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Properties of Range Sets of Continuous Functions in Reverse Mathematics

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Abstract. The range sets of continuous functions from $[0,1]$ into a complete separate metric space are closed, connected, compact, and bounded. The classification of these properties in reverse mathematics has not yet been fully explored prior to this work. Over RCA_0, the existence of closed set codes for the range sets of continuous functions is equivalent to WKL_0. The connectedness property is provable in RCA_0 and the compactness and boundedness properties are equivalent to WKL_0, as long as these properties are carefully defined to avoid the use of a closed code of $\mathsf{range}(\mathsf{f})$.

Keywords: Reverse Mathematics · Computability Theory · Computable Analysis

1 Introduction

1.1 Motivation

One of the first areas of interest in the early development of reverse mathematics was the classification of the most fundamental theorems of real analysis. A majority of these theorems are covered in great detail in Simpson's seminal text *Subsystems of Second Order Arithmetic* [6] and Brown's doctoral thesis *Functional Analysis in Weak Subsystems of Second Order Arithmetic* [1].

The Hahn-Mazurkiewicz Theorem (see [5]) is an early result in geometric analysis that has not yet been studied within the context of reverse mathematics.

Theorem 1 (Hahn-Mazurkiewicz). *A set $M \subseteq \mathbb{R}^n$ is the continuous image of $[0,1]$ if and only if it is compact, connected, and locally connected.*

The forward implication of the Hahn-Mazurkiewicz Theorem is, in standard mathematics, the simpler one. Proving that the image of $[0,1]$ is compact and connected is a fairly fundamental proof from a standard Real Analysis course. Local connectedness is generally not preserved by continuous functions, but if the domain is both locally connected and compact, the local connectedness is preserved. Regardless, proving a quality of the ranges of continuous functions (especially on a well-understood domain like $[0,1]$) is typically much simpler

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than constructing a continuous function given a potential range set, which is required for the backwards implication.

This backwards implication has been explored within the context of computable analysis by Couch, Daniel, and McNicholl [3] and Daniel and McNicholl [4]. The forward implication, despite its relative simplicity in standard mathematics, still requires careful consideration within reverse mathematics. These considerations are the focus of this paper.

We will first provide some preliminaries concerning how we represent and work with common features of real analysis within reverse mathematics.

1.2 Preliminaries

In reverse mathematics, we work within subsystems of second order arithmetic (Z_2). The three subsystems of interest to this paper are RCA_0, WKL_0, and ACA_0.

Definition 1. RCA_0 *is the typical base system for reverse mathematics, the subsystem of* Z_2 *consisting of* PA^-*, the* Σ^0_1 *induction scheme, and the* Δ^0_1 *comprehension scheme. (See Definition I.7.4 in Simpson [6]).*

Note that RCA_0 is sufficient to prove bounded Σ^0_1 comprehension (Theorem II.3.9 in Simpson [6])

Definition 2. WKL_0 *is the subsystem of* Z_2 *consisting of* RCA_0 *and Weak König's Lemma, which states the existence of infinite paths of infinite binary trees* $T \subseteq 2^{<\mathbb{N}}$*. (See Definition I.10.1 in Simpson [6]).*

Definition 3. ACA_0 *is the subsystem of* Z_2 *consisting of* PA^-*, the arithmetical induction scheme, and the arithmetical comprehension scheme. (See Definition I.3.2 in Simpson [6]).*

Note that WKL_0 is an intermediary system, strictly stronger than RCA_0 but strictly weaker than ACA_0 [6]. Later, we will utilize the fact that WKL_0 is equivalent over RCA_0 to Bounded König's Lemma, which states that "If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is an infinite tree and $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $\sigma \in T$ and $n < |\sigma|$, $\sigma(n) \leq f(n)$, then T has an infinite path". (See Lemma IV.1.4 in Simpson [6]).

When working with real numbers in reverse mathematics, we work with them via a representation by rapidly Cauchy sequences of rationals.

Definition 4. (RCA_0) *A real number* r *is a sequence of rational numbers* $\langle q_k : k \in \mathbb{N}\rangle$ *such that for all* $k, i \in \mathbb{N}$, $|q_k - q_{k+i}| \leq 2^{-k}$*. (See Definition II.4.4 in Simpson [6]).*

We specify that these sequences representing real numbers are rapidly Cauchy as the explicit convergence bound of 2^{-k} makes these representations much easier to use in weak subsystems like RCA_0. For reals $x = \langle x_k : k \in \mathbb{N}\rangle$ and $y = \langle y_k : k \in \mathbb{N}\rangle$, the statement $x \leq y$ is expressed by the Π^0_1 formula $\forall k(x_k \leq y_k + 2^{-k+1})$ and the statement $x < y$ is expressed by the Σ^0_1 formula $\exists k(x_k + 2^{-k+1} < y_k)$.

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As the statement $x \leq y$ is Π^0_1, it will be convenient to be able to reference statements of this form using a bounded quantifier. Given natural number m, we will state "$x \leq y$ as measured up to m" to mean $\forall k \leq m(x_k \leq y_k + 2^{-k+1})$.

We also use a similar coding for elements of any complete separable metric space. (See Definition II.5.1 in Simpson [6]).

Definition 5. (RCA_0) *A code for a complete separable metric space $\widehat{B}$ is given by a nonempty set $B \subseteq \mathbb{N}$ and a sequence $d : B \times B \to \mathbb{R}$ of reals satisfying the usual metric conditions.*

- $\forall a \in B(d(a, a) = 0)$
- $\forall a, b \in B(d(a, b) = d(b, a))$
- $\forall a, b, c \in B(d(a, c) \leq d(a, b) + d(b, c))$

A point $x \in \widehat{B}$ is a sequence $\langle b_k : k \in \mathbb{N}\rangle$ of elements of B such that for all $k, i \in \mathbb{N}$, $d(b_k, b_{k+i}) \leq 2^{-k}$.

Using this general definition, $\mathbb{R}$ is given by $\widehat{\mathbb{Q}}$.

