VIABILITY FOR LOCALLY MONOTONE EVOLUTION INCLUSIONS AND LOWER SEMICONTINUOUS SOLUTIONS OF HAMILTON–JACOBI–BELLMAN EQUATIONS IN INFINITE DIMENSIONS

JICHAO JIANG AND CHRISTIAN KELLER

ABSTRACT. We establish necessary and sufficient conditions for viability of evolution inclusions with locally monotone operators in the sense of Liu and Röckner [*J. Funct. Anal.*, 259 (2010), pp. 2902-2922]. This allows us to prove wellposedness of lower semicontinuous solutions of Hamilton–Jacobi–Bellman equations associated to the optimal control of evolution inclusions. Thereby, we generalize results in Bayraktar and Keller [*J. Funct. Anal.*, 275 (2018), pp. 2096-2161] on Hamilton–Jacobi equations in infinite dimensions with monotone operators in several ways. First, we permit locally monotone operators. This extends the applicability of our theory to a wider class of equations such as Burgers' equations, reaction-diffusion equations, and 2D Navier–Stokes equations. Second, our results apply to optimal control problems with state constraints. Third, we have uniqueness of viscosity solutions. Our results on viability and lower semicontinuous solutions are new even in the case of monotone operators.

CONTENTS

Date: November 26, 2024.
2010 *Mathematics Subject Classification.* 34G25, 47H05, 47J35, 49L25.
Key words and phrases. Evolution inclusions; viability; path-dependent partial differential equations; contingent solutions; viscosity solutions; optimal control.
This research was supported in part by NSF-grant DMS-2106077.

1. Introduction

We study viability problems involving evolution inclusions of the form

$$x'(t) + A(t, x(t)) \in F(t, x(t)) \quad \text{a.e. on } (0, T), \tag{1.1}$$

where A is a locally monotone operator on a Gelfand triple $V \subset H \subset V^*$. Evolution equations with locally monotone operators were introduced in [15, 16]. They generalize equations with monotone operators. Many important equations such as Navier–Stokes equations, reaction-diffusion equations, Burgers' equations, etc. can be written as abstract evolution equations involving locally monotone operators.

The main results of this work are necessary and sufficient criteria for some set K to be *viable* for (1.1), i.e., criteria that a solution x of (1.1) satisfies $x(t) \in K$ for all $t \in [0, T]$ provided it starts in K. Previous works on viability for evolution inclusions with monotone operators on Gelfand triples such as [2, 13, 19, 20] seem to only provide sufficient criteria. Note that there are results with strength similar than ours (necessary and sufficient conditions for viability) for evolution inclusions with operators that generate semigroups (see, e.g., [7, 8] and the references therein).

Thanks to our mentioned main results on viability for (1.1), we obtain applications for Hamilton–Jacobi equations in infinite dimensions. More precisely, we prove existence and uniqueness for appropriate nonsmooth solutions of the following path-dependent Hamilton–Jacobi–Bellman equation:

$$\begin{gathered} \partial_t u(t, x) + \langle A(t, x(t)), \partial_x u(t, x)\rangle + \inf_{f \in F(t, x(t))} (f, \partial_x u(t, x)) = 0, \\ (t, x) \in [0, T) \times C([0, T], H). \end{gathered} \tag{1.2}$$

This is a generalization of results in [4], where the operator A was required to be monotone. Moreover, our solutions of (1.2) need only be lower semicontinuous. This allows us to characterize value functions for optimal control problems with state constraints related to (1.1) as unique solutions to (1.2) in a similar way as it has been done in the finite-dimensional case in [9].

2. Setting

Let (V, H, V^*) be a Gelfand triple, i.e., we have $V \subset H \subset V^*$, where V is a reflexive and separable Banach space that is continuously and densily embedded into H, which is a Hilbert space, and V^* is the dual space of V. Also assume that the mentioned embedding is compact. We write $\|\cdot\| := \|\cdot\|_V$, $|\cdot| := \|\cdot\|_H$, and $\|\cdot\|_* := \|\cdot\|_{V^*}$ for the corresponding norms. We also use $|\cdot|$ for norms on Euclidean spaces. We write $(\cdot, \cdot)$ for the inner product in H and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V^*. Moreover, assume that $(h, v) = \langle h, v\rangle$ for each $h \in H$ and $v \in V$. For further details, see section 2 in [4], where the same setting is used.

2.1. **Notation.**

Balls in H centered at the origin are denoted by

$$B(0, r) := \{x \in H : |x| \le r\},\ r > 0.$$

Given two subsets E_1, E_2 of a group, put

$$E_1 + E_2 := \{e_1 + e_2 : e_1 \in E_1 \text{ and } e_2 \in E_2\}.$$

Given a function f from some set E to $\mathbb{R} \cup \{+\infty\}$, put

$$\operatorname{dom} f := \{x \in E : f(x) < +\infty\} \qquad (\textit{effective domain of } f),$$
$$\operatorname{epi} f := \{(x, y) \in E \times \mathbb{R} : y \geq f(x)\} \qquad (\textit{epigraph of } f).$$

We borrow the following notation from [6]:

$$E_{L^2} := \{f \in L^2(0, T; H) : f(t) \in E \text{ a.e. on } (0, T)\}, \quad E \subset H,$$

We write l.s.c. for lower semicontinuous and u.s.c. for upper semicontinuous.

2.2. **Path spaces.** We frequently use the space $C([0, T], H)$, which we consider to be equipped with the supremum norm $\|\cdot\|_\infty$. Subsets of $[0, T] \times C([0, T], H)$ are considered to be equipped with the pseudo-metric $\mathbf{d}_\infty$ defined by

$$\mathbf{d}_\infty(t_1, x_1; t_2, x_2) := |t_2 - t_1| + \|x_1(\cdot \wedge t_1) - x_2(\cdot \wedge t_2)\|_\infty.$$

Here, $\wedge$ means minimum, i.e., $a \wedge b := \min\{a, b\}$. Continuity and semicontinuity of functions defined on subsets of $[0, T] \times C([0, T], H)$ are always to be understood with respect to $\mathbf{d}_\infty$.

Remark 2.1. A semicontinuous function $u : [0, T] \times C([0, T]) \supset E \to \mathbb{R} \cup \{+\infty\}$ is *nonanticipating*, i.e., whenever $x_1 = x_2$ on $[0, t]$, then $u(t, x_1) = u(t, x_2)$. This follows immediately from the definition of $\mathbf{d}_\infty$.

2.3. **Standing assumptions.** Fix $T > 0$ and $p \geq 2$. Let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Fix an operator $A : [0, T] \times V \to V^*$ and a function $f^A \in L^1(0, T; \mathbb{R}_+)$. The following standing hypotheses (cf. [15, 16]) are always in force:

$\mathbf{H}(A)$: (i) For every x, $v \in V$, the map $t \mapsto \langle A(t, x), v\rangle$ is measurable.

(ii) Local monotonicity: There is a constant $c_0 \in \mathbb{R}$ and there are locally bounded functions ρ and η from V to $\mathbb{R}_+$ such that, for each x, $y \in V$, we have

$$\langle A(t, x) - A(t, y), x - y\rangle \geq -(c_0 + \rho(x) + \eta(y)) |x - y|^2 \qquad \text{a.e. on } (0, T).$$

Moreover,

$$\exists \beta > 0 : \forall x \in V : \rho(x) + \eta(x) \leq |c_0| (1 + \|x\|^p)(1 + |x|^\beta). \tag{2.1}$$

(iii) Hemicontinuity: For every x, y, $v \in V$ and a.e. $t \in (0, T)$, the map $s \mapsto \langle A(t, x + sy), v\rangle$, $[0, 1] \to \mathbb{R}$, is continuous.

(iv) Growth: There are constants c_1, $\alpha \in \mathbb{R}_+$ such that, for all $x \in V$ and a.a $t \in (0, T)$, we have

$$\|A(t, x)\|_* \leq \left(f^A(t)^{1/q} + c_1\|x\|^{p-1}\right)(1 + |x|^\alpha)$$

(v) Coercivity: There are constants $c_2 > 0$ and $c_3 \in \mathbb{R}_+$ such that, for all $x \in V$ and a.a $t \in (0, T)$, we have

$$\langle A(t, x), x\rangle \geq c_2\|x\|^p - c_3 |x|^2 - f^A(t).$$

Remark 2.2. Note that [15] has different signs in the monotonicity hypothesis $\mathbf{H}(A)$ (ii) and the coercivity hypothesis $\mathbf{H}(A)$ (iv). The reason is that in [15] equations of the form $x'(t) = A(t, x(t)) + f(t)$ are considered, whereas we deal with equations of the form $x'(t) + A(t, x(t)) = f(t)$.

Also, fix a multifunction $F : [0,T] \times H \rightsquigarrow H$ with non-empty, convex, and closed values. The following hypotheses are always in force.

$\mathbf{H}(F)$: (i) F is u.s.c. in the sense of Definition 2.6.2 of [7], i.e., for each $(t,x) \in [0,T] \times H$ and for each open neighborhood O of $F(t,x)$, there is an open neighborhood U of (t,x) such that $(s,y) \in U$ implies $F(s,y) \subset O$.

(ii) There is a constant $c_F \geq 0$ such that, for a.e. $t \in (0,T)$ and all $x \in H$,

$$|F(t,x)| := \sup\{|y| : y \in F(t,x)\} \leq c_F \cdot (1 + |x|). \tag{2.2}$$

2.4. **Function spaces.** We introduce several function spaces, which are frequently used throughout this work. To this end, we employ the sets

$$W_{pq}(t_0,T) := \{x \in L^p(t_0,T;V) : x' \in L^q(t_0,T;V^*)\}, \quad t_0 \in [0,T), \tag{2.3}$$

where x' denotes the generalized derivative of x (see p. 4 in [13] for more details).

Definition 2.3. Let $(t_0,x_0) \in [0,T) \times C([0,T],H)$.

(i) Define trajectory spaces related to our multifunction F by

$$\begin{aligned} \mathcal{X}^F(t_0,x_0) := \{&x \in C([0,T],H) \text{ with } x|_{(t_0,T)} \in W_{pq}(t_0,T): \\ &\exists f^x \in L^2(t_0,T;H) : x = x_0 \text{ on } [0,t_0] \text{ and} \\ &x'(t) + A(t,x(t)) = f^x(t) \in F(t,x(t)) \text{ a.e. on } (t_0,T)\}. \\ \mathcal{X}^F(T,x_0) := \{&x_0\} \end{aligned} \tag{2.4}$$

(ii) Given $E \subset H$, put

$$\begin{aligned} \mathcal{X}^E(t_0,x_0) := \{&x \in C([0,T],H) : x|_{(t_0,T)} \in W_{pq}(t_0,T) \text{ and } \exists f \in E_{L_2}: \\ &x'(t) + A(t,x(t)) = f(t) \text{ a.e. on } (t_0,T), x = x_0 \text{ on } [0,t_0]\}. \end{aligned} \tag{2.5}$$

(iii) Given $c \geq 0$, define the multifunction $B_c : [0,T] \times C([0,T],H) \rightsquigarrow H$ by

$$B_c(t_0,x_0) := \{y \in H : |y| \leq c \cdot (1 + \sup_{t \leq t_0} |x_0(t)|)\} \tag{2.6}$$

and put

$$\begin{aligned} \mathcal{X}^{B_c}(t_0,x_0) := \{&x \in C([0,T],H) \text{ with } x|_{(t_0,T)} \in W_{pq}(t_0,T): \\ &\exists f^x \in L^2(t_0,T;H) : x = x_0 \text{ on } [0,t_0] \text{ and} \\ &x'(t) + A(t,x(t)) = f^x(t) \in B_c(t,x) \text{ a.e. on } (t_0,T)\}. \end{aligned} \tag{2.7}$$

Remark 2.4. There should be no danger of confusion between (2.4) and (2.5), as in (2.4) the superscript F denotes a set-valued function whereas in (2.5) the superscript E denotes a set.

