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HOMOGENIZATION AND CORRECTOR RESULTS FOR A STOCHASTIC COUPLED THERMOELASTIC MODEL

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Abstract. The aim of this research is to explore new homogenization results for a stochastic linear coupled thermoelastic model. We focus on a stochastic equation of motion coupled with a stochastic heat equation, representing the linear thermoelastic behavior of composite anisotropic materials with a periodic heterogeneous structure. These materials are subjected to random external forces and heat sources. By employing the periodic unfolding method and leveraging probabilistic compactness results from Prokhorov and Skorokhod, we obtain homogenization, convergence of the associated stochastic energies, and corrector results.

1. Introduction.

Thermoelasticity, the study of the interplay between heat conduction and mechanical deformation, has captivated physicists, engineers, and applied mathematicians due to its broad spectrum of applications across various scientific and industrial domains. This theory examines the interdependence of temperature and strain fields, as evidenced by numerous physical experiments (see [24, 32] and references therein). The foundational mathematical model of thermoelasticity, introduced by Biot [4] and now known as the classical coupled theory of thermoelasticity, consists of two coupled partial differential equations: a hyperbolic equation governing motion and a parabolic equation describing energy transfer.

When applied to composite or perforated materials, the classical model exhibits an oscillatory behavior on a small scale, denoted by ϵ, with highly varying coefficients. These oscillations, often arising from periodic structures or perforations, necessitate the use of homogenization techniques to derive simplified "effective" models suitable for analytical or numerical analysis. Homogenization enables the replacement of equations with rapidly varying coefficients by ones with effective properties in a fixed domain.

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The first significant contributions to the homogenization of thermoelasticity were made by G. A. Francfort [13] in 1983, employing a semigroup approach. Since then, numerous advances have been made. For instance, W. J. Parnell [26] in 2006 used asymptotic expansions to homogenize a fully coupled one-dimensional linear thermoelastic model. V. L. Savatorova et al. [31] extended this work in 2013 through multi-scale modeling, examining the behavior of composite materials with periodic structures under thermal and mechanical stress. More recently, S. Nafiri [25] in 2023 applied two-scale convergence methods to homogenize a linear thermoelastic wave model.

While these studies focused on deterministic systems, incorporating stochastic effects into thermoelastic models is critical for addressing natural randomness in thermal processes and material behavior. In this work, we investigate a coupled hyperbolic-parabolic system describing the interaction between deformation and temperature fields in a composite material under heat sources and body forces, where stochasticity arises from environmental thermal interactions.

Our main goal is to establish homogenization results for this stochastic model. To achieve this, we employ the periodic unfolding method, a powerful tool in homogenization theory (see [7, 8, 9, 10]), along with probabilistic compactness techniques [3]. Although significant progress has been made in the deterministic homogenization of hyperbolic systems (see [11, 12, 15, 16]), the stochastic setting remains less explored. Pioneering work by M. Mohammed and M. Sango [20, 21, 22, 23] laid the foundation for the homogenization of stochastic systems. However, the homogenization of stochastic partial differential equations in perforated domains is still a developing field (see [29, 30, 17, 18, 14]), particularly for practical applications [2, 19].

To the best of our knowledge, this is the first work to analyze the asymptotic behavior of solutions for a coupled stochastic hyperbolic-parabolic system in a perforated domain.

This paper is organized as follows. Section 2 introduces the model, the functional framework, and the assumptions necessary for the analysis. In Section 3, we derive key a priori estimates essential for subsequent arguments. Section 4 presents the periodic unfolding operator, which forms the core of our homogenization framework. Section 5 addresses the tightness of probability measures associated with the solution sequence, enabling the application of Skorokhod's and Prokhorov's theorems to construct a limiting process that solves the homogenized system. Section 6 derives the homogenization results using the periodic unfolding method and probabilistic compactness tools. Finally, Section 7 establishes energy convergence results and corrector estimates. We conclude the paper with a summary of the research objectives, the main findings, and their implications.

2. **The model and functional setting.**

Let us go over some of the notations and functional spaces that are frequently utilized in this paper, see [11, 5] for detailed definitions:

- D is an open bounded subset of $\mathbb{R}^n$, $n = 1, 2$ or 3.
- $[0, T]$, $T > 0$ is the time interval.
- $Y = (0, 1) \times (0, 1) \times \cdots \times (0, 1) \subset \mathbb{R}^n$ is the reference cell.
- $Y_1 \subset Y$ such that $\bar{Y}_1 \cap \partial Y = \emptyset$ and ∂Y_1 is smooth enough. Furthermore, $Y_2 = Y \backslash \bar{Y}_1$ such that $\vartheta = \frac{|Y_2|}{|Y|} = |Y_2|$.
- $\{\epsilon\}$ is a sequence of positive integers such that ϵ goes to zero.

- $D_0^\epsilon = \bigcup_{k\in\mathbb{Z}^n} \left\{\epsilon\left(Y_1 + \sum_{j=1}^n k_j e_j\right) : \left(Y_1 + \sum_{j=1}^n k_j e_j\right) \subset D\right\}$ is the pore skeleton.
- $D^\epsilon = D\backslash D_0^\epsilon$ is the pore volume.
- $\partial D_0^\epsilon = \bigcup_{k\in\mathbb{Z}^n} \left\{\epsilon\left(\partial Y_1 + \sum_{j=1}^n k_j e_j\right) : \left(\partial Y_1 + \sum_{j=1}^n k_j e_j\right) \subset D\right\}$ is the skeleton surface.
- $\Gamma^\epsilon = \partial D_0^\epsilon \cup \partial D$.
- $\hat{D}^\epsilon$ is the interior of D^ϵ and $\Lambda^\epsilon = D\backslash\hat{D}^\epsilon$.
- For any open subset $\mathcal{O}$ of $\mathbb{R}^n$, $L^p(\mathcal{O})$ and $W^{1,p}(\mathcal{O})$ $(1 \le p \le \infty)$ (resp.) are the well known Lebesgue's and Sobolev's spaces (resp.). For $p = 2$, they are also known to be Hilbert's spaces.
- $L^p(0,T;L^q(\mathcal{O}))$ and $L^p(0,T;W^{1,q}(\mathcal{O}))$, $(1 \le p,q \le \infty)$ (resp.) are the time-space version of Lebesgue's and Soboleve's spaces (resp.).
- $L^2(\Omega;L^p(0,T;L^q(\mathcal{O})))$ and , $L^2(\Omega;L^p(0,T;W^{1,q}(\mathcal{O})))$, $(1 \le p,q \le \infty)$ (resp.) are the probabilistic time-space version of Lebesgue's and Sobolev's spaces (resp.).
- We define the following Hilbert's spaces
$$\mathcal{V}(D^\epsilon) = \left\{\mathbf{v}^\epsilon = (v_k^\epsilon)_{1\le k\le n} | \mathbf{v}^\epsilon \in [W^{1,2}(D^\epsilon)]^n; \mathbf{v}^\epsilon = 0 \text{ on } \partial D\right\},$$
and
$$\mathcal{S}(D^\epsilon) = \left\{\psi^\epsilon | \psi^\epsilon \in W^{1,2}(D^\epsilon); \psi^\epsilon = 0 \text{ on } \partial D\right\},$$
Equipped with the norms
$$\|\mathbf{v}^\epsilon\|^2_{\mathcal{V}(D^\epsilon)} = \int_{D^\epsilon} |\nabla \mathbf{v}^\epsilon|^2 dx \text{ and } \|\psi^\epsilon\|^2_{\mathcal{S}(D^\epsilon)} = \int_{D^\epsilon} |\nabla\psi^\epsilon|^2 dx.$$
We denote by $\mathcal{V}^{'}(D^\epsilon)$ and $\mathcal{S}^{'}(D^\epsilon)$ the dual space of $\mathcal{V}(D^\epsilon)$ and $\mathcal{S}(D^\epsilon)$ respectively, and $\langle\cdot,\cdot\rangle_{\mathcal{V}(D^\epsilon),\mathcal{V}'(D^\epsilon)}$ and $\langle\cdot,\cdot\rangle_{\mathcal{S}(D^\epsilon),\mathcal{S}'(D^\epsilon)}$ are the usual duality pairings.
- $\mathcal{M}_{\mathcal{O}}(f) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} f(x)dx$ for any function $f \in L^1(\mathcal{O})$ where $\mathcal{O}$ is open and bounded subset of $\mathbb{R}^n$.

In what follows, we introduce our model, which describes stochastic linear thermoelastic waves in anisotropic composite materials with highly heterogeneous coefficients.

$$\begin{cases} \rho^\epsilon d\left(\dfrac{\partial \mathbf{u}^\epsilon}{\partial t}\right) - [\operatorname{div}(A^\epsilon \nabla \mathbf{u}^\epsilon) - \nabla(\beta^\epsilon\theta^\epsilon)]dt \\ \qquad\qquad = \mathbf{f}_1^\epsilon dt + \mathbf{f}_2^\epsilon dW_1 \text{ in } \Omega\times D^\epsilon\times(0,T), \\ \rho^\epsilon c_v^\epsilon d\theta^\epsilon - \left[\operatorname{div}(\kappa^\epsilon\nabla\theta^\epsilon) - \beta^\epsilon \operatorname{div}\dfrac{\partial \mathbf{u}^\epsilon}{\partial t}\right]dt \\ \qquad\qquad = g_1^\epsilon dt + g_2^\epsilon dW_2 \text{ in } \Omega\times D^\epsilon\times(0,T), \\ \mathbf{u}^\epsilon = 0 = \theta^\epsilon \text{ on } \Omega\times\partial D\times(0,T), \\ A^\epsilon\nabla\mathbf{u}^\epsilon\cdot\nu = 0 = \kappa^\epsilon\nabla\theta^\epsilon\cdot\nu \text{ on } \Omega\times\partial D_0^\epsilon\times(0,T), \\ (A^\epsilon\nabla\mathbf{u}^\epsilon - \beta^\epsilon\theta^\epsilon)\cdot\nu = 0 \text{ on } \Omega\times\partial D_0^\epsilon\times(0,T), \\ \mathbf{u}^\epsilon(x,0) = \mathbf{h}_1^\epsilon(x), \quad \dfrac{\partial\mathbf{u}^\epsilon}{\partial t}(x,0) = \mathbf{h}_2^\epsilon(x), \theta^\epsilon(x,0) = h_3^\epsilon(x) \text{ in } D^\epsilon; \end{cases} \tag{1}$$

where $(x,t) \in D_T = D\times(0,T)$, $T > 0$, D^ϵ is the perforated domain. Here D and D^ϵ are bounded subset of $\mathbb{R}^n$. The vector $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon, \cdots, u_n^\epsilon)$, $n = 1,2$ or 3 represents the displacement, θ^ϵ stands for the increment in temperature, $\rho^\epsilon = \rho\left(\frac{x}{\epsilon}\right)$ is the material density, $c_v^\epsilon = c_v\left(\frac{x}{\epsilon}\right)$ is heat capacity $A^\epsilon = \left(a_{ijkl}\left(\frac{x}{\epsilon}\right)\right)_{1\le i,j,k,l\le n}$ is the stiffness (4-order) tensor, $\beta^\epsilon = \left(\beta_{i,j}\left(\frac{x}{\epsilon}\right)\right)_{1\le i,j\le n}$ is an nonzero (2-order) tensor, such that $\beta_{i,j} = a_{ijkl}b_{kl}$ where b_{kl} is the component of the thermal expansion , $\kappa^\epsilon = \left(\kappa_{ij}\left(\frac{x}{\epsilon}\right)\right)_{1\le i,j\le n}$ is the thermal conductivity tensor. $\mathbf{f}_i^\epsilon = \left(f_{ij}\left(\frac{x}{\epsilon}, t\right)\right)_{1\le j\le n}$, $i = 1, 2$ and $n = 1, 2$ or 3 and $\mathbf{f}_1^\epsilon dt + \mathbf{f}_2^\epsilon dW_1$ represents the body forces and $g_1^\epsilon dt + g_2^\epsilon dW_2$ represents the heat source, while W_1 and W_2 are one-dimensional Brownian motions defined on a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with a filtration $\mathcal{F}_t$, $t \in (0, T)$ and expectation $\mathbb{E}$. ν represents the unit outward normal vector to D_0^ϵ.

Assumptions.

A.1. *$\rho^\epsilon(x) = \rho(y)$ and $c_v^\epsilon(x) = c_v(y)$ are both $Y-$periodic and for some $\alpha > 0$ we have, $0 < \rho(y), c_v(y) \le \alpha$ for all $y \in Y$.*

A.2. *The stiffness tensor $A(y) = (a_{ijkl}(y))_{1\le i,j,k,l\le n}$ is $Y-$periodic and symmetric for all i, j, k and l. Furthermore $A(y)$ is bounded in $L^\infty(Y)$ and for some constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < \alpha_1 < \alpha_2$, we have*

$$\alpha_1|\eta|^2 \le A\eta\eta \le \alpha_2|\eta|^2, \tag{2}$$

for all symmetric tensor $\eta = (\eta_{ij})_{1\le i,j\le n} \in \mathbb{R}^n \times \mathbb{R}^n$, where $|\eta| = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{\frac{1}{2}}$.

A.3. *The thermal conductivity tensor $\kappa(y) = (\kappa_{ij}(y))_{1\le i,j\le n}$ is $Y-$periodic, and is bounded in $L^\infty(Y)$. Furthermore there exist $\alpha_3, \alpha_4 \in \mathbb{R}$ such that $0 < \alpha_3 < \alpha_4$ and*

$$\alpha_3|\xi|^2 \le \kappa\xi\xi \le \alpha_4|\xi|^2, \tag{3}$$

for all vector $\xi = (\xi_i)_{1\le i\le n} \in \mathbb{R}^n$.

A.4. *The functions $\boldsymbol{f}_1^\epsilon(x,t) = \boldsymbol{f}_1(x/\epsilon, t)$ and $\boldsymbol{f}_2^\epsilon(x,t) = \boldsymbol{f}_2(x/\epsilon, t)$ both in the space $L^2(0, T; [L^2(D^\epsilon)]^n)$ and $g_1^\epsilon(x,t) = g_1(x/\epsilon, t)$ and $g_2^\epsilon(x,t) = g_2(x/\epsilon, t)$ both in the space $L^2(0, T; L^2(D^\epsilon))$.*

A.5. *$\boldsymbol{h}_1^\epsilon(x) = \boldsymbol{h}_1(x/\epsilon) = (h_1^1(x/\epsilon), h_1^2(x/\epsilon), \cdots, h_1^n(x/\epsilon)) \in \mathcal{V}(D^\epsilon)$, $\boldsymbol{h}_2^\epsilon(x) = \boldsymbol{h}_2(x/\epsilon) = (h_2^1(x/\epsilon), h_2^2(x/\epsilon), \cdots, h_2^n(x/\epsilon)) \in [L^2(D^\epsilon)]^n$ and $h_3^\epsilon(x) = h_3(x/\epsilon) \in L^2(D^\epsilon)$.*

A.6. *$\beta^\epsilon(x)$ a non-zero symmetric tensor, where $\beta_{i,j} \in W_{per}^{1,2}(Y_2)$ for all $i, j = 1, 2, \cdots, n$ and*

$$|\beta^\epsilon\eta| \le \alpha_5|\eta| \text{ for positive constant } \alpha_5 \text{ and } \eta \in \mathbb{R}^n. \tag{4}$$

As is customary in elasticity, we write the strain and stress tensors as:

$$e(\mathbf{u}^\epsilon) = \frac{1}{2}\left(\nabla\mathbf{u}^\epsilon + (\nabla\mathbf{u}^\epsilon)^T\right) \text{ and } \sigma^\epsilon = A^\epsilon e(\mathbf{u}^\epsilon) - \beta^\epsilon\theta^\epsilon, \tag{5}$$

where $(A^\epsilon e(\mathbf{u}^\epsilon))_{i,j} = a_{ijkl}\frac{\partial u_k^\epsilon}{\partial x_l}$. By the symmetry of the tensor $e(\mathbf{u}^\epsilon)$, one sees that

$$\alpha_1|e(\mathbf{u}^\epsilon)|^2 \le A^\epsilon e(\mathbf{u}^\epsilon)e(\mathbf{u}^\epsilon) \le \alpha_2|e(\mathbf{u}^\epsilon)|^2. \tag{6}$$

From [11, Proposition 10.5], we have for some constants $c_1, c_2 > 0$

$$\int_{D^\epsilon} |e(\mathbf{u}^\epsilon)|^2 dx \le c_1\|\mathbf{u}^\epsilon\|_{\mathcal{V}(D^\epsilon)}^2 \text{ and } \|\mathbf{u}^\epsilon\|_{\mathcal{V}(D^\epsilon)}^2 \le c_2\int_{D^\epsilon} |e(\mathbf{u}^\epsilon)|^2 dx. \tag{7}$$

Furthermore, $\int_{D^\epsilon} |e(\mathbf{u}^\epsilon)|^2 dx$ defines an equivalent norm to that of $\mathcal{V}(D^\epsilon)$, lets denote it by $|||e(\mathbf{u}^\epsilon)|||_{\mathcal{V}(D^\epsilon)}^2$.