Since reverse mathematics concerns second order arithmetic, we cannot work directly with open and closed subsets of $\mathbb{R}$ (or any other complete separable metric space $\widehat{B}$). Instead we consider codes of these subsets, relying on countable collections of basic open sets from which we can build our desired subsets.

Definition 6. (RCA_0) *A basic open subset of $\widehat{B}$ is an open metric ball with center $a \in B$ and positive rational radius r, represented as (a, r) or $B_r(a)$. An open set $U \subseteq \widehat{B}$ is given by a code $U \subseteq \mathbb{N} \times B \times \mathbb{Q}^+$ where for $x \in \widehat{B}$, $x \in U \iff \exists n, a, r$ such that $(n, a, r) \in U$ and $d(x, a) < r$. (See Definition II.5.6 in Simpson [6]).*

Closed sets are coded by the same type of sets, but the condition on membership is different.

Definition 7. (RCA_0) *A closed set $C \subseteq \widehat{B}$ is given by a code $C \subseteq \mathbb{N} \times B \times \mathbb{Q}^+$, where $x \in C \iff \forall(n, a, r) \in C$, $d(x, a) \geq r$. (See Definition II.5.12 in Simpson [6]).*

Note that the code of an open set provides basic open sets contained within the open set, while the code of a closed set is a code for its open complement. Even though a subset and its code are different kinds of objects, we will sometimes refer to both with the same variable as long as the context makes it clear which use is relevant.

There are two equivalent versions of Definition 7 that will be useful moving forward.

Lemma 1. (RCA_0) *Every closed set $C \subseteq \widehat{B}$ has a sequence code $C = \langle\langle a_n, r_n\rangle : n \in \mathbb{N}\rangle$, where each $a_n \in B$ and for $x \in \widehat{B}$, $x \in C \iff \forall n(d(x, a_n) \geq r_n)$.*

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Lemma 2. (RCA_0) *Suppose $\varphi(x)$ is a Π^0_1 formula such that for all $x, y \in \widehat{B}$, if $x = y$ and $\varphi(x)$ then $\varphi(y)$. Then there is a closed $C \subseteq \widehat{B}$ such that for $x \in \widehat{B}$, $x \in C \iff \varphi(x)$. (See Lemma II.5.7 in Simpson [6]).*

Note that membership in a closed set is described by a Π^0_1 formula, while membership in an open set is described by a Σ^0_1 formula.

It is also necessary to work with codes for continuous functions on real numbers. We will code these functions as a set of quintuples following the approach in Simpson [6].

Definition 8. (RCA_0) *A code for a continuous function $f : [0,1] \to \widehat{B}$ is a set of quintuples $f \subseteq \mathbb{N} \times (\mathbb{Q} \cap [0,1]) \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which has the following properties. We use $(a,r)f(b,s)$ as short hand for $\exists n(n,a,r,b,s) \in f$.*

- $(a,r)f(b,s) \wedge (a,r)f(b',s') \implies d(b,b') \leq s + s'$
- $(a,r)f(b,s) \wedge (a',r') \subseteq (a,r) \implies (a',r')f(b,s)$
- $(a,r)f(b,s) \wedge (b,s) \subseteq (b',s') \implies (a,r)f(b',s')$
- *For all $x \in [0,1]$ and for all $\epsilon > 0$, there exists $(a,r)f(b,s)$ such that $d(x,a) < r$ and $s < \epsilon$.*

(See Definition II.6.1 in Simpson [6]).

Intuitively, $(a,r)f(b,s)$ represents that the open ball $B_r(a)$ in $[0,1]$ is mapped into the closed ball $\overline{B}_s(b)$ in $\widehat{B}$. Like earlier, while a function and its code are different kinds of objects, we will often refer to both with the same variable as long as the context makes it clear which sort of use is relevant.

When working with compact sets in reverse mathematics, some caution is necessary. The Heine-Borel Theorem is equivalent to WKL_0 and the Bolzano-Weierstrass Theorem is equivalent to ACA_0 [6]. So if we want to work with compact sets within RCA_0, we need to avoid a definition of compactness that implies either Heine-Borel or Bolzano-Weierstrass within RCA_0. Therefore, the standard definition of compactness in reverse mathematics is a relatively weak one. It is usually stated for complete separable metric spaces (including in Definition III.2.3 in Simpson [6]), but here we state it for closed subsets of complete separable metric spaces.

Definition 9. (RCA_0) *A closed set C is a compact subset of $\widehat{B}$ if there exists an infinite sequence of finite sequences $\langle\langle x_{i,j} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ (with $x_{i,j} \in \widehat{B}$) such that for all $j \in \mathbb{N}$ and $x \in C$, there exists some $i \leq n_j$ such that $d(x_{i,j}, x) < 2^{-j}$.*

When it comes to the completeness of $\mathbb{R}$, a similar caution is needed. If we are working in RCA_0, we cannot think of $\mathbb{R}$ being complete in the sense of the Least Upper Bound Property as that is equivalent to ACA_0. Instead, we will use the Nested Interval Completeness of $\mathbb{R}$, which is provable in RCA_0.

Theorem 2 (Nested Interval Completeness of $\mathbb{R}$). (RCA_0) *Let $\langle a_n : n \in \mathbb{N}\rangle$ and $\langle b_n : n \in \mathbb{N}\rangle$ be sequences of real numbers such that for all n, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, and $\lim_n |a_n - b_n| = 0$. Then there exists a real number x such that $x = \lim_n a_n = \lim_n b_n$. (See Theorem II.4.8 in [6]).*

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The nested intervals that are the name-sake of this theorem are the collection of $[a_n, b_n]$ for all n, and x is the unique intersection point of this collection.

Now we can move forward to the main properties of interest in this paper. In Sect. 2, we will show that the representation of $\mathsf{range(f)}$ as a closed set is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. This result will require us to adapt how we define properties of range sets to avoid the necessity of $\mathsf{WKL_0}$ to gain access to a code for $\mathsf{range(f)}$ as a closed set. In Sect. 3, we will show that a continuous function having connected range is provable in $\mathsf{RCA_0}$. In Sect. 4, we will show that a continuous function having compact or bounded range is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. In Sect. 5, we will discuss how we intend to use these results in our continued research.