We omit the proof of the following a-priori estimates, as it is nearly identical to the proof of Lemma A.1 in [4] (the only differences are our weaker growth and coercivity hypotheses $\mathbf{H}(A)$ (iv) and (v) compared to the ones in section 2.2 of [4]).

Lemma 2.5. *Let $c \geq 0$ and $r \geq 0$. Then there is a constant $C = C(c,r) > 0$ such that, for all $(t_0,x_0) \in [0,T) \times C([0,T],H)$ with $\sup_{t \leq t_0} |x_0(t)| \leq r$ and for all $x \in \mathcal{X}^{B_c}(t_0,x_0)$ with corresponding function $f^x \in L^2(t_0,T;H)$, we have*

$$\|x\|_\infty + \|x\|_{W_{pq}(t_0,T)} + \|\hat{A}x\|_{L^q(t_0,T;V^*)} + \|f^x\|_{L^2(t_0,T;H)} \leq C.$$

Here, $\hat{A}x : [t_0,T] \to V^*$ is defined by $\hat{A}x(t) = A(t,x(t))$.

Lemma 2.6. *Let $c \geq 0$ and $(t_0, x_0) \in [0,T) \times C([0,T],H)$. Then $\mathcal{X}^{B_c}(t_0, x_0)$ is precompact in $C([0,T],H)$. In particular, for each sequence $(x_n)_n$ in $\mathcal{X}^{B_c}(t_0,x_0)$ with corresponding sequence $(f^{x_n})_n$ in $L^2(t_0,T;H)$, there exist a pair $(x,f) \in C([0,T],H) \times L^2(t_0,T;H)$ with $x|_{(t_0,T)} \in W_{pq}(t_0,T)$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x$ in $C([0,T],H)$, $x_{n_k} \xrightarrow{w} x$ in $W_{pq}(t_0,T)$, $f^{x_{n_k}} \xrightarrow{w} f$ in $L^2(t_0,T;H)$, and $x'(t) + A(t,x(t)) = f(t)$ a.e. on (t_0,T).*

Proof. The proof is essentially the same as the proof of Lemma A.2 in [4] with the exception of the following two points (both due to A only being locally monotone as required in $\mathbf{H}(A)$ (ii), which is weaker than monotonicty). First, instead of the usual monotonicity trick (p. 474 in [22]), one should use the "modified monotonicity trick" (Lemma 2.5 in [15]). The second point is about the convergence of (x_{n_k}) to x in $C([0,T],H)$ assuming that (2.1) holds and that the remaining parts of the lemma's statement are already established. We show this convergence by slightly adjusting part (ii) of the proof of Theorem 1.1 in [15]. Comparing with the proof of Lemma A.2 in [4], we replace (A.9) in [4] by

$$
\begin{aligned}
\frac{1}{2}\left|x_{n_k}(t) - x(t)\right|^2 &= \int_{t_0}^{t} -\langle A(s, x_{n_k}(s)) - A(s,x(s)), x_{n_k}(s) - x(s)\rangle \\
&\qquad + (f^{x_{n_k}}(s) - f^{x}(s), x_{n_k}(s) - x(s))\, ds \\
&\leq \int_{t_0}^{t} (c_0 + \rho(x_{n_k}(s)) + \eta(x(s)))\left|x_{n_k}(s) - x(s)\right|^2\, ds \\
&\qquad + 2C\|x_{n_k} - x\|_{L^2(t_0,T;H)}
\end{aligned}
\tag{2.8}
$$

This is correct in our setting thanks to $\mathbf{H}(A)$ (ii). The constant C in (2.8) is the same as in Lemma 2.5. Since (2.8) holds for every $t \in [t_0,T]$, Gronwall's lemma together with (2.1) and Lemma 2.5 yield

$$
\begin{aligned}
\sup_{t\in[t_0,T]} \left|x_{n_k}(t) - x(t)\right|^2 &\leq 2C\|x_{n_k} - x\|_{L^2(t_0,T;H)} e^{\int_{t_0}^{T}(c_0+\rho(x_{n_k}(s))+\eta(x(s)))\,ds} \\
&\leq 2C\tilde{C}\|x_{n_k} - x\|_{L^2(t_0,T;H)} \to 0 \text{ as } k \to \infty
\end{aligned}
$$

for some constant $\tilde{C}$ that is independent from t and k. □

Theorem 2.7. *Fix $(t_*, x_*) \in [0,T] \times C([0,T],H)$. Let $(t_n,x_n)_n$ be a sequence in $[t_*,T] \times \mathcal{X}^{B_{(c_F)+1}}(t_*,x_*)$ that converges to some pair $(t_0,x_0) \in [t_*,T] \times C([0,T],H)$. Then every sequence $(\tilde{x}_n)_n$ with $\tilde{x}_n \in \mathcal{X}^F(t_n,x_n)$ has a convergent subsequence with limit in $\mathcal{X}^F(t_0,x_0)$.*

Proof. Fix a sequence $(\tilde{x}_n)_n$ with $\tilde{x}_n \in \mathcal{X}^F(t_n,x_n)$. By (2.2) in $\mathbf{H}(F)$ (ii), we have $\mathcal{X}^F(t_n,x_n) \subset \mathcal{X}^{B_{(c_F)+1}}(t_*,x_*)$, i.e., $\tilde{x}_n = x_*$ on $[0,t_*]$ and $\tilde{x}_n'(t) + A(t,\tilde{x}_n(t)) = f^{\tilde{x}_n}(t) \in B_{(c_F)+1}(t,\tilde{x}_n)$ a.e. on (t_*,T) for some $f^{\tilde{x}_n} \in L^2(t_*,T;H)$. Hence, by Lemma 2.6, $(t_n,x_n,\tilde{x}_n)_n$ has a subsequence, which we still denote by $(t_n,x_n,\tilde{x}_n)_n$, such that $\tilde{x}_n \to \tilde{x}_0$ in $C([0,T],H)$ and $f^{\tilde{x}_n} \xrightarrow{w} f^{\tilde{x}_0}$ in $L^2(t_*,T;H)$ for some pair $(\tilde{x}_0, f^{\tilde{x}_0}) \in C([0,T],H) \times L^2(t_*,T;H)$ that additionally satisfies $\tilde{x}_0'(t) + A(t,\tilde{x}_0(t)) = f^{\tilde{x}_0}(t)$ a.e. on (t_*,T), $\tilde{x}_0 = x_*$ on $[0,t_*]$, and $\tilde{x}_0|_{(t_*,T)} \in W_{pq}(t_*,T)$. Clearly, we also have $\tilde{x}_0 = x_0$ on $[0,t_0]$. Next, let $m \in \mathbb{N}$ with $1/m \in (0,T-t_0)$. Without loss of generality, assume that $t_n < t_0 + 1/m$ for all $n \geq m$, i.e., $f^{\tilde{x}_n}(t) \in F(t,\tilde{x}_n(t))$ a.e. on $(t_0+1/m,T)$ whenever $n \geq m$. Hence, by Lemma 2.6.2 in [7], $f^{\tilde{x}_0}(t) \in F(t,\tilde{x}_0(t))$ a.e. on $(t_0+1/m,T)$. Finally, the arbitrariness of m yields $\tilde{x}_0 \in \mathcal{X}^F(t_0,x_0)$. □

We will very often use the following regularity condition[1] for a function $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$ that is slightly weaker than lower semicontinuity:

$$\forall (t_*, x_*) \in [0,T] \times C([0,T],H) : \; u|_{[t_*,T] \times \mathcal{X}^{B(c_F)+1}(t_*,x_*)} \text{ is l.s.c.} \tag{2.9}$$

3. Viability

Given $(t_0, x_0) \in [0,T) \times C([0,T],H)$, consider the evolution inclusion

$$\begin{aligned} x'(t) + A(t, x(t)) &\in F(t, x(t)) \text{ a.e. on } (t_0, T), \\ x &= x_0 \text{ on } [0, t_0]. \end{aligned} \tag{3.1}$$

Definition 3.1. A set $K \subset H$ is *viable for* (3.1) if, for every $(t_0, x_0) \in [0,T) \times C([0,T],H)$ with $x_0(t_0) \in K$, there exists an $x \in \mathcal{X}^F(t_0, x_0)$ such $x(t) \in K$ for all $t \in [t_0, T]$.[2]

The next definition is adapted from Remark 10.1 of [8].

Definition 3.2. Fix $K \subset H$. Let $(t_0, x_0) \in [0,T] \times C([0,T],H)$ with $x_0(t_0) \in K$. A non-empty set $E \subset H$ is *A-quasi-tangent to K at* (t_0, x_0) if there are sequences $(\delta_n)_n$ in $\mathbb{R}_+$ with $\delta_n \downarrow 0$, $(b_n)_n$ in $L^2(0,T;H)$, $(p_n)_n$ in $B(0,1/n)_{L^2}$, and $(x_n)_n$ in $C([0,T],H)$ with $x_n|_{(t_0,T)} \in W_{pq}(t_0,T)$ such that

$$\begin{aligned} x_n'(t) + A(t, x_n(t)) &= b_n(t) + p_n(t) \text{ a.e. on } (t_0, T), \\ b_n(t) &\in E \text{ a.e. on } (t_0, t_0 + \delta_n), \\ x_n &= x_0 \text{ on } [0, t_0], \text{ and} \\ x_n(t_0 + \delta_n) &\in K. \end{aligned}$$

We write $\mathcal{QTS}_K^A(t_0, x_0)$ for the class of all A-quasi-tangent sets to K at (t_0, x_0).

Our main result on necessary and sufficient conditions of viability for (3.1) can be found below in subsection 3.3.

Next, we study viability for an extension of (3.1), which is useful for applications to partial differential equations in section 4. Given $(t_0, x_0, y_0) \in [0,T) \times C([0,T],H) \times \mathbb{R}$, consider the system

$$\begin{aligned} s'(t) &= 1 && \text{on } (t_0, T), \\ x'(t) + A(t, x(t)) &\in F(t, x(t)) && \text{a.e. on } (t_0, T), \\ y'(t) &= 0 && \text{on } (t_0, T), \\ (s(t_0), x|_{[0,t_0]}, y(t_0)) &= (t_0, x_0|_{[0,t_0]}, y_0). \end{aligned} \tag{3.2}$$

Definition 3.3. The epigraph of a function $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$ is *viable for* (3.2) if, for every $(t_0, x_0, y_0) \in \operatorname{epi} u$ with $t_0 < T$, there exists an $x \in \mathcal{X}^F(t_0, x_0)$ such that $(t, x, y_0) \in \operatorname{epi} u$ for all $t \in [t_0, T]$.

