Representation of random variables as Lebesgue integrals

SARA BIAGINI[1,a] and GORDAN ŽITKOVIĆ [2,b]

[1]*Department of Economics and Finance, LUISS University, Rome, Italy,* [a]*sbiagini@luiss.it*
[2]*Department of Mathematics, University of Texas at Austin, Texas, US ,* [b]*gordanz@math.utexas.edu*

We study representations of a random variable ξ as an integral of an adapted process with respect to the Lebesgue measure. The existence of such representations in two different regularity classes is characterized in terms of the quadratic variation of (local) martingales closed by ξ.

Keywords: Absolutely continuous representation; Girsanov theorem; martingale representation; quadratic variation.

1. Introduction

This paper focuses on the representation of a random variable as an adapted Lebesgue - as opposed to stochastic - integral. We start the analysis with statements of our main results and then place them in the extant literature while offering motivation for their study.

Let $(\Omega, (\mathcal{F}_t)_{t\leq T}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Given an $\mathcal{F}_T$-measurable random variable ξ, we ask whether there exists a progressively-measurable process β such that

$$\xi = \int_0^T \beta_u \, du, \text{ a.s.} \tag{1.1}$$

with β in a given integrability class. We focus on the Lebesgue measure on a finite time-horizon $[0,T]$ because other settings (alternative measures instead of the Lebesgue measure, alternative horizons, or the discrete time on an infinite horizon instead of the continuous time) lead to a similar analysis.

Our main results apply to two integrability classes for β, but we discuss interesting features of some other classes, too, in Section 4. We say that β is *weakly regular* if

$$\int_0^T \beta_u^2 \, du < \infty \ \ a.s.,$$

and *strongly regular* if

$$\mathbb{E}\left[\int_0^T \beta_u^2 \, du\right] < \infty.$$

Assuming throughout that all $\mathcal{F}$-local martingales are continuous, we show in Theorem 2.1 that the representation (1.1) holds for some strongly regular β if and only if $\xi \in \mathbb{L}^1$ and

$$\mathbb{E}\left[\int_0^T \frac{1}{T-t}\, d\langle M\rangle_t\right] < \infty \text{ where } M_t = \mathbb{E}[\xi \mid \mathcal{F}_t].$$

In a less restrictive, weakly regular case, our Theorem 3.1 states that (1.1) holds for a weakly regular β if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and a $\mathbb{Q}$ local martingale M with

$M_T = \xi$ such that

$$\int_0^T \frac{1}{T-t}\, d\langle M\rangle_t < \infty, \text{ a.s.} \tag{1.2}$$

Intuitively, an absolutely continuous representation of the form (1.1) with a weakly regular β exists if and only if ξ closes a local martingale whose quadratic variation grows slowly enough at T. This problem has a interesting link with the so-called "fundamental theorem of asset pricing" (see Theorem 1.1, p. 487 of Delbaen and Schachermayer (1994)). As is well known in the Mathematical Finance community, this Theorem states that a locally-bounded semimartingale M is a local martingale under some measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ if and only if it satisfies the condition of No Free Lunch with Vanishing Risk (NFLVR in the sequel). NFLVR is a slightly stronger version of the classical NA (No Arbitrage) condition of Mathematical Finance. We may think, informally, of a process that satisfies NFLVR as a measure-free version of a local martingale, or, similarly, as a semimartingale whose local-martingale part is everywhere more active than its finite-variation part.

When focusing on the representation (1.1) of ξ under the weaker, probability-free, condition on β, that question boils down to the relationship between ξ, the set of null events, and the filtration. Rephrased in financial terms, what we show is that (1.1) holds if and only if ξ closes a price process which has the property and moreover is a "slow" local martingale under a suitable $\mathbb{Q}$ - in the sense of (1.2). Such "slow" local martingale that converges to ξ can be used as a proxy for the good approximability of ξ by $\mathcal{F}_t$-adapted random variables as $t \nearrow T$.

Unlike in the case of martingale representation, the question of uniqueness of an absolutely continuous representation admits a trivially negative answer in many interesting integrability classes, including both weak and strong regularity discussed above. That fact served as a prompt to look for a canonical, rather than unique β. When $\mathbb{E}[\int_0^T \beta_u^2\, du] < \infty$ is required, the β that minimizes $\mathbb{E}[\int_0^T \beta_u^2\, du]$ admits an easy-to-verify explicit form, namely

$$\hat{\beta}_t = \frac{1}{T} M_0 + \int_0^t \frac{1}{T-u}\, dM_u,\ t \in [0, T),$$

where $M_t = \mathbb{E}[\xi \mid \mathcal{F}_t]$. Unfortunately, we could not identify an analogous natural notion of canonicity in the weakly regular case.

Absolutely continuous representation issues arise quite easily in applications. For instance, in Aïd and Biagini (2023) the authors deal with a linear-quadratic stochastic control problem on the Wiener space, arising from carbon regulation. In that problem, the controls are square integrable *rates*, i.e., state dynamics involve integrals of these controls with respect to dt. Furthermore, the objective function contains a terminal penalty term which is a function of an integral of one of the controls, β, so that the random variable $\xi = \int_0^T \beta_t\, dt$ appears in the objective function. Since the problem is not strictly convex in β, the authors of Aïd and Biagini (2023) were only able to obtain an explicit expression for the optimal $\hat{\xi}$, and for the associated martingale $\hat{M}_t = \mathbb{E}[\hat{\xi} \mid \mathcal{F}_t]$. They left the problem of finding an optimal, square integrable, rate $\hat{\beta}$ that *represents* the optimal $\hat{\xi}$ open (see Aïd and Biagini (2023), Remark 4.1).

Integrable-enough absolutely continuous representations come in handy in other contexts, as well. For example, they provide useful estimates when proving existence of solutions to stochastic differential equations. The interested reader can consult Chapter 6 of the Fabbri, Gozzi and Święch (2017) for a general treatment, or Biagini, Gozzi and Zanella (2022) for an application to stochastic delayed differential equations in an optimal investment problem.

The only existing result concerning absolutely-continuous representation we are aware of is the "factorization formula" of Da Prato and Zabczyk (see Theorem 5.2.5, p. 58 in Da Prato and Zabczyk

(1996)). Set on an abstract Wiener space, it provides an explicit absolutely continuous representation of a random variable given by a stochastic integral. It relies on a version of a stochastic Fubini theorem (see Theorem 4.18 of Da Prato and Zabczyk (2014)) but does not address the regularity of the representation itself, or provide any necessary conditions. A deeper discussion of why their approach, based on the stochastic Fubini theorem, does not lead to the kinds of results we are interested in is given in Remark 3.2.