Theorem 2.1. *For fixed ϵ and under the assumptions $(A1) - (A6)$, there exists a unique strong probabilistic solution $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon\right)$ such that:*

- $$\boldsymbol{u}^\epsilon \in C([0,T];\mathcal{V}(D^\epsilon)) \cap L^2(\Omega; L^2(0,T;\mathcal{V}(D^\epsilon))),$$
$$\frac{\partial \boldsymbol{u}^\epsilon}{\partial t} \in C([0,T];[L^2(D^\epsilon)]^n) \cap L^2(\Omega; L^2(0,T;[L^2(D^\epsilon)]^n)),$$
and
$$\theta^\epsilon \in C([0,T];\mathcal{S}(D^\epsilon)) \cap L^2(\Omega; L^2(0,T;\mathcal{S}(D^\epsilon))).$$
- $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon\right)$*-*$\mathcal{F}_t$*-measurable.*
- *For all* $t \in [0,T]$ *and* $(\boldsymbol{v},\psi) \in \mathcal{V}(D^\epsilon)\times\mathcal{S}(D^\epsilon)$*, we have that* $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon\right)$ *admit the following weak formulation*
$$\begin{aligned}\int_0^t \left(\rho^\epsilon d\left(\frac{\partial \boldsymbol{u}^\epsilon}{\partial t}\right), \boldsymbol{v}\right)_{L^2(D^\epsilon)} &+ \int_0^t (A^\epsilon e(\boldsymbol{u}^\epsilon), e(\boldsymbol{v}))_{L^2(D^\epsilon)} d\tau \\ &- \int_0^t (\beta^\epsilon \theta^\epsilon, \nabla \boldsymbol{v})_{L^2(D^\epsilon)} d\tau = \int_0^t (\boldsymbol{f}_1^\epsilon, \boldsymbol{v})_{L^2(D^\epsilon)} d\tau \\ &+ \int_0^t (\boldsymbol{f}_2^\epsilon, \boldsymbol{v})_{L^2(D^\epsilon)} dW_1(\tau), \end{aligned} \tag{8}$$
and
$$\begin{aligned}\int_0^t (\rho^\epsilon c_v^\epsilon d\theta^\epsilon, \psi)_{L^2(D^\epsilon)} &+ \int_0^t (\kappa^\epsilon \nabla\theta^\epsilon, \nabla\psi)_{L^2(D^\epsilon)} d\tau \\ &- \int_0^t (\beta^\epsilon \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \nabla\psi)_{L^2(D^\epsilon)} d\tau + \epsilon \int_0^t (\nabla\beta^\epsilon \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \psi)_{L^2(D^\epsilon)} d\tau \\ &= \int_0^t (g_1^\epsilon, \psi)_{L^2(D^\epsilon)} d\tau + \int_0^t (g_2^\epsilon, \psi)_{L^2(D^\epsilon)} dW_2(\tau). \end{aligned} \tag{9}$$

Sketch of the Proof. We give a sketch of the proof in the following steps.

Proof. 1. The first step is to construct finite-dimensional approximating subspaces $\mathcal{V}_n = \text{Span}\{\mathbf{v}_1^\epsilon, \mathbf{v}_2^\epsilon, \cdots, \mathbf{v}_n^\epsilon\}$ and $\mathcal{S}_n = \text{Span}\{\varphi_1^\epsilon, \varphi_2^\epsilon, \cdots, \varphi_n^\epsilon\}$, where $\mathbf{v}_k^\epsilon \in \mathcal{V}(D^\epsilon)$ and $\varphi_k \in \mathcal{S}(D^\epsilon)$ for all $k = 1, 2, \cdots, n$ and $(\mathbf{v}_k^\epsilon)_{1\le k\le n}$ and $(\varphi_k^\epsilon)_{1\le k\le n}$ are orthonormal basis of $[L^2(D^\epsilon)]^n$ and $L^2(D^\epsilon)$, respectively.

2. Using the above constructed spaces, one introduces the following finite dimensional approximating problem: Find $\mathbf{u}_n^\epsilon$, $\frac{\partial \mathbf{u}_n^\epsilon}{\partial t}$ and θ_n^ϵ such that
$$\mathbf{u}_n^\epsilon = \sum_{k=1}^n \ell_k^n(t)\mathbf{v}_k^\epsilon,\ \frac{\partial \mathbf{u}_n^\epsilon}{\partial t} = \sum_{k=1}^n \frac{\partial \ell_k^n}{\partial t}(t)\mathbf{v}_k^\epsilon \text{ and } \theta_n^\epsilon = \sum_{k=1}^n b_k^n(t)\varphi_k^\epsilon,$$
where $\ell_k^n(t) = (\mathbf{u}_n^\epsilon, \mathbf{v}_k^\epsilon)$ and $b_k^n(t) = (\theta_n^\epsilon, \varphi 6\epsilon_k)$ such that
$$\begin{aligned} d\left(\rho^\epsilon \frac{\partial \mathbf{u}_n^\epsilon}{\partial t}, \mathbf{v}_k\right)_{L^2(D^\epsilon)} &+ (A^\epsilon e(\mathbf{u}_n^\epsilon) e(\mathbf{v}_k))_{L^2(D^\epsilon)} dt - (\beta^\epsilon\theta_n^\epsilon, \nabla \mathbf{v}_k)_{L^2(D^\epsilon)}\, dt \\ &= (\mathbf{f}_1^\epsilon, \mathbf{v}_k)_{L^2(D^\epsilon)}\, dt + (\mathbf{f}_2^\epsilon dW_1(t), \mathbf{v}_k)_{L^2(D^\epsilon)}, \end{aligned} \tag{10}$$
and
$$\begin{aligned} d(\rho^\epsilon c_v^\epsilon \theta_n^\epsilon, \varphi_k)_{L^2(D^\epsilon)} &+ (\kappa^\epsilon \nabla\theta_n^\epsilon, \nabla\varphi_k)_{L^2(D^\epsilon)} dt - \left(\beta^\epsilon \frac{\partial \mathbf{u}_n^\epsilon}{\partial t}, \nabla\varphi_k\right)_{L^2(D^\epsilon)} dt \\ &+ \epsilon\left(\nabla\beta^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}, \varphi_k\right)_{L^2(D^\epsilon)} dt = (g_1^\epsilon, \varphi_k)_{L^2(D^\epsilon)} d\tau + (g_2^\epsilon dW_2(t), \varphi_k)_{L^2(D^\epsilon)}, \end{aligned} \tag{11}$$

with appropriate initial conditions that converge to the initial conditions of the original problem. This system is interpreted as a system of stochastic differential equations which has a unique solution by a classical result.

3. In this step, we prove some estimates on $\mathbf{u}_n^\epsilon$, $\frac{\partial \mathbf{u}_n^\epsilon}{\partial t}$ and θ_n^ϵ similar to the ones obtained in Theorem 3.1 below.
4. Using the previous uniform estimates, we pass to the limit on the weak formulations (10) and (11). Furthermore, we establish that the limits satisfy the same previous estimates.
5. Finally, we show a pathwise uniqueness, then together with the Yamada-Watanabe theorem we complete the proof of the theorem.

□

3. **A priori bounds and estimates.** Here, we prove key results for passing to the limits in the system above.

Theorem 3.1. *Under the assumptions A.1. − A.6. the following estimates hold true*

$$\mathbb{E}\left\|\frac{\partial \boldsymbol{u}^\epsilon}{\partial t}\right\|^2_{L^\infty(0,T;[L^2(D^\epsilon)]^n)} + \mathbb{E}\|\boldsymbol{u}^\epsilon\|^2_{L^\infty(0,T;\mathcal{V}(D^\epsilon))} \leq C \tag{12}$$

$$\mathbb{E}\|\theta^\epsilon\|^2_{L^\infty(0,T;L^2(D^\epsilon))} + \mathbb{E}\|\theta^\epsilon\|^2_{L^2(0,T;\mathcal{S}(D^\epsilon))} \leq C, \tag{13}$$

$$\mathbb{E}\|e(\boldsymbol{u}^\epsilon)\|^2_{L^\infty(0,T;\mathcal{V}(D^\epsilon))} \leq C. \tag{14}$$

for some $C > 0$.

Proof. We apply Itô's lemma to the function $\Phi(t, \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)) = \|\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\|^2_{L^2(D^\epsilon)}$ in the first equation of the system (1) and to the function $\Psi(t, \theta^\epsilon(t)) = \|\rho^\epsilon c_v^\epsilon \theta^\epsilon(t)\|^2_{L^2(D^\epsilon)}$ in the second equation of the system (1), we get:

$$\begin{aligned} d\left\|\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right\|^2_{L^2(D^\epsilon)} &= 2\left[\left(\operatorname{div}\left(A^\epsilon \nabla \mathbf{u}^\epsilon\right), \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} - \left(\nabla(\beta^\epsilon\theta^\epsilon), \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}\right] dt \\ &+ 2\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} + \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)} dt + 2\left(\mathbf{f}_2^\epsilon dW_1, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} d\|\rho^\epsilon c_v^\epsilon \theta^\epsilon\|^2_{L^2(D^\epsilon)} &= 2\left[(-\kappa^\epsilon \nabla\theta^\epsilon, \nabla\theta^\epsilon)_{L^2(D^\epsilon)} - \left(\beta^\epsilon \operatorname{div}\left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right), \theta^\epsilon\right)_{L^2(D^\epsilon)}\right] dt \\ &+ 2(g_1, \theta^\epsilon)_{L^2(D^\epsilon)} + \|g_2^\epsilon\|^2_{L^2(D^\epsilon)} dt + 2(g_2^\epsilon dW_2, \theta^\epsilon)_{L^2(D^\epsilon)}. \end{aligned} \tag{16}$$

Using integration by parts, the symmetry assumption of the stiffness tensor, and the fact that $\beta^\epsilon \operatorname{div}\left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right) = \operatorname{div}\left(\beta^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right) - \epsilon \nabla\beta^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}$, we have

$$\begin{aligned} &d\left[\left\|\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right\|^2_{L^2(D^\epsilon)} + (A^\epsilon e(\mathbf{u}^\epsilon), e(\mathbf{u}^\epsilon))_{L^2(D^\epsilon)}\right] \\ &= 2\left[\left(\beta^\epsilon \nabla\theta^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} - \epsilon\left(\theta^\epsilon \nabla\beta^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}\right] dt \\ &+ 2\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} + \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)} dt + 2\left(\mathbf{f}_2^\epsilon dW_1, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}, \end{aligned} \tag{17}$$

and

$$d\|\rho^\epsilon c_v^\epsilon \theta^\epsilon\|^2_{L^2(D^\epsilon)} + (\kappa^\epsilon \nabla\theta^\epsilon, \nabla\theta^\epsilon)_{L^2(D^\epsilon)}$$

$$= 2\left[-\left(\beta^\epsilon\frac{\partial \mathbf{u}^\epsilon}{\partial t}, \nabla\theta^\epsilon\right)_{L^2(D^\epsilon)} + \epsilon\left(\nabla\beta^\epsilon\frac{\partial \mathbf{u}^\epsilon}{\partial t}, \theta^\epsilon\right)_{L^2(D^\epsilon)}\right]dt$$
$$+ 2(g_1, \theta^\epsilon)_{L^2(D^\epsilon)} + \|g_2^\epsilon\|^2_{L^2(D^\epsilon)}dt + 2(g_2^\epsilon dW_2, \theta^\epsilon)_{L^2(D^\epsilon)}. \quad (18)$$

Adding together (17) and (18), we obtain

$$d\left[\left\|\rho^\epsilon\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right\|^2_{L^2(D^\epsilon)} + \|\rho^\epsilon c_v^\epsilon\theta^\epsilon\|^2_{L^2(D^\epsilon)} + (A^\epsilon e(\mathbf{u}^\epsilon), e(\mathbf{u}^\epsilon))_{L^2(D^\epsilon)}\right] + (\kappa^\epsilon\nabla\theta^\epsilon, \nabla\theta^\epsilon)_{L^2(D^\epsilon)}$$
$$= 2\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} + 2(g_1, \theta^\epsilon)_{L^2(D^\epsilon)}dt$$
$$+ \|g_2^\epsilon\|^2_{L^2(D^\epsilon)} + \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)}dt + 2\left(\mathbf{f}_2^\epsilon dW_1, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}$$
$$+ 2(g_2^\epsilon dW_2, \theta^\epsilon)_{L^2(D^\epsilon)}. \quad (19)$$

Using the assumptions A.1.-A.4., the relation in (7) and integrate from 0 to t where $t \in [0, T]$, we have

$$\alpha^2\left\|\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right\|^2_{L^2(D^\epsilon)} + \alpha^4\|\theta^\epsilon\|^2_{L^2(D^\epsilon)} + \alpha_2 c_1\|\mathbf{u}^\epsilon\|^2_{\mathcal{V}(D^\epsilon)} + \alpha_4\int_0^t \|\nabla\theta^\epsilon\|^2_{L^2(D^\epsilon)}ds$$
$$\leq C + 2\int_0^t\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}ds + 2\int_0^t (g_1, \theta^\epsilon)_{L^2(D^\epsilon)}ds$$
$$+ 2\int_0^t\left(\mathbf{f}_2^\epsilon dW_1(s), \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)} + 2\int_0^t (g_2^\epsilon dW_2(s), \theta^\epsilon)_{L^2(D^\epsilon)}. \quad (20)$$

Taking the sup over $0 \leq t \leq T$, followed by the expectation, we have

$$\alpha^2\mathbb{E}\sup_t\left\|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\right\|^2_{L^2(D^\epsilon)} + \alpha^4\mathbb{E}\sup_t\|\theta^\epsilon(t)\|^2_{L^2(D^\epsilon)} + \alpha_2 c_1\mathbb{E}\sup_t\|\mathbf{u}^\epsilon(t)\|^2_{\mathcal{V}(D^\epsilon)(D^\epsilon)}$$
$$+ \alpha_4\mathbb{E}\int_0^T \|\nabla\theta^\epsilon(t)\|^2_{L^2(D^\epsilon)}dt \leq C + 2\mathbb{E}\int_0^T\left|\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}\right|ds$$
$$+ 2\mathbb{E}\int_0^T\left|(g_1, \theta^\epsilon)_{L^2(D^\epsilon)}\right|ds + 2\mathbb{E}\sup_t\left|\int_0^t\left(\mathbf{f}_2^\epsilon dW_1(s), \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}\right|$$
$$+ 2\mathbb{E}\sup_t\left|\int_0^t (g_2^\epsilon dW_2(s), \theta^\epsilon)_{L^2(D^\epsilon)}\right|. \quad (21)$$

Cauchy's and Young's inequalities imply the following

$$\mathbb{E}\int_0^T\left|\left(\mathbf{f}_1^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)_{L^2(D^\epsilon)}\right|ds$$
$$\leq \delta\mathbb{E}\sup_t\left\|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\right\|^2_{L^2(D^\epsilon)} + C(\delta)\left(\int_0^T \|\mathbf{f}_1^\epsilon(t)\|^2_{L^2(D^\epsilon)}\right)^2, \quad (22)$$

where δ is sufficiently small. Similarly

$$\mathbb{E}\int_0^T\left|(g_1^\epsilon, \theta^\epsilon)_{L^2(D^\epsilon)}\right|ds$$

$$\leq \delta \mathbb{E} \sup_t \|\theta^\epsilon(t)\|^2_{L^2(D^\epsilon)} + C(\delta) \left(\int_0^T \|g_1^\epsilon(t)\|^2_{L^2(D^\epsilon)} \right)^2, \quad (23)$$

Following that, we can deduce from the Burkholder-Gundy-Davis inequality, followed by the Cauchy-Schwarz and Young inequalities, that

$$\begin{aligned}
&\mathbb{E} \sup_t \left| \int_0^t \left(\mathbf{f}_2^\epsilon dW_1(s), \frac{\partial \mathbf{u}^\epsilon}{\partial t} \right)_{L^2(D^\epsilon)} \right| \\
&\leq C\mathbb{E} \left(\int_0^T \left(\mathbf{f}_2^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t} \right)^2_{L^2(D^\epsilon)} dt \right)^{\frac{1}{2}} \\
&\leq C\mathbb{E} \left(\int_0^T \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)} \left\| \frac{\partial \mathbf{u}^\epsilon}{\partial t} \right\|^2_{L^2(D^\epsilon)} dt \right)^{\frac{1}{2}} \\
&\leq \mathbb{E} \sup_t \left\| \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t) \right\|_{L^2(D^\epsilon)} \left(\int_0^T \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)} dt \right)^{\frac{1}{2}} \\
&\leq \varsigma C\mathbb{E} \sup_t \left\| \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t) \right\|^2_{L^2(D^\epsilon)} + C(\varsigma) \int_0^T \|\mathbf{f}_2^\epsilon\|^2_{L^2(D^\epsilon)} dt, \quad (24)
\end{aligned}$$

where ς is sufficiently small. In a similar way we show that

$$\begin{aligned}
&\mathbb{E} \sup_t \left| \int_0^t (g_2^\epsilon dW_2(s), \theta^\epsilon)_{L^2(D^\epsilon)} \right| \\
&\qquad\qquad \leq \varrho C\mathbb{E} \sup_t \|\theta^\epsilon(t)\|^2_{L^2(D^\epsilon)} + C(\varrho) \int_0^T \|g_2^\epsilon\|^2_{L^2(D^\epsilon)} dt, \quad (25)
\end{aligned}$$

where ϱ is sufficiently small. Substituting (22), (23), (24) and (25) into (21), and using Grönwall's inequality, we obtain (12) and (13). The proof of (14) is an easy consequences of (7). □

Theorem 3.2. *Assume that the assumptions (A.1.-A.5.) are satisfied, with the additional assumptions* $\boldsymbol{f}_2^\epsilon \in L^2(0,T;[L^4(D^\epsilon)]^n)$ *and* $g_2^\epsilon \in L^2(0,T;L^4(D^\epsilon))$*. Then,*

$$\mathbb{E} \int_0^{T-s} \left\| \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}(t+s) - \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}(t) \right\|^2_{\mathcal{V}'(D^\epsilon)} dt \leq Cs, \quad (26)$$

and

$$\mathbb{E} \int_0^{T-s} \|\theta^\epsilon(t+s) - \theta^\epsilon(t)\|^2_{\mathcal{S}'(D^\epsilon)} dt \leq Cs, \quad (27)$$

where C, is a positive constant independent of ϵ.