2 range(f) as a Closed Set

One of the most familiar results in reverse mathematics about the range of functions is this common bridge theorem for $\mathsf{ACA_0}$.

Theorem 3. ($\mathsf{RCA_0}$) *The following are equivalent:*

- $\mathsf{ACA_0}$
- *For a one-to-one function $f : \mathbb{N} \to \mathbb{N}$, the range of f exists.*

(See Lemma III.1.3 in Simpson [6]).

Our main interest is in the range of continuous functions $f : [0, 1] \to \mathbb{R}^n$. Since $[0, 1]$ is compact, the range of f is closed. Therefore, the natural way to code $\mathsf{range(f)}$ in reverse mathematics is as a closed set.

Before we discuss our main result for this section, we will briefly discuss a special case when the code of the range set can exist within $\mathsf{RCA_0}$. The next lemma is an adaptation of the fact that $\mathsf{RCA_0}$ proves the existence of the range set of every increasing function $f : \mathbb{N} \to \mathbb{N}$.

Lemma 3. ($\mathsf{RCA_0}$) *Let $f : [0, 1] \to \mathbb{R}$ be increasing. Then there exists a code for* $\mathsf{range(f)}$ *as a closed set.*

Proof. Let f be as hypothesized. In $\mathsf{RCA_0}$ we can form the reals $f(0)$ and $f(1)$. Since f is increasing, $\mathsf{range(f)}$ is the closed set $[f(0), f(1)]$, which has a closed set code as defined by the following Π^0_1 formula.

$$y \in [f(0), f(1)] \iff f(0) \le y \le f(1) \tag{1}$$

□

In our proof that $\mathsf{WKL_0}$ suffices to form a code for $\mathsf{range(f)}$ as a closed set in the general case, we use $\mathsf{WKL_0}$ in two ways. First, we will use $\mathsf{WKL_0}$ to allow access to a modulus of uniform continuity of f.

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Definition 10. *Let $f : [0,1] \to \widehat{B}$ be continuous. A modulus of uniform continuity of f is a function $h : \mathbb{N} \to \mathbb{N}$ where for all $n \in \mathbb{N}$ and all $x, y, \in [0,1]$, if $d(x,y) < 2^{-h(n)}$, then $d(f(x), f(y)) < 2^{-n}$. (See Definition IV.2.1 in [6]).*

Theorem 4. ($\mathsf{WKL_0}$) *For any continuous $f : [0,1] \to \widehat{B}$, there exists some modulus of uniform continuity h of f. (See Theorem IV.2.2 in [6]).*

This modulus of uniform continuity will be instrumental in the proof, but we do need the modulus to have some additional properties, which follow from Simpson's proof of Theorem 4.

Lemma 4. ($\mathsf{WKL_0}$) *Every continuous $f : [0,1] \to \widehat{B}$ has a modulus of uniform continuity h that satisfies all three of the following conditions:*

1. *$h(0) \geq 1$*
2. *h is strictly increasing*
3. *For every n and $i \leq 2^{h(n)}$ there is a point $b \in B$ such that $(\frac{i}{2^{h(n)}}, 2^{-h(n)+2}) f(b, 2^{-n})$*

There is one more necessary lemma, which clarifies some of the properties of an element $x \in [0,1]$ that we will construct as part of the proof of the main result of this section.

Lemma 5. ($\mathsf{RCA_0}$) *Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with $g(0) \geq 1$. If $x = \langle x_n : n \in \mathbb{N} \rangle$ is a sequence of rationals such that for all n, $|x_{n+1} - x_n| < 2^{-g(n)}$, then $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$, $d(x, x_n) \leq 2^{-g(n)+1}$.*

Proof. First we will show that $x \in \mathbb{R}$. Let $n, i \in \mathbb{N}$ be arbitrary. Since $g(0) \geq 1$ and g is strictly increasing, we have $g(n) \geq n+1$ and $g(n+k) \geq g(n) + k$ for all n and k. From this, we have that

$$|x_{n+1} - x_n| < 2^{-g(n)} < 2^{-(n+1)} \tag{2}$$

Applying the triangle inequality results in the following:

$$|x_{n+i} - x_n| < \sum_{k=0}^{i-1} 2^{-(n+k+1)} = 2^{-n-1} \sum_{k=0}^{i-1} 2^{-k} \leq 2^{-n} \tag{3}$$

Therefore $x \in \mathbb{R}$. To complete the proof of the lemma, let $\epsilon > 0$, and let $i \in \mathbb{N}$ be sufficiently large such that $d(x, x_{n+i}) < \epsilon$. Applying the triangle inequality results in the following:

$$d(x, x_n) \leq d(x, x_{n+i}) + d(x_{n+i}, x_n) \leq \epsilon + 2^{-g(n)+1} \tag{4}$$

Since the above holds for all $\epsilon > 0$, $d(x, x_n) \leq 2^{-g(n)+1}$ as desired. □

Now we will prove the main theorem of interest in this section in two parts.

Theorem 5. ($\mathsf{WKL_0}$) *Every continuous $f : [0,1] \to \widehat{B}$ has a code for* $\mathsf{range(f)}$ *as a closed set.*

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Proof. Assume WKL_0 and let f be given as hypothesized. By Theorem 4, f has a modulus of uniform continuity h. By Lemma 4, we can assume that h is strictly increasing, $h(0) \geq 1$, and for every n and $i \leq 2^{h(n)}$ there is a point $b \in B$ such that $(\frac{i}{2^{h(n)}}, 2^{-h(n)+2})f(b, 2^{-n})$. For each $n \in \mathbb{N}$ and $i \leq 2^{h(n)}$, define $b_{n,i} \in B$ to be a point such that $(\frac{i}{2^{h(n)}}, 2^{-h(n)+2})f(b_{n,i}, 2^{-n})$.