The next definition extends Definition 3.2 in the spirit of Definition 4.4 of [14].

Definition 3.4. Fix $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$. Let $(t_0, x_0, y_0) \in \operatorname{epi} u$ with $t_0 < T$. A non-empty set $E \subset H$ is *A-quasi-tangent to* $\operatorname{epi} u$ *at* (t_0, x_0, y_0)

[1]Recall the constant c_F from (2.2) in $\mathbf{H}(F)$ (ii) and $\mathcal{X}^{B_c}$ from (2.7).

[2]Recall the definition of $\mathcal{X}^F(t_0, x_0)$ from (2.4).

if there are sequences $(\delta_n)_n$ in $\mathbb{R}_+$ with $\delta_n \downarrow 0$, $(b_n)_n$ in $L^2(0,T;H)$, $(p_n)_n$ in $B(0,1/n)_{L^2}$, and $(x_n)_n$ in $C([0,T],H)$ with $x_n|_{(t_0,T)} \in W_{pq}(t_0,T)$ such that

$$\begin{aligned} x_n'(t) + A(t,x_n(t)) &= b_n(t) + p_n(t) \text{ a.e. on } (t_0,T),\\ b_n(t) &\in E \text{ a.e. on } (t_0,t_0+\delta_n),\\ x_n &= x_0 \text{ on } [0,t_0], \text{ and}\\ u(t_0+\delta_n,x_n) &\leq y_0 + \delta_n/n. \end{aligned} \tag{3.3}$$

We write $\mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,y_0)$ for the class of all A-quasi-tangent sets to $\operatorname{epi} u$ at (t_0,x_0,y_0).

Lemma 3.5. *Let $K \subset H$. Consider $u : [0,T] \times C([0,T],H) \to \mathbb{R}$ defined by $u(t,x) := -\mathbf{1}_K(x(t))$.*[3] *Then the following holds:*

(i) The set K is viable for (3.1) *if and only if* $\operatorname{epi} u$ *is viable for* (3.2).

(ii) Let E be a non-empty subset of H. Then $E \in \mathcal{QTS}^A_K(t_0,x_0)$ for each $(t_0,x_0) \in [0,T)\times C([0,T],H)$ with $x_0(t_0) \in K$ if and only if $E \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,y_0)$ for each $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$.

Proof. (i) First, assume that K is viable for (3.1). Fix $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$. If $y_0 \geq 0$, then $(t,x,y_0) \in \operatorname{epi} u$ on $[t_0,T]$ for all $x \in \mathcal{X}^F(t_0,x_0)$. If $y_0 \in [-1,0)$, then $u(t_0,x_0) = -1$, i.e., $x_0 \in K$ and thus there is an $x \in \mathcal{X}^F(t_0,x_0)$ such that $x(t) \in K$ on $[t_0,T]$, which yields $y_0 \geq u(t,x)$ on $[t_0,T]$.

Next, assume that $\operatorname{epi} u$ is viable for (3.2). Fix $(t_0,x_0) \in [0,T) \times C([0,T],H)$ with $x_0(t_0) \in K$. Since $(t_0,x_0,u(t_0,x_0)) \in \operatorname{epi} u$, there is an $x \in \mathcal{X}^F(t_0,x_0)$ such that $-1 = u(t_0,x_0) \geq u(t,x) = -\mathbf{1}_K(x(t))$ on $[t_0,T]$, i.e., K is viable for (3.1).

(ii) First, let $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$ and assume that $E \in \mathcal{QTS}^A_K(t_*,x_*)$ for each $(t_*,x_*) \in [0,T) \times C([0,T],H)$ with $x_*(t_*) \in K$. If $x_0(t_0) \in K$, then $E \in \mathcal{QTS}^A_K(t_0,x_0)$ with corresponding sequences $(\delta_n)_n$, $(b_n)_n$, $(p_n)_n$, and $(x_n)_n$ from Definition 3.2, which satisfy $x_n(t_0+\delta_n) \in K$ for all $n \in \mathbb{N}$, and consequently, we have $-1 = u(t_0+\delta_n,x_n) \leq y_0 + \delta_n/n$ because $y_0 \geq -1$ due to $x_0(t_0) \in K$, i.e., $E \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,y_0)$. If $x_0(t_0) \notin K$, then $y_0 \geq 0$ and thus $u(t_0+\delta_n,x_n) \leq y_0 + \delta_n/n$ from (3.3) is automatically satisfied for all possible sequences $(\delta_n)_n$ and $(x_n)_n$ from Definition 3.4 (we only need to invoke standard existence results for evolution equations such as Theorem 1.1 in [15] to ensure the existence of at least one such sequence $(x_n)_n$). In any case, we have $E \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,y_0)$.

Now, let $(t_0,x_0) \in [0,T) \times C([0,T],H)$ with $x_0(t_0) \in K$ and assume that $E \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,u(t_0,x_0))$ with corresponding sequences $(\delta_n)_n$, $(b_n)_n$, $(p_n)_n$, and $(x_n)_n$ from Definition 3.4. By (3.3) and $x_0(t_0) \in K$, we have $u(t_0+\delta_n,x_n) \leq u(t_0,x_0)+\delta_n/n = -1+\delta_n/n$ for each $n \in \mathbb{N}$. Since $\delta_n \downarrow 0$, we have $u(t_n,x_n) < 0$ and thus $x_n(t_0+\delta_n) \in K$ for all sufficiently large $n \in \mathbb{N}$. Hence, $E \in \mathcal{QTS}^A_K(t_0,x_0)$. □

3.1. **Necessary condition for viability for** (3.2).

Theorem 3.6. *Let $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$. Suppose that* $\operatorname{epi} u$ *is viable for* (3.2). *Then, for every $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$, we have*

$$F(t_0,x_0(t_0)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,y_0). \tag{3.4}$$

[3] If K is closed, then u is l.s.c.

Proof. We proceed as in the proof of Theorem 10.1 in [8]. Fix $(t_0, x_0, y_0) \in \operatorname{epi} u$ with $t_0 < T$. Since $\operatorname{epi} u$ is viable for (3.2), there is an $x \in \mathcal{X}^F(t_0, x_0)$ with corresponding selector f^x of $F(\cdot, x)$ such that $(t, x, y_0) \in \operatorname{epi} u$ for all $t \in [t_0, T]$. Let $n \in \mathbb{N}$. By upper semicontinuity of F and continuity of x, there is a $\delta_n \in (0, 1/n]$ such that, for all $t \in [t_0, t_0 + \delta_n]$, we have

$$F(t, x(t)) \subset F(t_0, x_0(t_0)) + B(0, 1/n).$$

Thus, by Lemma 10.1 and Remark 10.5, both in [8], we have $f^x = b_n + p_n$ a.e. on (t_0, T) for some $b_n \in L^2(0, T; H)$ and $p_n \in B(0, 1/n)_{L^2}$ with $b_n(t) \in F(t_0, x_0)$ a.e. on $(t_0, t_0 + \delta_n)$. Since $(\delta_n, b_n, p_n, x_n)$ with $x_n = x$ satisfies (3.3), we have (3.4). □

3.2. **Sufficient condition for viability for** (3.2)**.** See section 3.5 for the proof of the next result.

Theorem 3.7. *Let* $u : [0, T] \times C([0, T], H) \to \mathbb{R} \cup \{+\infty\}$ *satisfy* (2.9)*. If, for every* $(t_0, x_0, y_0) \in \operatorname{epi} u$ *with* $t_0 < T$*, we have*

$$F(t_0, x_0(t_0)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0, x_0, y_0),$$

then $\operatorname{epi} u$ *is viable for* (3.2)*.*

3.3. **Necessary and sufficient condition for viability for** (3.1)**.** The next result follows immediately from Theorems 3.6, 3.7, and Lemma 3.5.

Corollary 3.8. *Let K be a closed subset of H. Then K is viable if and only if, for every* $(t_0, x_0) \in [0, T) \times C([0, T], H)$ *with* $x_0(t_0) \in K$*, we have*

$$F(t_0, x_0(t_0)) \in \mathcal{QTS}^A_K(t_0, x_0).$$

Remark 3.9. If $K = H$, then, by Theorem 1.1 of [15], which guarantees existence of solutions for locally monotone evolution equations, we have $F(t_0, x_0(t_0)) \in \mathcal{QTS}^A_K(t_0, x_0)$. Thus Corollary 3.8 is also an existence result for evolution inclusions.

3.4. ε**-approximate solutions.** The next definition is an appropriate modification of Definition 12.1 in [8] (see also Definition 3 in [12] and Definition 4.9 in [14] for stochastic cases).

Definition 3.10. Let $\varepsilon \in (0, 1]$, $u : [0, T] \times C([0, T], H) \to \mathbb{R} \cup \{+\infty\}$, and $(t_0, x_0, y_0) \in \operatorname{epi} u$ with $t_0 < T$. We call a quintuple (τ, ϱ, f, g, x) an ε*-approximate solution of* (3.2) *for* $\operatorname{epi} u$ *starting at* (t_0, x_0, y_0) if the following holds:

(i) $\tau \in (t_0, T]$.

(ii) $\varrho : [t_0, \tau] \to [t_0, \tau]$ is non-decreasing and we have $t - \varepsilon \le \varrho(t) \le t$ for all $t \in [t_0, \tau]$ as well as $\varrho(\tau) = \tau$.

(iii) $f \in L^2(0, T; H)$ with $f(t) = 0$ a.e. on (τ, T).

(iv) $g \in B(0, \varepsilon)_{L^2}$ with $g(t) = 0$ a.e. on (τ, T).

(v) $x \in C([0, T], H)$ satisfies $x|_{(t_0, T)} \in W_{pq}(t_0, T)$ and

$$\begin{aligned} x'(t) + A(t, x(t)) &= f(t) + g(t) \text{ a.e. on } (t_0, T), \\ f(t) &\in F(\varrho(t), x(\varrho(t))) \text{ a.e. on } (t_0, \tau), \\ x &= x_0 \text{ on } [0, t_0]. \end{aligned} \tag{3.5}$$

(vi) $u(\varrho(t), x) \le y_0 + \varepsilon \cdot (t - t_0)$ for all $t \in [t_0, \tau]$.

The next result is an adjustment of Lemma 12.1 in [8] (see Remark 3.12 for more details).

Proposition 3.11. *Let $\varepsilon > 0$. Let $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$ satisfy (2.9). Suppose that, for every $(t,x,y) \in \operatorname{epi} u$ with $t < T$,*

$$F(t,x(t)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t,x,y). \tag{3.6}$$

Then, for every $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$, there exists an ε-approximate solution (τ,ϱ,f,g,x) of (3.2) for $\operatorname{epi} u$ starting at (t_0,x_0,y_0) such that $\tau = T$.