Proof. Let us proof (26), we have

$$\begin{aligned}
\frac{\partial \mathbf{u}^\epsilon}{\partial t}(t+s) - \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t) &= \frac{1}{\rho^\epsilon} \int_t^{t+s} \operatorname{div} A^\epsilon \nabla \mathbf{u}^\epsilon(\tau) d\tau - \frac{1}{\rho^\epsilon} \int_t^{t+s} \nabla \beta^\epsilon \theta^\epsilon(\tau) d\tau \\
&\quad + \frac{1}{\rho^\epsilon} \int_t^{t+s} \mathbf{f}_1^\epsilon(\tau) d\tau + \frac{1}{\rho^\epsilon} \int_t^{t+s} \mathbf{f}_2^\epsilon(\tau) dW_1(\tau). \quad (28)
\end{aligned}$$

From this and the assumptions on the data, we have

$$\mathbb{E} \int_0^{T-s} \left\| \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t+s) - \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t) \right\|^2_{\mathcal{V}'(D^\epsilon)} dt$$

$$\leq C\mathbb{E}\int_0^{T-s}\left\|\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}dt$$
$$+C\mathbb{E}\int_0^{T-s}\left\|\int_t^{t+s}\nabla\beta^\epsilon\theta^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}dt$$
$$+C\int_0^{T-s}\left\|\int_t^{t+s}\mathbf{f}_1^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}dt$$
$$+C\mathbb{E}\int_0^{T-s}\left\|\int_t^{t+s}\mathbf{f}_2^\epsilon(\tau)dW_1(\tau)\right\|^2_{\mathcal{V}'(D^\epsilon)}dt. \tag{29}$$

Let us estimate the terms on the right hand side of (29), we let $\psi\in\mathcal{V}(D^\epsilon)$ such that $\|\psi\|_{\mathcal{V}(D^\epsilon)}=1$, then we have

$$\left\|\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}$$
$$\leq\left(\sup_\psi\left|\left\langle\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(\tau)d\tau,\psi\right\rangle_{\mathcal{V}'(D^\epsilon),\mathcal{V}(D^\epsilon)}\right|\right)^2$$
$$\leq\left(\sup_\psi\int_{D^\epsilon}\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(x,\tau)\psi(x)d\tau dx\right)^2. \tag{30}$$

We use Fubini's lemma, integration by parts, and (2) we get

$$\left\|\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}\leq\left(\sup_\psi\int_t^{t+s}\left(\int_{D^\epsilon}|\nabla\mathbf{u}^\epsilon(x,\tau)\nabla\psi(x)|dx\right)d\tau\right)^2$$
$$\leq\left(\int_t^{t+s}\|\nabla\mathbf{u}^\epsilon(\tau)\|_{L^2(D^\epsilon)}d\tau\right)^2.$$

From this and Cauchy-Schwartz's inequality, we have

$$\mathbb{E}\int_0^{T-s}\left\|\int_t^{t+s}\text{div}A^\epsilon\nabla\mathbf{u}^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}dt$$
$$\leq\mathbb{E}\int_0^{T-s}\left(\int_t^{t+s}\|\nabla\mathbf{u}^\epsilon(\tau)\|_{L^2(D^\epsilon)}d\tau\right)^2dt$$
$$\leq\mathbb{E}\int_0^{T-s}\left(\int_t^{t+s}1^2d\tau\right)\left(\int_t^{t+s}\|\nabla\mathbf{u}^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau\right)dt$$
$$\leq s\mathbb{E}\int_0^T\|\nabla\mathbf{u}^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}dt\leq Cs \tag{31}$$

For the second term, we use (4), to get

$$\left\|\int_t^{t+s}\nabla\beta^\epsilon\theta^\epsilon(\tau)d\tau\right\|^2_{\mathcal{V}'(D^\epsilon)}$$
$$\leq\left(\sup_\psi\left|\left\langle\int_t^{t+s}\nabla\beta^\epsilon\theta^\epsilon(\tau)d\tau,\psi\right\rangle_{\mathcal{V}'(D^\epsilon),\mathcal{V}(D^\epsilon)}\right|\right)^2$$

$$
\begin{aligned}
&\leq \alpha_5 \left(\sup_{\psi} \int_{D^\epsilon} \int_t^{t+s} \theta^\epsilon(x,\tau) \nabla \psi(x) d\tau dx \right)^2 \\
&\leq \alpha_5 \left(\sup_{\psi} \int_t^{t+s} \int_{D^\epsilon} \theta^\epsilon(x,\tau) \nabla \psi(x) dx d\tau \right)^2 \\
&\leq \alpha_5 \left(\int_t^{t+s} \|\theta^\epsilon(\tau)\|_{L^2(D^\epsilon)} d\tau \right)^2.
\end{aligned}
$$

From this and Cauchy-Schwartz's inequality, we have

$$
\begin{aligned}
&\mathbb{E} \int_0^{T-s} \left\| \int_t^{t+s} \nabla \beta^\epsilon \theta^\epsilon(\tau) d\tau \right\|^2_{\mathcal{V}'(D^\epsilon)} dt \\
&\quad \leq \alpha_5 \mathbb{E} \int_0^{T-s} \left(\int_t^{t+s} \|\theta^\epsilon(\tau)\|_{L^2(D^\epsilon)} d\tau \right)^2 dt \\
&\quad \leq \alpha_5 \mathbb{E} \int_0^{T-s} \left(\int_t^{t+s} 1^2 d\tau \right) \left(\int_t^{t+s} \|\theta^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} d\tau \right) dt \\
&\quad \leq s \alpha_5 \mathbb{E} \int_0^T \|\theta^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} dt \leq Cs. \qquad (32)
\end{aligned}
$$

Similarly, for the third term, we have

$$
\begin{aligned}
\int_0^{T-s} \left\| \int_t^{t+s} \mathbf{f}_1^\epsilon(\tau) d\tau \right\|^2_{\mathcal{V}'(D^\epsilon)} dt &\leq \int_0^{T-s} \left(\int_t^{t+s} \|\mathbf{f}_1^\epsilon(\tau)\|_{L^2(D^\epsilon)} d\tau \right)^2 dt \\
&\leq \int_0^{T-s} \left(\int_t^{t+s} 1^2 d\tau \right) \left(\int_t^{t+s} \|\mathbf{f}_1^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} d\tau \right) dt \\
&\leq s \int_0^T \|\mathbf{f}_1^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} dt \leq Cs. \qquad (33)
\end{aligned}
$$

Lastly, since $L^2(D^\epsilon)$ is continuously embedded in $\mathcal{V}'(D^\epsilon)$, we use Itô's isometry and Fubini's Lemma to obtain

$$
\begin{aligned}
&\mathbb{E} \int_0^{T-s} \left\| \int_t^{t+s} \mathbf{f}_2^\epsilon(\tau) dW_1(\tau) \right\|^2_{\mathcal{V}'(D^\epsilon)} \\
&\leq \mathbb{E} \int_0^{T-s} \int_t^{t+s} \|\mathbf{f}_2^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} d(\tau) \\
&\leq \mathbb{E} \int_0^{T-s} \left(\int_t^{t+s} 1^2 d\tau \right)^{\frac{1}{2}} \left(\int_t^{t+s} \|\mathbf{f}_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)} d(\tau) \right)^{\frac{1}{2}} \\
&\leq s^{\frac{1}{2}} \left(\int_0^T \|\mathbf{f}_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)} d(\tau) \right)^{\frac{1}{2}} \\
&\leq Cs^{\frac{1}{2}}. \qquad (34)
\end{aligned}
$$

Using estimates (31),(32), (33) and (34) into (29), we obtain (26). In a similar manner we can prove (27). □

4. **Unfolding operator.** In this part, we will quickly discuss the key definitions and properties associated with unfolding operators. The periodic unfolding mechanism was first presented in [6] (see also [8] for a complete description and thorough-depth proofs). It was broadened to perforated domains in [7, 9], and then further refined in [15] to address time-dependent problems (see also [12, 33, 18]). Let us decompose any $x \in \mathbb{R}^n$ as $x = \epsilon \left(\left[\frac{x}{\epsilon}\right]_Y + \left\{\frac{x}{\epsilon}\right\}_Y \right)$ where $\left[\frac{x}{\epsilon}\right]_Y$ represents the unique integer part for $\frac{x}{\epsilon}$ and $\left\{\frac{x}{\epsilon}\right\}_Y$ the non-integer part of $\frac{x}{\epsilon}$.

Definition 4.1. For a Lebesgue measurable vector $\mathbf{v} : (x,t) \in D^\epsilon \times (0,T) \to \mathbf{v}(x,t) \in \mathbb{R}^n$, $n = 1, 2$ or 3. We define

$$\mathbb{T}^\epsilon(\mathbf{v})(x,y,t) = \begin{cases} \mathbf{v}\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y, t\right) & a.e.\ (x,y,t) \in \widehat{D}^\epsilon \times Y_2 \times (0,T), \\ \\ 0 & a.e.\ (x,y,t) \in \Lambda^\epsilon \times Y_2 \times (0,T). \end{cases}$$

The next lemma states the main features of the unfolding operator within domains with periodic perforations.

Lemma 4.2. [12, 33, 18]. *The above constructed operator satisfies the following:*

1. $\mathbb{T}^\epsilon : L^p(0,T;[L^q(D^\epsilon)]^n) \to L^p(0,T;[L^q(D\times Y_2)]^n)$, $1 \le p,q \le \infty$, $n = 1,2$ *or* 3 *is continuous and linear.*
2. *For all vectors* $\boldsymbol{v}$*,* $\boldsymbol{u} \in L^p(0,T;[L^q(D^\epsilon)]^n)$*, we have* $\mathbb{T}^\epsilon(\boldsymbol{v}\cdot\boldsymbol{u}) = \mathbb{T}^\epsilon(\boldsymbol{v})\cdot\mathbb{T}^\epsilon(\boldsymbol{u})$.
3. *For all vectors* $\boldsymbol{v} \in L^p(0,T;[L^q(D)]^n)$*, we have*
$$\mathbb{T}^\epsilon(\boldsymbol{v}) \to \boldsymbol{v} \text{ strongly in } L^p(0,T;[L^q(D)]^n). \tag{35}$$
4. *For all vectors* $\boldsymbol{v} \in L^p(0,T;[L^q(D^\epsilon)]^n)$*, we have*
$$\begin{aligned} &\int_{\widehat{D}^\epsilon\times(0,T)} \boldsymbol{v}(x,t)dxdt \\ &\qquad = \int_{D^\epsilon\times(0,T)} \boldsymbol{v}(x,t)dxdt - \int_{\Lambda^\epsilon\times(0,T)} \boldsymbol{v}(x,t)dxdt \\ &\qquad = \frac{1}{|Y|}\int_{D\times Y_2\times(0,T)} \mathbb{T}^\epsilon(\boldsymbol{v})(x,y,t)dxdydt. \end{aligned} \tag{36}$$
5. *If* $\boldsymbol{v}^\epsilon \in L^p(0,T;[L^q(D)]^n)$ *where*
$$\boldsymbol{v}^\epsilon \to \boldsymbol{v} \text{ strongly in } L^p(0,T;[L^q(D)]^n).$$
Then
$$\mathbb{T}^\epsilon(\boldsymbol{v}^\epsilon) \to \boldsymbol{v} \text{ strongly in } L^p(0,T;[L^q(D\times Y_2)]^n).$$
6. *Let* $\boldsymbol{v} \in L^P(0,T;L^q(Y_2))$ *be a* $Y-$*periodic vector with* $\boldsymbol{v}^\epsilon(x,t) = \boldsymbol{v}(\frac{x}{\epsilon},t)$*, then* $\mathbb{T}^\epsilon(\boldsymbol{v}^\epsilon)(x,y,t) = \boldsymbol{v}(y,t)$ *a.e. in* $D\times Y_2\times(0,T)$.

5. **Compactness and convergences in probability.** Before we obtain some probabilistic compactness that leads to probabilistic convergences, we should note that we are working on a varying domain, which necessitates extra caution when passing to the limit; for this, we employ the concept of macro-micro operators. For all $\mathbf{v} \in L^p(0,T;[L^q(D^\epsilon)]^n)$, $n = 1,2$ or 3 and $1 \le p,q \le \infty$, we define the macro operator

$$\mathcal{Q}^\epsilon : \mathbf{v} \in L^p(0,T;[L^q(D^\epsilon)]^n) \longmapsto \mathcal{Q}^\epsilon(\mathbf{v}) \in L^p(0,T;[W^{1,\infty}(\widehat{D}^{\mathcal{Y}}_\epsilon)]^n),$$

as:

$$\mathcal{Q}^\epsilon(\mathbf{v})(\epsilon\xi,t) = \frac{1}{|Y_2|}\int_{Y_2}\mathbf{v}(\epsilon\,(\xi+y)\,,t)dy,$$

where the remainder $\mathcal{R}^\epsilon(\mathbf{v}) = \mathbf{v} - \mathcal{Q}^\epsilon(\mathbf{v})$ almost everywhere in $\left[\widehat{D}^\epsilon \cap \widehat{D}^{\mathcal{Y}}_\epsilon\right] \times (0,T)$, for more details on the set $\widehat{D}^{\mathcal{Y}}_\epsilon$, we refer to [12]. For simplicity, we write

$$\mathcal{Q}^\epsilon(\mathbf{v}) = \bar{\mathbf{v}} \text{ and } \mathcal{R}^\epsilon(\mathbf{v}) = \underline{\mathbf{v}}.$$

With the above setting, the oscillations resulting from perforations are shifted into a second variable y, which is related to a fixed domain Y_2, while the original variable x is in the domain D. Now, one uses the estimates in Theorem 3.1 and follow along the lines of [12, Proposition 2.15] to obtain the following result.

Theorem 5.1. *Assume that* $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon\right)$ *is the solution of system* (1) *and the assumptions A.1.-A.5. hold, then*

$$\mathbb{E}\left\|\frac{\partial \bar{\boldsymbol{u}}^\epsilon}{\partial t}\right\|^2_{L^2(0,T;[L^2(\widehat{D}^{\mathcal{Y}}_\epsilon)]^n)} + \mathbb{E}\|\bar{\boldsymbol{u}}^\epsilon\|^2_{L^2(0,T;\mathcal{V}(\widehat{D}^{\mathcal{Y}}_\epsilon))} + \mathbb{E}\|\bar{\theta}^\epsilon\|^2_{L^2(0,T;L^2(\widehat{D}^{\mathcal{Y}}_\epsilon))} + \mathbb{E}\|\bar{\theta}^\epsilon\|^2_{L^2(0,T;\mathcal{S}(\widehat{D}^{\mathcal{Y}}_\epsilon))} \le C, \tag{37}$$

$$\mathbb{E}\,\|\underline{\boldsymbol{u}}^\epsilon\|^2_{L^2(0,T;[L^2(\widehat{D}^\epsilon\cap\widehat{D}^{\mathcal{Y}}_\epsilon)]^n)} + \mathbb{E}\,\|\underline{\theta}^\epsilon\|^2_{L^2(0,T;L^2(\widehat{D}^\epsilon\cap\widehat{D}^{\mathcal{Y}}_\epsilon))} \le \epsilon C, \tag{38}$$

and

$$\mathbb{E}\,\|\underline{\boldsymbol{u}}^\epsilon\|^2_{L^2(0,T;\mathcal{V}(\widehat{D}^\epsilon\cap\widehat{D}^{\mathcal{Y}}_\epsilon))} + \mathbb{E}\,\|\underline{\theta}^\epsilon\|^2_{L^2(0,T;\mathcal{S}(\widehat{D}^\epsilon\cap\widehat{D}^{\mathcal{Y}}_\epsilon))} \le C. \tag{39}$$

Another bounds on the macro operator is needed before obtaining probabilistic compactness. This is the object of the following theorem.