Our results extend the existing ones in several directions. First, we give *necessary and sufficient* conditions on the random variable ξ for the representation to exist under both weak and strong regularity. Furthermore, in the strongly regular case we show that the unique *martingale* solution of the representation problem arises as the $\mathbb{L}^2$-norm minimizer on the product space.

The paper is organized as follows: Section 2 treats the strongly regular and Section 3 the weakly regular case; Section 4 contains further examples, results and comments.

Setup and notation. We consider a measurable space $(\Omega, \mathcal{F})$, together with a maximal family $\mathcal{P}$ of mutually equivalent probability measures on $\mathcal{F}$, as well as a right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, with $\mathcal{F}_0$ $\mathcal{P}$-trivial. When we write that a filtration is *generated by a Brownian motion W*, we always have the usual right-continuous and complete augmentation of the natural filtration in mind. On the other hand, a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is said to *support* a Brownian motion if there exists an $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion W.

We say that a process M is a *$\mathcal{P}$-local martingale* if it is a local martingale under some $\mathbb{P} \in \mathcal{P}$, and we denote the set of all $\mathcal{P}$-local martingales by $\mathcal{M}^{loc}$. We impose the following, standing, assumption throughout:

Assumption 1.1. Each $\mathcal{P}$-local martingale is continuous.

In particular, the above assumption implies that for each $M \in \mathcal{M}^{loc}$ there exists a unique process $\langle M\rangle$ such that $M^2 - \langle M\rangle$ is also in $\mathcal{M}^{loc}$.

Remark 1.2. According to Theorem 5.38, p. 155 in He, Wang and Yan (1992), continuity of all martingales on a filtered probability space is equivalent to the requirement that all $\{\mathcal{F}_t\}_{t\in[0,T]}$-stopping times be predictable. Since this property stays invariant under equivalent measure changes, we conclude that Assumption 1.1 holds if we only ask that there *exists* a single probability measure $\mathbb{P} \in \mathcal{P}$ such that all $\mathbb{P}$-local martingales are continuous.

For $\mathbb{P} \in \mathcal{P}$, $\mathbb{L}^p(\mathbb{P})$ is a shorthand for $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ while $\mathbb{L}^{p,q}(\mathbb{P})$, $q \in [0, \infty)$ denotes the set of all $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable processes β with $\int_0^T |\beta_u|^p\, du \in \mathbb{L}^q(\mathbb{P})$. When $p \geq 1$, the space $\mathbb{L}^{p,1}(\mathbb{P})$ comes with the norm:

$$||\beta||_{\mathbb{L}^{p,1}(\mathbb{P})} = \mathbb{E}^{\mathbb{P}}\left[\int_0^T |\beta_u|^p\, du\right]^{1/p},$$

while no topology on $\mathbb{L}^{p,0}(\mathbb{P})$ will be needed. Since the spaces $\mathbb{L}^0(\mathbb{P}), \mathbb{L}^{p,0}(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$ coincide, we omit the probability measure from the notation and simply write $\mathbb{L}^0, \mathbb{L}^{p,0}$.

For $\xi \in \mathcal{F}_T$ and $\mathbb{P} \in \mathcal{P}$, we set

$$\mathcal{B}^{p,q}(\xi, \mathbb{P}) := \left\{\beta \in \mathbb{L}^{p,q}(\mathbb{P}) \,:\, \int_0^T \beta_u\, du = \xi \text{ a.s.}\right\}.$$

When $q = 0$, we omit the measure $\mathbb{P}$ and write only $\mathcal{B}^{p,0}(\xi)$.

2. The strongly regular case

In this section we choose and fix a probability measure $\mathbb{P} \in \mathcal{P}$ and use it as the underlying measure in all probabilistic statements. In particular, we write: $\mathbb{L}^2$ and $\mathbb{L}^{2,1}$ for $\mathbb{L}^2(\mathbb{P})$ and $\mathbb{L}^{2,1}(\mathbb{P})$ respectively; and $\mathcal{B}^{2,1}(\xi)$ in place of $\mathcal{B}^{2,1}(\xi, \mathbb{P})$.

Theorem 2.1. *For $\xi \in \mathbb{L}^1$, let $\{M_t\}_{t\in[0,T]}$ and $\{\hat{\beta}_t\}_{t\in[0,T)}$ be defined by*

$$M_t = \mathbb{E}[\xi \mid \mathcal{F}_t],\ t \in [0,T] \tag{2.1}$$

$$\hat{\beta}_t = \frac{1}{T} M_0 + \int_0^t \frac{1}{T-u}\, dM_u,\ t \in [0,T). \tag{2.2}$$

The following statements are equivalent under Assumption 1.1:

1. $\mathcal{B}^{2,1}(\xi) \neq \emptyset$.
2. $\hat{\beta} \in \mathcal{B}^{2,1}(\xi)$.
3. $\hat{\beta} \in \mathbb{L}^{2,1}$.
4. $\mathbb{E}[\int_0^T \frac{1}{T-t}\, d\langle M\rangle_t] < \infty$.

When $\mathcal{B}^{2,1}(\xi) \neq \emptyset$, the process $\hat{\beta}$ given by (2.2) is, up to a version,

(a) *the unique martingale on $[0,T)$ in $\mathcal{B}^{2,1}(\xi)$*
(b) *the minimal $\mathbb{L}^{2,1}$-norm element in $\mathcal{B}^{2,1}(\xi)$.*

Proof. *1.→ 2.* Assuming that $\mathcal{B}^{2,1}(\xi)$ is nonempty consider the minimization problem

$$\inf_{\beta\in\mathcal{B}^{2,1}(\xi)} \mathbb{E}\left[\int_0^T \beta_u^2\, du\right] = \inf_{\beta\in\mathcal{B}^{2,1}(\xi)} \|\beta\|^2_{\mathbb{L}^{2,1}}. \tag{2.3}$$

The set $\mathcal{B}^{2,1}(\xi)$ is convex and closed in $\mathbb{L}^{2,1}$. By intersecting it with a large-enough ball in $\mathbb{L}^{2,1}$, we may assume that it is also bounded in $\mathbb{L}^{2,1}$. The Banach Alaoglu theorem ensures then that such restricted subset of $\mathcal{B}^{2,1}(\xi)$ is *weakly compact*. Since the $\mathbb{L}^{2,1}$ norm is a weakly lower semicontinuous function, so is its square and thus there exists a $\tilde{\beta}$ which attains the minimum in (2.3). This minimizer is also unique by strict convexity of the objective function.

The rest of the proof of this implication is organized as follows. We start by showing that the minimality of $\tilde{\beta}$ implies that $\tilde{\beta}$ is orthogonal in $\mathbb{L}^{2,1}$ to a sufficiently rich class of processes. Using this result, we prove that $\tilde{\beta}$ is a martingale. Finally, we apply Itô's formula and find that (a modification of) $\tilde{\beta}$ coincides with $\hat{\beta}$.

