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Fixed-Time Input-to-State Stability for Singularly Perturbed Systems via Composite Lyapunov Functions

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***Abstract*— We study singularly perturbed systems that exhibit input-to-state stability (ISS) with fixed-time properties in the presence of bounded disturbances. In these systems, solutions converge to the origin within a time frame independent of initial conditions when undisturbed, and to a vicinity of the origin when subjected to bounded disturbances. First, we extend the traditional composite Lyapunov method, commonly applied in singular perturbation theory to analyze asymptotic stability, to include fixed-time ISS. We demonstrate that if both the reduced system and the boundary layer system exhibit fixed-time ISS, and if certain interconnection conditions are met, the entire multi-time scale system retains this fixed-time ISS characteristic, provided the separation of time scales is sufficiently pronounced. Next, we illustrate our findings via analytical and numerical examples, including a novel application in fixed-time feedback optimization for dynamic plants with slowly varying cost functions.**

I. Introduction

Many complex dynamical systems can be decomposed into components operating on different time scales. Such multi-time-scale systems are common across a diverse array of engineering applications, including aerospace [1], chemical processing [2], smart grids [3], and biological and evolutionary systems [4], among others. Such systems can be studied using techniques from singular perturbation theory [5]–[7], where the methods typically focus on decomposing the system into lower order subsystems and studying the subsystems to draw conclusions on the behavior of the overall interconnection.

One simple yet powerful technique for analyzing singularly perturbed systems is the *composite Lyapunov method*, first introduced by Khalil in [8]. This method is particularly effective for assessing system stability, as it leverages the stability of the subsystems to construct a Lyapunov function for the overall system that is valid under sufficiently large time-scale separation. As demonstrated in [8], if the reduced and boundary layer systems admit certain quadratic-type Lyapunov functions, then under additional interconnection conditions, the interconnected system can be shown to be asymptotically stable, provided there is sufficient time-scale separation between the two subsystems. Moreover, if the Lyapunov functions satisfy specific quadratic bounds, these results can be extended to exponential stability. The composite Lyapunov method has found profound applications in control synthesis across various engineering domains, including power systems [9], biological systems [10], aerospace [11], etc. While the composite Lyapunov method has been widely used to establish asymptotic and exponential stability, it has found limited applications in settings which require stronger notions of stability. In [12], the authors leverage composite Lyapunov functions to establish finite-time stability for a special class of singularly perturbed homogeneous systems. However, to the best of the authors' knowledge, there does not exist a framework that studies fixed-time stability for a general class of singularly perturbed systems via composite Lyapunov techniques.

Fixed-time stability, which has been popularized through the introduction of Lyapunov conditions in [13], has garnered significant attention due to its ability to address challenges in control [14], optimization [15], and learning [16], [17]. Fixed-time stability ensures convergence to an equilibrium within a fixed time, regardless of the system's initial conditions. While the Lyapunov conditions simplify the verification of fixed-time stability in continuous-time dynamical systems, techniques for analyzing fixed-time stability in interconnected systems remain limited. Current state-of-the-art methods are primarily applicable to systems with specific homogeneity properties [18], restrictive structural requirements [19], or those satisfying certain small-gain conditions [20]. Moreover, in many practical applications, the systems of interest are also influenced by external disturbances or exogenous inputs. In particular, in this paper we study systems of the form

$$\dot{x} = f(x, z, u) \tag{1a}$$

$$\varepsilon \dot{z} = g(x, z, u), \tag{1b}$$

where $\varepsilon > 0$ is a small parameter that induces a time scale separation between the dynamics of x and the dynamics of z, and u is an exogenous input. One of the key tools for analyzing such systems is input-to-state stability (ISS), introduced by Sontag in [21] and later characterized through an equivalent Lyapunov framework in [22]. In simple terms, an ISS system converges asymptotically to a bounded set for any bounded input signal. A fixed-time counterpart, known as fixed-time ISS (FxT ISS), was introduced in [23] to address fixed-time stability in systems subject to disturbances. While standard ISS properties have been extensively studied in the context of singularly perturbed systems [24], [25], it remains an open question whether analogous results can be developed for fixed-

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time stability and whether the composite Lyapunov method introduced in [8] can be extended to analyze FxT ISS.

In this paper, we address the above questions by extending the popular composite Lyapunov method [8] to study FxT ISS in singularly perturbed systems with inputs. Specifically, under the assumptions that the respective "reduced" and "boundary-layer" dynamics of (1) are fixed-time stable on their own, we first derive interconnection conditions under which (1) renders the origin FxT ISS. These interconnection conditions parallel those developed in [8] for asymptotic and exponential stability. However, unlike the results of [8], absence of Lipschitz continuity at the origin results in a more complex paradigm for verifying our proposed interconnection conditions, which involve powers of the fixed-time Lyapunov functions of the reduced and boundary-layer dynamics. Subsequently, we employ our analytical tools to design and analyze a novel class of algorithms that achieve fixed-time feedback optimization under strongly convex cost functions and slowly varying inputs—a class of problems widely studied in the literature [26]–[28], but which, to the best of our knowledge, has not previously been addressed using algorithms with fixed-time stability properties. Finally, we present numerical and analytical examples to illustrate the key assumptions and the main stability results of the paper.

Earlier, preliminary results from this work were previously published in the proceedings of the American Control Conference [29]. However, the results of [29] focused exclusively on systems without inputs, addressing only fixed-time stability, and provided only proof sketches. In contrast, this paper extends the analysis to singularly perturbed systems *with inputs*, advancing the FxT stability results of [29] to encompass fixed-time input-to-state stability (FxT ISS). Furthermore, in this paper we present a comprehensive stability analysis, complete proofs, and novel analytical and numerical examples to illustrate the main findings. This includes an application to feedback optimization under slowly varying inputs in general fixed-time stable plants.

The rest of this paper is organized as follows. Section II presents the notation and preliminary results. Section III discusses the problem of interest, the main stability results, and showcases the applicability of our results in an illustrative example. Section IV studies fixed-time feedback optimization, and Section V ends with the conclusions.

