C_2 \beta^2 e^{C_2 \beta}}{\epsilon} = \mathcal{O}(1)\epsilon^{1/2} < \epsilon^{1/4},$

where each $\mathcal{O}(1) > 0$ is bounded by some universal constant, and the constants C_1 and C_2 refer to the universal constants from Prop. 3.1 and Prop. 4.1, respectively.

Next, the event $\widetilde{\Omega}$ of high probability we use for the proof of Theorem 1.1 is defined by

$$\widetilde{\Omega} := \Omega \cap \bigcap_{l=0}^{N-2} \left\{ a_{N-l-1}C_\beta < \epsilon \text{ and } a_{N-l} > 5a_{N-l-1} \right\}^c$$

$$= \Omega \cap \bigcap_{l=0}^{N-2} \left\{ a_{N-l-1}C_\beta \geq \epsilon \text{ or } a_{N-l} \leq 5a_{N-l-1} \right\},$$

where Ω denotes the set defined in (3.2). In particular, according to Prop. 3.1 combined with condition (i) from above as well as the fact that $\mathbb{P}(\Omega) \geq 1 - e^{-N}$, it holds true that

$$\mathbb{P}(\widetilde{\Omega}) \geq 1 - Ce^{-cN}$$

for universal constants $c, C > 0$.

Now, let us prove the following induction: if $a_{N-k-1} \leq 1 + 40\epsilon^{1/4}$ in $\widetilde{\Omega}$ for some $0 \leq k \leq N-2$, then we prove that this implies $a_{N-k} \leq 1 + 40\epsilon^{1/4}$ in $\widetilde{\Omega}$. Since $a_1 = 1$, the inductive assumption is clearly satisfied for $k = N-2$. Hence, Theorem 1.1 follows from the proof of the inductive step.

To prove the inductive step, by the inductive assumption on a_{N-k-1} and condition (ii) from above, we have that $a_{N-k-1}C_\beta < \epsilon$ in $\widetilde{\Omega}$. By definition of $\widetilde{\Omega}$, this implies that

$$a_{N-k} \leq 5a_{N-k-1} < 5(1 + 40\epsilon^{1/4})$$

in $\widetilde{\Omega}$. Therefore, Proposition 4.1 implies that

$$\left(1 - \frac{25C_2\beta^2 e^{C_2\beta}(1 + 40\epsilon^{1/4})^2}{\epsilon(N-k)}\right) a_{N-k} \leq \left(1 - \frac{1}{N-k}\right) a_{N-k-1} + \frac{(1 + 4\epsilon)^5}{N-k}$$

$$\leq \left(1 - \frac{1}{N-k}\right) a_{N-k-1} + \frac{1 + 32\epsilon^{1/4}}{N-k}.$$

Combining this with condition (iii) from above and using again $a_{N-k} \leq 5a_{N-k-1}$, we obtain that

$$a_{N-k} \leq \left(1 + \frac{5\epsilon^{1/4}}{N-k} - \frac{1}{N-k}\right) a_{N-k-1} + \frac{1 + 32\epsilon^{1/4}}{N-k}$$

$$\leq \left(1 + \frac{5\epsilon^{1/4}}{N-k} - \frac{1}{N-k}\right)(1 + 40\epsilon^{1/4}) + \frac{1 + 32\epsilon^{1/4}}{N-k}$$

$$= 1 + 40\epsilon^{1/4} + \frac{(5\epsilon^{1/4})(40\epsilon^{1/4}) - 3\epsilon^{1/4}}{N-k} \leq 1 + 40\epsilon^{1/4},$$

for all $\epsilon > 0$ sufficiently small. This proves the inductive step and therefore, arriving at $k = 0$, we have shown that on a set of probability at least $1 - Ce^{-cN}$, we have that $a_{H_N} \leq 1 + 40\epsilon^{1/4}$.   □

6 High probability estimates

In this section, we complete our arguments by providing the high probability estimates related to the set Ω, defined in (3.2), Lemma 3.2 as well as Lemma 4.2.