Proof. Fix $(t_0,x_0,y_0) \in \operatorname{epi} u$ with $t_0 < T$. Denote by $\mathcal{S}$ the set of ε-approximate solution of (3.2) for $\operatorname{epi} u$ starting at (t_0,x_0,y_0). Given $\mathfrak{s}_1 = (\tau_1,\varrho_1,f_1,g_1,x_1)$, $\mathfrak{s}_2 = (\tau_2,\varrho_2,f_2,g_2,x_2) \in \mathcal{S}$, we write $\mathfrak{s}_1 \preceq \mathfrak{s}_2$ if $\tau_1 \le \tau_2$, $(\varrho_1,x_1) = (\varrho_2,x_2)$ on $[t_0,\tau_1]$, and $(f_1,g_1) = (f_2,g_2)$ a.e. on (t_0,τ_1). Note that $\preceq$ defines a preorder on $\mathcal{S}$. In Steps 2 and 3 below, we shall also use the function $\mathcal{N} : \mathcal{S} \to [t_0,T]$ defined by $\mathcal{N}(\tau,\varrho,f,g,x) := \tau$. Moreover, we shall call $\mathfrak{s} \in \mathcal{S}$ an $\mathcal{N}$*-maximal* element of $\mathcal{S}$ if $\mathfrak{s} \preceq \mathfrak{s}_+ \in \mathcal{S}$ yields $\mathcal{N}(\mathfrak{s}) = \mathcal{N}(\mathfrak{s}_+)$ (section 2.1 in [7]).

Step 1 (existence of ε-approximate solutions). By (3.6) and Definition 3.4, there are $\delta \in (0,\varepsilon]$, $f \in L^2(0,T;H)$, $g \in B(0,\varepsilon)_{L_2}$, and $x \in C([0,T],H)$ with $x|_{(t_0,T)} \in W_{pq}(t_0,T)$ such that $u(t_0+\delta,x) \le y_0 + \varepsilon\delta$ and

$$\begin{aligned} x'(t) + A(t,x(t)) &= f(t) + g(t) \text{ a.e. on } (t_0,T), \\ f(t) &\in F(t_0,x_0(t_0)) \text{ a.e. on } (t_0,t_0+\delta), \\ x &= x_0 \text{ on } [0,t_0]. \end{aligned}$$

Put $\tau := t_0 + \delta$. Define $\varrho : [t_0,\tau] \to [t_0,\tau]$ by $\varrho(t) := t_0$ for $t \in [t_0,\tau)$ and $\varrho(\tau) := \tau$. Finally, consider a solution[4] $\tilde{x} \in C([0,T],H)$ with $\tilde{x}|_{(\tau,T)} \in W_{pq}(\tau,T)$ of

$$\begin{aligned} \tilde{x}'(t) + A(t,\tilde{x}(t)) &= 0 \text{ a.e. on } (\tau,T), \\ \tilde{x} &= x \text{ on } [0,\tau] \end{aligned}$$

Then $(\tau,\varrho,\mathbf{1}_{(0,\tau)} \cdot f, \mathbf{1}_{(0,\tau)} \cdot g, \mathbf{1}_{[0,\tau]} \cdot x + \mathbf{1}_{(\tau,T]} \cdot \tilde{x}) \in \mathcal{S}$.

Step 2 (existence of maximal ε-approximate solutions). Consider an increasing sequence $(\mathfrak{s}_n)_{n\ge 1} = (\tau_n,\varrho_n,f_n,g_n,x_n)_{n\ge 1}$ in $\mathcal{S}$. We show that this sequence is bounded from above. Then the Brezis–Browder principle (Theorem 2.1.1 in [7]) will yield the existence of an $\mathcal{N}$-maximal element in $\mathcal{S}$.

To establish boundedness of $(\mathfrak{s}_n)_n$, note first that

$$\begin{aligned} &\tau_n \uparrow \tau := \sup_m \tau_m, \\ &f_n(t) \to f(t) := \sum_{m=1}^{\infty} \mathbf{1}_{(\tau_{m-1},\tau_m]}(t)\, f_m(t) \text{ a.e. on } (0,T), \\ &g_n(t) \to g(t) := \sum_{m=1}^{\infty} \mathbf{1}_{(\tau_{m-1},\tau_m]}(t)\, g_m(t) \text{ a.e. on } (0,T), \end{aligned} \tag{3.7}$$

where $\tau_0 := t_0$. Next, note that $x_n \in \mathcal{X}^{B_c}(t_0,x_0)$ with $c = \varepsilon + c_F$ for each $n \in \mathbb{N}$ according to (3.5), (2.2), (2.6), and (2.7). Thus, by Lemma 2.6, $(\mathfrak{s}_n)_n$ has a subsequence $(\mathfrak{s}_{n_k})_k$ such that

$$\begin{aligned} &x_{n_k} \to x \text{ in } C([0,T],H) \text{ as well as in } W_{pq}(t_0,T) \text{ and} \\ &f_{n_k} + g_{n_k} \xrightarrow{w} \tilde{f} \text{ in } L^2(t_0,T;H) \text{ as } k \to \infty \end{aligned} \tag{3.8}$$

[4]Such a solution exists thanks to Theorem 1.1 in [15].

for some $(x,\tilde{f}) \in C([0,T],H) \times L^2(t_0,T;H)$ with $x = x_0$ on $[0,t_0]$, $x|_{(t_0,T)} \in W_{pq}(t_0,T)$, and $x'(t) + A(t,x(t)) = \tilde{f}(t)$ a.e. on (t_0,T). By (3.7), $\tilde{f} = f + g$ a.e. on (t_0,T). Next, define $\varrho : [t_0,\tau] \to [t_0,\tau]$ by

$$\varrho(t) := \mathbf{1}_{\{t_0\}}(t) \cdot \varrho_1(t) + \sum_{n=1}^{\infty} \mathbf{1}_{(\tau_{n-1},\tau_n] \cap \{\tau\}^c}(t) \cdot \varrho_n(t) + \mathbf{1}_{\{\tau\}}(t) \cdot \tau.$$

Now, we show that $f(t) \in F(\varrho(t), x(\varrho(t))$ a.e. on (t_0,τ). To this end, fix an arbitrary $\delta \in (0, \tau - t_0)$ and an $m \in \mathbb{N}$ such that $\tau_m > \tau - \delta$. As $f_n(t) \in F(\varrho_n(t), x(\varrho_n(t))$ a.e. on $(t_0, \tau - \delta)$ for each $n \geq m$ and as F is u.s.c. and is non-empty, convex, and closed valued, we only need, by Lemma 2.6.2 in [7], to show that

$$(3.9) \qquad (\varrho_{n_k}(t), x_{n_k}(\varrho_{n_k}(t)) \to (\varrho(t), x(\varrho(t)) \text{ a.e. on } (t_0, \tau - \delta) \text{ as } k \to \infty$$

in order to obtain $f(t) \in F(\varrho(t), x(\varrho(t))$ a.e. on $(t_0, \tau - \delta)$. Indeed, as we have $(\varrho_n(t), x_n(\varrho_n(t)) = (\varrho(t), x_n(\varrho(t))$ for every $t \in [t_0, \tau - \delta]$ and $n \geq m$, (3.8) yields (3.9). Thus, as δ was arbitrary in $(0, \tau - t_0)$, $f(t) \in F(\varrho(t), x(\varrho(t))$ a.e. on (t_0,τ). Also note that $x_{n_k} = x$ on $[0, \tau_{n_k}]$ for each $k \in \mathbb{N}$, as $(\mathfrak{s}_n)_n$ is increasing and because of (3.8). Thus $u(\varrho(t), x) \leq y_0 + \varepsilon \cdot (t - t_0)$ for all $t \in [t_0, \tau)$. By (2.9) and (2.2),

$$u(\varrho(\tau), x) = u(\tau, x) \leq \varliminf_{k} u(\tau_{n_k}, x_{n_k}) = \varliminf_{k} u(\varrho_{n_k}(\tau_{n_k}), x_{n_k}) \leq y_0 + \varepsilon \cdot (\tau - t_0).$$

We conclude that $\mathfrak{s} := (\tau, \varrho, f, g, x) \in \mathcal{S}$ and $\mathfrak{s}_{n_k} \preceq \mathfrak{s}$ for all $k \in \mathbb{N}$. Also note that, for each $m \in \mathbb{N}$, there is a $k \in \mathbb{N}$ with $n_k \geq m$ and thus $\mathfrak{s}_m \preceq \mathfrak{s}_{n_k} \preceq \mathfrak{s}$. Consequently, as pointed out at the beginning of Step 2, there exists an $\mathcal{N}$-maximal element in $\mathcal{S}$.

Step 3 (extension step). Let $\mathfrak{s}_0 = (\tau_0, \varrho_0, f_0, g_0, x_0)$ be an $\mathcal{N}$-maximal element of $\mathcal{S}$ with $\tau_0 < T$. In particular, we have $(\tau_0, x_0, y_0 + \varepsilon \cdot (\tau_0 - t_0)) \in \operatorname{epi} u$. Thus, by (3.6) and Definition 3.4, there is a quadruple

$$(\delta, b, p, x) \in (0, \varepsilon] \times L^2(0,T;H) \times B(0,\varepsilon)_{L^2} \times C([0,T],H)$$

with $x|_{(\tau_0,T)} \in W_{pq}(t_0,T)$, $x'(t) + A(t,x(t)) = b(t) + p(t)$ a.e. on (τ_0,T), $b(t) \in F(\tau_0, x_0(\tau_0))$ a.e. on $(\tau_0, \tau_0 + \delta)$, $x = x_0$ on $[0,\tau_0)$, and

$$u(\tau_0 + \delta, x) \leq y_0 + \varepsilon \cdot (\tau_0 - t_0) + \varepsilon \cdot \delta.$$

Next, just as in Step 1, let $\tilde{x} \in C([0,T],H)$ with $\tilde{x}|_{(\tau_0+\delta,T)} \in W_{pq}(\tau_0 + \delta, T)$ be a solution of $\tilde{x}'(t) + A(t, \tilde{x}(t)) = 0$ a.e. on $(\tau_0 + \delta, T)$ with initial condition $\tilde{x} = x$ on $[0, \tau_0 + \delta]$. Then

$$\begin{aligned}
\mathfrak{s}_0 \preceq \mathfrak{s}_+ := (&\tau_0 + \delta, \mathbf{1}_{[t_0,\tau_0]}(t) \cdot \varrho_0 + \mathbf{1}_{(\tau_0,\tau_0+\delta)} \cdot \tau_0 + \mathbf{1}_{\{\tau_0+\delta\}} \cdot (\tau_0 + \delta), \\
&\mathbf{1}_{(0,\tau_0)} \cdot f + \mathbf{1}_{(\tau_0,\tau_0+\delta)} \cdot b, \mathbf{1}_{(0,\tau_0)} \cdot g_0 + \mathbf{1}_{(\tau_0,\tau_0+\delta)} \cdot p, \\
&\mathbf{1}_{[0,\tau_0]} \cdot x_0 + \mathbf{1}_{(\tau_0,\tau_0+\delta]} \cdot x + \mathbf{1}_{(\tau_0+\delta,T]} \cdot \tilde{x}) \in \mathcal{S}
\end{aligned}$$

but $\mathcal{N}(\mathfrak{s}_0) \neq \mathcal{N}(\mathfrak{s}_+)$, which contradicts the $\mathcal{N}$-maximality of $\mathfrak{s}_0$. Thus, $\tau_0 = T$. $\square$

Remark 3.12. The main difference of the previous proof to corresponding places in [8] (see section 12 therein) can be found in Step 2 of the proof of Proposition 3.11, where we use a compactness argument to obtain an extension of the "x-components" of $(\tau_n, \varrho_n, f_n, g_n, x_n)_n$. This is possible thanks to our operator A being coercive, which is not assumed in [8], Note that in [8] the operator A generates a C_0-semigroup. This allows for different arguments in [8].