In order to show martingality of $\tilde{\beta}$, we perturb it in the direction of a process $\gamma \in \mathbb{L}^{2,1}$ with $\int_0^T \gamma_u\, du = 0$, a.s. By construction, the processes $\tilde{\beta} \pm \varepsilon\gamma$ belong to $\mathcal{B}^{2,1}(\xi)$, which implies that:

$$\|\tilde{\beta} \pm \varepsilon\gamma\|^2_{\mathbb{L}^{2,1}} \geq \|\tilde{\beta}\|^2_{\mathbb{L}^{2,1}},$$

for each $\varepsilon \in \mathbb{R}_+$. Writing down the relevant expectations, the inequality becomes

$$\|\tilde{\beta}\|^2_{\mathbb{L}^{2,1}} \pm 2\varepsilon\mathbb{E}\left[\int_0^T \tilde{\beta}_u\gamma_u\, du\right] + \varepsilon^2\mathbb{E}\left[\int_0^T \gamma_u^2\, du\right] \geq \|\tilde{\beta}\|^2_{\mathbb{L}^{2,1}}.$$

Simplifying and sending ε to zero, we get the following set of "first-order conditions"

$$\mathbb{E}\Big[\int_0^T \tilde\beta_u \gamma_u \, du\Big] = 0, \ \forall\, \gamma \in \mathbb{L}^{2,1} \text{ with } \int_0^T \gamma_u \, du = 0, \text{ a.s.} \tag{2.4}$$

Given $t < s$ in $[0,T)$, for each $\mathcal{F}_t$-measurable and random variable $\chi \in \mathbb{L}^2$ we define

$$\gamma_u^\chi = \begin{cases} 0 & u \in [0,t] \\ \chi \frac{1}{s-t}, & u \in (t,s], \\ -\chi \frac{1}{T-s}, & u \in (s,T]. \end{cases}$$

so that $\int_0^T \gamma_u^\chi \, du = 0$ for each χ. By applying the equality in (2.4) to γ^χ for all $\mathcal{F}_t$-measurable $\chi \in \mathbb{L}^2$ we obtain

$$\mathbb{E}\Big[\frac{\tilde A_s - \tilde A_t}{s-t} \,\Big|\, \mathcal{F}_t\Big] = \mathbb{E}\Big[\frac{\tilde A_T - \tilde A_s}{T-s} \,\Big|\, \mathcal{F}_t\Big] \text{ a.s.},$$

where $\tilde A_t = \int_0^t \tilde\beta_u \, du$. Since $\tilde\beta \in \mathcal{B}^{2,1}(\xi)$, $\tilde A_T = \xi = M_T$, with M given by (2.1). Slightly rearranging, we obtain:

$$\frac{1}{s-t}\Big(\mathbb{E}\big[\tilde A_s \,\big|\, \mathcal{F}_t\big] - \tilde A_t\Big) = \frac{1}{T-s}\Big(M_t - \mathbb{E}\big[\tilde A_s \,\big|\, \mathcal{F}_t\big]\Big) \text{ a.s.} \tag{2.5}$$

Since the right-hand side of (2.5) is a martingale in t on $[0,s)$, so is the left-hand side. In particular, the finite-variation part in its semimartingale decomposition, given via integration by parts by

$$\int_0^t \left(\frac{1}{(s-u)^2}\big(\mathbb{E}\big[\tilde A_s \,\big|\, \mathcal{F}_u\big] - \tilde A_u\big) - \frac{1}{s-u}\tilde\beta_u \right) du,$$

must vanish for all $t < s$, a.s. Consequently,

$$\tilde\beta_t = \frac{\mathbb{E}\big[\tilde A_s \,\big|\, \mathcal{F}_t\big] - \tilde A_t}{s-t} \text{ a.s., for almost all } t < s < T.$$

Passing to the limit $s \uparrow T$ on the right hand side above, we obtain

$$\tilde\beta_t = \frac{\mathbb{E}\big[\tilde A_T \,\big|\, \mathcal{F}_t\big] - \tilde A_t}{T-t} = \frac{M_t - \tilde A_t}{T-t} = \frac{M_t - \int_0^t \tilde\beta_u \, du}{T-t}. \tag{2.6}$$

It follows that $\tilde\beta$ has a continuous version on $[0,T)$, which we, from now on, adopt. Furthermore, the right-hand side of (2.6) is a semimartingale on $[0,T)$ so we can use Itô's formula once more to conclude that

$$\tilde\beta_t = \frac{1}{T}M_0 + \int_0^t \frac{1}{T-u} \, dM_u \text{ for } t \in [0,T).$$

Therefore, $\hat\beta = \tilde\beta$ and statement *2.* follows immediately.

2. $\rightarrow$ *3.* Immediate. *3.* $\rightarrow$ *4.* With $\tau_n = \inf\{t \geq 0 : \langle M \rangle_t \geq n\}$, Fubini's theorem implies that

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T (\hat\beta_{\tau_n\wedge u}-\hat\beta_0)^2\,du\Big] &= \int_0^T \mathbb{E}\Big[\int_0^{\tau_n\wedge u}\frac{1}{(T-t)^2}\,d\langle M\rangle_t\Big]\,du \\
&= \mathbb{E}\Big[\int_0^T\int_t^T 1_{\{t\le\tau_n\}}\frac{1}{(T-t)^2}\,du\,d\langle M\rangle_t\Big] \\
&= \mathbb{E}\Big[\int_0^{\tau_n} d\langle M\rangle_t \int_t^T \frac{1}{(T-t)^2}\,du\Big] = \mathbb{E}\Big[\int_0^{\tau_n}\frac{1}{T-t}\,d\langle M\rangle_t\Big].
\end{aligned}
\tag{2.7}
$$

Since $\hat\beta^{\tau_n}$ is an $\mathbb{L}^2$-bounded martingale on $[0,t]$ for each $t<T$, $(\hat\beta^{\tau_n}-\hat\beta_0)^2$ is a submartingale on the same domain, and the optional sampling theorem implies that

$$
\mathbb{E}\Big[(\hat\beta_{\tau_n\wedge t}-\hat\beta_0)^2\Big] \le \mathbb{E}\Big[(\hat\beta_t-\hat\beta_0)^2\Big] \text{ for each } t<T.
$$

Thus, by (2.7),

$$
\mathbb{E}\Big[\int_0^{\tau_n}\frac{1}{T-t}\,d\langle M\rangle_t\Big] = \int_0^T \mathbb{E}\Big[(\hat\beta_{\tau_n\wedge u}-\hat\beta_0)^2\Big]\,du \le \int_0^T \mathbb{E}\Big[(\hat\beta_u-\hat\beta_0)^2\Big]\,du = \|\hat\beta-\hat\beta_0\|^2_{\mathbb{L}^{2,1}} < \infty,
$$

and it suffices to let $n\to\infty$ and use the monotone convergence theorem.