Lemma 6.1 *Let Ω be defined as in (3.2). Then, $\mathbb{P}(\Omega) \geq 1 - e^{-N}$.*

Proof We recall from (2.1) that

$$
B_j^{[A,B]}(\sigma) = \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \left(\sum_{\substack{i_2,\dots,i_p \in B^c,}} g_{j i_2 \dots i_p} \sigma_{i_2} \dots \sigma_{i_p} + \dots \right.
$$

$$
\left. \dots + \sum_{i_1,\dots,i_{p-1} \in B^c} g_{i_1 i_2 \dots i_{p-1} j} \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_{p-1}} \right) + \eta_j
$$

for $\sigma \in \Sigma_{N-|A \cup B|}$ and σ_A understood to be fixed. It is then simple to see that an explicit expression for $\sigma_i \partial_i B_j^{[A,B]}$ is given by

$$
\sigma_i \partial_i B_j^{[A,B]}(\sigma) = \sum_{p \geq 2} \frac{\beta_p}{\sqrt{N}} \sum_{k_1,k_2,\dots,k_{p-2} \in B^c} g_{k_1 k_2 \dots k_{p-2}}^{ij} \frac{\sigma_{i_1}}{\sqrt{N}} \cdots \frac{\sigma_{i_2}}{\sqrt{N}} \cdots \frac{\sigma_{i_{p-2}}}{\sqrt{N}},
$$

where for each $i, j \in \{1, \dots, N - |A \cup B|\}$, the coupling $g_{k_1 k_2 \dots k_{p-2}}^{ij}$ is a sum of $p(p-1)$ i.i.d. standard Gaussian random variables (if the indices $k_u \neq k_s$ for $1 \leq u \neq s \leq p-2$ are all distinct; otherwise $g_{k_1 k_2 \dots k_{p-2}}^{ij} = 0$). Moreover, the couplings $g_{k_1 k_2 \dots k_{p-2}}^{ij}$ are independent over the superscripts $i, j \in \{1, \dots, N - |A \cup B|\}$. We can view this as

$$
\sigma_i \partial_i B_j^{[A,B]}(\sigma) = \sum_{p=2}^{\infty} \frac{\beta_p}{\sqrt{N}} G_p(\tau, \tau, \dots, \tau, \mathbf{e}_i, \mathbf{e}_j),
$$

where $G_p : (\mathbb{R}^N)^p \to \mathbb{R}$ denotes the multilinear map that is defined through $(G_p)_{k_1 k_2 \dots k_p} := g_{k_1,\dots,k_{p-2}}^{k_{p-1} k_p}$, the vector $\tau \in \{\mathbf{y} \in \mathbb{R}^N : \|\mathbf{y}\|_2 \leq 1\}$ has components $\tau_i = 0$ if $i \in B$ and $\tau_i = N^{-1/2}\sigma_i$ if $i \in \{1, \dots, N\}\backslash B$ and $(\mathbf{e}_k)_{k=1,\dots,N}$ denotes the standard basis of $\mathbb{R}^N$. In particular, we have that

$$
\sup_{\substack{A,B \subset \{1,\dots,N\}: \\ A \cap B = \emptyset}} \sup_{\sigma_A \in \Sigma_{|A|}} \sup_{\sigma \in \Sigma_{N-|A \cup B|}} \|\partial_i B_j^{[A,B]}(\sigma)\| \leq \sum_{p=2}^{\infty} \frac{\beta_p}{\sqrt{N}} \|G_p\|,
$$

where

$$\|G_p\| := \sup_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_p\in\mathbb{R}^N: \\ \|\mathbf{y}_i\|_2\leq 1\, \forall i=1,\dots,p}} G_p(\mathbf{y}_1\otimes\mathbf{y}_2\dots\otimes\mathbf{y}_p).$$

According to [14, Theorem 1], we have that

$$\mathbb{P}\left(\left\{N^{-1/2}\|G_p\| > \sqrt{p(p-1)[8\log(p/\log(3/2))p + \log t + \log 2]}\right\}\right) \leq e^{-N\log t}$$

for any $t > 1$. Choosing $t = 2p$ for $p \geq 2$, this implies

$$\mathbb{P}\left(\left\{N^{-1/2}\|G_p\| > 5\sqrt{p^3\log p}\right\}\right) \leq e^{-N\log 2p}.$$