II. Preliminaries

A. Notation

We use $\mathbb{R}_{\geq 0}$ to denote the set of nonnegative real numbers. We let $\text{sgn}: \mathbb{R} \to \{-1, 0, 1\}$ denote the sign function, i.e $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$ and $\text{sgn}(0) = 0$. For a continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we say $\alpha \in \mathcal{K}$ (i.e, α is of class $\mathcal{K}$) if $\alpha(0) = 0$ and α is strictly increasing. If $\alpha \in \mathcal{K}$ also satisfies $\lim_{s\to\infty} \alpha(s) = \infty$, we say $\alpha \in \mathcal{K}_\infty$. Given a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we say $\beta \in \mathcal{KL}$ if for each $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and for each $r \geq 0$, $\beta(r, \cdot)$ is non-increasing and asymptotically goes to 0. Furthermore, $\beta \in \mathcal{GKL}$ if $\beta(\cdot, 0) \in \mathcal{K}$ and for each fixed $r \geq 0$, $\beta(r, \cdot)$ is continuous, non-increasing and there exists a function $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(r, t) = 0$ for all $t \geq T(r)$. The mapping T is called a *settling time function*, which, in general, is not unique. Given a measurable function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ we denote $|u|_\infty = \operatorname{ess\,sup}_{t\geq 0} |u(t)|$. We use $\mathcal{L}_\infty^p$ to denote the set of measurable functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ satisfying $|u|_\infty < \infty$. Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, we use $\mathbf{J}_f(x) \in \mathbb{R}^{m\times n}$ to denote the Jacobian of f evaluated at $x \in \mathbb{R}^n$. If $m = 1$, we use $\nabla f(x) = \mathbf{J}_f(x)^\top$. If $\mathbf{J}_f(x)$ is continuous, we say f is $\mathcal{C}^1$.

B. Auxiliary Results

We first present some lemmas that will be instrumental for our results. Due to space limitations, the proofs can be found in the extended manuscript [30].

Lemma 1: Given $x, y \geq 0$ and $p_1, p_2 > 0$, the following inequality holds for all $c > 0$:

$$|x|^{p_1}|y|^{p_2} \leq c|x|^{p_1+p_2} + c^{-\frac{p_1}{p_2}}|y|^{p_1+p_2}.$$

□

Lemma 2: For $\xi_1 \in (0, 1)$ and $\xi_2 < 0$, the following inequalities hold for all $x, y \in \mathbb{R}^n$:

$$\left| \frac{x}{|x|^{\xi_1}} - \frac{y}{|y|^{\xi_1}} \right| \leq 2^{\xi_1}|x - y|^{1-\xi_1} \tag{2a}$$

$$\left| \frac{x}{|x|^{\xi_2}} - \frac{y}{|y|^{\xi_2}} \right| \leq K|y - x| \left(|x|^{-\xi_2} + |y - x|^{-\xi_2}\right). \tag{2b}$$

where $K := 1 + \max\left(1, -\xi_2 2^{-\xi_2 - 1}\right)$. □

C. Fixed-Time Input-to-State Stability

Consider a nonlinear dynamical system of the form

$$\dot{x} = f(x, u), \quad x(0) = x_0, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathcal{L}_\infty^p$ is an input signal. We assume the vector field f is continuous and satisfies $f(0, 0) = 0$. We will state some definitions from [23] that will be particularly relevant for our work.

Definition 1: System (3) is said to be *fixed-time input-to-state stable (FxT ISS)* if for each $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^p$, every solution $x(t)$ of (3) exists for $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x_0|, t) + \varrho(|u|_\infty), \tag{4}$$

where $\beta \in \mathcal{GKL}$, $\varrho \in \mathcal{K}$ and there exists a settling time function T of β that is continuous and uniformly bounded, with $T(0) = 0$.

Definition 2: A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *FxT ISS Lyapunov function* for (3) if there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{5}$$

and the following holds

$$\frac{\partial V}{\partial x} f(x, u) \leq -k_1 V^{a_1}(x) - k_2 V^{a_2}(x) + \rho(|u|), \tag{6}$$

for some $\rho \in \mathcal{K}_\infty$, $k_1, k_2 > 0$, $a_1 \in (0, 1)$ and $a_2 > 1$.

Remark 1: It can be verified that the "dissipation" formulation we use in (6) implies relation (10) in [23], which implies

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FxT ISS for (3) via [23, Thm 4]. Indeed, if (6) holds and we fix some $0 < \tilde{\varepsilon} < \min_i k_i$, we have

$$|x| > \chi(|u|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leq -\tilde{k}_1 V^{a_1}(x) - \tilde{k}_2 V^{a_2}(x),$$

where $\chi(\cdot) = \tilde{\chi}^{-1} \circ \rho(\cdot)$, $\tilde{\chi}(\cdot) = \tilde{\varepsilon}\alpha_2^{a_1}(\cdot) + \tilde{\varepsilon}\alpha_2^{a_2}(\cdot)$, and $\tilde{k}_i = k_i - \tilde{\varepsilon}$. By taking $\tilde{\varepsilon} \to 0$, it can be observed via [23, Corollary 2] that systems that admit a FxT ISS Lyapunov function that satisfy (6) also admit a settling time function satisfying the following bound

$$T(x_0) \leq \frac{1}{k_1(1 - a_1)} + \frac{1}{k_2(a_2 - 1)}, \tag{7}$$

for all $x_0 \in \mathbb{R}^n$.

III. Main Results

We consider singularly perturbed systems of the form (1), with states $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, input $u \in \mathbb{R}^p$, dynamics satisfying $f(0, z^*, 0) = g(0, z^*, 0) = 0$ for some $z^* \in \mathbb{R}^m$, and a small parameter $\varepsilon > 0$ that induces a time scale separation between the dynamics of x and z. Our main objective is to exploit the stability properties of the so-called *reduced system* and *boundary-layer system* associated with (1), as defined below, in order to derive Lyapunov-based sufficient conditions that ensure fixed-time input-to-state stability (FxT ISS) of (1), provided the time-scale separation is sufficiently large.

A. Assumptions

To study system (1) in the context of singular perturbations, we make the following standard assumption on (1b):

Assumption 1: There exists a $\mathcal{C}^1$ mapping $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x, z, 0) = 0$ if and only if $z = h(x)$. □

The map h is usually referred to as the *quasi-steady state mapping* [7] for the disturbance-free system (1b). By using this mapping, we can define the so-called *reduced system* from (1a):

$$\dot{x} = f(x, h(x), u). \tag{8}$$

Using the change of coordinates $y = z - h(x)$, we obtain the following error dynamics:

$$\dot{x} = f(x, y + h(x), u) \tag{9a}$$
$$\dot{y} = \frac{1}{\varepsilon} g(x, y + h(x), u) - \frac{\partial h}{\partial x} f(x, y + h(x), u). \tag{9b}$$

System (9b) is studied in the time scale $\tau = t/\varepsilon$ and taking $\varepsilon \to 0^+$ to obtain the *boundary layer system*:

$$\frac{dy}{d\tau} = g(x, y + h(x), u), \tag{10}$$

where $x \in \mathbb{R}^n$ is considered fixed. Since our goal is to study FxT ISS of (1), we make the following FxT ISS Lyapunov-based assumptions on the lower order systems (8) and (10):