4. $\to$ *3.* We let $n\to\infty$ in (2.7) and use Fatou's lemma on the left-hand side and the monotone convergence theorem on the right to conclude that

$$
\|\hat\beta\|_{\mathbb{L}^{2,1}} \le |\hat\beta_0| + \|\hat\beta-\hat\beta_0\|_{\mathbb{L}^{2,1}} \le |\hat\beta_0| + \mathbb{E}\Big[\int_0^T\frac{1}{T-t}\,d\langle M\rangle_t\Big]^{1/2} < \infty.
$$

3. $\to$ *1.* It suffices to show that $\int_0^T\hat\beta_t\,dt=\xi$, a.s. The definition of $\hat\beta$ in (2.2) and integration by parts imply that for $t\in[0,T)$ we have

$$
(T-t)\hat\beta_t = T\hat\beta_0 + \int_0^t dM_u - \int_0^t \hat\beta_u\,du = M_t - \int_0^t\hat\beta_u\,du. \tag{2.8}
$$

Another round of integration by parts, but this time applied to the stochastic integral $\int_0^t \frac{1}{T-u}\,dM_u$, implies that

$$
\hat\beta_t = \frac{1}{T-t}M_t + \int_0^t \frac{M_u}{(T-u)^2}\,du. \tag{2.9}
$$

Put together, identities (2.8) and (2.9), give

$$
\int_0^t\hat\beta_u\,du = \frac{\int_0^t\frac{M_u}{(T-u)^2}\,du,}{\frac{1}{T-t}} \text{ for } t\in[0,T),
$$

and the final step is to use l'Hôpital's rule and the fact that $M_t\to\xi$, as $t\to T$.

Concerning the last part of the statement of the theorem, (b) was established in the course of the proof of *1.* $\to$ *2.* above. For (a), we assume that there exists another martingale β^* in $\mathcal{B}^{2,1}(\xi)$ so that

$$
M_t = \mathbb{E}\Big[\int_0^T\beta^*_u\,du\,\Big|\,\mathcal{F}_t\Big] = \int_0^t\beta^*_u\,du + \beta^*_t(T-t).
$$

The equality (2.8) above implies that

$$0 = \int_0^t (\beta_u^* - \hat{\beta}_u)\, du + (\beta_t^* - \hat{\beta}_t)(T-t) \text{ for all } t \in [0,T), \text{ a.s.}$$

It follows that $\beta_t^* - \hat{\beta}_t$ is continuously differentiable for $t \in [0,T)$, and the conclusion $\beta^* = \hat{\beta}$ follows by differentiation. □

Remark 2.2. As the anonymous referee observes, equation (2.6) can be interpreted as a (pathwise) Volterra-type equation of the second kind:

$$u(t) = f(t) + \int_0^T K(t,s)u(s)\, ds, \text{ where } f(t) = \frac{M_t}{T-t} \text{ and } K(t,s) = -\frac{1}{T-t}1_{\{s\le t\}}.$$

A *formal* iterative solution (obtained by repeatedly replacing $u(\cdot)$ on the right-hand side by the whole right-hand side and then taking the limit) can be written as $\tilde{\beta}_t = \lim_n u^{(n)}(t)$, where

$$u^{(n)}(t) = f(t) + \sum_{i=0}^{n} \int K^i(t,s) f(s)\, ds$$

and K^i is the i-th composition power of K, i.e., $K^0 = K$ and

$$K^i(t,s) = \int_0^T \cdots \int_0^T K(t,s_1)K(s_1,s_2)\ldots K(s_i,s)\, ds_1 \ldots ds_i \text{ for } i \ge 1. \tag{2.10}$$

Substituting the explicit expression $K(t,s) = -(T-t)^{-1}1_{\{s\le t\}}$ into (2.10) above, we obtain

$$K^i(t,s) = (-1)^i (T-t)^{-1} \int_0^T \cdots \int_0^T \frac{1}{T-s_1} \cdots \frac{1}{T-s_i} 1_{\{t\ge s_1 \ge \cdots \ge s_i \ge s\}}\, ds_1 \ldots ds_i.$$

The last iterated integral is taken over the simplex $\Delta = \{(s_1,\ldots,s_i) \in [s,t]^{i-1} : s_1 \le s_2 \cdots \le s_i\}$, and the function $\prod_{j=1}^i (T-s_k)^{-1}$ inside the integral is symmetric in $s_1,\ldots,s_{i-1}$, so, for $s \le t$ we have

$$\begin{aligned} K^i(t,s) &= (-1)^i \frac{1}{T-t}\frac{1}{i!} \int_s^t \cdots \int_s^t (T-s_1)^{-1} \ldots (T-s_i)^{-1}\, ds_1 \ldots ds_i \\ &= (-1)^i \frac{1}{T-t}\frac{1}{i!}\Big(\int_s^t (T-u)^{-1}\, du\Big)^i = \frac{1}{T-t}\frac{1}{i!}\Big(\log\frac{T-t}{T-s}\Big)^i, \end{aligned}$$

and consequently,

$$\sum_{i=0}^{\infty} K^i(t,s) = \frac{1}{T-s}1_{\{s\le t\}}.$$

This implies that the formal solution $\tilde{\beta}_t$ takes the form

$$\tilde{\beta}_t = \frac{M_t}{T-t} + \int_0^t \frac{M_s}{(T-s)^2}\, ds, \tag{2.11}$$

which, after integrating by parts in the last integral, matches (2.2).

3. The weakly regular case

Fix an $\mathcal{F}_T$ measurable random variable $\xi \in \mathbb{L}^0$, and let $\mathcal{M}^{loc}(\xi)$ denote the set of all $M \in \mathcal{M}^{loc}$ such that $M_T = \xi$. Let $\mathcal{P}^1(\xi)$ be the set of probabilities in $\mathcal{P}$ which integrate ξ.[1] For $\mathbb{P} \in \mathcal{P}^1(\xi)$, we set

$$M_t^{\mathbb{P},\xi} = \mathbb{E}^{\mathbb{P}}[\xi \mid \mathcal{F}_t],\ t \in [0,T],$$

taken in its continuous version, so that $M^{\mathbb{P},\xi}$ is the unique $\mathbb{P}$-martingale in $\mathcal{M}^{loc}(\xi)$. Finally, for $M \in \mathcal{M}^{loc}$ we define

$$\hat{\beta}_t^M = \frac{1}{T} M_0 + \int_0^t \frac{1}{T-u}\, dM_u,\ t \in [0,T). \tag{3.1}$$

Theorem 3.1. *For an $\mathcal{F}_T$ measurable random variable $\xi \in \mathbb{L}^0$, the following are equivalent:*

1. $\mathcal{B}^{2,0}(\xi) \neq \emptyset$.
2. $\mathcal{B}^{2,1}(\xi,\mathbb{Q}) \neq \emptyset$ *for some* $\mathbb{Q} \in \mathcal{P}^1(\xi)$.
3. $\hat{\beta}^M \in \mathbb{L}^{2,0}$ *for some* $M \in \mathcal{M}^{loc}(\xi)$.
4. $\int_0^T \frac{1}{T-t}\, d\langle M\rangle_t < \infty$ *a.s., for some* $M \in \mathcal{M}^{loc}(\xi)$.
5. $\mathbb{E}^{\mathbb{Q}}[\int_0^T \frac{1}{T-t}\, d\langle M^{\mathbb{Q},\xi}\rangle_t] < \infty$, *for some* $\mathbb{Q} \in \mathcal{P}^1(\xi)$.