We define the closed set code C_f for $\mathsf{range(f)}$ with the following Π^0_1 formula. For $y \in \widehat{B}$, let $y \in C_f$ if and only if for all finite sequences $\langle\langle n_0, i_0\rangle, \dots \langle n_k, i_k\rangle\rangle$ such that $i_j \leq 2^{h(n_j)}$ and for all $j \leq k$ and the union of the balls $B_{2^{-h(n_j)}}(\frac{i_j}{2^{h(n_j)}})$ covers $[0, 1]$, there exists $j \leq k$ for which $d(y, b_{n_j,i_j}) \leq 2^{-n_j}$.

We need to verify that for $y \in \widehat{B}$, $y \in C_f$ if and only if there exists some $x \in [0, 1]$ such that $f(x) = y$.

First we will show that if $y \notin C_f$, then y is not in the range of f. Suppose $y \notin C_f$. Then there exists some sequence $\langle\langle n_0, i_0\rangle, \dots \langle n_k, i_k\rangle\rangle$ that witnesses this fact, where the union of the balls $B_{2^{-h(n_j)}}(\frac{i_j}{2^{h(n_j)}})$ covers $[0, 1]$ but for all $j \leq k$ we have that $d(y, b_{n_j,i_j}) > 2^{-n_j}$. For the sake of contradiction, suppose there exists some $x \in [0, 1]$ such that $f(x) = y$. Since the union of the balls $B_{2^{-h(n_j)}}(\frac{i_j}{2^{h(n_j)}})$ covers $[0, 1]$, there exists some $j \leq k$ such that $x \in B_{2^{-h(n_j)}}(\frac{i_j}{2^{h(n_j)}})$. As $(\frac{i}{2^{h(n_j)}}, 2^{-h(n)+2})f(b_{n_j,i_j}, 2^{-n_j})$, by the definition of image, $y \in \overline{B}_{2^{-n_j}}(b_{n_j,i_j})$. This is a contradiction to the fact that $d(y, b_{n_j,i_j}) > 2^{-n_j}$. Therefore y must not be in the range of f.

Lastly we will show that if $y \in C_f$, then y is in the range of f. Suppose $y \in C_f$. We will construct $x \in [0, 1]$ such that $f(x) = y$ using WKL_0, constructing x using an infinite path in an infinite bounded branching tree. First define a bounded branching tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as follows. Let $\sigma \in T$ if and only if for all $n < |\sigma|$ the following three conditions hold:

1. $\sigma(n) \leq 2^{h(n)}$
2. $d(y, b_{n,\sigma(n)}) \leq 2^{-n}$ as measured up to $|\sigma|$
3. If $n + 1 < |\sigma|$, then $d(\sigma(n + 1)/2^{h(n+1)}, \sigma(n)/2^{h(n)}) < 2^{-h(n)}$.

Note that, if T is infinite our desired result follows. Suppose T is infinite. By Condition 1, T has bounded branching by the function $2^{h(n)}$. So by WKL_0, T has some infinite path p. Define $x = \langle p(n)/2^{h(n)} : n \in \mathbb{N}\rangle$. Since $h(n)$ is strictly increasing, and $h(0) \geq 1$, it follows from Lemma 5 that $x \in \mathbb{R}$. Furthermore, $0 \leq p(n) \leq 2^{h(n)}$, so $x \in [0, 1]$. Finally, by Lemma 5, we have the following:

$$d(x, \frac{p(n)}{2^{h(n)}}) \leq 2^{-h(n)+1} < 2^{-h(n)+2} \tag{5}$$

Therefore $x \in B_{2^{-h(n)+2}}(\frac{p(n)}{2^{h(n)}})$ for all n. By Condition 2, $d(y, b_{n,\sigma(n)}) \leq 2^{-n}$ for all n. Since $(\frac{p(n)}{2^{h(n)}}, 2^{-h(n)+2})f(b_{n_j,p(n)}, 2^{-n})$ holds for all n, it follows that $f(x) = y$.

It remains to show that T is infinite. For the sake of contradiction, assume T is finite. Fix ℓ such that for all $\sigma \in T$, $|\sigma| < \ell$. Let $X = \{\langle n, i\rangle \ : \ n < \ell \wedge i \leq$

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$2^{h(n)} \wedge d(b_{n,i}, y) > 2^{-n}\}$. Since n (and with it, i) are bounded and $d(b_{n,i}, y) > 2^{-n}$ is a Σ^0_1 formula, X exists via Bounded Σ^0_1 comprehension.

Claim. For every $j \leq 2^{h(\ell)}$, there is a $\langle n, i\rangle \in X$ such that $\frac{j}{2^{h(\ell)}} \in B_{2^{-h(n)}}(\frac{i}{2^{h(n)}})$.

Before proving this claim, we will first discuss how this claim leads to a desired contradiction. If this claim were true, then the finite collection of balls $B_{2^{-h(n)}}(\frac{i}{2^{h(n)}})$ for $\langle n, i\rangle \in X$ would cover $[0, 1]$. Since $y \in C_f$, it follows that $d(y, b_{n,i}) \leq 2^{-n}$ for some $\langle n, i\rangle \in X$. This inequality contradicts the definition of $\langle n, i\rangle \in X$, which necessitates that $d(b_{n,i}, y) > 2^{-n}$.

Finally, we will prove the claim. For the sake of contradiction, suppose there exists some $j \leq 2^{h(\ell)}$, such that if $\frac{j}{2^{h(\ell)}} \in B_{2^{-h(n)}}(\frac{i}{2^{h(n)}})$ then $\langle n, i\rangle \notin X$. We will construct a string $\sigma \in T$ of length ℓ, which will contradict the choice of ℓ.

Let i_k be the largest number such that $i_k \leq 2^{h(k)}$ and $i_k/2^{h(k)} \leq j/2^{h(\ell)}$. By the contradiction assumption, $\langle k, i_k\rangle \notin X$ for all $k < \ell$.