Assumption 2: There exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \rho_R \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

and

$$\frac{\partial V}{\partial x} f(x, h(x), u) \leq -k_1 V^{a_1}(x) - k_2 V^{a_2}(x) + \rho_R(|u|),$$

where $k_1, k_2 > 0$, $a_1 \in (0, 1)$ and $a_2 > 1$. □

Assumption 3: There exists a $\mathcal{C}^1$ function $W : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $\tilde{\alpha}_1, \tilde{\alpha}_2, \rho_B \in \mathcal{K}_\infty$ such that

$$\tilde{\alpha}_1(|y|) \leq W(x, y) \leq \tilde{\alpha}_2(|y|),$$

and

$$\frac{\partial W}{\partial y} g(x, y + h(x), u) \leq -\kappa_1 W^{b_1}(x, y) - \kappa_2 W^{b_2}(x, y) + \rho_B(|u|),$$

where $\kappa_1, \kappa_2 > 0$, $b_1 \in (0, 1)$ and $b_2 > 1$. □

Assumptions 2 and 3 state, respectively, that the reduced system (8) and the boundary-layer system (10) admit individual FxT ISS Lyapunov functions and are, hence, FxT ISS. These assumptions mirror the classical Lyapunov-based conditions commonly used in the literature to study the asymptotic stability of singularly perturbed systems. However, as shown in [8] and [24], the mere stability or ISS properties of the individual reduced and boundary-layer systems are usually insufficient to guarantee that (1) is also stable or ISS, and additional interconnection conditions need to be examined.

B. Analysis

To assess the FxT stability of system (1) using a Lyapunov-based approach, we consider the following Lyapunov function candidate

$$\Psi_\zeta(x, y) = \zeta V(x) + (1 - \zeta) W(x, y), \quad \zeta \in (0, 1), \tag{11}$$

where V and W are from Assumptions 2 and 3 respectively. For historical reasons, we refer to (11) as a *composite Lyapunov function* [31], [32]. Evaluating the Lie derivative of (11) along the trajectories of (9) results in

$$\dot{\Psi}_\zeta = \zeta \left(\frac{\partial V}{\partial x} f(x, h(x), u) + I_1(x, y, u) \right) + (1 - \zeta) \left(\frac{1}{\varepsilon} \frac{\partial W}{\partial y} g(x, y + h(x), u) + I_2(x, y, u) \right), \tag{12}$$

where the *interconnection terms*, I_1 and I_2, are given by

$$I_1(x, y, u) = \frac{\partial V}{\partial x} \left(f(x, y + h(x), u) - f(x, h(x), u) \right) \tag{13a}$$
$$I_2(x, y, u) = \left(\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} \frac{\partial h}{\partial x} \right) f(x, y + h(x), u). \tag{13b}$$

Ideally, we aim to derive suitable bounds on these terms that would allow us to conclude that (1) is fixed-time ISS. Before we do so, we define the following terms:

$$\tilde{V}(x) := V^{\frac{a_1}{2}}(x) + V^{\frac{a_2}{2}}(x) \tag{14a}$$
$$\tilde{W}(x, y) := W^{\frac{b_1}{2}}(x, y) + W^{\frac{b_2}{2}}(x, y) \tag{14b}$$
$$\underline{k} := \min_i k_i, \quad \underline{\kappa} := \min_i \kappa_i, \tag{14c}$$

where $V, W, a_i, b_i, k_i, \kappa_i$ come from Assumptions 2-3.

With these definitions at hand, we can now state the first main result of the paper.

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Theorem 1: Consider system (9), and suppose that Assumptions 1-3 hold. Furthermore, suppose there exists $\nu_1, \nu_2, \omega_1, \omega_2 \in \mathbb{R}$ and $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that the interconnection terms in (13) satisfy

$$I_1(x,y,u) \leq \nu_1 \tilde{V}^2(x) + \omega_1 \tilde{W}^2(x,y) + \rho_1(|u|) \tag{15a}$$
$$I_2(x,y,u) \leq \nu_2 \tilde{V}^2(x) + \omega_2 \tilde{W}^2(x,y) + \rho_2(|u|) \tag{15b}$$
$$\nu_1 < \frac{1}{2}\underline{k}, \quad \text{or} \quad \nu_2 < 0. \tag{15c}$$

Then, there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*)$, the system (9) is FxT ISS. □

Proof: Continuing from (12), we can use Assumptions 2 and 3 to obtain:

$$\begin{aligned}\dot{\Psi}_\zeta &\leq -\zeta\underline{k}(V^{a_1}(x) + V^{a_2}(x)) + \zeta\rho_R(|u|) + \zeta I_1 + (1-\zeta)I_2 \\ &\quad - \frac{1-\zeta}{\varepsilon}\underline{\kappa}(W^{b_1}(x,y) + W^{b_2}(x,y)) + \frac{1-\zeta}{\varepsilon}\rho_B(|u|) \\ &\leq -\frac{\zeta}{2}\underline{k}\tilde{V}^2(x) - \frac{1-\zeta}{2\varepsilon}\underline{\kappa}\tilde{W}^2(x,y) + \zeta I_1 + (1-\zeta)I_2 \\ &\quad + \zeta\rho_R(|u|) + \frac{1-\zeta}{\varepsilon}\rho_B(|u|),\end{aligned}$$

where, for simplicity, we omit the arguments of I_1 and I_2. With conditions (15a) and (15b), we have:

$$\dot{\Psi}_\zeta \leq -\nu(\zeta)\tilde{V}^2(x) - \omega_\varepsilon(\zeta)\tilde{W}^2(x,y) + \rho_{\varepsilon,\zeta}(|u|),$$

where

$$\begin{aligned}\nu(\zeta) &= \zeta\left(\frac{1}{2}\underline{k} - \nu_1\right) - (1-\zeta)\nu_2 \\ \omega_\varepsilon(\zeta) &= \frac{1-\zeta}{2\varepsilon}\underline{\kappa} - \zeta\omega_1 - (1-\zeta)\omega_2 \\ \rho_{\varepsilon,\zeta}(s) &= \zeta\left(\rho_R(s) + \rho_1(s)\right) + (1-\zeta)\left(\frac{\rho_B(s)}{\varepsilon} + \rho_2(s)\right),\end{aligned}$$

and we clearly have $\rho_{\varepsilon,\zeta} \in \mathcal{K}_\infty$ for $\zeta \in (0,1)$ and $\varepsilon > 0$. By (15c) we can find $\zeta^* \in (0,1)$ such that $\nu^* := \nu(\zeta^*) > 0$, and then we can find $\varepsilon^* > 0$ such that $\omega_\varepsilon(\zeta^*) > \nu^*$ whenever $\varepsilon \in (0, \varepsilon^*)$. For these $\varepsilon \in (0, \varepsilon^*)$ we obtain:

$$\begin{aligned}\dot{\Psi}_{\zeta^*} &\leq -\nu^*\left(\tilde{V}^2(x) + \tilde{W}^2(x,y)\right) + \rho_{\varepsilon,\zeta^*}(|u|) \\ &\leq -\nu^*\left(V^{a_1}(x) + V^{a_2}(x) + W^{b_1}(x,y) + W^{b_2}(x,y)\right) \\ &\quad + \rho_{\varepsilon,\zeta^*}(|u|) \\ &= -\frac{\nu^*}{2}\left((V^{a_1}(x) + V^{a_2}(x)) + (V^{a_1}(x) + V^{a_2}(x))\right. \\ &\quad \left. + (W^{b_1}(x,y) + W^{b_2}(x,y)) + (W^{b_1}(x,y) + W^{b_2}(x,y))\right) \\ &\quad + \rho_{\varepsilon,\zeta^*}(|u|).\end{aligned}$$