Proof. *1. → 2.* We pick $\beta \in \mathcal{B}^{2,0}(\xi)$ and $\mathbb{P} \in \mathcal{P}$, and define $\mathbb{Q} \in \mathcal{P}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = c\frac{1}{1 + |\xi| + \int_0^T \beta_u^2\, du},$$

where c is the normalizing constant. This way $\mathbb{Q} \in \mathcal{P}^1(\xi)$ and the process β belongs to $\mathbb{L}^{2,1}(\mathbb{Q})$, and, hence, also to $\mathcal{B}^{2,1}(\mathbb{Q})$.

2. → 5. This is the content of the implication *1. → 4.* in Theorem 2.1, but, possibly, under an equivalent probability measure.

5. → 4. Immediate.

4. → 3. Let $M \in \mathcal{M}^{loc}(\xi)$ be as in the statement, and let $\mathbb{Q} \in \mathcal{P}$ be such that M is a $\mathbb{Q}$-local martingale. We define the nondecreasing sequence $\{T_m\}_{m\in\mathbb{N}}$ of stopping times by

$$T_m = \inf\left\{t \geq 0 : \int_0^t \frac{1}{T-u}\, d\langle M\rangle_u \geq m\right\},$$

so that $\mathbb{Q}[T_m < T] \to 0$, as $m \to \infty$. The process $\hat{\beta}^M$, given by (3.1) above, is a continuous local $\mathbb{Q}$-martingale on $[0,T)$ so there exists another nondecreasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping times with the property that $\tau_n \to T$, a.s., such that $(\hat{\beta}^M)^{\tau_n}$ is an $\mathbb{L}^2(\mathbb{Q})$-bounded martingale on $[0,T]$. In particular,

$$\mathbb{E}^{\mathbb{Q}}\Big[(\hat{\beta}^M_{u\wedge T_m\wedge\tau_n})^2\Big] = \mathbb{E}^{\mathbb{Q}}\big[\langle\hat{\beta}^M\rangle_{u\wedge T_m\wedge\tau_n}\big] \text{ for all } m,n \in \mathbb{N} \text{ and } u \in [0,T).$$

[1] The set $\mathcal{P}^1(\xi)$ is not empty, and it can be proved as in the proof of the next Theorem, arrow *1. → 2.*, by setting $\beta = 0$.

We let $n \to \infty$ and use Fatou's lemma together with the monotone convergence theorem to conclude that

$$\mathbb{E}^{\mathbb{Q}}\Big[(\hat{\beta}^M_{T_m\wedge u})^2\Big] \le \mathbb{E}^{\mathbb{Q}}\big[\langle\hat{\beta}^M\rangle_{T_m\wedge u}\big] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T_m\wedge u} \frac{1}{(T-r)^2}\, d\langle M\rangle_r\Big] \tag{3.2}$$

for $u < T$ and $m \in \mathbb{N}$. By using the inequality

$$\int_0^{T_m} (\hat{\beta}^M_u)^2\, du \le \int_0^T (\hat{\beta}^M_{T_m\wedge u})^2\, du \;\; a.s.,$$

integrating (3.2) above in u over $[0,T]$, and applying Fubini's Theorem under the product measure $du \otimes d\langle M\rangle_r$ we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T_m} \big(\hat{\beta}^M_u\big)^2\, du\Big] &\le \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \big(\hat{\beta}^M_{T_m\wedge u}\big)^2\, du\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \int_0^T 1_{\{r\le u\wedge T_m\}} \frac{1}{(T-r)^2}\, d\langle M\rangle_r\, du\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T_m} \frac{1}{(T-r)^2} d\langle M\rangle_r \int_r^T du\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T_m} \frac{1}{T-r}\, d\langle M\rangle_r\Big] \le m.\end{aligned}$$

Since $\mathbb{Q}[T_m = T] \to 1$ as $m \to \infty$, we conclude that $\int_0^T (\hat{\beta}^M_u)^2\, du < \infty$, a.s.

3. $\to$ *1.* The last argument in the proof of Theorem 2.1 is based only on the integration by parts formula and on the property $\hat{\beta} \in \mathbb{L}^{1,0}$. Therefore it can be applied here, since $\hat{\beta}^M \in \mathbb{L}^{2,0} \subseteq \mathbb{L}^{1,0}$. □

Remark 3.2. As it aims for generality, but also operates within specific regularity classes, our proof of Theorem 3.1 above does not use the *stochastic Fubini theorem*, a joint name for a class of statements about the permissibility of the interchange of a Lebesgue and a stochastic integral under different sets of conditions. We refer the reader to Theorem 4.18 in Da Prato and Zabczyk (2014) or Theorem 2.2 of Veraar (2012) for two versions referred to later in this paper.

To provide a more detailed explanation, let us start with a brief description of how an argument based on it would play out. Under Assumption 1.1, it would start with a choice of a measure $\mathbb{Q} \in \mathcal{P}$ as in Theorem 3.1, item *2.* and the associated martingale $M^{\mathbb{Q},\xi}$, where we assume, without loss of generality, that $M_0 = \mathbb{E}^{\mathbb{Q}}[\xi] = 0$. When its conditions are satisfied, the stochastic Fubini theorem, applied to the function $\psi(s,t,\omega) = \frac{1}{T-t} I_{[0,s]}(t)$ and with integrals with respect to dM and ds, yields

$$\int_0^T ds \int_0^s \frac{1}{T-t}\, dM_t = \int_0^T dM_t \frac{1}{T-t} \int_t^T ds = M_T = \xi,$$

making

$$\beta_s = \int_0^s \frac{1}{T-t}\, dM_t \tag{3.3}$$

an absolutely-continuous representation of ξ. One of the weakest conditions for the above to hold is due to Veraar (see Theorem 2.2 of Veraar (2012)), and it can be stated in our case as

$$\int_0^T \Big(\int_0^s \frac{1}{(T-t)^2}\, d\langle M\rangle_t\Big)^{\frac{1}{2}} ds < \infty, \text{ a.s.} \tag{3.4}$$

It is superficially related to our condition 4. of Theorem 3.1, but it does not automatically insert β into our (weak or strong) regularity classes. For example, assume that we are on a filtration generated by a Brownian motion W, and that $\xi = W_T$. In that case (3.4) is clearly satisfied, but, as we will see in Proposition 4.1 below, W_T does not admit an absolutely continuous representation with a weakly (or strongly) regular β. Put differently:

regularity conditions for the validity of the stochastic Fubini theorem do not correspond to our regularity classes.