3.5. **Proof of Theorem 3.7.** Fix $(t_0, x_0, y_0) \in \operatorname{epi} u$ with $t_0 < T$. Then, for each $n \in \mathbb{N}$, there exists, by Proposition 3.11, an $(1/n)$-approximate solution $\mathfrak{s}_n = (\tau_n, \varrho_n, f_n, g_n, x_n)$ of (3.2) for $\operatorname{epi} u$ starting at (t_0, x_0, y_0) with $\tau_n = T$. Note that

$$\varrho_n(t) \to t \text{ on } [t_0, T] \text{ and } g_n(t) \to 0 \text{ a.e. on } (t_0, T). \tag{3.10}$$

Lemma 2.6 yields (cf. Step 2 of the proof of Proposition 3.11) the existence of a subsequence $(\mathfrak{s}_{n_k})_k$ of $(\mathfrak{s}_n)_n$ such that $x_{n_k} \to x$ in $C([0,T], H)$ as well as in $W_{pq}(t_0, T)$ and $f_{n_k} + g_{n_k} \xrightarrow{w} f$ in $L^2(t_0, T; H)$ for some $(x, f) \in C([0,T], H) \times L^2(t_0, T; H)$ with $x|_{(t_0,T)} \in W_{pq}(t_0, T)$, $x = x_0$ on $[0, t_0]$, and $x'(t) + A(t, x(t)) = f(t)$ a.e. on (t_0, T). We can invoke now Lemma 2.6.2 in [7] (cf. Step 2 of the proof of Proposition 3.11) to obtain $f(t) \in F(t, x(t))$ a.e. on (t_0, T) because $f_{n_k} \xrightarrow{w} f$ in $L^2(t_0, T; H)$ and $(\varrho_{n_k}(t), x_{n_k}(\varrho_{n_k}(t))) \to (t, x(t))$ a.e. on (t_0, T) due to (3.10) and to

$$\begin{aligned} &\|x(\cdot \wedge t) - x_{n_k}(\cdot \wedge \varrho_{n_k}(t))\|_\infty \\ &\quad \le \|x(\cdot \wedge t) - x(\cdot \wedge \varrho_{n_k}(t))\|_\infty + \|x(\cdot \wedge \varrho_{n_k}(t)) - x_{n_k}(\cdot \wedge \varrho_{n_k}(t))\|_\infty \\ &\quad \le \omega(1/n_k) + \|x - x_{n_k}\|_\infty \to 0 \text{ for every } t \in [t_0, T], \end{aligned} \tag{3.11}$$

where ω is a modulus of continuity of x. Finally, by (2.9) and (2.2),

$$u(t, x) \le \varliminf_k u(\varrho_{n_k}(t), x_{n_k}) \le \varliminf_k [y + (1/n_k) \cdot (t - t_0)] = y,$$

as $\mathbf{d}_\infty(\varrho_{n_k}(t), x_{n_k}; t, x) \to 0$ due to (3.11). This concludes the proof. □

4. Hamilton–Jacobi–Bellman equations and optimal control

In this section, we additionally assume that F has bounded values.

Fix an l.s.c. *terminal cost function* $h : C([0,T], H) \to \mathbb{R} \cup \{+\infty\}$. We consider the following optimal control problem.

> **(OC).** Given $(t_0, x_0) \in [0, T) \times C([0,T], H)$, find a solution $\tilde{x}(\cdot)$ of evolution inclusion (1.1) with initial condition $\tilde{x} = x_0$ on $[0, t_0]$, i.e., find an element $\tilde{x} \in \mathcal{X}^F(t_0, x_0)$, such that
> $$h(\tilde{x}) = \inf\{h(x) : x \in \mathcal{X}^F(t_0, x_0)\}.$$

The *value function* $v : [0,T] \times C([0,T], H) \to \mathbb{R} \cup \{+\infty\}$ of **(OC)** is defined by

$$v(t_0, x_0) := \inf\{h(x) : x \in \mathcal{X}^F(t_0, x_0)\}. \tag{4.1}$$

Formally, v is a solution of the terminal-value problem

$$\begin{aligned} &-\partial_t u + \langle A(t, x(t)), \partial_x u\rangle - \inf_{f \in F(t, x(t))} (f, \partial_x u) = 0 \text{ in } [0, T) \times C([0,T], H), \\ &u(T, x) = h(x) \text{ on } C([0,T], H). \end{aligned} \tag{4.2}$$

Remark 4.1. Fix a topological space P, a function $f : [0,T] \times H \times P \to H$, and denote by $\mathcal{A}$ the set of all Borel measurable functions from $[0,T]$ to P. Consider a Mayer problem in the more usual form of finding a control $\tilde{a} \in \mathcal{A}$ such that

$$h(x^{t_0, x_0, \tilde{a}}) = \inf\{h(x^{t_0, x_0, a}) : a \in \mathcal{A}\} \quad \text{given } (t_0, x_0) \in [0, T) \times C([0,T], H).$$

Here, $x = x^{t_0, x_0, a}$ solves

$$\begin{aligned} x'(t) + A(t, x(t)) &= f(t, x(t), a(t)) \text{ a.e. on } (t_0, T), \\ x &= x_0 \text{ on } [0, t_0]. \end{aligned}$$

Such a problem can, under appropriate assumptions, be formulated as our optimal control problem **(OC)** (for more details, see section 1 in [9] for the finite-dimensional case and Chapter IV in [13] for the infinite-dimensional case).

4.1. **Properties of the value function.**

Theorem 4.2. *The following holds:*

(i) For every $(t_0, x_0) \in [0,T) \times C([0,T], H)$*, there exists an optimal trajectory* $x \in \mathcal{X}^F(t_0, x_0)$*, i.e.,* $v(t_0, x_0) = h(x)$*.*

(ii) v *satisfies* (2.9).

(iii) v *satisfies the dynamic programming principle, i.e., if* $0 \le t_0 < t \le T$*, then*

(4.3) $\qquad v(t_0, x_0) = v(t, \tilde{x}_0)$ *for some* $\tilde{x}_0 \in \mathcal{X}^F(t_0, x_0)$ *independent of* t *and*

(4.4) $\qquad v(t_0, x_0) \le v(t, x)$ *for all* $x \in \mathcal{X}^F(t_0, x_0)$*.*

Proof. (i) follows from the compactness of $\mathcal{X}^F(t_0, x_0)$ (Theorem 2.7) and from h being l.s.c.

(ii) Fix $(t_*, x_*) \in [0,T] \times C([0,T], H)$ and $(t_0, x_0) \in [t_*, T] \times \mathcal{X}^{B_{(c_F)+1}}(t_*, x_*)$. Let $(t_n, x_n)_n$ be a sequence in $[t_*, T] \times \mathcal{X}^{B_{(c_F)+1}}(t_*, x_*)$ that converges to (t_0, x_0). By part (i), there is a sequence $(\tilde{x}_n)_n$ with $\tilde{x}_n \in \mathcal{X}^F(t_n, x_n)$ such that $v(t_n, x_n) = h(\tilde{x}_n)$ for each $n \in \mathbb{N}$. By Theorem 2.7, this sequence has a subsequence $(\tilde{x}_{n_k})_k$ that converges to some $\tilde{x}_0 \in \mathcal{X}^F(t_0, x_0)$. Hence, thanks to h being l.s.c. and (4.1),

$$\varliminf_k v(t_{n_k}, x_{n_k}) = \varliminf_k h(\tilde{x}_{n_k}) \ge h(\tilde{x}_0) \ge v(t_0, x_0),$$

i.e., v satisfies (2.9).

(iii) By part (i), $v(t_0, x_0) = h(\tilde{x}_0)$ for some $\tilde{x}_0 \in \mathcal{X}^F(t_0, x_0)$ that does not depend on t. Let $x \in \mathcal{X}^F(t_0, x_0)$. Then, again by part (i), we have $v(t, x) = h(\tilde{x})$ for some $\tilde{x} \in \mathcal{X}^F(t, x) \subset \mathcal{X}^F(t_0, x_0)$. Thus, by (4.1), $v(t_0, x_0) = h(\tilde{x}_0) \le h(\tilde{x})$, i.e., (4.4) holds. Finally, note that, by part (i), $v(t, \tilde{x}_0) = h(\tilde{x}_1)$ for some $\tilde{x}_1 \in \mathcal{X}^F(t, \tilde{x}_0)$. Since $\tilde{x}_0 \in \mathcal{X}^F(t, \tilde{x}_0)$, we have, by (4.1), $v(t, \tilde{x}_0) = h(\tilde{x}_1) \le h(\tilde{x}_0) = v(t_0, x_0)$. Together with (4.4), (4.3) follows. □

4.2. **Quasi-contingent solutions.** We define the *A-contingent epiderivatives* $D^A_\uparrow$ of a function $u : [0,T] \times C([0,T], H) \to \mathbb{R} \cup \{+\infty\}$ at a point $(t, x) \in \operatorname{dom} u$ with $t < T$ in a multi-valued direction $E \subset H$ by[5]

$$D^A_\uparrow u(t,x)\,(E) := \sup_{\varepsilon > 0} \inf \left\{ \frac{u(t+\delta, \tilde{x}) - u(t,x)}{\delta} : \delta \in (0, \varepsilon],\, \tilde{x} \in \mathcal{X}^{E + B(0,\varepsilon)}(t,x) \right\}.$$

Remark 4.3. Similar (second-order) contingent epiderivatives with directions being sets of stochastic processes have been introduced in Definition 5.1 of [14]. Also note that a finite-dimensional counterpart of $D^A_\uparrow u(t,x)\,(E)$ with $A = 0$ plays an important role in [10] (see subsection 4.2 therein) in order to establish equivalence between viscosity and minimax solutions for path-dependent Hamilton–Jacobi

[5] Recall (2.5) for the definition of $\mathcal{X}^{E+B(0,\varepsilon)}$.

equations. Consider also the slightly different *lower and upper derivatives*

$$d_-^A\, u(t,x)\,(E) := \sup_{\varepsilon>0}\, \inf \left\{ \underline{\lim}_{\delta\downarrow 0} \frac{u(t+\delta,\tilde{x}) - u(t,x)}{\delta} \,:\, \tilde{x} \in \mathcal{X}^{E+B(0,\varepsilon)}(t,x) \right\},$$

$$d_+^A\, u(t,x)\,(E) := \inf_{\varepsilon>0}\, \sup \left\{ \overline{\lim}_{\delta\downarrow 0} \frac{u(t+\delta,\tilde{x}) - u(t,x)}{\delta} \,:\, \tilde{x} \in \mathcal{X}^{E+B(0,\varepsilon)}(t,x) \right\},$$

which are more closely linked to path derivatives (see Remark 4.2 in [10] and (8.4) in [17]). Appropriate counterparts of d_-^A and d_+^A (in finite dimensions with $A=0$) are used in the theory of minimax solutions of path-dependent Hamilton–Jacobi equations (see, e.g., [10, 11, 17, 18]). In this work, we mainly use $D_\uparrow^A$ because of their connection to quasi-contingent sets (see Lemma 4.4 below, which more or less corresponds to the relationship of contingent derivatives with contingent cones in Proposition 6.1.4 of [1]).