Theorem 5.2. *Let the assumptions of theorem 3.2 hold, then*

$$\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}\right\|^2_{l^2(\Omega;L^2(0,T;\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon))} \le C\left\|\frac{\partial \boldsymbol{u}^\epsilon}{\partial t}\right\|^2_{l^2(\Omega;L^2(0,T;\mathcal{V}'(D^\epsilon))} \tag{40}$$

and

$$\|\bar{\theta}^\epsilon\|^2_{l^2(\Omega;L^2(0,T;\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon))} \le C\,\|\theta^\epsilon\|^2_{l^2(\Omega;L^2(0,T;\mathcal{V}'(D^\epsilon))} \tag{41}$$

Proof. Let us first note that

$$\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}\right\|^2_{l^2(\Omega;L^2(0,T;\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon))} = \mathbb{E}\int_0^T\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}(t)\right\|^2_{\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon)}dt \tag{42}$$

Now, for all $\phi \in \mathcal{V}(\widehat{D}^{\mathcal{Y}}_\epsilon)$ with $\|\phi\|_{\mathcal{V}(\widehat{D}^{\mathcal{Y}}_\epsilon)} = 1$, we have

$$\begin{aligned}\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}\right\|^2_{\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon)} &= \sup_\phi\left|\left\langle\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t},\phi\right\rangle\right| = \sup_\phi\left|\int_{\widehat{D}^{\mathcal{Y}}_\epsilon}\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}\phi dx\right|,\\ &\le \sup_\phi\frac{1}{|Y_2|}\left|\int_{\widehat{D}^{\mathcal{Y}}_\epsilon}\left[\int_{Y_2}\left|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(\epsilon(\xi+y),t)\right|dy\right]\phi dx\right|,\\ &\le \sup_\phi\frac{1}{\epsilon^n|Y_2|}\left|\int_{\widehat{D}^{\mathcal{Y}}_\epsilon}\left[\int_{\epsilon(\xi+Y_2)}\left|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(x,t)\right|dx\right]\phi dx\right|.\end{aligned}$$

Using Fubini's theorem, we have

$$\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}\right\|^2_{\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon)} \leq \sup_\phi \frac{\epsilon(\xi+Y_2)}{\epsilon^n|Y_2|}\left|\int_{\widehat{D}^{\mathcal{Y}}_\epsilon}\left|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(x,t)\right|\phi dx\right|$$
$$\leq C\sup_\phi \int_{D^\epsilon}\left|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(x,t)\phi(x,t)\right|dx = C\left\|\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right\|^2_{\mathcal{V}'(D^\epsilon)}.$$

From this and (42), we obtain (40). Similarly we obtain the proof of (41). □

The results of Theorems 5.2 and 3.2 lead to the following finite difference estimate for the macro operators in the fixed domain $\widehat{D}^{\mathcal{Y}}_\epsilon$.

$$\mathbb{E}\int_0^{T-s}\left\|\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}(t+s)-\frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}(t)\right\|^2_{\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon)} dt \leq Cs, \tag{43}$$

and

$$\mathbb{E}\int_0^{T-s}\|\bar{\theta}^\epsilon(t+s)-\bar{\theta}^\epsilon(t)\|^2_{\mathcal{V}'(\widehat{D}^{\mathcal{Y}}_\epsilon)}dt \leq Cs. \tag{44}$$

The extension by zero of $\left(\bar{\mathbf{u}}^\epsilon, \frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}, \bar{\theta}^\epsilon\right)$ from $\widehat{D}^{\mathcal{Y}}_\epsilon$ to D and of $\left(\underline{\mathbf{u}}^\epsilon, \frac{\partial \underline{\mathbf{u}}^\epsilon}{\partial t}, \underline{\theta}^\epsilon\right)$ from $\widehat{D}^\epsilon \cap \widehat{D}^{\mathcal{Y}}_\epsilon$ to D satisfies the estimates (37), (38), (39), (43) and (44). We now define the set $\mathbb{K}^\omega = \mathbb{K}_1 \times \mathbb{K}_2 \times \mathbb{K}_3$, where

$$\mathbb{K}_1 = \left\{\mathbf{v} : \mathbf{v} \in l^2(\Omega; L^2(0,T;\mathcal{V}(D)) \text{ and } \frac{\partial \mathbf{v}}{\partial t} \in l^2(\Omega; L^2(0,T;[L^2(D)]^n))\right\},$$

$$\mathbb{K}_2 = \Bigg\{\mathbf{w} : \mathbf{w} \in L^2(\Omega; L^2(0,T;[L^2(D)]^n))$$
$$\text{and } \mathbb{E}\sup_n \frac{1}{a_n}\sup_{|s|\leq b_n}\left(\int_0^{T-s}\|\mathbf{w}(t+s)-\mathbf{w}(t)\|^2_{\mathcal{V}'(D)}dt\right) < \infty\Bigg\},$$

and

$$\mathbb{K}_3 = \Bigg\{\varphi : \varphi \in L^2(\Omega; L^2(0,T;L^2(D))) \cap L^2(\Omega; L^2(0,T;\mathcal{S}(D)))$$
$$\text{and } \mathbb{E}\sup_n \frac{1}{a_n}\sup_{|s|\leq b_n}\left(\int_0^{T-s}\|\varphi(t+s)-\varphi(t)\|^2_{\mathcal{S}'(D)}dt\right) < \infty\Bigg\},$$

where the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n, b_n \to 0$ when $n \to \infty$. It is clear that the extension (with the same notation) by zero of $\left(\bar{\mathbf{u}}^\epsilon, \frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}, \bar{\theta}^\epsilon\right)$ belongs to the set $\mathbb{K}^\omega$. Also, the set $\mathbb{K}_1$ is compactly embedded in the space $L^2(\Omega; L^2(0,T;L^2(D)))$. In [20] it was proved that $\mathbb{K}_2$ is compactly embedded in $L^2(\Omega; L^2(0,T;\mathcal{V}'(D)))$ for scalar functions, also in [1] it was shown that $\mathbb{K}_3$ is compact in $L^2(\Omega; L^2(0,T;L^2(D)))$. Let us denote to the probability law of $\left(\bar{\mathbf{u}}^\epsilon, \frac{\partial \bar{\mathbf{u}}^\epsilon}{\partial t}, \bar{\theta}^\epsilon, W_1, W_2\right)$ by Π^ϵ, then one can follow along the lines of [20, Lemma 7] to show that Π^ϵ is tight in $\mathbb{X}^\omega = \mathbb{K}^\omega \times C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$. This tightness leads to relative compactness (and hence weak convergence) by Prokhorov's theorem i.e. there exists a subsequence Π^{ϵ_n} of Π^ϵ and a probability low Π where $\Pi^\epsilon \rightharpoonup \Pi$ weakly in $\mathbb{X}^\omega$. Now, we are able to apply Skorokhod's representation theorem, by which we construct a new probability space $(\Omega_1, \mathbb{P}_1, \mathcal{F}_1)$ together with the sequence of random variables $\left(\mathbf{u}^{\bar{\epsilon}_n}, \frac{\partial \mathbf{u}^{\bar{\epsilon}_n}}{\partial t}, \theta^{\bar{\epsilon}_n}, W_1^{\epsilon_n}, W_2^{\epsilon_n}\right)$ and $\left(\bar{\mathbf{u}}, \frac{\partial \bar{\mathbf{u}}}{\partial t}, \bar{\theta}, \hat{W}_1, \hat{W}_2\right)$ such that

$$\left(\mathbf{u}^{\bar{\epsilon}_n}, \frac{\partial \mathbf{u}^{\bar{\epsilon}_n}}{\partial t}, \theta^{\bar{\epsilon}_n}, W_1^{\epsilon_n}, W_2^{\epsilon_n}\right) \to \left(\bar{\mathbf{u}}, \frac{\partial \bar{\mathbf{u}}}{\partial t}, \bar{\theta}, \hat{W}_1, \hat{W}_2\right) \text{ in } \mathcal{B}(\mathbb{X}^\omega)\, \mathbb{P}_1 - a.s., \tag{45}$$

as $n \to \infty$ and $\epsilon_n \to 0$, where $\mathcal{B}(\mathbb{X}^\omega)$ is the Borel set of $\mathbb{X}^\omega$. The system (8)-(9) is satisfied by $\left(\mathbf{u}^{\bar{\epsilon}_n}, \frac{\partial \mathbf{u}^{\bar{\epsilon}_n}}{\partial t}, \theta^{\bar{\epsilon}_n}, W_1^{\epsilon_n}, W_2^{\epsilon_n}\right)$, as we claim in the following lemma. As a result, we show that the sequence of solutions has better properties—that is, strong probabilistic convergence. For the proof, we adhere to the logic in [28, P.352-356].

Lemma 5.3. *Take the $\sigma-$algebra generated by $\left\{\boldsymbol{u}^{\bar{\epsilon}_n}, \frac{\partial \boldsymbol{u}^{\bar{\epsilon}_n}}{\partial t}, \theta^{\bar{\epsilon}_n}, W_1^{\epsilon_n}, W_2^{\epsilon_n}\right\}_{0\le s\le t}$ (denoted by $\mathcal{F}_{1t}$) as a filtration on the Skorokhod's probability space $(\Omega_1, \mathbb{P}_1, \mathcal{F}_1)$. Then*

(1) $\left(\hat{W}_1, \hat{W}_2\right)$ are correlated one-dimensional Wiener processes.

(2) $\left(\boldsymbol{u}^{\epsilon_n}, \frac{\partial \boldsymbol{u}^{\epsilon_n}}{\partial t}, \theta^{\epsilon_n}, W_1^{\epsilon_n}, W_2^{\epsilon_n}\right)$ admits with $\mathbb{P}_1-$almost surely the following:

$$\begin{aligned}
&\int_0^t (\rho^{\epsilon_n} d\frac{\partial \boldsymbol{u}^{\epsilon_n}}{\partial t}, \psi)_{L^2(D^{\epsilon_n})} d\tau + \int_0^t (A^{\epsilon_n} e(\boldsymbol{u}^{\epsilon_n}), e(\psi))_{L^2(D^{\epsilon_n})} d\tau \\
&\quad - \int_0^t (\beta^{\epsilon_n} \theta^{\epsilon_n}, \nabla\psi)_{L^2(D^{\epsilon_n})} d\tau \\
&= \int_0^t (\boldsymbol{f}_1^{\epsilon_n}, \psi)_{L^2(D^{\epsilon_n})} d\tau + \int_0^t (\boldsymbol{f}_2^{\epsilon_n}, \psi)_{L^2(D^{\epsilon_n})} dW_1(\tau),
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
&\int_0^t (\rho^{\epsilon_n} c_v^{\epsilon_n} d\theta^{\epsilon_n}, \phi)_{L^2(D^{\epsilon_n})} d\tau + \int_0^t (\kappa^{\epsilon_n} \nabla\theta^{\epsilon_n}, \nabla\phi)_{L^2(D^{\epsilon_n})} d\tau \\
&\quad - \int_0^t (\beta^{\epsilon_n} \frac{\partial \boldsymbol{u}^{\epsilon_n}}{\partial t}, \nabla\phi)_{L^2(D^{\epsilon_n})} d\tau + \epsilon_n \int_0^t (\nabla\beta^{\epsilon_n} \frac{\partial \boldsymbol{u}^{\epsilon_n}}{\partial t}, \phi)_{L^2(D^{\epsilon_n})} d\tau \\
&= \int_0^t (g_1^{\epsilon_n}, \psi)_{L^2(D^{\epsilon_n})} d\tau + \int_0^t (g_2^{\epsilon_n}, \psi)_{L^2(D^{\epsilon_n})} dW_2(\tau),
\end{aligned} \tag{47}$$

for all $(\psi, \phi) \in \mathcal{S}(D^{\epsilon_n}) \times \mathcal{V}(D^{\epsilon_n})$. With the initial conditions $\boldsymbol{u}^{\epsilon_n}(x, 0) = \boldsymbol{h}_1^{\epsilon_n}(x)$, $\frac{\partial \boldsymbol{u}^{\epsilon_n}}{\partial t}(x, 0) = \boldsymbol{h}_2^{\epsilon_n}(x)$ and $\theta^{\epsilon_n}(x, 0) = h_3^{\epsilon_n}(x)$.

6. Homogenization results.

For the sake of brevity, we will remove the index n from ϵ_n in this section. We have the following theorem

Theorem 6.1. *Suppose that the assumptions of Theorem 3.2 hold and $\|\boldsymbol{h}_1^\epsilon\|_{\mathcal{V}(D^\epsilon)} \le C$ such that*

$$\left(\widetilde{\boldsymbol{h}_1^\epsilon}, \widetilde{\boldsymbol{h}_2^\epsilon}\right) \rightharpoonup \vartheta\,(\boldsymbol{h}_1, \boldsymbol{h}_2) \text{ weakly in } [L^2(D)]^n \times [L^2(D)]^n, \tag{48}$$

$$\left(\widetilde{\boldsymbol{f}_1^\epsilon}, \widetilde{\boldsymbol{f}_2^\epsilon}\right) \rightharpoonup \vartheta\,(\boldsymbol{f}_1, \boldsymbol{f}_2) \text{ weakly in } L^2(0,T;[L^2(D)]^n) \times L^2(0,T;[L^2(D)]^n), \tag{49}$$

$$\left(\widetilde{g_1^\epsilon}, \widetilde{g_2^\epsilon},\right) \rightharpoonup \vartheta\,(g_1, g_2) \text{ weakly in } L^2(0,T;L^2(D)) \times L^2(0,T;L^2(D)), \tag{50}$$

$$\widetilde{h_3^\epsilon} \rightharpoonup \vartheta h_3 \text{ weakly in } L^2(D). \tag{51}$$

Furthermore, let $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon, W_1^\epsilon, W_2^\epsilon\right)$ *be the solution of system* (46)-(47)*. Then there exists a set of random variables* $\left\{\boldsymbol{u}, \hat{\boldsymbol{u}}, \frac{\partial \boldsymbol{u}}{\partial t}, \theta, \hat{\theta}, \hat{W}_1, \hat{W}_2\right\}$ *defined on the probability space* $(\Omega_1, \mathbb{P}_1, \mathcal{F}_1)$ *such that*

1. $\mathbb{T}^\epsilon(\boldsymbol{u}^\epsilon) \rightharpoonup \boldsymbol{u}$ *weakly in* $L^2(\Omega, L^(0,T; [L^2(D; W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s.$.
2. $\mathbb{T}^\epsilon(\boldsymbol{u}^\epsilon) \to \boldsymbol{u}$ *strongly in* $L^2(\Omega, L^(0,T; [L^2(D; W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s.$.
3. $\mathbb{T}^\epsilon(\nabla \boldsymbol{u}^\epsilon) \rightharpoonup \nabla_x \boldsymbol{u} + \nabla_y \hat{\boldsymbol{u}}$ *weakly in* $L^2(\Omega, L^2(0,T; [L^2(D \times Y_2)]^{n\times n})),\ \mathbb{P}_1 - a.s.$.
4. $\mathbb{T}^\epsilon\left(\frac{\partial \boldsymbol{u}^\epsilon}{\partial t}\right) \rightharpoonup \frac{\partial \boldsymbol{u}}{\partial t}$ *weakly in* $L^2(\Omega, L^2(0,T; [L^2(D \times Y_2)]^n)),\ \mathbb{P}_1 - a.s.$.
5. $\mathbb{T}^\epsilon(\theta^\epsilon) \rightharpoonup \theta$ *weakly in* $L^2(\Omega, L^(0,T; [L^2(D; W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s.$.
6. $\mathbb{T}^\epsilon(\theta^\epsilon) \to \theta$ *strongly in* $L^2(\Omega, L^(0,T; [L^2(D; W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s.$.
7. $\mathbb{T}^\epsilon(\nabla \theta^\epsilon) \rightharpoonup \nabla_x \theta + \nabla_y \hat{\theta}$ *weakly in* $L^2(\Omega, L^2(0,T; [L^2(D \times Y_2)]^n)),\ \mathbb{P}_1 - a.s.,$

where

$$\boldsymbol{u} \in L^2(\Omega; L^2(0,T; \mathcal{V}(D)),\ \frac{\partial \boldsymbol{u}}{\partial t} \in L^2(\Omega; L^2(0,T; [L^2(D)]^n)),$$
$$\theta \in L^2(\Omega; L^2(0,T; \mathcal{V}(D)) \cap L^2(\Omega; L^2(0,T; L^2(D))),$$
$$\hat{\boldsymbol{u}} \in L^2(\Omega; L^2(0,T; [L^2(D; W^{1,2}_{per}(Y_2))]^n)),\ \hat{\theta} \in L^2(\Omega; L^2(0,T; L^2(D; W^{1,2}_{per}(Y_2))),$$