A natural question is whether a condition such as:

4'. $\int_0^T \frac{1}{T-t}\, d\langle M\rangle_t < \infty$ a.s., for all $M \in \mathcal{M}^{loc}(\xi)$

can be inserted in Theorem 3.1. An equivalent question is whether the condition 4. of Theorem 3.1 implies the condition $4'$. above. We only have a partial (positive) answer to this problem. It states that under certain regularity conditions, if $M^{\mathbb{P}}$ satisfies condition 4 in Theorem 2.1, then all the probability measures $\mathbb{Q} \sim \mathbb{P}$ with a finite relative entropy share the property, namely $\mathbb{E}^{\mathbb{Q}}[\int_0^T \frac{1}{T-t}\, d\langle M^{\mathbb{Q}}\rangle_t] < \infty$.

Proposition 3.3. *Suppose that the filtration is generated by a $\mathbb{P}$-Brownian motion W and that ξ is of the form*

$$\xi = \int_0^T \sigma_u \, dW_u, \text{ for some bounded } \sigma.$$

If

1. *the martingale $M_t^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[\xi \mid \mathcal{F}_t]$ satisfies*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \frac{1}{T-t}\, d\langle M^{\mathbb{P}}\rangle_t\right] < \infty, \text{ and}$$

2. *$\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ with a finite relative entropy, i.e.,*

$$\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty, \tag{3.5}$$

then

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{1}{T-t}\, d\langle M^{\mathbb{Q}}\rangle_t\right] < \infty. \tag{3.6}$$

Proof. With $\mathbb{P} \sim \mathbb{Q}$ as in the statement, let θ be such that the dynamics of the density process $Z_t = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$ is given by

$$dZ_t = -Z_t\, \theta_t \, dW_t.$$

Let us show, first, that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \theta_u^2 \, du\right] < \infty. \tag{3.7}$$

Condition (3.5) above, together with the fact that the function $x \mapsto x\log(x)$ is convex and bounded from below, implies that the process $Z\log(Z)$ is a continuous $\mathbb{P}$-submartingale on $[0,T]$. Hence, there exists a finite constant C such that

$$\mathbb{E}^{\mathbb{P}}[Z_\tau \log(Z_\tau)] \leq C \text{ for each } [0,T]\text{-valued stopping time } \tau, \tag{3.8}$$

Itô's formula applied to the semimartingale $Z\log(Z)$ yields

$$Z_\tau \log(Z_\tau) = N_\tau + \tfrac{1}{2}\int_0^\tau Z_u \theta_u^2 \, du \text{ where } N \text{ is a local } \mathbb{P}\text{-martingale.}$$

Let $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times that reduces the process N. The upper bound of (3.8) above implies that

$$C \geq \mathbb{E}^{\mathbb{P}}\big[Z_{\tau_n}\log(Z_{\tau_n})\big] = \tfrac{1}{2}\mathbb{E}^{\mathbb{P}}\bigg[\int_0^{\tau_n} Z_u\theta_u^2\,du\bigg] = \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}\bigg[\int_0^{\tau_n}\theta_u^2\,du\bigg],$$

and it remains to use the monotone-convergence theorem to conclude that (3.7) holds.

We continue the proof by using Girsanov's theorem and the boundedness of σ to conclude that the process:

$$M'_t = \int_0^t \sigma_u\,(dW_u + \theta_u\,du) = M_t^{\mathbb{P}} + \int_0^t \sigma_u\theta_u\,du$$

is a $\mathbb{Q}$-martingale. Boundedness of the process σ, together with (3.7), implies that the random variable $\xi = M_T^{\mathbb{P}} = M'_T - \int_0^T \sigma_u\theta_u\,du$ is $\mathbb{Q}$-integrable, so that the $\mathbb{Q}$-martingale $M_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\xi \mid \mathcal{F}_t]$ is well defined. Moreover, we have

$$M_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}\bigg[M'_T - \int_0^T \sigma_u\theta_u\,du \,\bigg|\, \mathcal{F}_t\bigg] = M_t^{\mathbb{P}} + \int_0^t \sigma_u\theta_u\,du - L_t,$$

where

$$L_t = \mathbb{E}^{\mathbb{Q}}\bigg[\int_0^T \sigma_u\theta_u\,du \,\bigg|\, \mathcal{F}_t\bigg].$$

It follows that

$$\langle M^{\mathbb{Q}}\rangle_t = \langle M^{\mathbb{P}} - L\rangle_t.$$

Furthermore, since

$$\langle M^{\mathbb{P}} - L\rangle_t \leq \langle M^{\mathbb{P}} - L\rangle_t + \langle M^{\mathbb{P}} + L\rangle_t = 2\langle M^{\mathbb{P}}\rangle_t + 2\langle L\rangle_t,$$

our final goal, namely (3.6), will be reached if we can prove that

$$\mathbb{E}^{\mathbb{Q}}\bigg[\int_0^T \frac{1}{T-t}\,d\langle L\rangle_t\bigg] < \infty. \tag{3.9}$$

For that, we note that $\xi' = L_T$ admits the absolutely continuous representation $L_T = \int_0^T \beta'_t\,dt$, where $\beta'_t = \sigma_t\theta_t$, by its very definition. Since the boundedness of σ and (3.7) above imply that $\beta' \in \mathbb{L}^{2,1}(\mathbb{Q})$, equation (3.9) follows from the implication 1. → 4. of Theorem 2.1. □

4. Examples and further remarks

4.1. Functions of the terminal value of a Brownian motion W

Our first subsection focuses on the (lack of) absolutely continuous representation property for the random variables of the form $g(W_T)$. In contrast with what happens with the martingale representation, we show that such a ξ admits an absolutely continuous representation if and only if it is constant. Informally speaking: in order to have regular-enough representation, ξ must be sufficiently path dependent.

