

EXISTENCE OF THE FREE ENERGY FOR HEAVY-TAILED SPIN GLASSES

AUKOSH JAGANNATH AND PATRICK LOPATTO

ABSTRACT. We study the free energy of a mean-field spin glass whose coupling distribution has power-law tails. For couplings with infinite variance and finite mean, we show that the thermodynamic limit of the quenched free energy exists and that the free energy is self-averaging.

1. INTRODUCTION

We study a mean-field spin glass with heavy-tailed (infinite variance) couplings. This model was introduced by Cizeau and Bouchaud over 30 years ago to understand surprising experimental results on dilute spin glasses with dipolar interactions [8]. It has been extensively studied by physicists [17, 11, 5, 4, 1, 16, 18, 15, 20], who have also drawn connections to random matrix theory and finance [9, 3, 13]. However, to our knowledge, nothing is known rigorously, as the proof techniques developed for lighter-tailed couplings do not apply. Indeed, even the existence of the thermodynamic limit of the quenched free energy has not been established. In this note, we prove this existence result and, in addition, prove that the free energy is self-averaging.

Let us now be more precise. Fix $\alpha \in (0, 2)$. Let J be a symmetric random variable such that

$$\mathbb{P}(|J| \geq t) = \frac{C_0}{t^\alpha} \quad (1.1)$$

for all $|t| > 1$, for some constant $C_0 > 0$, and such that $\mathbb{E}[|J| \mathbb{1}_{|J| \leq 1}] < \infty$. Let $\{J_{ij}\}_{1 \leq i < j \leq N}$ be a collection of independent, identically distributed random variables with the same distribution as J . We consider the Hamiltonian

$$H(\sigma) = \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j, \quad \sigma \in \Sigma_N = \{-1, +1\}^N. \quad (1.2)$$

The partition function and the quenched average of the free energy at inverse temperature $\beta > 0$ are given by

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} e^{\beta H(\sigma)}, \quad F_N(\beta) = \frac{1}{N} \mathbb{E}[\log Z_N]. \quad (1.3)$$

Our first main result establishes that the limit of $F_N(\beta)$ as N grows large exists when $\alpha > 1$. (It is straightforward to see that $F_N(\beta)$ is infinite when $\alpha < 1$.)

Theorem 1.1. *For every $\alpha \in (1, 2)$ and $\beta > 0$, the limit $\lim_{N \rightarrow \infty} F_N(\beta)$ exists and is finite.*

Our second main result establishes that the free energy concentrates around its quenched average (after taking $t = N^{-\delta/2}$).

Theorem 1.2. *For every $\alpha \in (1, 2)$, $\beta > 0$, and $\delta > 0$, there exists a constant $C(\alpha, \beta, \delta, C_0) > 1$ such that*

$$\mathbb{P}\left(N^{-1} \left| \log Z_N(\beta) - \mathbb{E}[\log Z_N(\beta)] \right| > t\right) \leq \frac{C N^{1-\alpha+2\delta}}{t^{\alpha-\delta}}. \quad (1.4)$$

We now comment on the ideas of the proofs, beginning with Theorem 1.1. It is well known that given a collection of N independent variables with a distribution of the form (1.1), there will be approximately a constant number of them of order $N^{1/\alpha}$, with the rest lower order in N (with high

probability). This explains the normalization in (1.2): after normalizing by $N^{-1/\alpha}$, for each spin σ_i there will be a constant number of constant order couplings, with the others much smaller. This configuration of couplings resembles a sparse weighted graph, after neglecting the small couplings. With this heuristic in mind, we begin by showing that the limit of the quenched free energy for the heavy-tailed spin glass is equal to that of a sparse spin glass model. We then apply a technique from the literature on such models, the combinatorial interpolation of Bayati, Gamarnik, and Tetali [2], to show that the limit of the quenched free energy exists for the sparse model and complete the proof of Theorem 1.1. This interpolation argument has its roots in earlier work of Franz and Leone [12]. The proof of Theorem 1.2 is through a short martingale argument, which establishes a bound on the second moment of the difference considered in (1.4). Then Theorem 1.2 follows from an application of Markov's inequality.