with $\mathcal{M}_{Y_2}(\hat{\boldsymbol{u}}) = \mathcal{M}_{Y_2}(\hat{\theta}) = 0$ *is a unique probabilistic solution to the system:*

$$\begin{aligned}
&|Y_2|\mathcal{M}_{Y_2}(\rho) \int_D d\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)\varphi dx \\
&\quad + \int_{D\times Y_2} A(y)(\nabla_x \boldsymbol{u} + \nabla_y \hat{\boldsymbol{u}}) \cdot (\nabla_x \varphi + \nabla_y \hat{\varphi}) dxdydt \\
&\quad + |Y_2|\mathcal{M}_{Y_2}(\beta) \int_D \nabla\theta \varphi dxdt = |Y_2| \int_D \boldsymbol{f}_1 \varphi dxdt \\
&\quad + |Y_2| \int_D \boldsymbol{f}_2 \varphi d\hat{W}_1,
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&|Y_2|\mathcal{M}_{Y_2}(\rho c_v) \int_D d\theta \psi dx \\
&\quad + \int_{D\times Y_2} (\kappa(y)\nabla_x \theta + \nabla_y \hat{\theta}) \cdot (\nabla_x \psi + \nabla_y \hat{\psi}) dxdydt \\
&\quad + |Y_2|\mathcal{M}_{Y_2}(\beta) \int_D div\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)\psi dxdt = |Y_2| \int_D g_1 \psi dxdt \\
&\quad + |Y_2| \int_D g_2 \psi d\hat{W}_2,
\end{aligned} \tag{53}$$

where

$$\hat{\boldsymbol{u}}(t,x,y) = \Phi(y)\nabla_x \boldsymbol{u}(x,t) + V(y)\theta(x,t),$$

and

$$\hat{\theta}(t,x,y) = \Psi(y) \cdot \nabla_x \theta(x,t) + V(y)\frac{\partial \boldsymbol{u}}{\partial t}(x,t),$$

such that $\Phi = (\Phi_{klm})_{1\le k,l,m\le n}$*,* $V = ((V_k)_{1\le k\le n})$ *and* $\Psi = ((\Psi_k)_{1\le k\le n})$ *solve uniquely the following cell problems*

$$\begin{cases}
div(A(y)\nabla_y(\Phi_{klm} + y_m)) = 0 & \text{in } Y_2, \\
A(y)\nabla_y(\Phi_{klm} + y_m) \cdot \nu = 0 & \text{on } \partial Y_1, \\
\mathcal{M}_{Y_2}(\Phi_{klm}) = 0 & \Phi_{klm} \text{ is } Y\text{−periodic},
\end{cases} \tag{54}$$

$$\begin{cases} div_y\,(A(y)(\nabla_y V(y)) - \beta(y)) = 0 & in\ Y_2, \\ A(y)\,[(\nabla_y V(y)) - \beta(y)]\cdot \nu = 0 & on\ \partial Y_1, \\ \mathcal{M}_{Y_2}(V) = 0 & V\ is\ Y-periodic, \end{cases} \tag{55}$$

and

$$\begin{cases} div\,(\kappa(y)\nabla_y(\Psi_k + y_k)) = 0 & in\ Y_2, \\ \kappa(y)\nabla_y(\Psi_k + y_k)\cdot \nu = 0 & on\ \partial Y_1, \\ \mathcal{M}_{Y_2}(\Psi_k) = 0 & \Psi_k\ is\ Y-periodic. \end{cases} \tag{56}$$

Moreover, $\left(\boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial t}, \theta\right)$ *a unique solution to the strong formulation:*

$$\begin{cases} \mathcal{M}_{Y_2}(\rho) d\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)(x,t) - A_0 \Delta \boldsymbol{u}(x,t)dt + \beta_0 \nabla\theta(x,t)dt \\ = \boldsymbol{f}_1(x,t))dt + \boldsymbol{f}_2(x,t)d\hat{W}_1(t),\ \ in\ \Omega\times D\times(0,T), \\ \\ \mathcal{M}_{Y_2}(\rho c_v) d\theta(x,t) - \kappa_0\Delta\theta(x,t)dt + \beta_0\, div\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)(x,t)dt \\ = g_1(x,t))dt + g_2(x,t)d\hat{W}_2(t),\ \ in\ \Omega\times D\times(0,T), \\ \\ \boldsymbol{u}(x,t) = 0 = \theta(x,t)\ \ on\ \Omega\times\partial D\times(0,T), \\ \boldsymbol{u}(x,0) = \boldsymbol{h}_1(x),\ \ \frac{\partial \boldsymbol{u}}{\partial t}(x,0) = \boldsymbol{h}_2(x)\ \ and\ \theta(x,0) = h_3(x)\ \ in\ D, \end{cases} \tag{57}$$

where the constant tensor , $A_0 = (a^0_{ijkl})_{(1\le i,j,k,l\le n)}$ *is symmetric for all* i, j, k *and* l *and elliptic where*

$$a^0_{ijkl} = \frac{1}{|Y_2|}\int_{Y_2}\left(a_{ijkl}(y) + a_{ijmn}(y)\frac{\partial \Phi_{klm}}{\partial y_n}(y)\right)dy, \tag{58}$$

$0 \neq \beta_0 = (\beta^0_{ij})_{(1\le i,j\le n)}$ *such that*

$$\beta^0_{ij} = \frac{1}{|Y_2|}\int_{Y_2}\left(\beta_{ij}(y) - a_{ijmn}(y)\frac{\partial V_m}{\partial y_n}(y)\right)dy, \tag{59}$$

and $\kappa_0 = (\kappa^0_{ij})_{(1\le i,j\le n)}$ *is elliptic such that*

$$\kappa^0_{ij} = \frac{1}{|Y_2|}\int_{Y_2}\left(\kappa_{ij}(y) + \kappa_{ik}(y)\frac{\partial \Psi_j}{\partial y_k}(y)\right)dy. \tag{60}$$

Proof. By considering the a priori bounds found in Theorem 3.1 and arguing as in [12, Lemmas 2.16 and 2.17], we establish that the solution of system (1) fulfills the following convergences (up to subsequence) $\mathbb{P}_1-$a.s.

$$\mathbb{T}^\epsilon(\bar{\mathbf{u}}^\epsilon) \rightharpoonup \mathbf{u} \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D;W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s., \tag{61}$$

$$\mathbb{T}^\epsilon(\nabla\bar{\mathbf{u}}^\epsilon) \rightharpoonup \nabla_x\mathbf{u} \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D\times Y_2)]^{n\times n})),\ \mathbb{P}_1 - a.s., \tag{62}$$

$$\frac{1}{\epsilon}\mathbb{T}^\epsilon(\underline{\mathbf{u}}^\epsilon) \rightharpoonup \hat{\mathbf{u}} \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D;W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s., \tag{63}$$

$$\mathbb{T}^\epsilon(\nabla\underline{\mathbf{u}}^\epsilon) \rightharpoonup \nabla_y\hat{\mathbf{u}} \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D;W^{1,2}(Y_2))]^{n\times n}))\ \mathbb{P}_1 - a.s., \tag{64}$$

$$\mathbb{T}^\epsilon(\underline{\mathbf{u}}^\epsilon) \to 0 \text{ strongly in } L^2(\Omega, L^2(0,T;[L^2(D;W^{1,2}(Y_2))]^n)),\ \mathbb{P}_1 - a.s., \tag{65}$$

$$\mathbb{T}^\epsilon\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right) \rightharpoonup \frac{\partial\mathbf{u}}{\partial t} \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D\times Y_2)]^{n\times n})),\ \mathbb{P}_1 - a.s., \tag{66}$$

$$\mathbb{T}^\epsilon(\bar{\theta}^\epsilon) \rightharpoonup \theta \text{ weakly in } L^2(\Omega, L^2(0,T;L^2(D;W^{1,2}(Y_2)))),\ \mathbb{P}_1 - a.s., \tag{67}$$

$$\mathbb{T}^\epsilon(\nabla\bar{\theta}^\epsilon) \rightharpoonup \nabla_x\theta \text{ weakly in } L^2(\Omega, L^2(0,T;[L^2(D\times Y_2)]^n)),\ \mathbb{P}_1 - a.s., \tag{68}$$

$$\frac{1}{\epsilon}\mathbb{T}^\epsilon(\underline{\theta}^\epsilon) \rightharpoonup \hat{\theta} \text{ weakly in } L^2(\Omega, L^2(0,T;L^2(D;W^{1,2}(Y_2)))),\ \mathbb{P}_1 - a.s., \tag{69}$$

$$\mathbb{T}^\epsilon(\nabla\underline{\theta}^\epsilon) \rightharpoonup \nabla_y\hat{\theta} \text{ weakly in } L^2(\Omega, L^(0,T;[L^2(D;W^{1,2}(Y_2))]^n))\ \mathbb{P}_1 - a.s., \tag{70}$$

$$\mathbb{T}^\epsilon(\underline{\theta}^\epsilon) \to 0 \text{ strongly in } L^2(\Omega, L^2(0,T;L^2(D;W^{1,2}(Y_2)))),\ \mathbb{P}_1 - a.s., \tag{71}$$

where

$$\mathbf{u} \in L^2(\Omega;L^2(0,T;\mathcal{V}(D)),\ \frac{\partial \mathbf{u}}{\partial t} \in L^2(\Omega;L^2(0,T;[L^2(D)]^n)),$$

$$\theta \in L^2(\Omega;L^2(0,T;\mathcal{V}(D)) \cap L^2(\Omega;L^2(0,T;L^2(D))),$$

$$\hat{\mathbf{u}} \in L^2(\Omega;L^2(0,T;[L^2(D;W^{1,2}_{\text{per}}(Y_2))]^n)),\ \hat{\theta} \in L^2(\Omega;L^2(0,T;L^2(D;W^{1,2}_{\text{per}}(Y_2))),$$

with $\mathcal{M}_{Y_2}(\hat{\mathbf{u}}) = \mathcal{M}_{Y_2}(\hat{\theta}) = 0$. Apart from the aforementioned convergences, the strong convergence (45) that is derived from Skorokhod's representation theorem and Lemma 4.2 (5) gives us the following:

$$\mathbb{T}^\epsilon(\bar{\mathbf{u}}^\epsilon) \to \mathbf{u} \text{ strongly in } [\mathbb{H}(D\times Y_2)]^n,\ \mathbb{P}_1 - a.s., \tag{72}$$

$$\mathbb{T}^\epsilon(\bar{\theta}^\epsilon) \to \theta \text{ strongly in } \mathbb{H}(D\times Y)2),\ \mathbb{P}_1 - a.s. \tag{73}$$

Convergences (61)-(73), gives the desired convergences (1)-(7). Let's now test equations (46) and (47) by $v(x)w(t)$, where $(v,w) \in C_0^\infty(D)\times C_0^\infty([0,T])$ and in each of its terms, we pass to the limit using the above convergences and the properties of the unfolding operator. We have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\rho^\epsilon d\left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)(x,t)v(x)w(t)dxdt \\
&\qquad = -\lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\rho(y)\mathbb{T}^\epsilon\left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)\mathbb{T}^\epsilon(v)\frac{dw}{dt}dxdydt \\
&\qquad = -|Y_2|\mathcal{M}_{Y_2}(\rho)\int_0^T\int_D\frac{\partial \mathbf{u}}{\partial t}(x,t)v(x)\frac{dw}{dt}(t)dxdt.
\end{aligned}\tag{74}$$

For the second term and third term, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}A^\epsilon\nabla\mathbf{u}^\epsilon(x,t)\nabla v(x)w(t)dxdt \\
&\quad = \lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\mathbb{T}^\epsilon\left(A^\epsilon\right)\mathbb{T}^\epsilon\left(\nabla\mathbf{u}^\epsilon\right)\mathbb{T}^\epsilon(\nabla v)w dxdydt \\
&\quad = \int_0^T\int_{D\times Y_2}A(y)\left(\nabla\mathbf{u}(x,t)+\nabla_y\hat{\mathbf{u}}(x,y,t)\right)\nabla v(x)w(t)dxdydt.
\end{aligned}\tag{75}$$

Similarly, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\nabla\left(\beta^\epsilon\theta^\epsilon\right)v(x)w(t)dxdt \\
&\qquad = -\lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\beta(y)\mathbb{T}^\epsilon\left(\theta^\epsilon\right)\mathbb{T}^\epsilon\left(\nabla v\right)w(t)dxdt \\
&\qquad = -\int_0^T\int_{D\times Y_2}\beta(y)\theta(x,t)\nabla v(x)w(t)dxdydt.
\end{aligned}\tag{76}$$

For the first term on the right hand side of (46), we easily use the weak limit (49) to get

$$\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_1^\epsilon(x,t)v(x)w(t)dxdt = |Y_2|\int_0^T\int_D \mathbf{f}_1(x,t)v(x)w(t)dxdt. \tag{77}$$

As for the stochastic integral, we write

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxdW_1^\epsilon(t) &= \lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxd[W_1^\epsilon(t)-\hat{W}_1(t)]\\ &+\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxd\hat{W}_1(t).\end{aligned} \tag{78}$$

For the second term on the right hand side of (78), we use the weak convergence (49) and stochastic convergence theorem by Rozovskii [27, Theorem 4, pg 63] to get

$$\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxd\hat{W}_1(t) = |Y_2|\int_0^T\int_D \mathbf{f}_2(x,t)v(x)w(t)dxd\hat{W}_1(t).$$

For the first term on the right hand side of (78), we define a regularization function to the intensity $\widetilde{\mathbf{f}}_2^\epsilon$ (denoted by $\widetilde{\mathbf{f}}_{2\delta}^\epsilon$), where $\widetilde{\mathbf{f}}_{2\delta}^\epsilon$ is differentiable in time, $\|\widetilde{\mathbf{f}}_{2\delta}^\epsilon\|_{L^2(0,T;L^2(D))}\le C\|\widetilde{\mathbf{f}}_2^\epsilon\|_{L^2(0,T;L^2(D))}$ and

$$\widetilde{\mathbf{f}}_{2\delta}^\epsilon \to \widetilde{\mathbf{f}}_2^\epsilon \text{ strongly in } L^2(0,T;L^2(D)) \text{ as } \delta\to 0. \tag{79}$$

With this setting, we rewrite the first term on the right hand side of (78) as

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxd[W_1^\epsilon(t)-\hat{W}_1(t)] &= \lim_{\epsilon\to 0}\int_0^T\int_D [\widetilde{\mathbf{f}}_2^\epsilon(x,t)-\widetilde{\mathbf{f}}_{2\delta}^\epsilon(x,t)]v(x)w(t)dxd[W_1^\epsilon(t)-\hat{W}_1(t)]\\ &+\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_{2\delta}^\epsilon(x,t)v(x)w(t)dxd[W_1^\epsilon(t)-\hat{W}_1(t)].\end{aligned} \tag{80}$$

Because of the strong convergence (79) and Burkholder-Davis-Gundy's inequality, the first term on the right-hand side of (80) approaches to zero with probability almost surly. We use the integration by parts (because the regularization is differentiable in t) and the strong convergence (45) for the second term. As a result, the second term also attends to zero. Thus

$$\lim_{\epsilon\to 0}\int_0^T\int_D \widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)w(t)dxdW_1^\epsilon(t) = |Y_2|\int_0^T\int_D \mathbf{f}_2(x,t)v(x)w(t)dxd\hat{W}_1(t). \tag{81}$$

The following is obtained by combining convergences (74), (75), (76), (77) and (81)

$$\begin{aligned}&|Y_2|\mathcal{M}_{Y_2}(\rho)\int_0^T\int_D d\left(\frac{\partial \mathbf{u}}{\partial t}\right)(x,t)v(x)w(t)dxdt\\&\quad+\int_0^T\int_{D\times Y_2}A(y)\left(\nabla\mathbf{u}(x,t)+\nabla_y\hat{\mathbf{u}}(x,y,t)\right)\cdot\nabla v(x)w(t)dxdydt\\&\quad-\int_0^T\int_{D\times Y_2}\beta(y)\theta(x,t)\nabla v(x)w(t)dxdydt\\&=|Y_2|\int_0^T\int_D\mathbf{f}_1(x,t)v(x)w(t)dxdt\\&\quad+|Y_2|\int_0^T\int_D\mathbf{f}_1(x,t)v(x)w(t)dxd\hat{W}_1(t).\end{aligned}\tag{82}$$

In a similar way we pass to the limit into the heat equation, we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\rho^\epsilon c_v^\epsilon d\theta^\epsilon(x,t)v(x)w(t)dx\\&\qquad=-|Y_2|\mathcal{M}_{Y_2}(\rho c_v)\int_0^T\int_D\theta(x,t)v(x)\frac{dw}{dt}(t)dxdt,\end{aligned}\tag{83}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\kappa^\epsilon\nabla\theta^\epsilon(x,t)\nabla v(x)w(t)dxdt\\&\quad=\int_0^T\int_{D\times Y_2}\kappa(y)\left(\nabla\theta(x,t)+\nabla_y\hat{\theta}(x,y,t)\right)\nabla v(x)w(t)dxdydt,\end{aligned}\tag{84}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\beta^\epsilon(x)\frac{\partial\mathbf{u}^\epsilon}{\partial t}(x,t)\nabla v(x)w(t)dxdt\\&\qquad=\int_0^T\int_{D\times Y_2}\beta(y)\frac{\partial\mathbf{u}}{\partial t}(x,t)\nabla v(x)w(t)dxdydt.\end{aligned}\tag{85}$$