We pick $\gamma_1 \in [\max\{a_1, b_1\}, 1)$ and $\gamma_2 \in (1, \min\{a_2, b_2\}]$ to obtain

$$\begin{aligned}\dot{\Psi}_{\zeta^*} &\leq -\frac{\nu^*}{2}\left(V^{\gamma_1}(x) + V^{\gamma_2}(x) + W^{\gamma_1}(x,y) + W^{\gamma_2}(x,y)\right) \\ &\quad + \rho_{\varepsilon,\zeta^*}(|u|) \\ &\leq -\frac{\nu^*}{2}\left((V(x) + W(x,y))^{\gamma_1} + 2^{1-\gamma_2}(V(x) + W(x,y))^{\gamma_2}\right) \\ &\quad + \rho_{\varepsilon,\zeta^*}(|u|) \\ &\leq -\frac{\nu^*}{2}\left(\Psi_{\zeta^*}^{\gamma_1} + 2^{1-\gamma_2}\Psi_{\zeta^*}^{\gamma_2}\right) + \rho_{\varepsilon,\zeta^*}(|u|).\end{aligned}$$

Hence, (9) is FxT ISS for $\varepsilon \in (0, \varepsilon^*)$. ■

We would like to note that the "or" condition in (15c) is non-exclusive, i.e it is acceptable for both conditions to be satisfied. In this case, any choice of the weight $\zeta \in (0,1)$ will result in a valid composite Lyapunov function candidate.

Remark 2: The functions $\tilde{V}(x)$ and $\tilde{W}(x,y)$ are highly analogous to the functions $\psi_1(x)$ and $\psi_2(y)$, respectively, from [7, Chapter 11.5]. For the asymptotic stability (resp. FxT ISS) result from [7, Theorem 11.3] (resp. Theorem 1 in this paper), the Lie derivatives of V and W along the reduced and boundary layer dynamics are assumed to be upper bounded by negative multiples of $\psi_1^2(x)$ and $\psi_2^2(y)$ (resp. $\tilde{V}^2(x)$ and $\tilde{W}^2(x,y)$), respectively. Hence, we can observe that the structure of the interconnection conditions in our paper are actually more forgiving than those from [7, Chapter 11.5] in the sense that we allow for extra $\tilde{V}^2(x)$ and $\tilde{W}^2(x,y)$ terms. This extra degree of freedom is particularly useful since it allows us to apply our results to a variety of interesting systems, such as those presented later in this paper. However, our conditions are also more restrictive in the sense that we now require very specific forms for the expressions ψ_1 and ψ_2. But this is expected, since Theorem 1 considers fixed-time ISS, which is a much stronger notion of stability.

We have shown that under suitable assumptions, the inequalities in (15) imply that system (9) is FxT ISS provided there is a sufficiently large time scale separation. The proof of Theorem 1 provides an efficient methodology for computing a somewhat conservative estimate of the required timescale separation, i.e ε^*, for FxT ISS. It can also be seen from the proof of Theorem 1 that if the fast dynamics have no input, then the derived gain ϱ in the FxT ISS bound (4) can be made independent of ε. This is further detailed in the following Corollary.

Corollary 1: Consider (9) and suppose the conditions of Theorem 1 are satisfied, but with $\rho_B \equiv 0$ and the vector field g is independent of u. Then, there exists $\beta \in \mathcal{GKL}$, $\varrho \in \mathcal{K}$ and $\varepsilon^* > 0$ such that the following holds

$$|s(t)| \leq \beta\left(|s_0|, t\right) + \varrho(|u|_\infty), \tag{17}$$

for all $t \geq 0$, $u \in \mathcal{L}_\infty^p$, and $\varepsilon \in (0, \varepsilon^*)$, where $s(t) := [x(t), y(t)]^\top$.

Proof: We follow the same steps from the proof of Theorem 1, but with $\tilde{\rho}_\zeta(s) = \zeta\left(\rho_R(s) + \rho_1(s)\right) + (1-\zeta)\rho_2(s)$ instead of $\rho_{\varepsilon,\zeta}$. This results in an upper bound on $\dot{\Psi}_{\zeta^*}$ that holds uniformly for $\varepsilon \in (0, \varepsilon^*)$. ■

While the results of Theorem 1 and Corollary 1 hold for system (9), the question of whether (1) is FxT ISS may also be of interest. In other words: we ask if the transformation $y = z - h(x)$ preserves the FxT ISS property. Fortunately, as long as the quasi steady state map satisfies a mild boundedness assumption, (9) being FxT ISS implies (1) is FxT ISS. This is further detailed in the following result.

Theorem 2: Suppose (9) is FxT ISS and $|h(x)| \leq \tilde{\alpha}(|x|)$ for some $\tilde{\alpha} \in \mathcal{K}$, then (1) is also fixed time ISS. □

Proof: Since (9) is FxT ISS, there exists $\beta \in \mathcal{GKL}$ and $\varrho \in \mathcal{K}$ such that

$$\left|\begin{bmatrix} x(t) \\ z(t) - h(x(t)) \end{bmatrix}\right| \leq \beta\left(\left|\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right|, t\right) + \varrho(|u|_\infty), \tag{18}$$

where $\beta(r,t) = 0$ for each $t > T(r)$, and T is a continuous function that satisfies $\sup_{r \geq 0} T(r) < \infty$ and $T(0) = 0$.