Let σ be defined as $\sigma(k) = i_k$ for all $k < \ell$. Thus $|\sigma| = \ell$. It is left to show that $\sigma \in T$, by verifying all three conditions. Condition 1 holds since $i_k \leq 2^{h(k)}$. Since $\langle k, i_k\rangle \notin X$, k and i_k cannot satisfy the conditions of the definition of X and we must have instead that $d(b_{k,i_k}, y) \leq 2^{-k}$. This satisfies Condition 2.

To verify Condition 3, let $k+1 < \ell$. Note that, if it is the case that $j/2^{h(\ell)} = i_{k+1}/2^{h(k+1)}$ then we have that

$$\frac{i_{k+1}}{2^{h(k+1)}} = \frac{j}{2^{h(\ell)}} \in B_{2^{-h(k)}}(\frac{i_k}{2^{h(k)}}) \tag{6}$$

Thus $d(i_{k+1}/2^{h(k+1)}, i_k/2^{h(k)}) < 2^{-h(k)}$ as desired.

So consider instead, the case that $j/2^{h(\ell)} \neq i_{k+1}/2^{h(k+1)}$. By definition of i_k, we know that both $i_k/2^{h(k)}$ and $i_{k+1}/2^{h(k+1)}$ are less than or equal to $j/2^{h(\ell)}$. Since i_{k+1} is chosen to be the largest possible value satisfying the relevant requirements, we have the following:

$$\frac{i_k}{2^{h(k)}} \leq \frac{i_{k+1}}{2^{h(k+1)}} \leq \frac{j}{2^{h(\ell)}} \tag{7}$$

Since $\frac{j}{2^{h(\ell)}} \in B_{2^{-h(k)}}(\frac{i_k}{2^{h(k)}})$, $i_{k+1}/2^{h(k+1)}$ is also contained in $B_{2^{-h(k)}}(\frac{i_k}{2^{h(k)}})$. Thus $d(i_{k+1}/2^{h(k+1)}, i_k/2^{h(k)}) < 2^{-h(k)}$ as desired. This completes the proof of the claim and with it, the proof of the theorem. □

Now we move on to the reversal of this result. This proof relies on the contrapositive approach from Simpson's proof that several properties on continuous functions on $[0, 1]$ reverse to WKL_0 as part of Theorem IV.2.3 in [6].

Theorem 6. (RCA_0) *The statement that for continuous $f : [0, 1] \to \widehat{B}$ there is a code for* range(f) *as a closed set implies* WKL_0.

Proof. We will demonstrate this implication by contrapositive in the case that $\widehat{B} = \mathbb{R}$. Assume $\neg WKL_0$. We will construct a continuous function $f : [0, 1] \to \mathbb{R}$ where range(f) cannot have a closed set code, following the method used by Simpson in his proof of Theorem IV.2.3 [6]. Since $\neg WKL_0$, we also have $\neg ACA_0$. Therefore by Theorem 3, there is some one-to-one function $g : \mathbb{N} \to \mathbb{N}$ such that the range of g does not exist. Define c_n as follows:

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$$c_n = \begin{cases} 0 & n = 0 \\ \sum_{i=0}^{n} 2^{-g(i)} & n > 0 \end{cases} \tag{8}$$

Note that c_n is a bounded increasing sequence of rational numbers $c_0 < c_1 < \cdots < c_n < \cdots < 2$, and since the range of g does not exist, $\sup_{n\in\mathbb{N}} c_n$ also does not exist.

In the proof of Theorem IV.2.3, Simpson uses the sequence $\langle c_n \rangle$ and the existence of an infinite tree $T \subseteq 2^{<\mathbb{N}}$ with no infinite path to define a continuous function $\phi_4 : [0,1] \to \mathbb{R}$ with three crucial properties. Letting $f = \phi_4$ for clarity of notation, these properties are:

1. $f(0) = 0$
2. $\forall n(0 \le x \le c_n \Rightarrow x \in \mathsf{range(f)})$
3. $\forall n(c_n < x) \Rightarrow x \notin \mathsf{range(f)}$

(For full details on the construction of f, reference the proof of Theorem IV.2.3 in [6]. However, we will only need the three properties above for the remainder of this proof).

Classically, the range set of this function f should be $[0, \sup_{n\in\mathbb{N}} c_n]$, however such a supremum does not exist by assumption. We claim that the range of f cannot have a closed set code.

For the sake of contradiction, let $C = \langle\langle a_n, r_n\rangle : n \in \mathbb{N}\rangle$ be a closed set code for $\mathsf{range(f)}$ (see Lemma 1). Recall that the balls $B_{r_n}(a_n)$ enumerated by this code cover the open complement of $\mathsf{range(f)}$. Define a sequence of real numbers $\langle \rho_n : n \in \mathbb{N}\rangle$ where

$$\rho_n = \begin{cases} a_n - r_n & a_n - r_n > 0 \\ 0 & a_n - r_n \le 0 \end{cases} \tag{9}$$

Essentially, if $\overline{B}_{r_n}(a_n)$ contains any positive numbers, ρ_n is the left-most boundary point of that closed ball in $\mathbb{R}$. If not, we let ρ_n be 0. Since $f(0) = 0$, we know 0 cannot be contained in $B_{r_n}(a_n)$ for any n. So, if $\overline{B}_{r_n}(a_n)$ contains any positive numbers, it only contains positive numbers.

We now introduce and prove two claims that are necessary for the rest of the proof.

Claim. If q is an upper bound of $\langle c_n : n \in \mathbb{N}\rangle$, then there exists some k such that $0 < \rho_k < q$.

Let q be as hypothesized. Note that q must be positive. Since $c_n < q$ for all n, it must be that $q \notin \mathsf{range(f)}$ by Property 3. So there exists some k such that $q \in B_{r_k}(a_k)$. Thus $0 < a_k - r_k < q < a_k + r_k$. So $\rho_k = a_k - r_k$ is as desired.

Claim. For all m, k, if $\rho_k \ne 0$, then $c_m < \rho_k$.

For the sake of contradiction, suppose there exists m and k such that $\rho_k \ne 0$ but $\rho_k \le c_m$. We will prove this in two cases regarding the position of c_m relative to the ball $B_{r_k}(a_k)$.