Finally, since $\sup_{\substack{A,B\subset\{1,\dots,N\}: \\ A\cap B=\emptyset}} \sup_{\sigma_A\in\Sigma_{|A|}} \sup_{\sigma\in\Sigma_{N-|A\cup B|}} \|\partial_i B_j^{[A,B]}(\sigma)\| > 5\beta$ for $\beta = \sum_{p=2}^{\infty}\sqrt{p^3\log p}\beta_p$, defined in (3.1), implies that $N^{-1/2}\|G_{p_*}\| > 5\sqrt{p_*^3\log p_*}$ for some $p_* \geq 2$, we conclude that $\mathbb{P}(\Omega) \geq 1 - e^{-N\log 4}\sum_{p=2}\left(\frac{2}{p}\right)^N \geq 1 - e^{-N}$. $\qquad\square$

We continue with the proof of Lemma 4.2.

Proof of Lemma 4.2 Assuming the first claim $I_q^{[A,B]} \leq (5\beta)^q$ for $q \geq 2$ for the moment, the bounds on the matrices $X_q^{[A,B]}$ are simple. Indeed, for $q = 1$, we have $X_{1,ij}^{[A,B]} = \partial_i B_j^{[A,B]}$, so $\|X_1^{[A,B]}\| \leq 5\beta$ in Ω, by definition of Ω. For $q \geq 2$, on the other hand, we have that $\|X_q^{[A,B]}\| \leq I_q^{[A,B]}$ by standard matrix properties and hence the claim follows if $I_q^{[A,B]} \leq (5\beta)^q$ for $q \geq 2$. To this end, notice for $q = 2$

$$\begin{aligned}
I_2^{[A,B]} &= \max_{1\leq i\leq N-|A\cup B|} \max\left([(X_1^{[A,B]})^t(X_1^{[A,B]})]_{ii}, [(X_1^{[A,B]})(X_1^{[A,B]})^t]_{ii}\right) \\
&\leq \|X_1^{[A,B]}\|^2 \leq (5\beta)^2
\end{aligned}$$

and for $q > 2$ that $I_q \leq I_2 \max_{1\leq i,j\leq N-|A\cup B|} |\partial_i B_j^{[A,B]}|^{q-2} \leq (5\beta)^q$ in Ω. $\qquad\square$

Finally, it remains to prove Lemma 3.2 which follows as a simple corollary from the next Lemma.

Lemma 6.2 *Let $\epsilon \in (0, 10^{-2})$ and $K \in \mathbb{R}$. Suppose that $A, B \subset \{1,\dots,N\}$ are disjoint and let $j \notin A \cup B$. Then, if β is sufficiently small (depending on K) and if*

$$C_{K,\beta}a_{[A,B\cup\{j\}]} := (20K\beta)^2\exp(20K\beta)a_{[A,B\cup\{j\}]} < \epsilon,$$

we have in Ω that

$$\langle\exp(KB_j^{[A,B]})\rangle_{[A,B\cup\{j\}]} \leq (1+4\epsilon)\exp\left(K\langle B_j^{[A,B]}\rangle_{[A,B\cup\{j\}]}\right).$$

Proof Writing $\langle f^2 \rangle_{[A,B\cup\{j\}]} = \langle f; f \rangle_{[A,B\cup\{j\}]} + \langle f \rangle^2_{[A,B\cup\{j\}]}$. Since $\cosh(x) \geq 1$ (see also Remark 2 after Theorem 1.1), we bound

$$\langle f^2 \rangle_{[A,B\cup\{j\}]} \leq a_{[A,B\cup\{j\}]} \sum_{k \notin A \cup B \cup \{j\}} \langle (\partial_k f)^2 \rangle_{[A,B\cup\{j\}]} + \langle f \rangle^2_{[A,B\cup\{j\}]},$$

using the spectral gap inequality. Choosing $f = \exp\left(K B_j^{[A,B]}\right)$, we have $\partial_k f = \exp\left(K B_j^{[A,B]}\right)\left(1 - \exp\left(2K \partial_k B_j^{[A,B]}\right)\right)$ and Lemma 4.2 implies

$$\sum_{k \notin A \cup B \cup \{j\}} \left(1 - \exp\left(2K \partial_k B_j^{[A,B]}\right)\right)^2 \leq \sum_{q \geq 2} \frac{(4K)^q}{(q-2)!} \sum_{k \notin A \cup B} |\partial_k B_j|^q$$

$$\leq \sum_{q \geq 2} \frac{(4K)^q}{(q-2)!} I_q^{[A,B]} \leq (20K\beta)^2 \exp(20K\beta)$$