Lemma 4.4. *Let* $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$*. Then, for every* $(t_0,x_0) \in \operatorname{dom} u$ *with* $t_0 < T$*, we have*

$$D_\uparrow^A\, u(t_0,x_0)\,(F(t_0,x_0(t_0))) \le 0 \iff F(t_0,x_0(t_0)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,u(t_0,x_0)).$$

Proof. First, let $D_\uparrow^A\, u(t_0,x_0)\,(F(t_0,x_0(t_0))) \le 0$, i.e., for each $n \in \mathbb{N}$, there are $x_n \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0,x_0)$ and $\delta_n \in (0,1/n]$ with $u(t_0+\delta_n,x_n) \le u(t_0,x_0)+\delta_n/n$. Then, together with Lemma 10.1 of [8] (cf. the proof of Theorem 3.6), we can deduce that, for each $n \in \mathbb{N}$, we have $x_n'(t) + A(t,x_n(t)) = b_n(t) + p_n(t)$ a.e. on (t_0,T) for some $b_n \in F(t_0,x_0(t_0))_{L_2}$ and $p_n \in B(0,1/n)_{L_2}$. Thus, recalling Definition 3.4, we can see that $F(t_0,x_0(t_0)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,u(t_0,x_0))$ holds.

Next, assume that $F(t_0,x_0(t_0)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t_0,x_0,u(t_0,x_0))$ holds with corresponding sequences $(\delta_n)_n$, $(b_n)_n$, $(p_n)_n$, and $(x_n)_n$ from Definition 3.4, which satisfy (3.3) with $E = F(t_0,x_0(t_0))$ and $y_0 = u(t_0,x_0)$. Fix an arbitrary $f \in F(t_0,x_0(t_0))$. Next, for each $n \in \mathbb{N}$, fix an arbitrary $x_n^f \in \mathcal{X}^{\{f\}}(t_0+\delta_n,x_n)$ (which is possible by Theorem 1.1 in [15]). Then, for each $n \in \mathbb{N}$, the functions

$$\tilde{b}_n := \mathbf{1}_{[t_0,t_0+\delta_n]}\, b_n + \mathbf{1}_{[0,t_0)\cup(t_0+\delta_n,T]}\, f \quad \text{and} \quad \tilde{x}_n := \mathbf{1}_{[0,t_0+\delta_n]}\, x_n + \mathbf{1}_{(t_0+\delta_n,T]}\, x_n^f$$

satisfy $\tilde{b}_n + p_n \in (F(t_0,x_0(t_0)) + B(0,1/n))_{L^2}$ and, by Theorem 1.16 on p. 6 in [13], $\tilde{x}_n \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0,x_0)$, i.e., $D_\uparrow^A\, u(t_0,x_0)\,(F(t_0,x_0(t_0))) \le 0$ by (3.3). ☐

Definition 4.5. Let $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$ be a function.

(i) We call u a *quasi-contingent supersolution of* (4.2) if u satisfies (2.9), $u(T,\cdot) \ge h$, and, for all $(t_0,x_0) \in \operatorname{dom} u$ with $t_0 < T$, we have

$$D_\uparrow^A u(t_0,x_0)\,(F(t_0,x_0(t_0))) \le 0. \tag{4.5}$$

(ii) We call u an *l.s.c.*[6] *quasi-contingent subsolution of* (4.2) if u satisfies (2.9), $u(T,\cdot) \le h$, and, for all $(t_*,x_*) \in [0,T) \times C([0,T],H)$, $t_0 \in (t_*,T]$, and $x_0 \in \mathcal{X}^{F(t_*,x_*)}(t_*,x_*)$ with $(t_0,x_0) \in \operatorname{dom} u$, we have

$$\underline{\lim}_{\delta\downarrow 0} \frac{u(t_0-\delta,x_0) - u(t_0,x_0)}{\delta} \le 0. \tag{4.6}$$

(iii) We call u an *l.s.c. quasi-contingent solution of* (4.2) if u is a quasi-contingent super- and an l.s.c. quasi-contingent subsolution of (4.2).

[6] Note that only appropriate restrictions of u are required to be l.s.c. For details, see (2.9).

Theorem 4.6. *The value function v is an l.s.c. quasi-contingent solution of* (4.2).

Proof. (i) (Regularity). By Theorem 4.2, v satisfies (2.9).

(ii) (Quasi-contingent supersolution property). We proceed very similarly to the proof of Theorem 8.1 in [17]. Fix $(t_0, x_0) \in \operatorname{dom} v$ with $t_0 < T$. By (4.3), there is an $x \in \mathcal{X}^F(t_0, x_0)$ with selector f^x such that $\underline{\lim}_{\delta\downarrow 0} \delta^{-1} [v(t_0 + \delta, x) - v(t_0, x_0)] \leq 0$. Since, in addition, for every $n \in \mathbb{N}$, there is a $\delta_n \in (0, 1/n]$ such that $f^x = b_n + p_n$ a.e. on $(t_0, t_0 + \delta_n)$ for some $b_n \in F(t_0, x_0(t_0))_{L^2}$ and $p_n \in B(0, 1/n)_{L^2}$ (cf. the proof of Theorem 3.6), we can deduce that there is a sequence $(x_n)_n$ such that, for every $n \in \mathbb{N}$, we have $x_n \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0 + \delta_n, x)$ and $x_n \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0, x_0)$ as well. Note that $x_n = x$ on $[0, t_0 + \delta_n]$. Hence,

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \inf \left\{ \frac{v(t_0 + \delta, \tilde{x}) - v(t_0, x_0)}{\delta} : 0 < \delta < \frac{1}{n}, \tilde{x} \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0, x_0) \right\} \\
&\leq \lim_{n\to\infty} \inf \left\{ \underline{\lim}_{\delta\downarrow 0} \frac{v(t_0 + \delta, \tilde{x}) - v(t_0, x_0)}{\delta} : \tilde{x} \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,1/n)}(t_0, x_0) \right\} \\
&\leq \overline{\lim_{n\to\infty}}\, \underline{\lim}_{\delta\downarrow 0} \frac{v(t_0 + \delta, x_n) - v(t_0, x_0)}{\delta} = \overline{\lim_{n\to\infty}}\, \underline{\lim}_{\delta\downarrow 0} \frac{v(t_0 + \delta, x) - v(t_0, x_0)}{\delta} \leq 0,
\end{aligned}$$

which yields (4.5).

(iii) (L.s.c. quasi-contingent subsolution property). By (4.4), we have (4.6). □

Theorem 4.7. *Let u be a quasi-contingent supersolution of* (4.2)*. Let $(t_0, x_0) \in \operatorname{dom} u$. Then there is an $x \in \mathcal{X}^F(t_0, x_0)$ with $u(t, x) \leq u(t_0, x_0)$ for all $t \in [t_0, T]$.*

Proof. For every $(t, x, y) \in \operatorname{epi} u$ with $t < T$, we have $D^A_\uparrow u(t, x)\,(F(t, x(t))) \leq 0$, which, by Lemma 4.4, implies $F(t, x(t)) \in \mathcal{QTS}^A_{\operatorname{epi} u}(t, x, y)$. Thus Theorem 3.7 yields the viability of epi u for (3.2). Taking $y = u(t_0, x_0)$ concludes the proof. □

Theorem 4.8. *Let u be an l.s.c. quasi-contingent subsolution of* (4.2)*. Let $(t_0, x_0) \in \operatorname{dom} u$. Then $u(t, x) \geq u(t_0, x_0)$ for all $t \in [t_0, T]$ and $x \in \mathcal{X}^F(t_0, x_0)$.*

Proof. One can proceed (nearly exactly) as in the proof of in Lemma 8.2 in [5]. □

Theorem 4.9. *Let u_- be an l.s.c. quasi-contingent subsolution of* (4.2) *and let u_+ be a quasi-contingent supersolution of* (4.2)*. Then $u_- \leq u_+$.*

Proof. Let $(t_0, x_0) \in [0, T] \times C([0, T], H)$. By Theorem 4.7, there is an $x \in \mathcal{X}^F(t_0, x_0)$ with $u_+(T, x) \leq u_+(t_0, x_0)$. Thus, by Definition 4.5 and Theorem 4.8,

$$u_-(t_0, x_0) \leq u_-(T, x) \leq h(x) \leq u_+(T, x) \leq u_+(t_0, x_0)$$

This concludes the proof. □

The next result follows immediately from Theorems 4.6 and 4.9.

Corollary 4.10. *The value function v is the unique l.s.c. quasi-contingent solution of* (4.2).

4.3. **Viscosity solutions.** First, we introduce spaces of smooth functions on path spaces. To this end, we rephrase Definition 2.16 in [4].

Definition 4.11. Fix $t_0 \in [0,T)$. We denote by $\mathcal{C}_V^{1,1}([t_0,T]\times C([0,T],H))$ the set of all continuous functions $\varphi : [t_0,T]\times C([0,T],H) \to \mathbb{R}$ for which there exist continuous functions $\partial_t\varphi : [t_0,T] \times C([0,T],H) \to \mathbb{R}$ and $\partial_x\varphi : [t_0,T] \times C([0,T],H) \to H$, which we call *path derivatives* of φ, such that, for each t_1, $t_2 \in [t_0,T]$ with $t_1 < t_2$, and each $x \in C([0,T],H)$ with $x|_{(t_1,t_2)} \in W_{pq}(t_1,t_2)$, we have $x(t) \in V$ implies $\partial_t v(t,x(t)) \in V$ a.e. on (t_1,t_2) and

$$\varphi(t_2,x) - \varphi(t_1,x) = \int_{t_1}^{t_2} \partial_t\varphi(t,x) + \langle x'(t), \partial_x\varphi(t,x)\rangle \, dt.$$

Now, we are able to define test function spaces needed for our definition of viscosity solutions.

Definition 4.12. Given $E \subset H$, $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$, and $(t_0,x_0) \in [0,T) \times C([0,T],H)$, put

$$\begin{aligned}\overline{\Phi}_+^E\, u(t_0,x_0) := \{&\varphi \in \mathcal{C}_V^{1,1}([t_0,T] \times C([0,T],H)) : \\ &\exists\varepsilon > 0 : \forall t \in [t_0,t_0+\varepsilon] : \forall x \in \mathcal{X}^{E+B(0,\varepsilon)}(t_0,x_0) : \\ &0 = (\varphi - u)(t_0,x_0) \geq (\varphi - u)(t,x)\},\\ \overline{\Phi}_-\, u(t_0,x_0) := \{&\varphi \in \mathcal{C}_V^{1,1}([0,t_0] \times C([0,T],H)) : \exists\varepsilon > 0 : \forall t \in [t_0-\varepsilon,t_0] : \\ &0 = (\varphi - u)(t_0,x_0) \geq (\varphi - u)(t,x_0)\}.\end{aligned}$$

Definition 4.13. Let $u : [0,T] \times C([0,T],H) \to \mathbb{R} \cup \{+\infty\}$ be a function.