For the following term we note that $\epsilon\mathbb{T}^\epsilon\left(\nabla\beta^\epsilon\right)=\mathbb{T}^\epsilon\left(\nabla_y\beta^\epsilon\right)=\nabla_y\beta(y)$ and from the assumption A.6. and [11, Proposition 3.49], we have $\mathcal{M}_{Y_2}(\nabla_y\beta)=0$. Then

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\epsilon\nabla\beta^\epsilon(x)\frac{\partial\mathbf{u}^\epsilon}{\partial t}(x,t)v(x)w(t)dxdt\\&\qquad=\lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\epsilon\mathbb{T}^\epsilon\left(\nabla\beta^\epsilon\right)\mathbb{T}^\epsilon\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)\mathbb{T}^\epsilon\left(v\right)w(t)dxdt\\&\qquad=\lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\epsilon\mathbb{T}^\epsilon\left(\nabla\beta^\epsilon\right)\mathbb{T}^\epsilon\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)\mathbb{T}^\epsilon\left(v\right)w(t)dxdt\\&\qquad=\lim_{\epsilon\to 0}\int_0^T\int_{D\times Y_2}\mathbb{T}^\epsilon\left(\nabla_y\beta^\epsilon\right)\mathbb{T}^\epsilon\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)\mathbb{T}^\epsilon\left(v\right)w(t)dxdt\\&\qquad=|Y_2|\mathcal{M}_{Y_2}(\nabla_y\beta)\int_0^T\int_D\frac{\partial\mathbf{u}}{\partial t}(x,t)v(x)w(t)dxdt=0.\end{aligned}\tag{86}$$

As in (77) and (81), we obtain the corresponding limits and thus we have:

$$\begin{aligned}
|Y_2|\mathcal{M}_{Y_2}(\rho c_v)\int_0^T\int_D d\theta(x,t)v(x)w(t)dx& \\
+\int_0^T\int_{D\times Y_2}\kappa(y)(\nabla_x\theta(x,t)+\nabla_y\hat{\theta}(x,y,t))\cdot\nabla v(x)w(t)dxdydt& \\
-\int_0^T\int_{D\times Y_2}\beta(y)\frac{\partial\mathbf{u}}{\partial t}(x,t)\nabla v(x)w(t)dxdydt& \\
=|Y_2|\int_0^T\int_D g_1(x,t)v(x)w(t)dxdt& \\
+|Y_2|\int_0^T\int_D g_2(x,t)v(x)w(t)dxd\hat{W}_2.& \qquad (87)
\end{aligned}$$

The aim now is to identify the functions $\hat{\mathbf{u}}$ and $\hat{\theta}$. To do this, $\epsilon v(x)\psi^\epsilon(x)w(t)$ is substituted for the test function $v(x)w(t)$, where $\psi\in W^{1,2}_{\text{per}}(Y_2)$. It is clear that

$$\mathbb{T}^\epsilon(\epsilon v\psi^\epsilon)\to 0 \text{ strongly in } L^2(D\times Y_2), \qquad (88)$$

and

$$\mathbb{T}^\epsilon\left(\nabla\left(\epsilon v\psi^\epsilon\right)\right)=\epsilon(\mathbb{T}^\epsilon(\nabla v))+\mathbb{T}^\epsilon(v\nabla_y\psi)\to v\nabla_y\psi \text{ strongly in } L^2(D\times Y_2). \qquad (89)$$

The following convergences hold:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{D^\epsilon}\rho^\epsilon d\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)(x,t)v(x)\psi^\epsilon(x)w(t)dxdt \\
&=-\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{D\times Y_2}\rho(y)\mathbb{T}^\epsilon\left(\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)\mathbb{T}^\epsilon(v)\mathbb{T}^\epsilon(\psi^\epsilon)\frac{\partial w}{\partial t}dxdydt=0. \qquad (90)
\end{aligned}$$

In similar way and in view of (89), we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{D^\epsilon}A^\epsilon\nabla\mathbf{u}^\epsilon(x,t)\nabla\left(v(x)\psi^\epsilon(x)\right)w(t)dxdt \\
&=\int_0^T\int_{D\times Y_2}A(y)\left(\nabla\mathbf{u}+\nabla_y\hat{\mathbf{u}}\right)(x,t)v(x)\nabla_y\psi(y)w(t)dxdydt, \qquad (91)
\end{aligned}$$

$$\begin{aligned}
&-\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{D^\epsilon}\nabla\left(\beta^\epsilon\theta^\epsilon\right)(x,t)v(x)\psi^\epsilon(x)w(t)dxdt \\
&=\lim_{\epsilon\to 0}\int_0^T\int_{D^\epsilon}\beta^\epsilon\theta^\epsilon(x,t)\nabla\left(\epsilon v(x)\psi^\epsilon(x)\right)w(t)dxdt \\
&=\int_0^T\int_{D\times Y_2}\beta(y)\theta(x,t)v(x)\nabla_y\psi(y)w(t)dxdydt, \qquad (92)
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}\epsilon\int_0^T\int_D\widetilde{\mathbf{f}}_1^\epsilon(x,t)v(x)\psi^\epsilon(x)w(t)dxdt=0. \qquad (93)$$

The stochastic integral treated using Burkhlder-Davis-Gundy's inequality as follows

$$\lim_{\epsilon\to 0}\epsilon\mathbb{E}_1\sup_t\left|\int_0^T\int_D\widetilde{\mathbf{f}}_2^\epsilon(x,t)v(x)\psi^\epsilon(x)w(t)dxdW_1^\epsilon(t)\right|$$

$$\leq C \lim_{\epsilon\to 0} \epsilon \left(\int_0^T w(t) \left(\int_D \widetilde{\mathbf{f}}_2^{\epsilon}(x,t) v(x) \psi^{\epsilon}(x) dx \right)^2 dt \right)^{\frac{1}{2}} = 0. \tag{94}$$

Convergences (90)-(94) and the weak formulation (46) lead to

$$\int_0^T \int_{D\times Y_2} \left[A(y) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}(x,y,t) \right) - \beta(y)\theta \right] v \nabla_y \psi(y) w dx dy dt = 0. \tag{95}$$

This gives,

$$\mathrm{div}_y \left[A(y) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}(x,y,t) \right) - \beta(y)\theta \right] = 0, \tag{96}$$

Following [31] and because $\mathbf{u}$ and θ are independent of y and $\mathcal{M}_{Y_2}(\hat{\mathbf{u}}) = 0$, we may express the solution of (96) as:

$$\hat{\mathbf{u}}(t,x,y) = \Phi(y) \nabla_x \mathbf{u}(x,t) + V(y)\theta(x,t), \tag{97}$$

where $\Phi = (\Phi_{klm})_{1\leq k,l,m\leq n}$ is a 3-dimensional tensor and $V = (V_m)_{1\leq m\leq n}$ solve uniquely the following cell problems

$$\begin{cases} \mathrm{div}_y \left(A(y) \nabla_y (\Phi_{klm} + y_m) \right) = 0 & \text{in } Y_2, \\ A(y) \nabla_y (\Phi_{klm} + y_m) \cdot \nu = 0 & \text{on } \partial Y_1, \\ \mathcal{M}_{Y_2}(\Phi_{klm}) = 0 & \Phi_{klm} \text{ is } Y\text{-periodic,} \end{cases}$$

and

$$\begin{cases} \mathrm{div}_y \left(A(y) (\nabla_y V_m(y)) - \beta(y) \right) = 0 & \text{in } Y_2, \\ A(y) \left[(\nabla_y V(y)) - \beta(y) \right] \cdot \nu = 0 & \text{on } \partial Y_1, \\ \mathcal{M}_{Y_2}(V) = 0 & V \text{ is } Y\text{-periodic,} \end{cases}$$

Following the same process as above, we get

$$\int_0^T \int_{D\times Y_2} \left[\kappa(y) \left(\nabla\theta + \nabla_y \hat{\theta}(x,y,t) \right) - \beta(y) \frac{\partial \mathbf{u}}{\partial t} \right] v \nabla_y \psi(y) w dx dy dt = 0, \tag{98}$$

from which we have

$$\mathrm{div}_y \left[\kappa(y) \left(\nabla\theta + \nabla_y \hat{\theta}(x,y,t) \right) - \beta(y) \frac{\partial \mathbf{u}}{\partial t} \right] = 0, \tag{99}$$

Similar to the above we write

$$\hat{\theta}(t,x,y) = \Psi(y) \cdot \nabla_x \theta(x,t) + V(y) \frac{\partial \mathbf{u}}{\partial t}(x,t), \tag{100}$$

where $V = (V_m)_{1\leq m\leq n}$ is defined as above and $\Psi = (\Psi_k)_{1\leq k\leq n}$ solve uniquely the following problem:

$$\begin{cases} \mathrm{div}_y \left(\kappa(y) \nabla_y (\Psi_k + y_k) \right) = 0 & \text{in } Y_2, \\ \kappa(y) \nabla_y (\Psi_k + y_k) \cdot \nu = 0 & \text{on } \partial Y_1, \\ \mathcal{M}_{Y_2}(\Psi_k) = 0 & \Psi_k \text{ is } Y\text{-periodic,} \end{cases}$$

Using (97) into (82) and (100) into (87), we get

$$\begin{aligned} \mathcal{M}_{Y_2}(\rho) d \left(\frac{\partial \mathbf{u}}{\partial t} \right)(x,t) - A_0 \Delta \mathbf{u}(x,t) dt + \beta_0 \nabla\theta(x,t) dt \\ = \mathbf{f}_1(x,t)) dt + \mathbf{f}_2(x,t) d\hat{W}_1(t), \end{aligned} \tag{101}$$

and

$$\mathcal{M}_{Y_2}(\rho c_v) d\theta(x,t) - \kappa_0 \Delta\theta(x,t) dt + \beta_0 \mathrm{div} \left(\frac{\partial \mathbf{u}}{\partial t} \right)(x,t) dt$$

$$= g_1(x,t))dt + g_2(x,t)d\hat{W}_2(t), \tag{102}$$

where A_0 and κ_0 are constant elliptic matrices given by (58) and (60) respectively. Now, let's show that the initial conditions are satisfied. We choose as a test function $v(x)w(t)$ where $(v,w) \in C_0^\infty(D) \times C^\infty([0,T])$ and $w(0) = 1$ and $w(T) = 0$. We have

$$\begin{aligned}
&- \int_0^T \int_{D^\epsilon} \rho^\epsilon \left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)(x,t)v(x)\frac{dw}{dt}(t)dxdt - \int_{D^\epsilon} \rho^\epsilon \mathbf{h}_2^\epsilon(x)v(x)dx \\
&+ \int_{D^\epsilon} A^\epsilon \nabla \mathbf{u}^\epsilon(x,t)\nabla v(x)w(t)dxdt \\
&+ \int_0^T \int_{D^\epsilon} \nabla\left(\beta^\epsilon \theta^\epsilon\right)(x,t)v(x)w(t)dxdt \\
= &\int_0^T \int_{D^\epsilon} \mathbf{f}_1^\epsilon(x,t)v(x)w(t)dxdt + \int_0^T \int_{D^\epsilon} \mathbf{f}_2^\epsilon(x,t)v(x)w(t)dxdW_1^\epsilon(t).
\end{aligned}$$

Making passage to the limit in the above equation, we get

$$\begin{aligned}
&- |Y_2|\mathcal{M}_{Y_2}(\rho) \int_0^T \int_D \frac{\partial \mathbf{u}}{\partial t}(x,t)v(x)\frac{dw}{dt}(t)dxdt \\
&- |Y_2|\mathcal{M}_{Y_2}(\rho) \int_D \mathbf{h}_2(x)v(x)dx \\
&+ \int_0^T \int_{D\times Y_2} A(y)\left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right)(x,t)\nabla v(x)w(t)dxdydt \\
&+ |Y_2|\mathcal{M}_{Y_2}(\beta) \int_0^T \int_D \nabla\theta(x,t)v(x)w(t)dxdt \\
= &|Y_2| \int_0^T \int_D \mathbf{f}_1(x,t)v(x)w(t)dxdt \\
&+ |Y_2| \int_0^T \int_D \mathbf{f}_1(x,t)v(x)w(t)dxd\hat{W}_1(t).
\end{aligned} \tag{103}$$

Integration by parts in the first term in (103) gives

$$\begin{aligned}
&|Y_2|\mathcal{M}_{Y_2}(\rho) \int_0^T \int_D d\left(\frac{\partial \mathbf{u}}{\partial t}(x,t)\right)v(x)\frac{dw}{dt}(t)dxdt \\
&+ |Y_2|\mathcal{M}_{Y_2}(\rho) \int_0^T \int_D \frac{\partial \mathbf{u}}{\partial t}(x,0)v(x)dxdt - |Y_2|\mathcal{M}_{Y_2}(\rho) \int_D \mathbf{h}_2(x)v(x)dx \\
&+ \int_0^T \int_{D\times Y_2} A(y)\left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right)(x,t)\nabla v(x)w(t)dxdydt \\
&+ |Y_2|\mathcal{M}_{Y_2}(\beta) \int_0^T \int_D \nabla\theta(x,t)v(x)w(t)dxdt \\
= &|Y_2| \int_0^T \int_D \mathbf{f}_1(x,t)v(x)w(t)dxdt \\
&+ |Y_2| \int_0^T \int_D \mathbf{f}_1(x,t)v(x)w(t)dxd\hat{W}_1(t).
\end{aligned}$$

From this and the limit problem (82), we have $\frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{h}_2(x)$. With the same test function one can easily show that $\theta(x,0) = h_3(x)$. If we choose $w(t)$ such that $w(0) = w(T) = \frac{dw}{dt}(T) = 0$ and $\frac{dw}{dt}(0) = 1$, we show that $\mathbf{u}(x,0) = \mathbf{h}_1(x)$.

As a result, we were able to formulate the limit problem in both the weak and strong formulations. However, the convergences were limited to sub-sequences; in order for the entire sequence $\left(\mathbf{u}^\epsilon, \frac{\partial \mathbf{u}^\epsilon}{\partial t}, \theta^\epsilon, W_1^\epsilon, W_2^\epsilon\right)$ to converge, it was necessary to demonstrate that the limit $\left(\mathbf{u}, \hat{\mathbf{u}}, \frac{\partial \mathbf{u}}{\partial t}, \theta, \hat{\theta}, \hat{W}_1, \hat{W}_2\right)$ was uniquely determined. We have the following theorem for this. $\square$

Theorem 6.2. *The system* (52) *and* (53) *has at most one solution.*

Proof. We first state that by standard argument, see for example [11] the functions $\hat{\mathbf{u}}(t,x,y)$ and $\hat{\theta}(t,x,y)$ are unique solutions to (95) and (98) respectively.