Previous work. The existence of the thermodynamic limit of the quenched free energy for a mean-field spin glass with Gaussian spins (the Sherrington–Kirkpatrick model) was established by Guerra and Toninelli [14]. Carmona and Hu showed that the limit exists for identically distributed couplings with finite third moment [6]. Chatterjee then showed that finite variance suffices (which addresses the case of $\alpha > 2$ in (1.1)) [7]. Additionally, Starr and Vermesi proved a formula for the difference of the free energies of two mean-field spin glass models with infinitely divisible coupling distributions in terms of expectations of multi-spin overlaps [21]. This class of distributions includes α -stable laws, which have infinite variance for $\alpha < 2$. At present, it seems difficult to usefully apply this formula in the heavy-tailed context, since we lack information about the overlaps of such models.

Outline. In Section 2, we reduce the proof of Theorem 1.1 to the proof of an analogous theorem for a certain sparse spin glass model. In Section 3, we prove Theorem 1.1 assuming an interpolation result, given as Lemma 3.1. In Section 4, we prove Lemma 3.1, completing the argument for Theorem 1.1. Section 5 contains the proof of Theorem 1.2.

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Notation. For brevity, we take $\beta = 1$ throughout, though the arguments are identical for $\beta > 0$. Further, we let $C, c > 0$ denote constants that may change from line to line, and may depend on C_0 . The notation $\llbracket a, b \rrbracket$ denotes the set of integers k such that $a \leq k \leq b$.

2. REDUCTION TO SPARSE HAMILTONIAN

In this section, we reduce the problem of establishing the limit of the quenched average of the free energy for the Hamiltonian H to the analogous one for a certain sparse Hamiltonian $H_{u,v,m}^\circ$ (defined below in Section 2.3), for particular choices of parameters u, v, m . This reduction proceeds in a series of steps. In Section 2.1, we begin by showing that this problem for H is equivalent to one for a sparse Hamiltonian \hat{H} , which omits all couplings smaller than a certain threshold. In Section 2.2, we further reduce the problem to studying another sparse Hamiltonian \tilde{H} , and in Section 2.3, we complete the reduction to $H_{u,v,m}^\circ$.

2.1. The truncated model. We begin by viewing H as a perturbation of the Hamiltonian for a model that omits couplings smaller than a certain threshold, and showing that the perturbation term provides a negligible contribution to the free energy. To this end, let $\varepsilon > 0$ be a parameter that will be chosen later and $R = N^{1/\alpha-\varepsilon}$. We may then write $H(\sigma) = \widehat{H}(\sigma) + \widehat{p}(\sigma)$, where

$$\widehat{H}(\sigma) = \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} J_{ij} \mathbb{1}_{|J_{ij}| \geq R} \sigma_i \sigma_j, \quad \widehat{p}(\sigma) = \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} J_{ij} \mathbb{1}_{|J_{ij}| < R} \sigma_i \sigma_j. \quad (2.1)$$

Denote the partition function and free energy for $\widehat{H}(\sigma)$ by $\widehat{Z}_N = \sum_{\sigma \in \Sigma_N} e^{\widehat{H}(\sigma)}$ and $\widehat{F}_N = \frac{1}{N} \mathbb{E}[\log \widehat{Z}_N]$ respectively. Our goal is to prove the following:

Lemma 2.1. *There exists $c > 0$ such that for all $\varepsilon \in (0, c)$, we have that $\lim_{N \rightarrow \infty} \widehat{F}_N = \lim_{N \rightarrow \infty} F_N$, if the limit on the left exists.*

The proof of this result will follow by applying the following elementary fact whose proof follows by repeatedly applying Jensen's inequality.

Lemma 2.2. *Suppose that $x(\sigma)$ and $y(\sigma)$ are random processes on some finite set Σ , and let $\langle \cdot \rangle$ denote expectation with respect to the Gibbs measure $\pi(\{\sigma\}) \propto \exp(x(\sigma))$. Then*

$$\mathbb{E} \log \sum_{\sigma \in \Sigma} e^{x(\sigma)} + \mathbb{E} \langle y(\sigma) \rangle \leq \mathbb{E} \log \sum_{\sigma \in \Sigma} e^{x(\sigma) + y(\sigma)} \leq \mathbb{E} \log \sum_{\sigma \in \Sigma} e^{x(\sigma)} + \log \mathbb{E} \langle e^{y(\sigma)} \rangle, \quad (2.2)$$

provided all the expectations are finite.