Author Proof

Case 1. $\rho_k < c_m$.

In this case, $(0, c_m) \cap B_{r_k}(a_k) \neq \emptyset$. However $(0, c_m) \subset \mathsf{range(f)}$, while $B_{r_k}(a_k)$ is disjoint from $\mathsf{range(f)}$. This is a contradiction.

Case 2. $\rho_k = c_m$.

In this case, since the sequence $\langle c_n \rangle$ is increasing, $\rho_k < c_{m+1}$. Thus we can proceed as in Case 1 relative to c_{m+1} as opposed to c_m to find a contradiction.

Note that by this claim, every $\rho_k \neq 0$ is an upper bound of $\langle c_n \rangle$. Thus, by the first claim, for all such k, there exists some ℓ such that $\rho_\ell < \rho_k$.

With these two claims proved, we now recursively define a function $g : \mathbb{N} \to \mathbb{N}$ that picks a decreasing subsequence of $\langle p_n \rangle$ in the following way:

$$g(k) = \begin{cases} \text{least n such that } \rho_n > 0 & k = 0 \\ \text{least } n > g(\ell) \text{ such that } 0 < \rho_n < \rho_{g(\ell)} & k = \ell + 1 \text{ for some } \ell \in \mathbb{N} \end{cases} \tag{10}$$

By the previous two claims, g is defined for all $k \in \mathbb{N}$. Thus $\langle \rho_{g(k)} \rangle$ is a decreasing sequence, and by the second claim, for all n, $c_n < c_{n+1} < \rho_{g(n+1)} < \rho_{g(n)}$.

The last claim follows immediately from the definition of g and the first claim.

Claim. If q is an upper bound of $\langle c_n \rangle$, then there exists some k such that $\rho_{g(k)} < q$.

In pursuit of using the Nested Interval Completeness of $\mathbb{R}$, we now claim that $\lim_n \rho_{g(n)} - c_n = 0$. For the sake of contradiction, suppose not. Then there exists some M such that $\rho_{g(n)} - c_n > M$ for all n. Let $q \in \mathbb{Q}$ be an upper bound of $\langle c_n \rangle$ for which there exists j such that $q - c_j < M$. By the last claim, there is a $k \geq j$ such that $\rho_{g(k)} < q$. However, then $\rho_{g(k)} - c_k \leq \rho_{g(k)} - c_j < q - c_j < M$, which contradicts the choice of M.

Thus, by Nested Interval Completeness, there exists some real number x such that $x = \lim c_n = \lim \rho_{g(n)}$. However, this x would be the least upper bound of $\langle c_n \rangle$, which is a contradiction. Therefore there must not exist any closed set code C for $\mathsf{range(f)}$. Thus this theorem holds by contrapositive. □

Combining the two main theorems in this section, we get the following result.

Theorem 7. ($\mathsf{RCA_0}$) *The following are equivalent:*

- $\mathsf{WKL_0}$
- *For continuous $f : [0, 1] \to \widehat{B}$, there is a code for* $\mathsf{range(f)}$ *as a closed set.*

The proof of Lemma 4 generalizes from $[0, 1]$ to any compact complete separable metric space. This leads to a more general result.

Theorem 8. ($\mathsf{RCA_0}$) *The following are equivalent:*

- $\mathsf{WKL_0}$
- *For continuous $f : [0, 1] \to \widehat{B}$, there is a code for* $\mathsf{range(f)}$ *as a closed set.*
- *For complete separable metric spaces $\hat{A}$ and $\widehat{B}$, where $\widehat{A}$ is compact, if $f : \widehat{A} \to \widehat{B}$ is continuous, there is a code for* $\mathsf{range(f)}$ *as a closed set.*

Author Proof

3 The Connectedness of range(f)

Since using a closed set code for range(f) requires WKL_0, we will carefully define our sense of having a connected range such that it implies that the range set of a function is connected without relying on a code for the range set itself. This will allow us to work with the theorem of interest in this section within RCA_0 without the complications that we just discussed in Sect. 2.

Definition 11. *Let $\widehat{C}$ be a complete separable metric space. An open or closed set $U \subseteq \widehat{C}$ is connected in $\widehat{C}$ if and only if for all open sets $A, B \subseteq \widehat{C}$, if $U \subseteq A \cup B$ and $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, then $A \cap B \neq \emptyset$.*

Definition 12. *For continuous function $f : [0,1] \to \widehat{C}$, f has connected range if and only if for all open sets $A, B \subseteq \widehat{C}$, if*

1. $\forall x \in [0,1](f(x) \in A \vee f(x) \in B)$,
2. $\exists x \in [0,1](f(x) \in A)$, *and*
3. $\exists x \in [0,1](f(x) \in B)$

then there exists some $x \in [0,1]$ such that $f(x) \in A \cap B$.

We need two lemmas for the main result in this section. The first is Lemma 1.24 in Brown [1]. The second will allow us to use the connectedness of the domain of f in the main theorem of the section.

Lemma 6. (RCA_0) *If $f : [0,1] \to \widehat{C}$ is continuous, then for all nonempty open $U \subseteq \widehat{C}$, $f^{-1}(U) = \{x \in [0,1] \mid f(x) \in U\}$ is open in $[0,1]$.*

Lemma 7. (RCA_0) $[0,1]$ *is connected in* $\mathbb{R}$

Proof. For the sake of contradiction, assume $[0,1]$ is not connected. Then there exist open sets $A, B \subseteq \mathbb{R}$ such that $[0,1] \subseteq A \cup B$, $[0,1] \cap A \neq \emptyset$, and $[0,1] \cap B \neq \emptyset$, but $A \cap B = \emptyset$. Let A be coded by $\langle\langle a_n, r_n\rangle : n \in \mathbb{N}\rangle$ and B be coded by $\langle\langle b_n, s_n\rangle : n \in \mathbb{N}\rangle$. Let $x_0 \in A \cap [0,1]$ and $y_0 \in B \cap [0,1]$ where $x_0 \neq y_0$. Without loss of generality, assume $x_0 < y_0$.