Without loss of generality, we can assume $\beta(\cdot,t)$ is non-decreasing for each $t \geq 0$. Note that (18) implies

$$|x(t)| \leq \beta\left(\left|\begin{bmatrix}x_0\\y_0\end{bmatrix}\right|, t\right) + \varrho(|u|_\infty) \tag{19a}$$

$$|z(t)| \leq \beta\left(\left|\begin{bmatrix}x_0\\y_0\end{bmatrix}\right|, t\right) + \varrho(|u|_\infty) + |h(x(t))|. \tag{19b}$$

But we also have

$$\beta\left(\left|\begin{bmatrix}x_0\\y_0\end{bmatrix}\right|, t\right) = \beta\left(\left|\begin{bmatrix}x_0\\z_0 - h(x_0)\end{bmatrix}\right|, t\right) \leq \overline{\beta}\left(\left|\begin{bmatrix}x_0\\z_0\end{bmatrix}\right|, t\right),$$

where $\overline{\beta}(r,t) := \beta(2r,t) + \beta(2\tilde{\alpha}(r),t) \in \mathcal{GKL}$. Moreover,

$$|h(x(t))| \leq \tilde{\alpha}(|x(t)|) \leq \tilde{\beta}\left(\left|\begin{bmatrix}x_0\\z_0\end{bmatrix}\right|, t\right) + \tilde{\varrho}(|u|_\infty), \tag{20}$$

where $\tilde{\beta}(r,t) := \tilde{\alpha}\left(2\overline{\beta}(r,t)\right) \in \mathcal{GKL}$, and $\tilde{\varrho}(\cdot) := \tilde{\alpha}(2\varrho(\cdot)) \in \mathcal{K}$. Combining (19) and (20) yields the result. ■

C. A Stylized Example

To illustrate our results, we first consider a singularly perturbed system with scalar reduced and boundary layer systems. In particular, consider the plant

$$\dot{x} = -\lceil z \rfloor^{r_1} - \lceil z \rfloor^{r_2} + u_1 \tag{21a}$$

$$\varepsilon\dot{z} = -\lceil z - x - u_1 \rfloor^{q_1} - \lceil z - x - u_2 \rfloor^{q_2} + u_1 u_2, \tag{21b}$$

where $\lceil \cdot \rfloor^q := |\cdot|^q \text{sgn}(\cdot)$, $x, z, u_1, u_2 \in \mathbb{R}$, $0 < q_1 \leq r_1 < 1$ and $1 < r_2 \leq q_2$. System (21) has the quasi steady state $h(x) = x$, with reduced system:

$$\dot{x} = -\lceil x \rfloor^{r_1} - \lceil x \rfloor^{r_2} + u_1, \tag{22}$$

and boundary layer system

$$\frac{dy}{d\tau} = -\lceil y - u_1 \rfloor^{q_1} - \lceil y - u_2 \rfloor^{q_2} + u_1 u_2. \tag{23}$$

Furthermore, it is easy to see that h satisfies the class $\mathcal{K}$ bound assumption from Theorem 2. To verify that (22) and (23) satisfy Assumptions 2 and 3, we will use the Lyapunov functions $V(x) = x^2$ and $W(y) = y^2$. For the reduced system (22) we have:

$$\frac{\partial V}{\partial x} f(x, h(x), u) \leq -V^{\tilde{r}_1}(x) - V^{\tilde{r}_2}(x) + |u|^2, \tag{24}$$

where $\tilde{r}_i := \frac{1}{2}(r_i + 1)$. Thus, Assumption 2 is satisfied.

Similarly, we can differentiate W along the trajectories of (23) to obtain the following:

$$\dot{W} \leq -(2^{-q_1} - 2^{-1-q_2}) W^{\tilde{q}_1}(y) - 2^{-1-q_2} W^{\tilde{q}_2}(y),$$
$$\forall\ |y| > \max\{\varrho_B^{-1}(2^{q_2}|u|^4), 2|u|\},$$

where $\tilde{q}_i = \frac{1}{2}(q_i + 1)$ and

$$\varrho_B(s) := (2^{-q_1} - 2^{-1-q_2})|s|^{q_1+1} + 2^{-1-q_2}|s|^{q_2+1}.$$

This implies that system (23) is FxT ISS uniformly in x, and hence the assumptions are satisfied. To verify if system (21) is FxT ISS, it remains to check if the interconnection terms associated with system (21) satisfy the interconnection conditions (15). Let $\tilde{V}(x) = |x|^{\tilde{r}_1} + |x|^{\tilde{r}_2}$ and $\tilde{W}(y) = |y|^{\tilde{q}_1} + |y|^{\tilde{q}_2}$. Then, we can compute I_1 and apply Lemmas 1-2 to obtain the following estimate:

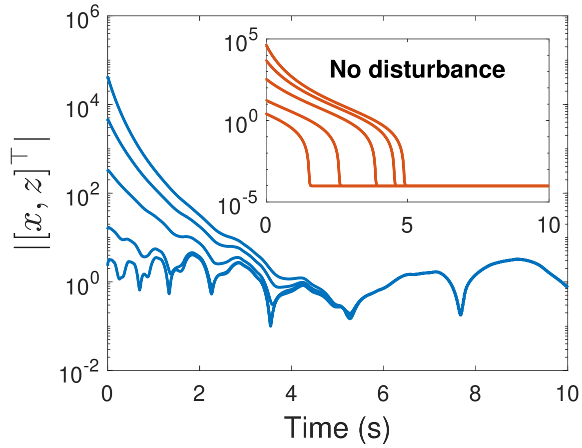

Fig. 1: Trajectories of system (21) with and without the disturbance $u(t)$, where we use $\varepsilon = 0.01$. The theoretically computed settling time bound of 18.15 obtained using (7) is conservative compared to the observed trajectories.

$$I_1 = 2x\Big(-\lceil y + x \rfloor^{r_1} - \lceil y + x \rfloor^{r_2} + \lceil x \rfloor^{r_1} + \lceil x \rfloor^{r_2}\Big)$$
$$\leq c\tilde{V}^2(x) + 2c^{-\sigma_1}\tilde{W}^2(y),$$

for all $c \in (0,1)$, where $\sigma_1 > 0$ is obtained by applying Lemma 1 twice and upper bounding appropriately. By taking $c > 0$ sufficiently small we can satisfy conditions (15a) and (15c). Thus, it remains to check if I_2 also satisfies (15b). Indeed, by computing I_2 we obtain:

$$I_2 = 2y\left(\lceil y + x \rfloor^{r_1} + \lceil y + x \rfloor^{r_2} - u_1\right)$$
$$\leq c\tilde{V}^2(x) + (2c^{-\sigma_2} + 3 + 2^{r_2})\tilde{W}^2(y) + |u|^2,$$

for all $c \in (0,1)$, where $\sigma_2 > 0$ is again obtained from Lemma 1. We conclude that the conditions of Theorem 1 and 2 are satisfied, and hence there exists ε^* such that the singularly perturbed system (21) is FxT ISS for $\varepsilon \in (0, \varepsilon^*)$. We simulate the system using $r_1 = \frac{2}{5}, r_2 = \frac{6}{5}, q_1 = \frac{1}{3}, q_2 = \frac{9}{7}$ and the disturbances $u_1(t) = e^{\sin t}, u_2(t) = \sin(19\log(t+1)) - 0.21$, where we clearly have $u(t) = [u_1(t), u_2(t)]^\top \in \mathcal{L}_\infty^2$. The trajectories of this system with and without the input are shown in Figure 1, illustrating the FxT ISS property