We will apply this result with $x = \widehat{H}$ and $y = \widehat{p}$.

It will be helpful to notice that while $\widehat{H}(\sigma)$ and $\widehat{p}(\sigma)$ are nominally dependent due to the common coefficients J_{ij} , we can introduce independence through the following two-step resampling procedure for the J_{ij} . Let $\{L_{ij}\}_{1 \leq i < j \leq N}$ be mutually independent random variables such that

$$\mathbb{P}(L_{ij} = 1) = p, \quad \mathbb{P}(L_{ij} = 0) = 1 - p, \quad p = p_N = \mathbb{P}(|J| \geq R). \quad (2.3)$$

Let $\{a_{ij}, b_{ij}\}_{1 \leq i < j \leq N}$ be a collection of mutually independent random variables (which are also independent from the L_{ij} variables) such that for every interval $I \subset \mathbb{R}$, we have

$$\begin{aligned} \mathbb{P}(a_{ij} \in I) &= \mathbb{P}(J \in I \mid |J| < R) \\ \mathbb{P}(b_{ij} \in I) &= \mathbb{P}(J \in I \mid |J| \geq R). \end{aligned} \quad (2.4)$$

Then we have the distributional equalities

$$\begin{aligned} \{J_{ij}\}_{1 \leq i < j \leq N} &\stackrel{(d)}{=} \{(1 - L_{ij})a_{ij} + L_{ij}b_{ij}\}_{1 \leq i < j \leq N} \\ \widehat{H}(\sigma) &\stackrel{(d)}{=} \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} L_{ij} b_{ij} \sigma_i \sigma_j, \quad \widehat{p}(\sigma) \stackrel{(d)}{=} \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} (1 - L_{ij}) a_{ij} \sigma_i \sigma_j, \end{aligned} \quad (2.5)$$

with the dependence between $\widehat{H}(\sigma)$ and $\widehat{p}(\sigma)$ expressed through the L_{ij} .¹ Observe that after conditioning on L , the sums $\widehat{H}(\sigma)$ and $\widehat{p}(\sigma)$ are independent.

Before proving Lemma 2.1, we note the following useful moment bound. Under the assumption $|x| \leq 1$, we have $e^x \leq 1 + x + x^2$. Then for any random variable X such that $\mathbb{E}[X] = 0$ and $|X| \leq 1$, we have

$$\mathbb{E}[\exp(X)] \leq 1 + \mathbb{E}[X^2] \leq \exp(\mathbb{E}[X^2]), \quad (2.6)$$

where we used used $1 + x \leq e^x$ in the second inequality.

¹This construction can be viewed as implementing the “total variation” coupling between a_{ij} and J_{ij} . See, e.g., [19, (17.30)].

Lemma 2.3. *There exists $C(\varepsilon) < \infty$ such that*

$$\mathbb{E}[\exp(N^{-1/\alpha}(1 - L_{ij})a_{ij})] \leq \exp(CR^{2-\alpha}N^{-2/\alpha})$$

for all $1 \leq i < j \leq N$.

Proof. Since $|N^{-1/\alpha}(1 - L_{ij})a_{ij}| \leq 1$ by definition, (2.6) yields

$$\mathbb{E}[\exp(N^{-1/\alpha}(1 - L_{ij})a_{ij})] \leq \exp(N^{-2/\alpha}\mathbb{E}[(1 - L_{ij})^2a_{ij}^2]). \quad (2.7)$$

Using (2.4), $|L_{ij}| \leq 1$, and $p_N = o(1)$ (from (1.1)), we have

$$\mathbb{E}[(1 - L_{ij})^2a_{ij}^2] \leq 2 \cdot \mathbb{E}[J_{ij}^2 \mathbb{1}_{|J_{ij}| < R}] \leq C \left(1 + \int_1^R t^{1-\alpha} dt \right) \leq CR^{2-\alpha}, \quad (2.8)$$

where the first inequality holds for sufficiently large N (depending on α , ε , and C_0), the second inequality follows from (1.1), and we increased the constant C in its second appearance. Inserting (2.8) into (2.7) completes the proof. \square

We now turn to our reduction.