By primitive recursion we define sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ as follows, to satisfy the conditions in the Nested Interval Completeness Theorem.

$$x_{n+1} = \begin{cases} \frac{x_n+y_n}{2} & \text{if } \frac{x_n+y_n}{2} \in A \\ x_n & \text{otherwise} \end{cases} \tag{11}$$

$$y_{n+1} = \begin{cases} \frac{x_n+y_n}{2} & \text{if } \frac{x_n+y_n}{2} \in B \\ y_n & \text{otherwise} \end{cases} \tag{12}$$

From the definitions above, for all n we have that $x_n \in A$, $y_n \in B$, and $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$. In addition:

$$\lim_n |x_n - y_n| = \lim_n \frac{|x_0 - y_0|}{2^n} = 0 \tag{13}$$

Author Proof

With these properties of $\langle x_n \rangle$ and $\langle y_n \rangle$, in RCA_0, by the Nested Interval Completeness of $\mathbb{R}$ there exists a real number x such that $x = \lim_n x_n = \lim_n y_n$. Note that $x \in [0,1]$. Since A and B are disjoint either $x \in A$ or $x \in B$ but not both.

First, suppose $x \in A$. Since A is open, there exists $\langle a_n, r_n \rangle$ in the code of A such that $x \in B_{r_n}(a_n)$. However, since A and B are disjoint, $B_{r_n}(a_n)$ and B are disjoint. Since $\lim_n y_n \to x$, there exists some k such that $d(x, y_k) < r_n$. However, that implies that $y_k \in B_{r_n}(a_n)$. This is a contradiction.

If it is supposed instead that $x \in B$, we derive a contradiction in the same manner as above. As there is a contradiction in both cases, $[0,1]$ must be connected. □

With the lemmas above, we can now prove the main theorem of this section.

Theorem 9. (RCA_0) *For all continuous $f : [0,1] \to \widehat{C}$, f has connected range.*

Proof. Fix open sets $A, B \subseteq \widehat{C}$ that satisfy Conditions 1 through 3 in Definition 12. By Lemma 6, $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of $[0,1]$. By Condition 1, $[0,1] \subseteq f^{-1}(A) \cup f^{-1}(B)$. By Conditions 2 and 3, $f^{-1}(A) \cap [0,1]$ and $f^{-1}(B) \cap [0,1]$ are nonempty. Thus, by Lemma 7, there is a real $x \in [0,1]$ such that $x \in f^{-1}(A) \cap f^{-1}(B)$. By applying f, we have that $f(x) \in A \cap B$. Therefore f has connected range. □

As in the previous section, this result extends to a more general case, as stated below:

Theorem 10. (RCA_0) *Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces and let $\widehat{A}$ be connected. For all continuous $f : \widehat{A} \to \widehat{B}$, f has connected range.*

Using the results of this section, we can now provide a simpler proof that WKL_0 is equivalent to the existence of a closed set code for $\mathsf{range(f)}$ in the special case when $f : [0,1] \to \mathbb{R}$.

Lemma 8. (RCA_0) *Let $f : [0,1] \to \mathbb{R}$ be continuous. If $a <_{\mathbb{R}} b$ and there exist $x_a, x_b \in [0,1]$ such that $f(x_a) = a$ and $f(x_b) = b$, then for all $y \in [a,b]$, there exists $x \in [0,1]$ such that $f(x) = y$.*

Proof. Let f, a, b, x_a, and x_b be as hypothesized. Let $y \in [a,b]$ be given. For the sake of contradiction suppose $f(x) \neq y$ for all $x \in [0,1]$. By Theorem 9, f has connected range. However, the open sets $A = (-\infty, y)$ and $B = (y, \infty)$ satisfy Conditions 1–3 in Definition 12 and are disjoint, providing the desired contradiction. □

Theorem 11. (RCA_0) *The following are equivalent.*

(1) WKL_0
(2) *For continuous $f : [0,1] \to \mathbb{R}$, if f is non-constant, there exist $a <_{\mathbb{R}} b$ such that* $\mathsf{range(f)}$ *is $[a,b]$.*
(3) *For continuous $f : [0,1] \to \mathbb{R}$, there is a code for* $\mathsf{range(f)}$ *as a closed set.*

Author Proof

Proof. Note that (3) ⇒ (1) follows by Theorem 6. (2) ⇒ (3) follows from the fact that RCA_0 suffices to prove there is a closed set code for $[a, b]$ whenever $a <_{\mathbb{R}} b$. In the case where f is constant, RCA_0 also suffices to show that singleton sets have closed set codes.

It remains to show that (1) ⇒ (2). Let non-constant continuous $f : [0, 1] \to \mathbb{R}$ be given. Assume WKL_0. By Theorem IV.2.3 in Simpson [6], f has and attains its supremum, and by an equivalent argument, f also has and attains its infimum. Let a denote the infimum of f and let b denote the supremum of f. Fix $x_a, x_b \in [0, 1]$ such that $f(x_a) = a$ and $f(x_b) = b$. The range of f is contained in $[a, b]$, and by Lemma 8, $\mathsf{range}(\mathsf{f})$ must be equal to $[a, b]$. □

4 The Compactness and Boundedness of range(f)

Similarly to Sect. 3, we need to ensure our proof for the compactness of the range set does not rely on a code for the range set itself. So we will first state an adapted definition for compactness. While compactness is our main property of interest here, boundedness is essential for the proof and is a useful property to examine in its own right. So we will also provide the boundedness definition that is appropriate for our setting.