IV. Fixed-Time Feedback Optimization with Time-varying Costs

In this section, we leverage the results of Theorems 1-2 to study a practical problem of interest in the context of singular perturbations: feedback optimization under slowly-varying cost functions [26]. In contrast to the existing asymptotic results [27], [28], [33], [34], we introduce an optimization-based controller able to achieve FxT stability via non-Lipschitz feedback. In particular, we consider plants of the form

$$\dot{z} = g(\hat{x}, z), \tag{25}$$

where $z \in \mathbb{R}^m$ is the state, $\hat{x} \in \mathcal{L}_\infty^n$ is a measurable and bounded control input, and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function satisfying the following condition:

Assumption 4: There exists a continuously differentiable, globally Lipschitz mapping $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(\hat{x}, h(\hat{x})) = 0$ for all $\hat{x} \in \mathbb{R}^n$. Moreover, there exists a $\mathcal{C}^1$ function $W : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $c_1, c_2 > 0$ such that

$$c_1|y|^2 \leq W(y) \leq c_2|y|^2, \tag{26a}$$

$$\nabla W(y)^\top g(\hat{x}, y + h(\hat{x})) \leq -\kappa_1 W^{b_1}(y) - \kappa_2 W^{b_2}(y), \tag{26b}$$

for all $y \in \mathbb{R}^m$, where $\kappa_1, \kappa_2 > 0$, $b_1 \in (0, 1)$ and $b_2 > 1$. There also exists $\eta > 0$ such that the function W satisfies

$$|\nabla W(y)| \leq \eta |y|, \tag{27}$$

for all $y \in \mathbb{R}^m$. □

Assumption 4 is the "fixed-time" version of standard open-loop stability assumptions considered in the setting of feedback optimization [7], [27]. In particular, by taking $y = z - h(\hat{x})$ to quantify the deviation of z from its steady-state approximation, the conditions of Assumption 4 simply ask that such deviation converges to zero by a fixed-time, for each fixed $\hat{x} \in \mathbb{R}^n$. For a general class of linear and nonlinear plants, this property can be achieved using different types of non-smooth controllers that combine super-linear and sub-linear feedback [35], homogeneity tools [36], or the implicit Lyapunov technique [37], to name just a few examples.

Our primary goal is to design a control law on the input $\hat{x}$ that stabilizes (25) in a fixed time the solution of the following time-varying optimization problem

$$\min_{\hat{x}, z} \ \phi_\theta(\hat{x}, z) \tag{28a}$$

$$\text{subject to:} \quad z = h(\hat{x}). \tag{28b}$$

where the time variation on the cost functions $\phi_\theta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is induced by a dynamic parameter $\theta \in \mathbb{R}^q$ that evolves according to the dynamics

$$\dot{\theta} = \varepsilon \varepsilon_0 \Pi(\theta), \quad \theta \in \Theta, \tag{29}$$

where $\varepsilon_0 \geq 0$, and $\varepsilon > 0$ is a small parameter that captures the rate of change of θ, and $\Pi : \mathbb{R}^q \to \mathbb{R}^q$ and $\Theta \subset \mathbb{R}^q$ satisfy the following mild conditions, which can be used to cover a broad class of time-varying signals $t \mapsto \theta(t)$:

Assumption 5: The function $\Pi(\cdot)$ is Lipschitz continuous, and the set Θ is compact and forward invariant under the dynamics (29). □

We can observe that, by substituting (28b) into (28a), we arrive at the unconstrained parameterized optimization problem:

$$\min_{\hat{x}} \Phi_\theta(\hat{x}), \tag{30}$$

where $\Phi_\theta(\hat{x}) := \phi_\theta(\hat{x}, h(\hat{x}))$. To guarantee that (30) is a well-defined optimization problem with a unique solution for each $\theta \in \mathbb{R}^q$, we consider cost functions that satisfy the following assumptions, which are fairly standard in the time-varying feedback optimization literature [33], [34]:

Assumption 6: The function $\hat{x} \mapsto \Phi_\theta(\hat{x})$ is L-smooth and κ-strongly convex, uniformly in θ. Moreover, there exists a $\mathcal{C}^1$ function $\varphi : \mathbb{R}^q \to \mathbb{R}^n$ such that $\varphi(\theta) = \arg\min_{\hat{x}} \Phi_\theta(\hat{x})$. □

Note that when $\varepsilon_0 = 0$ in (5), the parameter θ remains constant for all time, yielding a constant solution $\varphi(\theta)$ to (30). In contrast, when $\varepsilon_0 \gg 0$, the function $t \mapsto \varphi(\theta(t))$ may exhibit fast time variations that are difficult to track without additional information about the functions Π, h, and ϕ_θ. Therefore, to address the optimization problem (28a) using real-time gradient feedback, we can regard $\varepsilon_0 \Pi(\theta)$ as the "input" to the system and study FxT ISS with respect to the *tracking error* of $\varphi(\theta(\cdot))$.

A. Fixed-Time Gradient-Based Feedback

Given $\xi_1 \in (0, 1)$ and $\xi_2 < 0$, we define the following function $\mathcal{F}_{\xi_1, \xi_2} : \mathbb{R}^p \to \mathbb{R}^p$:

$$\mathcal{F}_{\xi_1, \xi_2}(x) = \frac{x}{|x|^{\xi_1}} + \frac{x}{|x|^{\xi_2}}, \tag{31}$$

which is continuous at $x = 0$, see [15]. To solve (30) in fixed-time, we draw inspiration from the asymptotic counterpart [27] and propose a *fixed-time* gradient flow on $\Phi_\theta(\hat{x})$ with time scale separation:

$$\dot{\hat{x}} = -\varepsilon \mathcal{F}_{\xi_1, \xi_2}(\hat{P}_\theta(\hat{x}, h(\hat{x}))), \tag{32}$$

where

$$\hat{P}_\theta(\hat{x}, z) := H(\hat{x})^\top \nabla \phi_\theta(\hat{x}, z), \quad H(x)^\top = [\mathbb{I}_n \quad \mathbf{J}_h(x)^\top]. \tag{33}$$

It can be verified, using the chain rule, that

$$\hat{P}_\theta(\hat{x}, h(\hat{x})) = \nabla \Phi_\theta(\hat{x}),$$

and hence, for each θ, when the plant dynamics (25) are negligible, the dynamics (32) converge to the solution of (30) in fixed time [15], [16]. Since in our setting the dynamics (25) cannot be neglected, we can obtain a real-time feedback controller by replacing $h(\hat{x})$ in (32) with the measured value of z to obtain the following closed-loop system:

$$\dot{z} = g(\hat{x}, z) \tag{34a}$$

$$\dot{\hat{x}} = -\varepsilon \mathcal{F}_{\xi_1, \xi_2}(\hat{P}_\theta(\hat{x}, z)). \tag{34b}$$