Assume that $\left(\mathbf{u}_1, \frac{\partial \mathbf{u}_1}{\partial t}, \theta_1, \hat{W}_1, \hat{W}_2\right)$ and $\left(\mathbf{u}_2, \frac{\partial \mathbf{u}_2}{\partial t}, \theta_2, \hat{W}_1, \hat{W}_2\right)$ are two solutions to the system (101) and (102) and write $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2$, $\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{u}_1}{\partial t} - \frac{\partial \mathbf{u}_2}{\partial t}$ and $V = \theta_1 - \theta_2$. Applying Ito's formula to the functions $\phi\left(t, \frac{\partial \mathbf{U}}{\partial t}(t)\right) = \left\|\frac{\partial \mathbf{U}}{\partial t}(t)\right\|^2$ and $\psi(t,V) = \|V(t)\|^2$, we get after integration by parts

$$d\left[\mathcal{M}_{Y_2}(\rho^2)\left\|\frac{\partial \mathbf{U}}{\partial t}\right\|^2 + (A_0 e(\mathbf{U}), e(\mathbf{U}))\right] = -\beta_0\left(\nabla V, \frac{\partial \mathbf{U}}{\partial t}\right)dt \tag{104}$$

and

$$d\mathcal{M}_{Y_2}(\rho^2 c_v^2)\|V\|^2 + (\kappa_0 \nabla V, \nabla V)dt = \beta_0\left(\frac{\partial \mathbf{U}}{\partial t}, \nabla V\right)dt. \tag{105}$$

Adding together (104) and (105), we obtain

$$\begin{aligned} d\left[\mathcal{M}_{Y_2}(\rho^2)\left\|\frac{\partial \mathbf{U}}{\partial t}\right\|^2 + (A_0 e(\mathbf{U}), e(\mathbf{U})) + \mathcal{M}_{Y_2}(\rho^2 c_v^2)\|V\|^2\right] \\ + (\kappa_0 \nabla V, \nabla V)dt = 0. \end{aligned} \tag{106}$$

Using the ellipticity of the matrices A_0 and κ_0, the assumptions on the data and integration over $(0,t)$, we complete the proof. $\square$

7. Convergence of associated energies and corrector results.

In this section, we study the asymptotic behaviour of the energy associated to the problem (1) to the energy associated to the limit problem (57). Let us define the associated energies for the problems (1) and (57). We have

$$\begin{aligned} \mathcal{E}^\epsilon(\mathbf{u}^\epsilon, \theta^\epsilon)(t) = {} & \mathbb{E}_1\left\|\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\right\|^2_{L^2(D^\epsilon)} + \mathbb{E}_1\|\rho^\epsilon c_v^\epsilon \theta^\epsilon(t)\|^2_{L^2(D^\epsilon)} \\ & + \mathbb{E}_1\int_{D^\epsilon} A^\epsilon e(\mathbf{u}^\epsilon)(t) e(\mathbf{u}^\epsilon)(t)dx \\ & + 2\mathbb{E}_1\int_0^t\int_{D^\epsilon} \kappa^\epsilon \nabla\theta^\epsilon(\tau)\cdot\nabla\theta^\epsilon(\tau)dxd\tau, \end{aligned} \tag{107}$$

and

$$\begin{aligned} \mathcal{E}(\mathbf{u}, \theta)(t) = {} & \mathcal{M}_{Y_2}(\rho^2)\mathbb{E}_1\left\|\frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\right\|^2_{L^2(D)} + \mathcal{M}_{Y_2}(\rho^2 c_v^2)\mathbb{E}_1\|\theta(t)\|^2_{L^2(D)} \\ & + \mathbb{E}_1\int_D A_0 e(\mathbf{u})(t)e(\mathbf{u})(t)dx \\ & + \mathbb{E}_1\int_0^t\int_D \kappa_0\nabla\theta(\tau)\nabla\theta(\tau)dxd\tau. \end{aligned} \tag{108}$$

As before, we apply Itô's lemma to the function $\Phi(t, \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)) = \|\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}(t)\|^2_{L^2(D^\epsilon)}$ in the first equation of system (1) and to the function $\Psi(t, \theta^\epsilon(t)) = \|\rho^\epsilon c_v^\epsilon \theta^\epsilon(t)\|^2_{L^2(D^\epsilon)}$ in the second equation of system (1). Next, we add up the resulting equations, take the expectation and integrate from 0 to $t \le T$, we obtain

$$\begin{aligned}\mathcal{E}^\epsilon(\mathbf{u}^\epsilon, \theta^\epsilon)(t) &= \|\rho^\epsilon \mathbf{h}_2^\epsilon\|^2_{L^2(D^\epsilon)} + \|\rho^\epsilon c_v^\epsilon h_3^\epsilon\|^2_{L^2(D^\epsilon)} + \int_{D^\epsilon} A^\epsilon e(\mathbf{h}_1^\epsilon) e(\mathbf{h}_1^\epsilon) dx \\ &+ 2\mathbb{E}_1 \int_0^t \left(\mathbf{f}_1^\epsilon(\tau), \frac{\partial \mathbf{u}^\epsilon}{\partial t}(\tau)\right)_{L^2(D^\epsilon)} d\tau + 2\mathbb{E}_1 \int_0^t \left(g_1^\epsilon(\tau), \theta^\epsilon(\tau)\right)_{L^2(D^\epsilon)} d\tau \\ &+ \int_0^t \|\mathbf{f}_2^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} d\tau + \int_0^t \|g_2^\epsilon(\tau)\|^2_{L^2(D^\epsilon)} d\tau. \end{aligned} \tag{109}$$

Following the same process, we obtain

$$\begin{aligned}\mathcal{E}(\mathbf{u}, \theta)(t) &= \mathcal{M}_{Y_2}(\rho^2) \|\mathbf{h}_2\|^2_{L^2(D)} + \mathcal{M}_{Y_2}(\rho^2 c_v^2) \|h_3\|^2_{L^2(D)} \\ &+ \int_D A_0 e(\mathbf{h}_1) e(\mathbf{h}_1) dx + 2\mathbb{E}_1 \int_0^t \left(\mathbf{f}_1(\tau), \frac{\partial \mathbf{u}}{\partial t}(\tau)\right)_{L^2(D)} d\tau \\ &+ 2|\mathbb{E}_1 \int_0^t \left(g_1(\tau), \theta(\tau)\right)_{L^2(D)} d\tau + \int_0^t \|\mathbf{f}_2(\tau)\|^2_{L^2(D)} d\tau \\ &+ \int_0^t \|g_2(\tau)\|^2_{L^2(D)} d\tau. \end{aligned} \tag{110}$$

As usual when proving convergence of energies, some stronger assumptions on the data are needed, see for example [12, 17]. So, we have the following assumptions:

$$\|\mathbf{h}_2^\epsilon - \mathbf{h}_2\|_{[L^2(D^\epsilon)]^n} \to 0 \text{ as } \epsilon \to 0, \tag{111}$$

$$\|h_3^\epsilon - h_3\|_{L^2(D^\epsilon)} \to 0 \text{ as } \epsilon \to 0, \tag{112}$$

$$\|\mathbf{f}_1^\epsilon - \mathbf{f}_1\|_{L^2(0,T;[L^2(D^\epsilon)]^n)} \to 0 \text{ as } \epsilon \to 0, \tag{113}$$

$$\|\mathbf{f}_2^\epsilon - \mathbf{f}_2\|_{L^2(0,T;[L^2(D^\epsilon)]^n)} \to 0 \text{ as } \epsilon \to 0, \tag{114}$$

$$\|g_2^\epsilon - g_2\|_{L^2(0,T;[L^2(D^\epsilon)]^n)} \to 0 \text{ as } \epsilon \to 0, \tag{115}$$

where $(\mathbf{h}_2, h_3) \in [L^2(D)]^n \times L^2(D)$, $(\mathbf{f}_1, \mathbf{f}_2) \in L^2(0,T;[L^2(D^\epsilon)]^n) \times L^2(0,T;[L^2(D^\epsilon)]^n)$ and $(g_1, g_2) \in L^2(0,T;[L^2(D^\epsilon)]^n) \times L^2(0,T;[L^2(D^\epsilon)]^n)$. As for the function $\mathbf{h}_1^\epsilon$, we assume that it satisfies the following problem:

$$\begin{cases} -\text{div}\,(A^\epsilon e(\mathbf{h}_1^\epsilon)) = \mathcal{L}_\epsilon\,(A_0 e(\mathbf{h}_1)) & \text{in } D^\epsilon \\ A^\epsilon e(\mathbf{h}_1^\epsilon) \cdot \nu = 0 & \text{on } \partial D_0^\epsilon, \\ \mathbf{h}_1^\epsilon = 0 & \text{on } \partial D, \end{cases}$$

such that $\mathbf{h}_1$ is given by (48) and the operator $\mathcal{L}_\epsilon\,(A_0 e(\mathbf{h}_1)) : \mathcal{V}(D^\epsilon) \to \mathbb{R}$ is defined as:

$$\mathcal{L}_\epsilon\,(A_0 e(\mathbf{h}_1))(v) = \int_{D^\epsilon} A_0 e(\mathbf{h}_1)(\widetilde{\widetilde{v}})|_{D^\epsilon} dx, \text{ for all } v \in \mathcal{V}(D^\epsilon),$$

where, $\widetilde{\widetilde{v}}$ is the extension by zero to the macro operator of the function v. With this assumptions, see [12, Proposition 4.2.], it is easy to see that, $\|\mathbf{h}_1^\epsilon\|_{\mathcal{V}(D^\epsilon)} \le C$ for some constant independent of ϵ, (48) hold true, and:

$$\int_{D^\epsilon} A^\epsilon e(\mathbf{h}_1^\epsilon) e(\mathbf{h}_1^\epsilon) dx \to |Y_2| \int_D A_0 e(\mathbf{h}_1) e(\mathbf{h}_1) dx. \tag{116}$$

We now state and prove the convergence result for the energy associated with the micro-model to the one associated with the macro-model.

Theorem 7.1. *Let* $\left(\boldsymbol{u}^\epsilon, \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}, \theta^\epsilon, W_1^\epsilon, W_2^\epsilon\right)$ *be the solution of system* (46)-(47) *and assume that* (111)-(116) *hold, then*

$$\mathcal{E}^\epsilon\left(\boldsymbol{u}^\epsilon, \theta^\epsilon\right)(t) \to |Y_2|\mathcal{E}\left(\boldsymbol{u}, \theta\right)(t) \text{ strongly in } C([0,T]). \tag{117}$$

Proof. We use the assumptions (111)-(115) together with [12, Proposition 2.12.] to obtain

$$\mathbb{T}^\epsilon(\mathbf{h}_2^\epsilon) \to \mathbf{h}_2 \text{ strongly in } [L^2(D\times Y_2)]^n, \tag{118}$$

$$\mathbb{T}^\epsilon(h_3^\epsilon) \to h_3 \text{ strongly in } L^2(D\times Y_2), \tag{119}$$

$$\mathbb{T}^\epsilon(\mathbf{f}_1^\epsilon) \to \mathbf{f}_1 \text{ strongly in } L^2(0,T;[L^2(D\times Y_2)]^n), \tag{120}$$

$$\mathbb{T}^\epsilon(\mathbf{f}_2^\epsilon) \to \mathbf{f}_2 \text{ strongly in } L^2(0,T;[L^2(D\times Y_2)]^n), \tag{121}$$

$$\mathbb{T}^\epsilon(g_2^\epsilon) \to g_2 \text{ strongly in } L^2(0,T;L^2(D\times Y_2)). \tag{122}$$

Also, the integral on Λ^ϵ for the square of the functions $\mathbf{h}_2^\epsilon$, h_3^ϵ, $\mathbf{f}_1^\epsilon$, $\mathbf{f}_2^\epsilon$, g_1^ϵ, and g_2^ϵ converges to zero. We now pass to the limit on (109), from convergences (118) and (119) and the definition of ρ^ϵ and A^ϵ we have the following two limits:

$$\lim_{\epsilon\to 0}\int_{D^\epsilon}\left(\rho^\epsilon \mathbf{h}_2^\epsilon\right)^2 dx = \lim_{\epsilon\to 0}\int_{D\times Y_2}\left(\mathbb{T}^\epsilon(\rho^\epsilon)\mathbb{T}^\epsilon(\mathbf{h}_2^\epsilon)\right)^2 dxdy$$

$$\lim_{\epsilon\to 0}\int_{D\times Y_2}\rho^2(y)\left(\mathbb{T}^\epsilon(\mathbf{h}_2^\epsilon)\right)^2 dxdy = |Y_2|\mathcal{M}_{Y_2}(\rho^2)\left\|\mathbf{h}_2\right\|^2_{L^2(D)}, \tag{123}$$

and

$$\lim_{\epsilon\to 0}\int_{D^\epsilon}\left(\rho^\epsilon c_v^\epsilon h_3^\epsilon\right)^2 dx = \lim_{\epsilon\to 0}\int_{D\times Y_2}\left(\mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon)\mathbb{T}^\epsilon(h_3^\epsilon)\right)^2 dxdy$$

$$\lim_{\epsilon\to 0}\int_{D\times Y_2}\rho^2(y)c_v^2(y)\left(\mathbb{T}^\epsilon(h_3^\epsilon)\right)^2 dxdy = |Y_2|\mathcal{M}_{Y_2}(\rho^2 c_v^2)\left\|h_3\right\|^2_{L^2(D)}. \tag{124}$$

From the weak convergence (66) and the strong convergence (120), we have

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D^\epsilon}\mathbf{f}_1^\epsilon\frac{\partial \mathbf{u}^\epsilon}{\partial t}dxd\tau &= \lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D\times Y_2}\mathbb{T}^\epsilon(\mathbf{f}_1^\epsilon)\mathbb{T}^\epsilon\left(\frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)dxdyd\tau\\ &= |Y_2|\mathbb{E}_1\int_0^t\int_D \mathbf{f}_1\frac{\partial \mathbf{u}}{\partial t}dxd\tau. \end{aligned}\tag{125}$$

From the weak convergence (50) and the strong convergence (6), we have

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D^\epsilon}g_1^\epsilon\theta^\epsilon dxd\tau &= \lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D\times Y_2}\mathbb{T}^\epsilon(g_1^\epsilon)\mathbb{T}^\epsilon\left(\theta^\epsilon\right)dxdyd\tau\\ &= |Y_2|\mathbb{E}_1\int_0^t\int_D g_1\theta dxd\tau. \end{aligned}\tag{126}$$

From convergences (121) and (122), we have

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D^\epsilon}\mathbf{f}_2^\epsilon\mathbf{f}_2^\epsilon dxd\tau &= \lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D\times Y_2}\mathbb{T}^\epsilon(\mathbf{f}_2^\epsilon)\mathbb{T}^\epsilon\left(\mathbf{f}_2^\epsilon\right)dxdyd\tau\\ &= |Y_2|\mathbb{E}_1\int_0^t\int_D |\mathbf{f}_2|^2 dxd\tau, \end{aligned}\tag{127}$$

and

$$\lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D^\epsilon} g_2^\epsilon g_2^\epsilon dx d\tau = \lim_{\epsilon\to 0}\mathbb{E}_1\int_0^t\int_{D\times Y_2}\mathbb{T}^\epsilon(g_2^\epsilon)\mathbb{T}^\epsilon(g_2^\epsilon)\, dxdyd\tau$$
$$= |Y_2|\mathbb{E}_1\int_0^t\int_D |g_2|^2 dx d\tau. \tag{128}$$

Combining convergences (123)-(128) together with (116), we see that

$$\lim_{\epsilon\to 0}\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t) = |Y_2|\mathcal{E}(\mathbf{u},\theta)(t) \text{ for all } t\in[0,T].$$

So far, we have proved the point wise convergence of $\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)$ to $\mathcal{E}(\mathbf{u},\theta)$, we must now demonstrate that the sequence $\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)$ belongs to a compact subset of C([0,T]) by demonstrating that $\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)$ is uniformly bounded and equicontinuous on $[0,T]$, and so Arzela-Ascoli's theorem will entail the proof. A direct application of Theorem 3.1 in the definition (107) on easily see that $|\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t)|\le C$ for all $t\in[0,T]$. For the equicontinuity, we see that

$$\begin{aligned}
&|\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t+s)-\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t)|\\
&\le 2\mathbb{E}_1\int_t^{t+s}\int_{D^\epsilon}\left|\mathbf{f}_1^\epsilon(\tau)\frac{\partial\mathbf{u}^\epsilon}{\partial t}(\tau)\right| dxd\tau\\
&\quad + 2\mathbb{E}_1\int_t^{t+s}\int_{D^\epsilon}|g_1^\epsilon(\tau),\theta^\epsilon(\tau)|\, dxd\tau\\
&\quad + \int_t^{t+s}\|\mathbf{f}_2^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau + \int_t^{t+s}\|g_2^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau\\
&\le 2\mathbb{E}_1\left(\int_t^{t+s}\left\|\frac{\partial\mathbf{u}^\epsilon}{\partial t}(\tau)\right\|^2_{[L^2(D^\epsilon)]^n}d\tau\right)^{\frac12}\left(\int_t^{t+s}\|\mathbf{f}_1^\epsilon(\tau)\|^2_{[L^2(D^\epsilon)]^n}d\tau\right)^{\frac12}\\
&\quad + 2\mathbb{E}_1\left(\int_t^{t+s}\|\theta^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\left(\int_t^{t+s}\|g_1^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\\
&\quad + \left(\int_t^{t+s}1^2 d\tau\right)^{\frac12}\left(\int_t^{t+s}\|\mathbf{f}_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\\
&\quad + \left(\int_t^{t+s}1^2 d\tau\right)^{\frac12}\left(\int_t^{t+s}\|g_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\\
&\le 2s^{\frac12}\mathbb{E}_1\sup_t\left\|\frac{\partial\mathbf{u}^\epsilon}{\partial t}(t)\right\|_{[L^2(D^\epsilon)]^n}\left(\int_t^{t+s}\|\mathbf{f}_1^\epsilon(\tau)\|^2_{[L^2(D^\epsilon)]^n}d\tau\right)^{\frac12}\\
&\quad + 2s^{\frac12}\mathbb{E}_1\sup_t\|\theta^\epsilon(t)\|_{L^2(D^\epsilon)}\left(\int_t^{t+s}\|g_1^\epsilon(\tau)\|^2_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\\
&\quad + s^{\frac12}\left(\int_t^{t+s}\|\mathbf{f}_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)}d\tau\right)^{\frac12}\\
&\quad + s^{\frac12}\left(\int_t^{t+s}\|g_2^\epsilon(\tau)\|^4_{L^2(D^\epsilon)}d\tau\right)^{\frac12}.
\end{aligned}\tag{129}$$