(i) We call u a *viscosity supersolution* of

$$-\partial_t u + \langle A(t,x(t)), \partial_x u\rangle - \inf_{f\in F(t,x(t))} (f, \partial_x u) = 0 \text{ on } [0,T) \times C([0,T],H) \tag{4.7}$$

if u satisfies (2.9) and, for every $(t_0,x_0) \in [0,T)\times C([0,T],H)$ with $(t_0,x_0) \in \operatorname{dom} u$ and every test function $\varphi \in \overline{\Phi}_+^{F(t_0,x_0(t_0))}\, u(t_0,x_0)$ with corresponding number $\varepsilon > 0$, there exists an $x \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,\varepsilon)}(t_0,x_0)$ such that

$$\begin{aligned}-\partial_t\varphi(t_0,x_0) + \varliminf_{\delta\downarrow 0} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \langle A(t,x(t)), \partial_x\varphi(t,x)\rangle \, dt \\ - \inf_{f\in F(t_0,x_0(t_0))} (f, \partial_x\varphi(t_0,x_0)) \geq 0.\end{aligned} \tag{4.8}$$

(ii) We call u a *viscosity supersolution* of

$$\partial_t u - \langle A(t,x(t)), \partial_x u\rangle + \inf_{f\in F(t,x(t))} (f, \partial_x u) = 0 \text{ on } (0,T] \times C([0,T],H) \tag{4.9}$$

if u satisfies (2.9) and, for all $(t_*,x_*) \in [0,T) \times C([0,T],H)$, all $t_0 \in (t_*,T]$, all $x_0 \in \mathcal{X}^F(t_*,x_*)$ with selector $t \mapsto f^{x_0}(t) \in F(t,x(t))$ and with $(t_0,x_0) \in \operatorname{dom} u$, and all $\varphi \in \overline{\Phi}_-\, u(t_0,x_0)$, we have

$$\partial_t\varphi(t_0,x_0) + \varlimsup_{\delta\downarrow 0} \frac{1}{\delta}\int_{t_0-\delta}^{t_0} \langle -A(t,x_0(t)) + f^{x_0}(t), \partial_x\varphi(t,x_0)\rangle \, dt \geq 0. \tag{4.10}$$

(iii) We call u an *l.s.c.*[7] *viscosity solution* (or *bilateral supersolution*) of (4.2) if $u(T,\cdot) = h$ and if u is a viscosity supersolution of (4.7) as well as of (4.9).

[7]Note that only appropriate restrictions of u are required to be l.s.c. For details, see (2.9).

Remark 4.14. Roughly speaking, satisfying the viscosity supersolution property for (4.9) with the test function spaces $\overline{\Phi}_-\, u(t_0,x_0)$ can be thought of as satisfying the viscosity supersolution property for all linear equations of the form

$$\partial_t u(t_0,x_0) - \langle A(t_0,x_0(t_0)), \partial_x u(t_0,x_0)\rangle + (f, \partial_x u(t_0,x_0)) = 0$$

for every $f \in F(t_0,x_0(t_0))$, $(t_0,x_0) \in (0,T] \times C([0,T],H)$, i.e., formally

$$\partial_t u(t_0,x_0) - \langle A(t_0,x_0(t_0)), \partial_x u(t_0,x_0)\rangle + \inf_{f\in F(t_0,x_0(t_0))} (f, \partial_x u(t_0,x_0)) \geq 0$$

holds in the viscosity sense. Hence, identifying the selectors f^x in (2.4) with admissible controls $t \mapsto a(t)$, we can say that for an l.s.c. viscosity solution u of (4.2) the following holds in the viscosity sense for suitably defined operators $\mathcal{L}_a^+$ and $\mathcal{L}_a^-$:

$$\begin{aligned} -\inf_{a\in F(t_0,x(t_0))} \mathcal{L}_a^+\, u(t_0,x_0) \geq 0, \qquad & (t_0,x_0) \in [0,T) \times C([0,T],H),\\ \mathcal{L}_a^- u(t_0,x_0) \geq 0 \qquad & \text{for all } t \mapsto a(t) \in F(t,x_0(t)),\\ & (t_0,x_0) \in (t_*,T] \times \mathcal{X}^F(t_*,x_*),\\ & (t_*,x_*) \in [0,T) \times C([0,T],H) \end{aligned}$$

(cf. Definition 6.5 and Remark 6.6 in [14]).

Remark 4.15. Typically (see [3]), bilateral supersolutions are related to a backward dynamic programming principle. E.g., consider a problem of the form

$$\tilde{v}(t_0,x_0) := \inf\{\tilde{h}(x^{t_0,x_0,a}(T)) :\, a \in \mathcal{A}), \quad (t_0,x_0) \in [0,T] \times \mathbb{R},$$

where $x = x^{t_0,x_0,a}$ solves $x'(t) = f(t,x(t),a(t))$ on (t_0,T) with initial condition $x(t_0) = x_0$. Then $\tilde{v}(t_0,x_0) \leq \tilde{v}(t, x^{t_0,x_0,a}(t))$ for all admissible controls $a \in \mathcal{A}$ and $t \in [t_0,T]$. Similarly, a backward version holds, i.e., $\tilde{v}(t, x_-^{t_0,x_0,a}(t))) \leq \tilde{v}(t_0,x_0))$ for all $a \in \mathcal{A}$ and $t \in [0,t_0]$, where $x_- = x_-^{t_0,x_0,a}$ solves $x_-'(t) = f(t,x_-(t),a(t))$ on $(0,t_0)$ with terminal condition $x_-(t_0) = x_0$. This backward version leads to viscosity supersolutions of the corresponding HJB equation (but with opposite sign). In our context, the situation is slightly different. First, due to the operator A, we cannot expect solutions of backward evolution equations to exist, in general. Second, in the path-dependent case, the terminal condition for a backward evolution equation with history-dependent data means that $x^{t_0,x_0,a}|_{[0,t_0]} = x_0|_{[0,t_0]}$ should hold,[8] which means that we should only consider “terminal data” x_0 that already satisfy our evolution equation at least on some time interval $(t_0-\delta,t_0)$. This naturally leads to Definition 4.13 (ii) with the test function spaces $\overline{\Phi}_-\, u(t_0,x_0)$.

4.3.1. *Viscosity solutions: Comparison principle.*

Theorem 4.16. *Let u be viscosity supersolution of* (4.7) *with $u(T,\cdot) \geq h$. Then u is a quasi-contingent supersolution of* (4.2).

Proof. Fix $(t_0,x_0) \in \operatorname{dom} u$ with $t_0 < T$. Put $E := F(t_0,x_0(t_0))$. Assume that $D_\uparrow^A\, u(t_0,x_0)\,(E) > 0$, i.e., there are $c > 0$ and $\varepsilon > 0$ such that

$$u(t_0+\delta,x) - u(t_0,x_0) > c\cdot\delta$$

for all $\delta \in (0,\varepsilon]$ and all $x \in \mathcal{X}^{E+B(0,\varepsilon)}(t_0,x_0)$. Now, we can easily (just as in the proof of Theorem 6.8 in [14]) obtain a test function $\varphi \in \overline{\Phi}_+^E\, u(t_0,x_0)$ that does

[8]Note that here $x_0 \in C([0,T],H)$.

not satisfy (4.8), i.e., we have a contradiction. Thus our assumption is wrong, i.e., $D_{\uparrow}^{A}\, u(t_0, x_0)\,(E) \leq 0$ holds. This concludes the proof. $\square$

Theorem 4.17. *Let u be viscosity supersolution of* (4.9) *with $u(T,\cdot) \leq h$. Then u is an l.s.c. quasi-contingent subsolution of* (4.2).

Proof. Fix $(t_*, x_*) \in [0,T) \times C([0,T], H)$, $t_0 \in (t_*, T]$, and $x_0 \in \mathcal{X}^F(t_*, x_*)$ with selector $f^{x_0} \in F(\cdot, x(\cdot))$. Suppose that $(t_0, x_0) \in \operatorname{dom} u$. Assume that there are $c > 0$ and $\varepsilon > 0$ such that

$$u(t_0 - \delta, x_0) - u(t_0, x_0) > \delta \cdot c$$

for all $\delta \in (0, \varepsilon]$. Define $\varphi : [0, t_0] \times C([0,T], H) \to \mathbb{R}$ by

$$\varphi(t,x) := u(t_0, x_0) + (t_0 - t) \cdot c.$$

Then $\varphi \in \mathcal{C}_V^{1,1}([0,t_0] \times C([0,T],H))$ with $\partial_x \varphi(t,x) = 0$ and $\partial_t \varphi(t,x) = -c$, and thus $\varphi \in \overline{\Phi}_-\, u(t_0,x_0)$, which contradicts (4.10). Hence our assumption is wrong and thus u is an l.s.c. quasi-contingent supersolution of (4.2). $\square$

The next result follows from Theorems 4.9, 4.16, and 4.17.

Theorem 4.18. *If u_- is a viscosity supersolution of* (4.9) *with $u_-(T,\cdot) \leq h$ and u_+ is a viscosity supersolution of* (4.7) *with $u_+(T,\cdot) \geq h$, then $u_- \leq u_+$.*

4.3.2. *Viscosity solutions: Existence and uniqueness.*

Theorem 4.19. *The value function v is the unique l.s.c. viscosity solution of* (4.2).

Proof. (i) *Regularity*: By Theorem 4.2, v satisfies (2.9).

(ii) *Viscosity supersolution property for* (4.7): Let $(t_0, x_0) \in [0,T) \times C([0,T],H)$ with $(t_0, x_0) \in \operatorname{dom} v$ and $\varphi \in \overline{\Phi}_+^{F(t_0,x_0)}\, v(t_0, x_0)$ with corresponding $\varepsilon > 0$. By (4.3), there is an $x \in \mathcal{X}^F(t_0, x_0)$ such that $v(t,x) = v(t_0,x_0)$ for all $t \in [t_0, T]$. From the proof of Theorem 3.6, we can deduce that there are $\delta_0 > 0$ and $\tilde{x} \in \mathcal{X}^{F(t_0,x_0(t_0))+B(0,\varepsilon)}(t_0,x_0)$ with corresponding selector $f^{\tilde{x}} = \tilde{b} + \tilde{p}$ a.e. on $(t_0, t_0+\delta_0)$ for some $\tilde{b} \in F(t_0, x(t_0))_{L^2}$ and $\tilde{p} \in B(0,\varepsilon)_{L^2}$ such that that $x = \tilde{x}$ on $[0, t_0 + \delta_0]$. Thus, thanks to the upper semicontinuity of F, for every $n \in \mathbb{N}$, there are $\delta_n \in (0, \delta_0)$, $\tilde{b}_n \in F(t_0, x(t_0))_{L^2}$ and $\tilde{p}_n \in B(0, 1/n)_{L^2}$ such that, for every $\delta \in (0, \delta_n)$, we have

$$\begin{aligned}
0 &= v(t_0+\delta, x) - v(t_0, x_0) = v(t_0 + \delta, \tilde{x}) - v(t_0, x_0)\\
&\geq \varphi(t_0 + \delta, \tilde{x}) - \varphi(t_0, x_0)\\
&\geq \int_{t_0}^{t_0+\delta} \partial_t \varphi(t,x) - \langle A(t,x(t)), \partial_x \varphi(t,x)\rangle + (\tilde{b}_n(t) + \tilde{p}_n(t), \partial_x \varphi(t,x))\, dt\\
&\geq \int_{t_0}^{t_0+\delta} \partial_t \varphi(t,x) - \langle A(t,x(t)), \partial_x \varphi(t,x)\rangle + (\tilde{b}_n(t), \partial_x \varphi(t,x) - \partial_x \varphi(t_0,x_0))\\
&\qquad\qquad + \inf_{f \in F(t_0, x(t_0))} (f, \partial_x \varphi(t_0, x_0)) + (\tilde{p}_n(t), \partial_x \varphi(t,x))\, dt.
\end{aligned}$$