Proof of Lemma 2.1. Using the representation (2.5), we see that if $\langle \cdot \rangle$ denotes expectation with respect to \widehat{H} , then integrating first in the variables a_{ij} yields $\mathbb{E}\langle \widehat{p} \rangle = \mathbb{E}\langle \mathbb{E}_a p \rangle = 0$. Thus by Lemma 2.2, we have that

$$\widehat{F}_N \leq F_N \leq \widehat{F}_N + \log \mathbb{E}\langle \mathbb{E}_a e^{\widehat{p}} \rangle.$$

For a fixed $\sigma \in \Sigma_N$, the expectation $\mathbb{E}_a[e^{\widehat{p}(\sigma)}]$ does not depend on the values of the spins σ_i because the a_{ij} are symmetric. Then

$$\mathbb{E}_a[e^{\widehat{p}(\sigma)}] = \mathbb{E}_a \left[\prod e^{N^{-1/\alpha}(1 - L_{ij})a_{ij}} \right] \leq \left(\exp(CR^{2-\alpha}N^{-2/\alpha}) \right)^{N^2} = \exp(CN^{2-2/\alpha}R^{2-\alpha}), \quad (2.9)$$

where we used the independence of the $(1 - L_{ij})a_{ij}$ variables and Lemma 2.3 for the inequality. By (1.1) and our choice $R = N^{1/\alpha-\varepsilon}$, we have $N^{2-2/\alpha}R^{2-\alpha} = O(N^{1+\varepsilon(\alpha-2)})$. Using this bound in (2.9) completes the proof. \square

2.2. Model with fixed edge number. We next consider a sparse Hamiltonian where the total number of couplings is fixed. Before defining this model, we note the following preliminary lemma.

Let $M = M_N = \sum_{1 \leq i < j \leq N} L_{ij}$, which represents the number of nonzero couplings in $\widehat{H}(\sigma)$. Recalling that $p = C_0N^{-1+\alpha\varepsilon}$ by (2.3) and applying the multiplicative Chernoff inequality yields the following concentration bound on M_N .

Lemma 2.4. *For every $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that*

$$\left| \mathbb{E}[M] - \frac{C_0}{2}N^{1+\alpha\varepsilon} \right| \leq c^{-1}N^{\alpha\varepsilon}, \quad \mathbb{P}\left(\left|M - \mathbb{E}[M]\right| > c^{-1}N^{1/2+\alpha\varepsilon/2+\varepsilon}\right) \leq 2e^{-cN^\varepsilon}. \quad (2.10)$$

We now introduce a new Hamiltonian \widetilde{H} . Recall the constant C_0 from (1.1), and set

$$S_N = \left\lfloor \frac{C_0}{2}N^{1+\alpha\varepsilon} \right\rfloor. \quad (2.11)$$

Let $\{\widetilde{L}_{ij}\}_{1 \leq i,j \leq N}$ denote the adjacency matrix of a graph drawn uniformly at random from graphs with precisely S_N edges on N vertices, so that each \widetilde{L}_{ij} is 0 or 1. By definition, exactly S_N of the \widetilde{L}_{ij} are nonzero. Set

$$\{\widetilde{J}_{ij}\}_{i < j} = \{\widetilde{L}_{ij}b_{ij}\}_{i < j}, \quad (2.12)$$

where we recall that the variables $\{b_{ij}\}_{i < j}$ were defined in (2.4), and we require that the variables $\{\tilde{L}_{ij}, a_{ij}, b_{ij}\}_{i < j}$ are all mutually independent. Then define the Hamiltonian

$$\tilde{H}(\sigma) = \frac{1}{N^{1/\alpha}} \sum_{1 \leq i < j \leq N} \tilde{J}_{ij} \sigma_i \sigma_j,$$

and set $\tilde{Z}_N = \sum_{\sigma \in \Sigma_N} e^{\tilde{H}(\sigma)}$ and $\tilde{F}_N = \frac{1}{N} \mathbb{E}[\log \tilde{Z}_N]$. We now show that \tilde{F} is a good approximation to \hat{F} .