From the estimates obtained in Theorem 3.1 and the assumptions on $\mathbf{f}_2^\epsilon$ and g_2^ϵ, we have

$$|\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t+s)-\mathcal{E}^\epsilon(\mathbf{u}^\epsilon,\theta^\epsilon)(t)|\le Cs^{\frac12},\ \text{ for all } t\in[0,T-s], \tag{130}$$

which implies that $\mathcal{E}^\epsilon(\mathbf{u}^\epsilon, \theta^\epsilon)$ is equicontinuous and therefore the proof is complete. □

We are now in the position to prove the following corrector results

Theorem 7.2. *Assume that the assumptions in Theorems 3.1, 3.2, and 7.1 are correct. Then the following strong convergences apply*

$$\left(\left\| \rho^\epsilon \frac{\partial \boldsymbol{u}^\epsilon}{\partial t} \right\|_{L^2(\Omega;L^2(0,T;[L^2(\Lambda^\epsilon)]^n))}, \| \nabla \boldsymbol{u}^\epsilon \|_{L^2(\Omega;L^2(0,T;[L^2(\Lambda^\epsilon)]^n\times n))}, \right.$$
$$\left. \| A^\epsilon \theta^\epsilon \|_{L^2(\Omega;L^2(0,T;L^2(\Lambda^\epsilon)))}, \| \nabla \theta^\epsilon \|_{L^2(\Omega;L^2(0,T;L^2(\Lambda^\epsilon)))} \right) \to 0, \tag{131}$$

$$\mathbb{T}^\epsilon\left(\rho^\epsilon \frac{\partial \boldsymbol{u}^\epsilon}{\partial t}\right) \to \mathcal{M}_{Y_2}(\rho)\frac{\partial \boldsymbol{u}}{\partial t} \text{ strongly in } L^2(\Omega; L^2(0,T;[L^2(D\times Y_2]^n)), \tag{132}$$
$$\mathbb{T}^\epsilon(\nabla \boldsymbol{u}^\epsilon) \to \nabla \boldsymbol{u} + \nabla_y \hat{\boldsymbol{u}} \text{ strongly in } L^2(\Omega; L^2(0,T;[L^2(D\times Y_2]^{n\times n})), \tag{133}$$
$$\mathbb{T}^\epsilon(A^\epsilon \theta^\epsilon) \to \mathcal{M}_{Y_2}(c)\theta \text{ strongly in } L^2(\Omega; L^2(0,T;[L^2(D\times Y_2]^n)), \tag{134}$$
$$\mathbb{T}^\epsilon(\nabla \theta^\epsilon) \to \nabla\theta + \nabla_y \hat{\theta} \text{ strongly in } L^2(\Omega; L^2(0,T;[L^2(D\times Y_2]^n)). \tag{135}$$

Proof. Let us first notice that from (58), we have

$$\int_0^T \int_D A_0 \nabla \mathbf{u} \nabla \mathbf{u} dx dt = \frac{1}{|Y_2|} \int_0^T \int_{D\times Y_2} A(y) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right) \nabla \mathbf{u} dx dy dt,$$

from (95), we also have

$$\int_0^T \int_{D\times Y_2} A(y) \left(\nabla \mathbf{u}(x,t) + \nabla_y \hat{\mathbf{u}}(x,y,t)\right) \nabla_y \hat{\mathbf{u}}(x,y,t) dx dy dt = 0.$$

Thus,

$$\int_0^T \int_D A_0 \nabla \mathbf{u} \cdot \nabla \mathbf{u} dx dt = \frac{1}{|Y_2|} \int_0^T \int_{D\times Y_2} A(y) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right) dx dy dt.$$

In a similar way, we have

$$\int_0^T \int_D \kappa_0 \nabla \theta \cdot \nabla \theta dx dt = \frac{1}{|Y_2|} \int_0^T \int_{D\times Y_2} \kappa(y) \left(\nabla \theta + \nabla_y \hat{\theta}\right) \cdot \left(\nabla \theta + \nabla_y \hat{\theta}\right) dx dy dt.$$

Considering the stochastic energy associated with the limit problem, i.e. equation (108) and classical result (Lower semi-continuity), we may see that:

$$\begin{aligned}
\int_0^T \mathcal{E}(\mathbf{u},\theta)(t) dt &= \frac{1}{|Y_2|}\mathbb{E}_1 \int_0^T \int_{D\times Y_2} \left(\rho(y)\frac{\partial \mathbf{u}}{\partial t}\right)^2 dx dy dt \\
&+ \frac{1}{|Y_2|}\mathbb{E}_1 \int_0^T \int_{D\times Y_2} A(y) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right) \left(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}\right) dx dy dt \\
&+ \frac{1}{|Y_2|}\mathbb{E}_1 \int_0^T \int_{D\times Y_2} \left(\rho(y) c_v(y)\theta\right)^2 dx dy dt \\
&+ \frac{1}{|Y_2|}\mathbb{E}_1 \int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa(y) \left(\nabla \theta + \nabla_y \hat{\theta}\right) \cdot \left(\nabla \theta + \nabla_y \hat{\theta}\right) dx dy d\tau\right] dt \\
&\le \liminf_{\epsilon\to 0} \frac{1}{|Y_2|}\mathbb{E}_1 \int_0^T \int_{D\times Y_2} \left(\mathbb{T}^\epsilon\left(\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)\right)^2 dx dy dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|Y_2|}\mathbb{E}_1\int_0^T\int_{D\times Y_2}A(y)(\mathbb{T}^\epsilon(\nabla\mathbf{u}^\epsilon)\cdot\mathbb{T}^\epsilon(\nabla\mathbf{u}^\epsilon)\,dxdydt\\
&+\frac{1}{|Y_2|}\mathbb{E}_1\int_0^T\int_{D\times Y_2}(\mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon\theta^\epsilon))^2\,dxdydt\\
&+\frac{1}{|Y_2|}\mathbb{E}_1\int_0^T\left[\int_0^t\int_{D\times Y_2}\kappa(y)\mathbb{T}^\epsilon(\nabla\theta\epsilon)\cdot\mathbb{T}^\epsilon(\nabla\theta^\epsilon)\,dxdyd\tau\right]dt.
\end{aligned}
$$

From (36), we write

$$
\int_0^T\mathcal{E}(\mathbf{u},\theta)(t)dt\le\liminf_{\epsilon\to0}\frac{1}{|Y_2|}\int_0^T\widehat{\mathcal{E}}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt, \tag{136}
$$

where

$$
\begin{aligned}
\widehat{\mathcal{E}}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)=&\;\mathbb{E}_1\int_{\hat{D}^\epsilon}\left(\rho^\epsilon\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)^2dx+\mathbb{E}_1\int_{\hat{D}^\epsilon}A^\epsilon e\mathbf{u}^\epsilon)e(\mathbf{u}^\epsilon)dx\\
&+\mathbb{E}_1\int_{\hat{D}^\epsilon}(\rho^\epsilon A^\epsilon\theta^\epsilon)^2\,dx+\mathbb{E}_1\int_0^t\int_{\hat{D}^\epsilon}\kappa^\epsilon\nabla\theta^\epsilon\nabla\theta^\epsilon dxdt.
\end{aligned}
$$

From (136) and Theorem 7.1, we have

$$
\begin{aligned}
\int_0^T\mathcal{E}(\mathbf{u},\theta)(t)dt&\le\liminf_{\epsilon\to0}\frac{1}{|Y_2|}\int_0^T\widehat{\mathcal{E}}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt\\
&\le\limsup_{\epsilon\to0}\frac{1}{|Y_2|}\int_0^T\widehat{\mathcal{E}}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt\\
&\le\limsup_{\epsilon\to0}\frac{1}{|Y_2|}\int_0^T\mathcal{E}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt=\int_0^T\mathcal{E}(\mathbf{u},\theta)(t)dt,
\end{aligned}
$$

which implies that

$$
\lim_{\epsilon\to0}\int_0^T\widehat{\mathcal{E}}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt=\lim_{\epsilon\to0}\int_0^T\mathcal{E}(\mathbf{u}^\epsilon,\theta^\epsilon)(t)dt=|Y_2|\int_0^T\mathcal{E}(\mathbf{u},\theta)(t)dt.
$$

From this and (36), we get

$$
\begin{aligned}
&\mathbb{E}_1\int_{\hat{\Lambda}^\epsilon}\left(\rho^\epsilon\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)^2dx\to0,\quad\mathbb{E}_1\int_{\hat{\Lambda}^\epsilon}A^\epsilon e\mathbf{u}^\epsilon)e(\mathbf{u}^\epsilon)dx\to0\\
&\mathbb{E}_1\int_{\hat{\Lambda}^\epsilon}(\rho^\epsilon c_v^\epsilon\theta^\epsilon)^2\,dx\to0,\quad\mathbb{E}_1\int_0^t\int_{\hat{\Lambda}^\epsilon}\kappa^\epsilon\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon dxdt\to0.
\end{aligned}
$$

This gives the proof of (131). In order to prove (132)-(135), we first note that:

$$
\begin{aligned}
\lim_{\epsilon\to0}\frac{1}{|Y_2|}\mathbb{E}_1\Bigg[&\int_0^T\int_{D\times Y_2}\left(\mathbb{T}^\epsilon\left(\rho^\epsilon\frac{\partial\mathbf{u}^\epsilon}{\partial t}\right)\right)^2dxdydt\\
&+\int_0^T\int_{D\times Y_2}A(y)\mathbb{T}^\epsilon(\nabla\mathbf{u}^\epsilon)\,\mathbb{T}^\epsilon(\nabla\mathbf{u}^\epsilon)\,dxdydt\\
&+\int_0^T\int_{D\times Y_2}(\mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon\theta^\epsilon))^2\,dxdydt\\
&+\int_0^T\left[\int_0^t\int_{D\times Y_2}\kappa(y)\mathbb{T}^\epsilon(\nabla\theta^\epsilon)\cdot\mathbb{T}^\epsilon(\nabla\theta^\epsilon)\,dxdyd\tau\right]dt\Bigg]
\end{aligned}
$$

$$= \int_0^T \mathcal{E}(\mathbf{u}, \theta)(t)dt. \tag{137}$$

It is easy to see that:

$$\begin{aligned}
&\frac{1}{|Y_2|}\mathbb{E}_1\Bigg[\int_0^T \int_{D\times Y_2} \left(\mathbb{T}^\epsilon \left(\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right) - \rho(y)\frac{\partial \mathbf{u}}{\partial t}\right)^2 dxdydt \\
&\quad + \int_0^T \int_{D\times Y_2} A(y)[\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon) - \nabla \mathbf{u} - \nabla_y \hat{\mathbf{u}}] \\
&\quad \cdot [\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon) - \nabla \mathbf{u} - \nabla_y \hat{\mathbf{u}}]dxdydt \\
&\quad + \int_0^T \int_{D\times Y_2} (\mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon \theta^\epsilon) - \rho(y)c_v(y)\theta)^2 dxdydt \\
&\quad + \int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa(y)[\mathbb{T}^\epsilon(\nabla \theta^\epsilon) - \nabla\theta - \nabla_y\hat{\theta}]\right. \\
&\quad \left. \cdot [\mathbb{T}^\epsilon(\nabla \theta^\epsilon) - \nabla\theta - \nabla_y\hat{\theta}]dxdydt\right] \\
&= I_1 - I_2 - I_3 + I_4,
\end{aligned} \tag{138}$$

where

$$\begin{aligned}
I_1 =& \frac{1}{|Y_2|}\mathbb{E}_1\Bigg[\int_0^T \int_{D\times Y_2} \left(\mathbb{T}^\epsilon \left(\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)\right)^2 dxdydt \\
& + \int_0^T \int_{D\times Y_2} A(y)\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon)\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon) dxdydt \\
& + \int_0^T \int_{D\times Y_2} (\mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon \theta^\epsilon))^2 dxdydt \\
& + \int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa(y)\mathbb{T}^\epsilon(\nabla \theta\epsilon)\cdot \mathbb{T}^\epsilon(\nabla\theta^\epsilon) dxdyd\tau\right] dt\Bigg].
\end{aligned} \tag{139}$$

$$\begin{aligned}
I_2 =& \frac{1}{|Y_2|}\mathbb{E}_1\Bigg[\int_0^T \int_{D\times Y_2} \mathbb{T}^\epsilon \left(\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right)\left(\rho(y)\frac{\partial \mathbf{u}}{\partial t}\right) dxdydt \\
& + \int_0^T \int_{D\times Y_2} A(y)\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon)(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}}) dxdydt \\
& + \int_0^T \int_{D\times Y_2} \mathbb{T}^\epsilon(\rho^\epsilon c_v^\epsilon \theta^\epsilon)(\rho(y)c_v(y)\theta)dxdydt \\
& + \kappa\int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa(y)\mathbb{T}^\epsilon(\nabla \theta\epsilon)\cdot \left(\nabla\theta + \nabla_y\hat{\theta}\right) dxdyd\tau\right] dt\Bigg].
\end{aligned} \tag{140}$$

$$\begin{aligned}
I_3 =& \frac{1}{|Y_2|}\mathbb{E}_1\Bigg[\int_0^T \int_{D\times Y_2} \left(\rho(y)\frac{\partial \mathbf{u}}{\partial t}\right)\mathbb{T}^\epsilon \left(\rho^\epsilon \frac{\partial \mathbf{u}^\epsilon}{\partial t}\right) dxdydt \\
& + \int_0^T \int_{D\times Y_2} A(y)(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}})(\mathbb{T}^\epsilon(\nabla \mathbf{u}^\epsilon) dxdydt
\end{aligned}$$

$$+\int_0^T \int_{D\times Y_2} \mathbb{T}^\epsilon\left(\rho^\epsilon c_v^\epsilon \theta^\epsilon\right)\left(\rho(y)c_v(y)\theta\right) dxdydt$$
$$\left.+\int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa(y)\left(\nabla\theta+\nabla_y\hat{\theta}\right)\cdot \mathbb{T}^\epsilon\left(\nabla\theta\epsilon\right) dxdyd\tau\right] dt\right]. \tag{141}$$

$$I_4 = \frac{1}{|Y_2|}\mathbb{E}_1\left[\int_0^T \int_{D\times Y_2} \left(\rho(y)\frac{\partial \mathbf{u}}{\partial t}\right)^2 dxdydt\right.$$
$$+\int_0^T \int_{D\times Y_2} A(y)\left(\nabla \mathbf{u}+\nabla_y\hat{\mathbf{u}}\right)\left(\nabla \mathbf{u}+\nabla_y\hat{\mathbf{u}}\right) dxdydt$$
$$+\int_0^T \int_{D\times Y_2} (\rho(y)c_v(y)\theta)^2 dxdydt$$
$$\left.+\int_0^T \left[\int_0^t \int_{D\times Y_2} \kappa\left(\nabla\theta+\nabla_y\hat{\theta}\right)\cdot\left(\nabla\theta+\nabla_y\hat{\theta}\right) dxdyd\tau\right] dt\right]. \tag{142}$$

We can observe from the limits obtained in Theorem 6.1 that:

$$\begin{aligned}&\lim_{\epsilon\to 0} I_1 - I_2 - I_3 + I_4\\ &= \int_0^T \mathcal{E}(\mathbf{u},\theta)(t)dt - \int_0^T \mathcal{E}(\mathbf{u},\theta)(t)dt\\ &- \int_0^T \mathcal{E}(\mathbf{u},\theta)(t)dt + \int_0^T \mathcal{E}(\mathbf{u},\theta)(t)dt = 0.\end{aligned} \tag{143}$$

This implies (132)-(135) and completes the proof. □

8. **Conclusion.**

In this study, we have developed a linear model that describes the interaction between deformation and temperature fields in a thermoelastic composite material with a highly heterogeneous, anisotropic, and $\epsilon-$periodic structure. The material is subject to external heat sources and body forces, and its behavior is influenced by natural randomness arising from thermal interactions with the environment. The model in question is governed by a system of coupled linear stochastic equations: a wave equation of motion and a heat equation. Both equations contain highly oscillatory coefficients that reflect the composite material's properties, and the material domain is assumed to be perforated. We used the periodic unfolding approach to study the asymptotic behavior of the model's solution when the characteristic length scale ϵ tends to zero. We developed an effective model described by stochastic linear thermoelastic waves with constant effective coefficients in a fixed domain. These coefficients represent the material's homogenized properties, averaged over its microstructure. We established cell problems (54)-(56) that capture the material's local behavior at small scales and are crucial for computing homogenized material properties. We have shown that the energy of the original model converges to the energy of the homogenized material, providing a solid basis for the validity of the effective model. Furthermore, we proved strong convergence results, including corrector terms, which refine the approximation of the macroscopic material response at different scales.

For future work, we plan to extend this framework to nonlinear models and develop numerical methods for implementation and validation. Specifically, we aim to explore nonlinear heat conduction and nonlinear interactions between mechanical and thermal fields, including phenomena such as thermally induced buckling, dynamic instability, nonlinear damping, or external random forces. We will also develop numerical algorithms to solve the homogenized equations and compare the results with experimental data for real composite materials.

In conclusion, these results contribute to a deeper understanding of the interplay between deformation, temperature, and randomness in composite thermoelastic materials, offering valuable insights into their homogenization and effective properties for practical applications.

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