We study system (34) under the following mild Lipschitz assumption, which is standard in the literature [27]:

Assumption 7: For the function $\hat{P}_\theta(x, z)$ defined in (33), there exists $\ell > 0$ such that

$$|\hat{P}_\theta(\hat{x}, z') - \hat{P}_\theta(\hat{x}, z)| \leq \ell |z' - z|, \tag{35}$$

for all $\hat{x} \in \mathbb{R}^n$, $z', z \in \mathbb{R}^m$ and $\theta \in \mathbb{R}^q$. □

To put system (34) into the form (1), let $x = \hat{x} - \varphi(\theta)$ and $\tau = \varepsilon t$, which leads to the following dynamics in the τ-time scale:

$$\varepsilon \frac{dz}{d\tau} = g(x + \varphi(\theta), z) \tag{36a}$$

$$\frac{dx}{d\tau} = -\mathcal{F}_{\xi_1, \xi_2}(\hat{P}_\theta(x + \varphi(\theta), z)) - \mathbf{J}_\varphi(\theta) u(\tau) \tag{36b}$$

$$\frac{d\theta}{d\tau} = \varepsilon_0 \Pi(\theta), \quad \theta \in \Theta, \tag{36c}$$

where $u(\tau) = \varepsilon_0 \Pi(\theta(\tau))$ can be thought of as the "input" in (36b). Note that, since Θ is compact and forward invariant, and the trajectories of (36c) are restricted to evolve in Θ, we only need to consider the stability properties of system (36a)-(36b). The following result establishes FxT ISS for this system:

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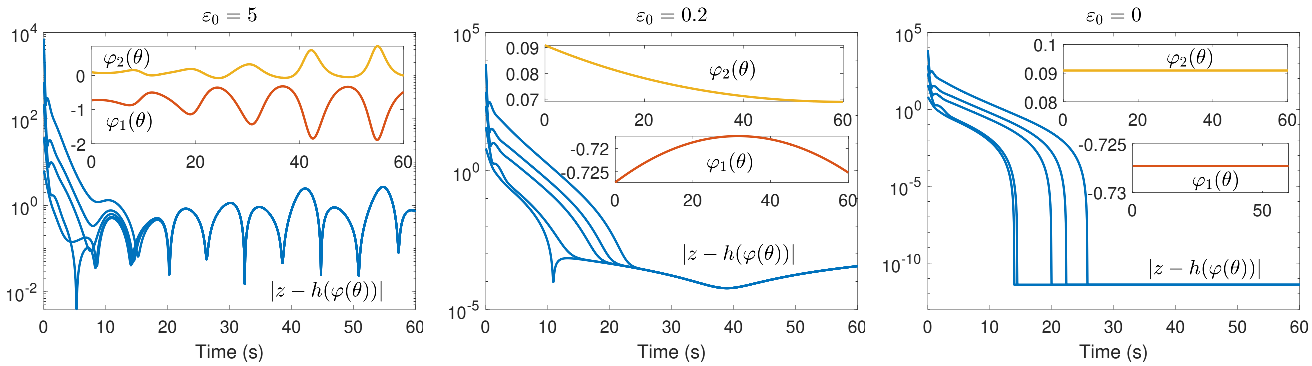

Fig. 2: Trajectories of the FxT time-varying feedback optimization example, with $\varepsilon_0 = 5, 0.2, 0$ and varying initial conditions.

Theorem 3: Suppose that Assumptions 4-6 hold, and consider the singularly perturbed system (36). Then, for all $\xi_1 \in (0, \min(2-2b_1, 1))$ and $\xi_2 \in (2-2b_2, 0)$, there exists $\beta \in \mathcal{GKL}$, $\varrho \in \mathcal{K}$ and $\varepsilon^* > 0$ such that

$$\left|\begin{bmatrix} x(\tau) \\ z(\tau) - h(x(\tau) + \varphi(\theta(\tau))) \end{bmatrix}\right| \leq \beta\left(\left|\begin{bmatrix} x_0 \\ z_0 - h(x_0 + \varphi(\theta_0)) \end{bmatrix}\right|, \tau\right) + \varrho(|u(\tau)|_\infty),$$

for all $\varepsilon \in (0, \varepsilon^*)$, $\tau \geq 0$ and $\varepsilon_0 \geq 0$. □

Proof: Consider the τ-time scale system (36a). This system has a quasi-steady state $h(x + \varphi(\theta))$, which results in the following reduced system:

$$\frac{dx}{d\tau} = -\mathcal{F}_{\xi_1,\xi_2}(\nabla\Phi_\theta(x + \varphi(\theta))) - \varepsilon_0 \mathbf{J}_\varphi(\theta)\Pi(\theta). \tag{37}$$

Let $M = \sup_{\theta\in\Theta} |\mathbf{J}_\varphi(\theta)|$, which is bounded since Θ is compact. Moreover, from Assumption 6, we have $\hat{x}^\top \nabla\Phi_\theta(\hat{x} + \varphi(\theta)) \geq \frac{1}{L}|\nabla\Phi_\theta(\hat{x} + \varphi(\theta))|^2$ and $|\nabla\Phi_\theta(\hat{x} + \varphi(\theta))| \geq \kappa|\hat{x}|$. Then, with the Lyapunov function $V(x) = |x|^2$, we can use Lemma 1 to obtain

$$\begin{aligned} \frac{dV}{d\tau} &\leq -\frac{2}{L}|\nabla\Phi_\theta(x + \varphi(\theta))|^{2-\xi_1} - \frac{2}{L}|\nabla\Phi_\theta(x + \varphi(\theta))|^{2-\xi_2} \\ &\quad + c|x|^2 + \frac{M^2}{c}|\varepsilon_0\Pi(\theta)|^2 \\ &\leq -\frac{\kappa^{2-\xi_1}}{L}V^{1-\frac{1}{2}\xi_1}(x) - \frac{\kappa^{2-\xi_2}}{L}V^{1-\frac{1}{2}\xi_2}(x) + \frac{M^2}{c}|\varepsilon_0\Pi(\theta)|^2, \end{aligned}$$

where $0 < c < \frac{1}{L}\min\left\{\kappa^{2-\xi_1}, \kappa^{2-\xi_2}\right\}$, which shows that Assumption 2 is satisfied. Next, let $y = z - h(x + \varphi(\theta))$, which leads to the following dynamics:

$$\begin{aligned} \frac{dy}{d\tau} &= \frac{1}{\varepsilon}g(x + \varphi(\theta), y + h(x + \varphi(\theta))) \\ &\quad - \mathbf{J}_h(x + \varphi(\theta))\left(\frac{dx}{d\tau} + \varepsilon_0\mathbf{J}_\varphi(\theta)\Pi(\theta)\right). \end{aligned} \tag{38}$$