Here, we used the elementary bound $(e^x - 1)^2 \leq |x|^2 e^{2|x|}$ for $x \in \mathbb{R}$. We conclude

$$\sum_{k \notin A \cup B \cup \{j\}} \langle (\partial_k f)^2 \rangle_{[A,B\cup\{j\}]} \leq C_{K,\beta} \langle f^2 \rangle_{[A,B\cup\{j\}]},$$

for $f = \exp\left(K B_j^{[A,B]}\right)$ and since $C_{K,\beta} a_{[A,B\cup\{j\}]} < \epsilon < 1$, this implies

$$\langle f^2 \rangle_{[A,B\cup\{j\}]} \leq \left(1 - C_{K,\beta} a_{[A,B\cup\{j\}]}\right)^{-1} \langle f \rangle^2_{[A,B\cup\{j\}]}.$$

Now, $f^{1/2} = \exp\left(\frac{K}{2} B_j^{[A,B]}\right)$ so we can iterate the previous step and find that

$$\left\langle \exp\left(K B_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]}$$

$$\leq \prod_{l=0}^{\infty} \frac{1}{\left(1 - C_{2^{-l}K,\beta} a_{[A,B\cup\{j\}]}\right)^{2^l}} \lim_{l \to \infty} \left\langle \exp\left(2^{-l} K B_j^{[A,B]}\right)\right\rangle^{2^l}_{[A,B\cup\{j\}]}.$$

Noting that $\lim_{l \to \infty} \left\langle \exp\left(2^{-l} K B_j^{[A,B]}\right)\right\rangle^{2^l}_{[A,B\cup\{j\}]} = \exp\left(K \langle B_j^{[A,B]} \rangle_{[A,B\cup\{j\}]}\right)$ and

$$\log \prod_{l=0}^{\infty} \frac{1}{\left(1 - C_{2^{-l}K,\beta} a_{[A,B\cup\{j\}]}\right)^{2^l}}$$

$$\leq \sum_{l=0}^{\infty} 2^l (-\log)\left(1 - 2^{-2l} \exp\left(20K\beta(2^{-l} - 1)\right) C_{K,\beta} a_{[A,B\cup\{j\}]}\right) \leq 3\epsilon$$

for β small enough (depending on K, so that $\exp(|20K\beta|)$ is sufficiently close to one). All in all, we conclude that

$$\left\langle \exp\left(KB_j^{[A,B]}\right)\right\rangle_{[A,B\cup\{j\}]} \leq (1+4\epsilon)\exp\left(K\langle B_j^{[A,B]}\rangle_{[A,B\cup\{j\}]}\right).$$

□

Proof of Lemma 3.2 This is a simple application of the previous lemma, noting that the definition of C_β in (3.3) and the assumption $C_\beta a_{[A,B\cup\{j\}]} < \epsilon$ allows us to apply Lemma 6.2 uniformly in $K \in [-20, 20]$, if β is small enough (so that $\exp(20^2\beta)$ is sufficiently close to one). Indeed, with this observation, we obtain directly (3.7) and (3.8). The second inequality in (3.6) follows similarly, realizing first that by Jensen's inequality and the identities $m_i^{[A,B]} = \left\langle \tanh(B_i^{[A,B]})\right\rangle_{[A,B]}$ and (3.10), we have that

$$\frac{1}{1 - (m_j^{[A,B]})^2} \leq \left\langle \frac{1}{1 - \tanh^2(B_j^{[A,B]})}\right\rangle_{[A,B]} = \left\langle \cosh^2(B_j^{[A,B]})\right\rangle_{[A,B]}$$

$$= \frac{\left\langle \cosh^3(B_j^{[A,B]})\right\rangle_{[A,B\cup\{j\}]}}{\left\langle \cosh(B_j^{[A,B]})\right\rangle_{[A,B\cup\{j\}]}}$$

and then applying (3.8) three times. The first inequality in (3.6) is a consequence of Cauchy-Schwarz inequality. □

Declarations

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