We start with a simple argument of limited scope just to provide some intuition. Let ξ be the terminal value W_T itself, and let β be its absolutely continuous representation, i.e., $\xi = \int_0^T \beta_u\, du$. Since the process $W_t^\beta = W_t - \int_0^t \beta_u\, du$ satisfies $W_T^\beta = 0$, it cannot be a martingale under any equivalent measure. This means that β cannot be too regular in the sense that the stochastic exponential $\exp(\int_0^t \beta_u\, dW_u - \frac{1}{2}\int_0^t \beta_u^2\, du)$ cannot be a martingale, if it is well-defined at all. In fact, as our next Proposition shows, such a β cannot be weakly regular, i.e., $\int_0^T \beta_u^2\, du = +\infty$ with positive probability.

Proposition 4.1. *Suppose that the filtration supports a Brownian motion W, and that $\xi = g(W_T)$ for some $g \in C^2(\mathbb{R})$. Then $\mathcal{B}^{2,0}(\xi) \neq \emptyset$ if and only if g is constant.*

Proof. The only implication which requires a proof is ($\Rightarrow$).

We focus, first, on the case where the filtration is generated by the Brownian motion. Since $g \in C^2(\mathbb{R})$, the process $g''(W)$ is locally bounded so that $\int_0^T (g''(W_u))^2\, du < \infty$, a.s. Itô's formula then implies that ξ admits an absolutely continuous representation under weak regularity, i.e., that $\mathcal{B}^{2,0}(\xi) \neq \emptyset$, if and only if $\mathcal{B}^{2,0}(\bar{\xi}) \neq \emptyset$, where

$$\bar{\xi} = \int_0^T g'(W_t)\, dW_t.$$

Fix a $\mathbb{Q} \in \mathcal{P}$ and call $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$ its density process. Since the filtration is assumed to be Brownian, there exists a progressively measurable process θ such that $Z = \mathcal{E}(\int_0^\cdot \theta_t\, dW_t)$. Thus,

$$\bar{\xi} = \int_0^T g'(W_t)\, dW_t = \int_0^T g'(W_t)\, dW_t^Z + \int_0^T g'(W_t)\theta_t\, dt$$

where $W^Z = W - \int_0^\cdot \theta_t\, dt$ is a $\mathbb{Q}$-Brownian motion. By the continuity of g' and the paths of W, we have $\int_0^T (g'(W_t)\theta_t)^2\, dt < \infty$, a.s. Being deterministic, the quadratic variation of the Brownian motion has the same distribution across all the probabilities in $\mathcal{P}$, so, by Theorem 3.1, $\bar{\xi}$ is representable if and only if:

$$\int_0^T \frac{(g'(W_t))^2}{T-t}\, dt < \infty, \text{ a.s.} \tag{4.1}$$

Continuity of g' and W force $g'(W_t) \to 0$ when $t \to T$, i.e., $g'(W_T) = 0$, a.s., whenever (4.1) holds. Since W_T has full support under any $\mathbb{P} \in \mathcal{P}$, we conclude that $g' \equiv 0$.

The next step in the proof is to relax the assumption that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by a Brownian motion. In preparation, let $\{\mathcal{F}_t^W\}_{t\in[0,T]}$ denote the subfiltration of $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the Brownian motion W, and let Prog and Prog^W denote the progressive σ-algebras on $[0,T]\times\mathcal{F}$ corresponding to $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\{\mathcal{F}_t^W\}_{t\in[0,T]}$, respectively.

We argue by contradiction and assume that some $\xi = g(W_T)$, with non constant g, can be represented as $\int_0^T \beta_u\,du$ for some $\beta\in\text{Prog}$ with $\int_0^T \beta_u^2\,du < \infty$, a.s. With $\mathbb{Q}$ denoting a probability measure equivalent to $\mathbb{P}$ with the property that $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_u^2\,du] < \infty$, we observe that β, seen as a Prog-measurable function on the product space $[0,T]\times\Omega$, is square integrable with respect to the product probability measure $\mu = \frac{1}{T}\lambda\otimes\mathbb{Q}$ (where λ denotes the Lebesgue measure on $[0,T]$). Hence, the conditional expectation β^W of β, taken on the probability space $([0,T]\times\Omega, \text{Prog}, \mu)$, given the σ-algebra Prog^W, is well defined and satisfies

$$\int_B \beta_u^W(\omega)\,d\mu(u,\omega) = \int_B \beta_u(\omega)\,d\mu(u,\omega) \text{ for all } B\in\text{Prog}^W.$$

If we set $B = [0,T]\times A$, for $A\in\mathcal{F}_T^W$ and remember that $\xi\in\mathcal{F}_T^W$, we obtain immediately that $\int_0^T \beta_u^W\,du = \xi$, a.s. Moreover, since the conditional expectation preserves square integrability, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\beta_u^W)^2\,du\right] < \infty,$$

and, consequently, $\int_0^T (\beta_u^W)^2\,du < \infty$, a.s. It remains to observe that the existence of such β^W contradicts the conclusion of the first part of the proof, and completes the argument. □

4.2. Representations in $\mathbb{L}^{p,1}$ for $p < 2$

The "factorization formula" of Da Prato and Zabczyk when specialized to the case $S(t) = \text{Id}$ and $U = H = \mathbb{R}$ (see Theorem 5.2.5, p. 58 in Da Prato and Zabczyk (1996) for the statement and the paragraph that precedes it for the necessary definitions and notation) states the following: whenever $\alpha\in[0,1)$, and Ψ is a progressively measurable process such that

$$\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-r)^{-2\alpha}\mathbb{E}\left[\Psi(r)^2\right]dr\right)^{1/2} ds < \infty \text{ a.s.,} \tag{4.2}$$

we have

$$\int_0^t \Psi(s)\,dW_s = \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-1} Y_\alpha^\Psi(s)\,ds \text{ a.s., for all } t\in[0,T], \tag{4.3}$$

where

$$Y_\alpha^\Psi(s) = \int_0^t (s-r)^{-\alpha}\Psi(r)\,dW_r.$$

To see how (4.3) above leads to an interesting absolutely continuous representation, we pick a bounded progressively measurable process σ and apply the factorization formula (4.3) above with $t = T$ and $\Psi = \sigma$. When $\alpha\in[0,1/2)$, so that (4.2) is satisfied, (4.3) yields directly the following absolutely-continuous representation for the last element $\xi = M_T$ of the martingale $M_t = \int_0^t \sigma_u\,dW_u$:

$$\xi = \int_0^T \beta_t\,dt \text{ where } \beta_t = \frac{\sin(\alpha\pi)}{\pi}(T-t)^{\alpha-1}R_t \text{ and } R_t = \int_0^t (t-u)^{-\alpha}\,dM_u. \tag{4.4}$$

Note that we can interpret R_t as a formal Riemann-Liouville fractional integral of order α of the noise dM, up to a multiplicative constant. To complete the factorization formula, we then integrate the result multiplied by $(T-t)^{\alpha-1}$ with respect to dt, i.e., compute the Riemann-Liouville integral of the complementary order $1-\alpha$ of the result. The semigroup property of Riemann-Liouville integration would suggest that the result should coincide with the integral of order $\alpha + (1-\alpha) = 1$ of dM, i.e., it should yield $M_T = \xi$, which is precisely the case. We refer the reader to Samko, Kilbas and Marichev (1993) for details on fractional integration.