Lemma 2.5. *There exists $c > 0$ such that the following holds for all $\varepsilon \in (0, c)$. We have $\lim_{N \rightarrow \infty} \tilde{F}_N = \lim_{N \rightarrow \infty} \hat{F}_N$, if the limit on the left exists.*

Proof. Given a realization of the $\{\tilde{J}_{ij}\}$, we consider the effect on \tilde{Z} of fixing some (i, j) and changing \tilde{L}_{ij} to $1 - \tilde{L}_{ij}$ (that is, adding or deleting a coupling). If we denote the partition function that results from this change by \tilde{Z}_{new} , we see by direct calculation that

$$|\log \tilde{Z} - \log \tilde{Z}_{\text{new}}| \leq N^{-1/\alpha} |b_{ij}|. \quad (2.13)$$

Using (2.4), we find $\mathbb{E}[|b_{ij}|] = O(N^{1/\alpha - \varepsilon})$, so (2.13) yields

$$|\mathbb{E}[\log \tilde{Z}] - \mathbb{E}[\log \tilde{Z}_{\text{new}}]| \leq CN^{-\varepsilon}. \quad (2.14)$$

for some constant $C > 0$. Therefore, adding or removing k couplings in this way results in a change of at most Ck to the expected log-partition function.

Let \mathcal{A} be the event on which $|M - S_N| \leq c^{-1} N^{1/2 + \alpha\varepsilon/2 + \varepsilon}$ holds. We write

$$|\hat{F}_N - \tilde{F}_N| \leq \frac{1}{N} \left| \mathbb{E}[(\log \hat{Z} - \log \tilde{Z}) \mathbb{1}_{\mathcal{A}}] \right| + \frac{1}{N} \left| \mathbb{E}[(\log \hat{Z} - \log \tilde{Z}) \mathbb{1}_{\mathcal{A}^c}] \right|. \quad (2.15)$$

We now consider an alternative sampling scheme for \hat{H} . Begin by sampling the mutually independent variables $\{L_{ij}, \tilde{L}_{ij}, a_{ij}, b_{ij}\}$, and let ℓ be the (random) number of nonzero L_{ij} . From these realizations, we obtain a realization of \tilde{H} ; we will use this realization of \tilde{H} to produce a coupled realization of \hat{H} in the following way. If $\ell > S_N$, choose uniformly at random $\ell - S_N$ distinct index pairs $\mathcal{P} = \{(i_a, j_a)\}_{a=1}^{\ell - S_N}$ from the set $\{(i, j) : \tilde{L}_{ij} = 0\}$. For $1 \leq i < j \leq N$, we set $\hat{L}_{ij} = \tilde{L}_{ij}$ if $(i, j) \notin \mathcal{P}$, and $\hat{L}_{ij} = 1 - \tilde{L}_{ij}$ if $(i, j) \in \mathcal{P}$. Likewise, if $\ell < S_N$, we choose uniformly at random $S_N - \ell$ index pairs $\mathcal{P} = \{(i_a, j_a)\}_{a=1}^{\ell - S_N}$ from the set $\{(i, j) : \tilde{L}_{ij} = 1\}$; we set $\hat{L}_{ij} = \tilde{L}_{ij}$ if $(i, j) \notin \mathcal{P}$, and $\hat{L}_{ij} = 1 - \tilde{L}_{ij}$ if $(i, j) \in \mathcal{P}$. Finally, if $\ell = S_N$, we set $\hat{L}_{ij} = \tilde{L}_{ij}$ for all (i, j) such that $1 \leq i < j \leq N$.