Dividing by δ and applying $\underline{\lim}_{\delta\downarrow 0}$ yields

$$\begin{aligned}
&\partial_t\varphi(t_0,x_0)+\underline{\lim}_{\delta\downarrow 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\langle -A(t,x(t)),\partial_x\varphi(t,x)\rangle\,dt+\inf_{f\in F(t_0,x_0(t_0))}(f,\partial_x\varphi(t_0,x_0))\\
&\le -\underline{\lim}_{\delta\downarrow 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}[(\tilde{p}_n(t),\partial_x\varphi(t,x))+(\tilde{b}_n(t),\partial_x\varphi(t,x)-\partial_x\varphi(t_0,x_0))]\,dt\\
&\le \frac{1}{n}\cdot\sup_{t\in[t_0,T]}|\partial_x\varphi(t,x)|+|F(t_0,x_0)|\cdot\overline{\lim_{\delta\downarrow 0}}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}|\partial_x\varphi(t,x)-\partial_x\varphi(t_0,x_0))|\;dt\\
&=\frac{1}{n}\cdot\sup_{t\in[t_0,T]}|\partial_x\varphi(t,x)|
\end{aligned}$$

thanks to the superadditivitiy of $\underline{\lim}$ and the boundedness of $F(t_0,x_0)$. Since $n\in\mathbb{N}$ was arbitrary, we have (4.8).

(iii) *Viscosity supersolution property for* (4.9): Let $(t_*,x_*)\in[0,T)\times C([0,T],H)$, $t_0\in(0,T]$, $x_0\in\mathcal{X}^F(t_*,x_*)$ with $(t_0,x_0)\in\operatorname{dom}v$, and $\varphi\in\overline{\Phi}_-(t_0,x_0)$ with corresponding number $\varepsilon>0$. Noting that $x_0\in\mathcal{X}^F(t,x_0)$ for every $t\in[t_*,T]$, we can deduce with (4.4) that, for every $\delta\in(0,\varepsilon]$ with $t_0-\delta\ge t_*$, we have

$$0\ge v(t_0-\delta,x_0)-v(t_0,x_0)\ge\varphi(t_0-\delta,x_0)-\varphi(t_0,x_0).$$

Thus

$$\begin{aligned}
0&\le\overline{\lim_{\delta\downarrow 0}}\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\partial_t\varphi(t,x_0)-\langle A(t,x_0(t))\,\partial_x\varphi(t,x_0)\rangle+(f^{x_0}(t),\partial_x\varphi(t,x_0))\,dt\\
&\le\partial_t\varphi(t_0,x_0)+\overline{\lim_{\delta\downarrow 0}}\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\langle -A(t,x_0(t)),\partial_x\varphi(t,x_0)\rangle+(f^{x_0}(t),\partial_x\varphi(t,x_0))\,dt.
\end{aligned}$$

(iv) Uniqueness follows from Theorem 4.18. This concludes the proof. □

4.4. **An example.** We consider a distributed control problem for the heat equation with l.s.c. terminal cost. To this end, we fix a bounded domain G in $\mathbb{R}^n$ with smooth boundary ∂G, a constant $C_P>0$, and the set

$$P:=\{z\in L^2(G):\,\|z\|_{L^2(G)}\le C_P\},$$

which will be used as control set.

Given $(t_0,x_0)\in[0,T)\times C([0,T],L^2(G))$ and a Borel-measurable function $a:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ with $\int_G|a(t,\xi)|^2\;d\xi\le(C_P)^2$ for all $t\in[0,T]$, consider the Cauchy–Dirichlet problem

$$\begin{aligned}
x_t(t,\xi)-\Delta_\xi x(t,\xi)&=a(t,\xi),\quad (t,\xi)\in(t_0,T)\times G,\\
x(t,\xi)&=0,\quad (t,\xi)\in(t_0,T]\times\partial G,\\
x(t,\xi)&=x_0(t),\quad (t,\xi)\in[0,t_0]\times G.
\end{aligned}\tag{4.11}$$

As in Example 1.3 in [4] (see also Chapter 23 in [21] for a detailed treatment) we can formulate (4.11) as an abstract evolution equation on (V,H,V^*) of the form

$$x'(t)+Ax(t)=\mathbf{a}(t)\text{ a.e. on }(t_0,T)\text{ with }x=x_0\text{ on }[0,t_0],\tag{4.12}$$

where $H=L^2(G)$, $V=H_0^1(G)$, and $(t,y)\mapsto Ay$, $[0,T]\times V\to V^*$, satisfies $\mathbf{H}(A)$. Here, A corresponds to $-\Delta_\xi$ and $\mathbf{a}:[0,T]\to P$ to the function $a=a(t,\xi)$ above. We write $x^{t_0,x_0,\mathbf{a}}$ for the solution of (4.12) in $C([0,T],L^2(G))\cap W_{pq}(t_0,T)$. Note that here $p = 2$. Existence as well as uniqueness of our solution $x^{t_0,x_0,\mathbf{a}}$ are standard results (see, e.g., [21]).

Next, we specify the remaining data for our control problem. As terminal cost, we use $h : C([0,T], L^2(G)) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x(t) \in K \text{ for all } t \in [0,T], \\ +\infty & \text{otherwise,} \end{cases},$$

where

$$K = \{z \in L^2(G) : \|z\|_{L^2(G)} \leq C_K\}$$

and $C_K >$ is a constant. Note that h is l.s.c. As class of admissible controls, we use the set of all Borel-measurable functions from $[0,T]$ to P, which we denote by $\mathcal{A}$.

Given $(t_0, x_0) \in [0,T) \times C([0,T], L^2(G))$, our optimal control problem is to find a control $\tilde{\mathbf{a}} \in \mathcal{A}$ such that

$$h(x^{t_0,x_0,\tilde{\mathbf{a}}}) = \inf\{h(x^{t_0,x_0,\mathbf{a}}) : \mathbf{a} \in \mathcal{A}\}.$$

The corresponding value function $v : [0,T] \times C([0,T], L^2(G)) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$v(t_0, x_0) := \inf\{h(x^{t_0,x_0,\mathbf{a}}) : \mathbf{a} \in \mathcal{A}\} = \inf\{h(x) : x \in \mathcal{X}^F(t_0, x_0)\},$$

where $F(t,z) = P$ for all $(t,z) \in [0,T] \times L^2(G)$.

Note that v is not continuous. In particular, $v(t_0, x_0) = +\infty$ whenever $x_0(s) \notin K$ for some $s \in [0, t_0]$ and $v(t_0, x_0) = 0$ whenever $x_0(t) = 0$ for all $t \in [0, t_0]$. Nevertheless, v can be characterized as a unique nonsmooth solution of the corresponding Hamilton–Jacobi–Bellman equation.

Indeed, by Theorem 4.19, v is the unique l.s.c. viscosity solution of

$$\begin{aligned} &-\partial_t u + \langle Ax(t), \partial_x u\rangle - \inf_{\mathbf{p}\in P} (\mathbf{p}, \partial_x u) = 0 \text{ in } [0,T) \times C([0,T], L^2(G)), \\ &u(T,x) = h(x) \text{ on } C([0,T], L^2(G)). \end{aligned}$$

References

[1] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition.

[2] E. P. Avgerinos and N. S. Papageorgiou, *Viable solutions for evolution inclusions*, Yokohama Math. J. **37** (1989), no. 2, 101–108.

[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1997. With appendices by M. Falcone and P. Soravia.

[4] E. Bayraktar and C. Keller, *Path-dependent Hamilton-Jacobi equations in infinite dimensions*, J. Funct. Anal. **275** (2018), no. 8, 2096–2161.

[5] ———, *Path-dependent Hamilton-Jacobi equations with super-quadratic growth in the gradient and the vanishing viscosity method*, SIAM J. Control Optim. **60** (2022), no. 3, 1690–1711.

[6] O. Carja, *The minimum time function for semilinear evolutions*, SIAM J. Control Optim. **50** (2012), no. 3, 1265–1282.

[7] O. Cârjă, M. Necula, and I. I. Vrabie, *Viability, invariance and applications*, North-Holland Mathematics Studies, vol. 207, Elsevier Science B.V., Amsterdam, 2007.

[8] ———, *Necessary and sufficient conditions for viability for semilinear differential inclusions*, Trans. Amer. Math. Soc. **361** (2009), no. 1, 343–390.

[9] H. Frankowska, *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optim. **31** (1993), no. 1, 257–272. MR1200233
[10] M. I. Gomoyunov and A. R. Plaksin, *Equivalence of minimax and viscosity solutions of path-dependent Hamilton-Jacobi equations*, J. Funct. Anal. **285** (2023), no. 11, Paper No. 110155, 41 pp.
[11] M. I. Gomoyunov, N. Yu. Lukoyanov, and A. R. Plaksin, *Path-dependent Hamilton-Jacobi equations: The minimax solutions revised*, Appl. Math. Optim. **84** (2021), S1087–S1117.
[12] D. Goreac, *Viability of stochastic semi-linear control systems via the quasi-tangency condition*, IMA J. Math. Control Inform. **28** (2011), no. 3, 391–415.
[13] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. II: Applications*, Mathematics and its Applications, vol. 500, Kluwer Academic Publishers, Dordrecht, 2000.
[14] C. Keller, *Mean viability theorems and second-order Hamilton-Jacobi equations*, SIAM J. Control Optim. **62** (2024), no. 3, 1615–1642.
[15] W. Liu, *Existence and uniqueness of solutions to nonlinear evolution equations with locally monotone operators*, Nonlinear Anal. **74** (2011), no. 18, 7543–7561.
[16] W. Liu and M. Röckner, *SPDE in Hilbert space with locally monotone coefficients*, J. Funct. Anal. **259** (2010), no. 11, 2902–2922.
[17] N. Yu. Lukoyanov, *Functional Hamilton-Jacobi type equations in ci-derivatives for systems with distributed delays*, Nonlinear Funct. Anal. Appl. **8** (2003), no. 3, 365–397.
[18] ______, *Differential inequalities for a nonsmooth value functional in control systems with an aftereffect*, Proc. Steklov Inst. Math. (2006), S103–S114.
[19] N. S. Papageorgiou, *A viability result for nonlinear time dependent evolution inclusions*, Yokohama Math. J. **40** (1992), no. 1, 73–86.
[20] S. Z. Shi, *Nagumo type condition for partial differential inclusions*, Nonlinear Anal. **12** (1988), no. 9, 951–967.
[21] E. Zeidler, *Nonlinear functional analysis and its applications. II/A*, Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.
[22] ______, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, 1990.

Department of Mathematics, University of Central Florida, Orlando, FL 32816, United States

Email address: jichao.jiang@ucf.edu
Email address: christian.keller@ucf.edu