Since no restriction other than boundedness is imposed on σ it may appear at first glance that (4.4) contradicts Proposition 4.1 above, in that it provides a representation for $\xi = W_T$, for example. The difference, as in the discussion of the stochastic Fubini theorem in Remark 3.2 above, lies in the regularity class. As an easy illustration, let us show that in the special case $\sigma \equiv 1$, the representation β given by (4.4) does not even belong to weak regularity class $\mathcal{B}^{2,0}(\xi)$. In fact, we show below that it satisfies $\int_0^T \beta_u^2\, du = +\infty$, a.s. To do that, we focus on the process R from (4.4) which, in this case, takes the form

$$R_t = \int_0^t (t-s)^{-\alpha}\, dW_s. \tag{4.5}$$

Known as the *Riemann-Liouville process* (up to a multiplicative constant), the process R from (4.5) above is Gaussian and admits a continuous modification (see Marinucci and Robinson (1999) for a survey of its properties and its relation to the fractional Brownian motion). Its value R_T at T is normally distributed with a nonzero variance, so that, by continuity, we have $\lim_{t\to T} R_t = R_T \neq 0$, a.s. It follows that there exists a random variable K such that $K > 0$, a.s., and

$$|\beta_t| \geq K(T-t)^{\alpha-1}, \text{ a.s.,}$$

for all t in a (random) neighborhood of T. Since $\alpha < 1/2$, i.e., $2\alpha - 2 < -1$, we necessarily have $\int_0^T \beta_t^2\, dt = \infty$, a.s.

On the other hand, as our next result states, the factorization theorem allows us to construct absolutely continuous representations of a wide variety of random variables in any one of slightly less regular classes $\mathbb{L}^{p,1}$, $p \in [1,2)$.

Proposition 4.2. *Suppose that the filtration supports a Brownian motion W and that $\xi = \int_0^T \sigma_u\, dW_u$, with σ bounded. Then ξ admits an absolutely continuous representation in $\mathbb{L}^{p,1}$ for each $p \in [1,2)$.*

Proof. Let $p \in [1,2)$ be given. We claim that that the representation (4.4) above, with $M_t = \int_0^t \sigma_u\, dW_u$, and with a suitably chosen $\alpha \in (0,1/2)$ belongs to $\mathbb{L}^{p,1}$. To see that we use Doob's maximal inequality followed by the Burkholder-Davis-Gundy theorem to obtain the following:

$$\begin{aligned}
\mathbb{E}[|\beta_t|^p] &= (T-t)^{p(\alpha-1)}\mathbb{E}\left[\left|\int_0^t (t-u)^{-\alpha}\, dM_u\right|^p\right] \\
&\leq (T-t)^{p(\alpha-1)}\mathbb{E}\left[\sup_{s\leq t}\left|\int_0^s (t-u)^{-\alpha}\, dM_u\right|^p\right] \\
&\lesssim (T-t)^{p(\alpha-1)}\mathbb{E}\left[\left(\int_0^t (t-u)^{-2\alpha}\, d\langle M\rangle_u\right)^{p/2}\right]
\end{aligned}$$

where $a \lessapprox b$ is a shorthand for $a \le C\, b$, for some constant $C > 0$ which depends only on p. Allowing C to depend on σ as well, we can go on to conclude that

$$\begin{aligned}\mathbb{E}\left[\int_0^T |\beta_t|^p \, dt\right] &\le \int_0^T (T-t)^{p(\alpha-1)} \mathbb{E}\left[\left(\int_0^t (t-u)^{-2\alpha} \, d\langle M\rangle_u\right)^{p/2}\right] dt \\ &\lessapprox \int_0^T (T-t)^{p(\alpha-1)} \left(\int_0^t (t-u)^{-2\alpha} \, du\right)^{p/2} dt \\ &= \int_0^T (T-t)^{p(\alpha-1)} \frac{t^{p(1/2-\alpha)}}{1-2\alpha} \, dt.\end{aligned} \tag{4.6}$$

The last integral is finite if and only if $\alpha \in (1 - 1/p, 1/2)$. Therefore, the choice $\alpha = \frac{3}{4} - \frac{1}{2p}$ in (4.4) provides an absolutely continuous representation of ξ in $\mathbb{L}^{p,1}$ for $p \in (1, 2)$. □

Remark 4.3. The result of Proposition 4.2 above should be contrasted with the findings in our Theorem 2.1, which implies that under the same assumptions, a representation of $\xi = \int_0^T \sigma_u \, dW_u$ is possible in $\mathbb{L}^{2,1}$ if and only if, additionally,

$$\mathbb{E}\left[\int_0^T \frac{\sigma_u^2}{T-t} \, dt\right] < \infty.$$

Acknowledgments

The authors would like to thank Fausto Gozzi for suggesting some related literature. They are also grateful to the anonymous referees, and the Editors for a number of constructive comments that improved the quality of this paper.

During the preparation of this work the second named author was supported by the National Science Foundation under Grants No. DMS-1815017 No. DMS-2307729 Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation (NSF).

References

AÏD, R. and BIAGINI, S. (2023). Optimal dynamic regulation of carbon emissions market. *Math. Finance* **33** 80–115.

BIAGINI, S., GOZZI, F. and ZANELLA, M. (2022). Robust portfolio choice with sticky wages. *SIAM J. Financial Math.* **13** 1004–1039.

DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for infinite-dimensional systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge University Press, Cambridge.

DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic equations in infinite dimensions*, second ed. *Encyclopedia of Mathematics and its Applications* **152**. Cambridge University Press, Cambridge.

DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

FABBRI, G., GOZZI, F. and ŚWIECH, A. (2017). *Stochastic optimal control in infinite dimension*. *Probability Theory and Stochastic Modelling* **82**. Springer, Cham Dynamic programming and HJB equations, With a contribution by Marco Fuhrman and Gianmario Tessitore.

HE, S. W., WANG, J. G. and YAN, J. A. (1992). *Semimartingale theory and stochastic calculus*. Kexue Chubanshe (Science Press), Beijing; CRC Press, Boca Raton, FL.

MARINUCCI, D. and ROBINSON, P. M. (1999). Alternative forms of fractional Brownian motion. *J. Statist. Plann. Inference* **80** 111–122.

SAMKO, S. G., KILBAS, A. A. and MARICHEV, O. I. (1993). *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publishers.

VERAAR, M. (2012). The stochastic Fubini theorem revisited. *Stochastics* **84** 543–551.