This yields the following boundary layer system:

$$\dot{y} = g(x + \varphi(\theta), y + h(x + \varphi(\theta))). \tag{39}$$

It is easy to verify that Assumption 3 holds with Lyapunov function $W(y)$ obtained from Assumption 4, so it remains to check the interconnection conditions (15a)-(15b). Indeed, if we denote $P_\theta(x, y) := \hat{P}_\theta(x + \varphi(\theta), y + h(x + \varphi(\theta)))$, we have:

$$\begin{aligned} |I_1| &\leq 2|x|\left(\left|\frac{P_\theta(x,y)}{|P_\theta(x,y)|^{\xi_1}} - \frac{P_\theta(x,0)}{|P_\theta(x,0)|^{\xi_1}}\right|\right. \\ &\quad \left. + \left|\frac{P_\theta(x,y)}{|P_\theta(x,y)|^{\xi_2}} - \frac{P_\theta(x,0)}{|P_\theta(x,0)|^{\xi_2}}\right|\right) \\ &\leq 2^{\xi_1+1}\ell^{1-\xi_1}|x||y|^{1-\xi_1} \\ &\quad + 2K|x||y|\left(\ell^{-\xi_2}|y|^{-\xi_2} + L^{-\xi_2}|x|^{-\xi_2}\right) \\ &\leq c\left(|x|^{2-\xi_1} + |x|^{2-\xi_2}\right) + \frac{1}{\mu_1(c)}\left(|y|^{2-\xi_1} + |y|^{2-\xi_2}\right) \\ &\leq c\tilde{V}^2(x) + \frac{c_1^{\frac{1}{2}\xi_1 - 1} + c_1^{\frac{1}{2}\xi_2 - 1}}{\mu_1(c)}\tilde{W}^2(y), \end{aligned}$$

for all $c > 0$, and $\mu_1 \in \mathcal{K}_\infty$ is given by $\mu_1(c) := m^{-1}\left(c^{-\sigma_1} + c^{-\sigma_2} + c^{-\frac{1}{\sigma_2}}\right)^{-1}$, with $\sigma_i = \frac{1}{1-\xi_i}$ and $m := \max\{(2^{\xi_1+1}\ell^{1-\xi_1})^{1+\sigma_1}, (4K\ell^{-\xi_2})^{1+\sigma_2}, (4K\ell^{-\xi_2})^{1+\frac{1}{\sigma_2}}\}$. Picking $c < \frac{1}{L}\min\left\{\kappa^{2-\xi_1}, \kappa^{2-\xi_2}\right\}$ establishes (15a) and (15c). To verify (15b), we have

$$I_2 = \nabla W(y)^\top \mathbf{J}_h(x + \varphi(\theta))\mathcal{F}_{\xi_1,\xi_2}(P_\theta(x, y)). \tag{40}$$

Let $h^* := \sup_x |\mathbf{J}_h(x)|$, which exists and is finite since h is $\mathcal{C}^1$ and globally Lipschitz. We can proceed from (40) as follows:

$$\begin{aligned} |I_2| &\leq \eta h^*|y|\left((\ell|y| + L|x|)^{1-\xi_1} + (\ell|y| + L|x|)^{1-\xi_2}\right) \\ &\leq \tilde{L}\left(|x|^{2-\xi_1} + |x|^{2-\xi_2} + |y|^{2-\xi_1} + |y|^{2-\xi_2}\right) \\ &\leq \tilde{L}\left(\tilde{V}^2 + 2\tilde{W}^2\right), \end{aligned}$$

where $\tilde{L} := \eta h^* \max\{\ell^{1-\xi_1}, L^{1-\xi_1}, 2^{-\xi_2}\ell^{1-\xi_2}, 2^{-\xi_2}L^{1-\xi_2}\}$. By applying Corollary 1 we obtain the result. ■

To illustrate Theorem 3 via a numerical example, we simulate system (34) with the plant dynamics given by $\dot{z} = -\mathcal{F}_{\frac{2}{5},-\frac{2}{7}}(z - 2\hat{x})$, which has the quasi-steady state $h(\hat{x}) = 2\hat{x}$. The cost function ϕ_θ takes the quadratic form $\phi_\theta(\hat{x}, z) = \frac{1}{2}\hat{x}^\top Q_\theta z + b_\theta^\top z$, where Q_θ and b_θ are given by

$$Q_\theta := \begin{bmatrix} 3 + d_1(t) & 2 \\ 2 & 5 + d_2(t) \end{bmatrix}, \quad b_\theta := \begin{bmatrix} 2 + d_3(t) \\ 1 \end{bmatrix}.$$

The parameters d_i are given by $d_1(t) = 0.8\sin(2.2\varepsilon\varepsilon_0 t)$, $d_2(t) = 1.8\sin(1.7\varepsilon\varepsilon_0 t)$, $d_3(t) = 0.66\sin(1.9\varepsilon\varepsilon_0 t)$. These signals can be generated by a system of the form (29) by setting $\theta(t) \in \mathbb{R}^6$, $d_i(t) = \theta_{2i}(t)$, and $\Pi(\theta) = \mathcal{R}\theta$, where

$\mathcal{R} \in \mathbb{R}^{6\times 6}$ is a block diagonal matrix with rotation matrices on the diagonal. Moreover, in accordance with Proposition 3 we set $\xi_1 = \frac{1}{3}, \xi_2 = -\frac{1}{5}$. It can be verified that $Q_\theta \succ 0$ for all $t \geq 0$, and the optimizer of Φ_θ is given by $\varphi(\theta) = -Q_\theta^{-1} b_\theta$. We interconnect (34b) with the plant dynamics, where $\hat{P}_\theta(\hat{x}, z) = \frac{1}{2} Q_\theta z + Q_\theta \hat{x} + 2b_\theta$. The trajectories of the system are shown in Figure 2, with $\varepsilon = 0.05$ and different values of ε_0. As observed in the plot, the state z converges in fixed-time to a neighborhood of the time-varying optimizer, whose size shrinks as $\varepsilon_0 \to 0^+$.

V. Conclusion

We establish sufficient Lyapunov conditions for the study of FxT ISS properties in singularly perturbed systems. The results were applied to two illustrative examples: a particular nonsmooth second-order interconnection of systems, and a general fixed-time feedback optimization problem with time-varying cost functions, which has not been addressed before using fixed-time stability tools. Our method of verifying the interconnection conditions establishes an efficient paradigm for applying our results to other classes of algorithms and feedback schemes that exhibit multiple time scales. Future research directions include applying our results to a broader range of systems, including systems with more than two time scales. Moreover, it is also of interest to identify a more general class of systems and quasi-steady state mappings for which our interconnection conditions hold, including characterizations based on homogeneity.

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