Definition 13. (RCA_0) *For a continuous function $f : [0, 1] \to \widehat{B}$, f has compact range if and only if there is a sequence of finite sequences $\langle\langle b_{i,j} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ (with $b_{i,j} \in \widehat{B}$) such that for all $j \in \mathbb{N}$ and all $x \in [0, 1]$, there exists $i \leq n_j$ such that $d(b_{i,j}, f(x)) < 2^{-j}$.*

Definition 14. (RCA_0) *For a continuous function $f : [0, 1] \to \widehat{B}$, f is bounded if and only if there exists $M \in \mathbb{Q}^+$ such that for all $x, y \in [0, 1]$, $d(f(x), f(y)) \leq M$.*

In the special case that $f : [0, 1] \to \mathbb{R}$, note that RCA_0 is sufficient to show that the closed interval $[a, b]$ is compact and bounded. (See Examples III.2.6 in [6]). Thus, by Theorem 11, WKL_0 suffices to show that for continuous functions $f : [0, 1] \to \mathbb{R}$, f has compact and bounded range.

We will prove the main result of this section by extending some standard results from Simpson, returning again to Theorem IV.2.3 [6].

Theorem 12. (RCA_0) *The following are equivalent:*

(1) WKL_0.
(2) If $f : [0, 1] \to \widehat{B}$ is continuous, then f is bounded.
(3) If $f : [0, 1] \to \widehat{B}$ is continuous, then f has compact range.

Proof. Note that Simpson proved (1) ⇔ (2). (See Theorem IV.2.3 [6]). Thus, proving that (2) ⇔ (3) is sufficient to prove the theorem.

First we will show that (2) ⇒ (3). Since (1) ⇔ (2), assume WKL_0. By Theorem 4, let h be a modulus of uniform continuity of f. By Lemma 4, for each $j \in \mathbb{N}$ and $i \leq 2^{h(j)}$, let $b_{i,j} \in B$ be a point such that $(\frac{i}{2^{h(j)}}, 2^{-h(j)})f(b_{i,j}, 2^{-j})$.

Author Proof

We claim that $\langle\langle b_{i,j+1} : i \leq 2^{h(j+1)}\rangle : j \in \mathbb{N}\rangle$ witnesses that f has compact range. Let $j \in \mathbb{N}$ and $x \in [0,1]$ be given. There exists some $i \leq 2^{h(j+1)}$ such that $x \in B_{2^{-h(j+1)}}(\frac{i}{2^{h(j+1)}})$. Therefore $d(b_{i,j}, f(x)) \leq 2^{-(j+1)} < 2^{-j}$. Thus f has compact range.

Next we will show that (3) ⇒ (2), reasoning in RCA_0. Let $\langle\langle b_{i,j} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ be given demonstrating that f has compact range. Consider the first finite sequence $\langle b_{i,0} : i \leq n_0\rangle$ of this infinite sequence. We claim that any $M \in \mathbb{Q}$ such that $M \geq 2+\max_{j,k\leq n_0} d(b_{j,0}, b_{k,0})$ is a bound for the range of f as in Definition 14. Let $x, y \in [0,1]$ be arbitrary and fix $j, k \leq n_0$ such that $d(b_{j,0}, f(x)) < 1$ and $d(b_{k,0}, f(y)) < 1$. By applying the triangle inequality we find the following:

$$d(f(x), f(y)) \leq d(f(x), b_{j,0}) + d(b_{j,0}, b_{k,0}) + d(b_{k,0}, f(y)) \leq 2 + \max_{j,k\leq n_0} d(b_{j,0}, b_{k,0}) \leq M \tag{14}$$

Thus f is bounded. □

Brown proved something similar to (1) ⇒ (3) for functions between complete separable metric spaces. (See Theorem 3.20 in Brown [1]). However, Brown's proof invoked the code for the range set, so further analysis was needed to determine whether or not the implication held without the strength of the implicit existence of the code.

As in the previous sections, our result extends to the more general case with some adapted definitions.

Theorem 13. (RCA_0) *Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces, and let $\widehat{A}$ be compact. The following are equivalent:*

(1) WKL_0.
(2) If $f : \widehat{A} \to \widehat{B}$ is continuous, then f is bounded.
(3) If $f : \widehat{A} \to \widehat{B}$ is continuous, then f has compact range.

5 Future Directions

In this paper, we have classified several fundamental theorems about range sets of continuous functions. However, there are several questions still unanswered. First, there is an alternative coding scheme for closed sets in reverse mathematics. Sets with these codes are called separably closed sets to distinguish them from those coded using the definition we discussed in this paper. Since Brown [2] proved that all closed subsets of compact spaces being separably closed is equivalent to ACA_0 over RCA_0, the results in Sect. 2 and 4 imply that ACA_0 is sufficient for the range set of a continuous function on $[0,1]$ to have a separably closed code. It is currently unclear whether ACA_0 is necessary or if a weaker subsystem is sufficient to prove this.

However, the main interest of our current line of research is local connectedness: the remaining property of range sets in the forward direction of the Hahn-Mazurkiewicz Theorem. It is still unknown which subsystem is required to prove that the range of continuous functions on $[0,1]$ are locally connected. Complicating this analysis is the variety of equivalent definitions for local connectedness.

Author Proof

It is still unclear how best to adapt the definition of local connectedness to suit the context of reverse mathematics. Once these questions are answered, the next topic of focus is the backwards implication of the Hahn-Mazurkiewicz Theorem, and further still, the classification of additional theorems related to space-filling curves. Space-filling curves are an active area of research in geometric analysis with many cross-disciplinary applications in computer science.

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References

1. Brown, D.: Functional analysis in weak subsystems of second order arithmetic. Doctoral Dissertation, The Pennsylvania State University (1987)
2. Brown, D.: Notions of closed sets of a complete separable metric space in weak subsystems of second order arithmetic. Contemp. Math. **106**, 39–50 (1990)
3. Couch, P.J., Daniel, B.D., McNicholl, T.: Computing space filling curves. Theory Comput. Syst. **50**, 370–386 (2012)
4. Daniel, D., McNicholl, T.: Effective versions of local connectivity properties. Theory Comput. Syst. **50**, 621–640 (2012)
5. Sagan, H.: Space-Filling Curves, 1st edn. Springer, New York (1994)
6. Simpson, S.: Subsystems of Second Order Arithmetic, 2nd edn. Cambridge University Press, Cambridge (2010)