Since this procedure is symmetric with respect to the edges (i, j) , and ℓ equals the number of nonzero \hat{L}_{ij} labels, we find that

$$\hat{H}(\sigma) \stackrel{(d)}{=} \frac{1}{N^{1/\alpha}} \sum_{i < j} \hat{L}_{ij} b_{ij} \sigma_i \sigma_j,$$

and we have produced a coupling between $\hat{H}(\sigma)$ and $\tilde{H}(\sigma)$ through the addition or subtraction of $|\ell - S_N|$ couplings from $\tilde{H}(\sigma)$.

On the event \mathcal{A} , we have $|\ell - S_N| \leq c^{-1} N^{1/2 + \alpha\varepsilon/2 + \varepsilon}$ by definition. Therefore we obtain from (2.14) that

$$\left| \mathbb{E}[(\log \hat{Z} - \log \tilde{Z}) \mathbb{1}_{\mathcal{A}}] \right| \leq CN^{1/2 + \alpha\varepsilon/2 + \varepsilon}. \quad (2.16)$$

For the other term in (2.15), we use Hölder's inequality to show that

$$\left| \mathbb{E}[(\log \hat{Z} - \log \tilde{Z}) \mathbb{1}_{\mathcal{A}^c}] \right| \leq \mathbb{P}(\mathcal{A}^c)^{\frac{\varepsilon}{1+\varepsilon}} \mathbb{E}[|\log \hat{Z}|^{1+\varepsilon}]^{\frac{1}{1+\varepsilon}} + \mathbb{P}(\mathcal{A}^c)^{\frac{\varepsilon}{1+\varepsilon}} \mathbb{E}[|\log \tilde{Z}|^{1+\varepsilon}]^{\frac{1}{1+\varepsilon}}.$$

We give details only for the bound on $\mathbb{E}[|\log \hat{Z}|^{1+\varepsilon}]$, as the bound for $\mathbb{E}[|\log \tilde{Z}|^{1+\varepsilon}]$ is similar.

By definition, we have $|\widehat{H}(\sigma)| \leq N^{-1/\alpha} \sum_{1 \leq i < j \leq N} |b_{ij}|$, which implies

$$2^N \exp\left(-N^{-1/\alpha} \sum_{i < j} |b_{ij}|\right) \leq \widehat{Z}_N \leq 2^N \exp\left(N^{-1/\alpha} \sum_{i < j} |b_{ij}|\right).$$

This in turn implies that $|\log \widehat{Z}| \leq N \log 2 + N^{-1/\alpha} \sum_{i < j} |b_{ij}|$. Therefore

$$\mathbb{E}[|\log \widehat{Z}|^{1+\varepsilon}] \leq \mathbb{E}[(N \log 2 + \sum |b_{ij}|)^{1+\varepsilon}] \leq CN^{2\varepsilon} \mathbb{E}[(N^{1+\varepsilon} + \sum |b_{ij}|^{1+\varepsilon})] \leq CN^5, \quad (2.17)$$

for some constant $C(\varepsilon) > 1$. In the second inequality, we used the elementary inequality

$$\left(\sum_{i=1}^k x_i\right)^{1+\varepsilon} \leq k^\varepsilon \sum_{i=1}^k |x_i|^{1+\varepsilon},$$

which follows from Hölder's inequality. In the third inequality, we used that the $(1 + \varepsilon)$ -th moment of $|b_{ij}|$ exists for ε small enough (depending only on $\alpha > 1$) and is less than CN^2 for some $C(\varepsilon) > 1$. We conclude using (2.10) and (2.17) that there exists $c(\varepsilon) > 0$ such that

$$\mathbb{P}(\mathcal{A}^c)^{\frac{\varepsilon}{1+\varepsilon}} \mathbb{E}[|\log \widehat{Z}|^{1+\varepsilon}]^{\frac{1}{1+\varepsilon}} \leq c^{-1} N^5 \exp(-cN^\varepsilon). \quad (2.18)$$

The conclusion now follows from combining (2.15), (2.16), and (2.18). \square

2.3. Model with multi-edges. The Hamiltonian \tilde{H} may be thought of as arising from a collection of S_N weighted edges on a simple random graph. However, it will be convenient to consider instead a similar model where the edges are sampled with replacement from pairs of vertices $\{(i, j)\}_{1 \leq i \leq j \leq N}$. In particular, self-edges of the form (i, i) are allowed, as well as multi-edges, meaning that an edge (i, j) may appear two or more times.

We define the Hamiltonian $H_{u,v,m}^\circ(\sigma)$ as follows. The parameters u, v, m denote the edge number, vertex number, and effective normalization, respectively, as we now explain. Let $\{c_a\}_{1 \leq a \leq u}$ be a collection of identically distributed, independent random variables, where each c_a is an ordered pair (i, j) chosen uniformly at random from the set $\{(i, j)\}_{1 \leq i \leq j \leq v}$. We write $c_a(1)$ and $c_a(2)$ for the first and second coordinates of c_a , respectively. While the definition of the c_a variables depends on both u and v , we omit this from the notation.

We set $q = \mathbb{P}(|J| \geq m^{1/\alpha-\varepsilon})$ and define identically distributed, independent random variables $\{d_a\}_{1 \leq a \leq u} = \{d_a(m)\}_{1 \leq a \leq u}$ by requiring that

$$\mathbb{P}(d_a \in I) = q^{-1} \mathbb{P}\left(J \in I \cap ((-\infty, -m^{1/\alpha-\varepsilon}] \cup [m^{1/\alpha-\varepsilon}, \infty))\right) \quad (2.19)$$

for every interval $I \subset \mathbb{R}$. We define

$$H_{u,v,m}^\circ(\sigma) = \frac{1}{m^{1/\alpha}} \sum_{1 \leq a \leq u} d_a \sigma_{c_a(1)} \sigma_{c_a(2)},$$

and set $Z_v^\circ(u, m) = \sum_{\sigma \in \Sigma_N} \exp(H_{u,v,m}^\circ(\sigma))$ and $F_{u,v,m}^\circ = \frac{1}{m} \mathbb{E}[\log Z_v^\circ(u, m)]$.

We now define a quantity that represents the total number of loops and multi-edges among the c_a . We set

$$f_{u,v} = \sum_{1 \leq i \leq v} \sum_{1 \leq a \leq u} \mathbb{1}_{c_a=(i,i)} + \sum_{1 \leq i < j \leq v} \mathbb{1} \left\{ \left(\sum_{1 \leq a \leq u} \mathbb{1}_{c_a=(i,j)} \right) > 1 \right\} \cdot \left(\sum_{1 \leq a \leq u} \mathbb{1}_{c_a=(i,j)} - 1 \right).$$

Lemma 2.6. *For every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that*

$$\mathbb{P}(f_{S_N, N} \geq N^{3\alpha\varepsilon}) \leq \exp(-cN^\varepsilon). \quad (2.20)$$

Proof. Consider a sequence of random multi-graphs \mathbb{G}_t on the vertex set $\{1, \dots, N\}$ built in the following way. Let \mathbb{G}_0 be the graph with no edges, and for each $t \in \llbracket 1, S_N \rrbracket$, let \mathbb{G}_t be the graph with edge set $\{c_a\}_{1 \leq a \leq t}$. The definition of \mathbb{G}_t naturally extends to $t > S_N$ by choosing additional multi-edges uniformly at random. Let $T = \min \{t : |\mathcal{E}_t| \geq S_N - N^{3\alpha\varepsilon}\}$, where

$$\mathcal{E}_t = \{(i, j) : 1 \leq i < j \leq N, (i, j) = c_a \text{ for some } a \leq t\}$$

is the set of (non-loop) edges that have been added at time t (after removing duplicates). Observe that

$$\mathbb{P}(f_{S_N, N} \geq N^{3\alpha\varepsilon}) = \mathbb{P}(T \leq S_N),$$

so it suffices to bound $\mathbb{P}(T \leq S_N)$.

Let $T_1 = \min\{t : |\mathcal{E}_t| \geq 1\}$ and define T_i for $i \geq 2$ by

$$T_i = \min\{t : |\mathcal{E}_t| \geq i\} - \min\{t : |\mathcal{E}_t| \geq i-1\}.$$

Then, by definition,

$$T = \sum_{i=1}^{S_N - N^{3\alpha\varepsilon}} T_i. \quad (2.21)$$

We say that X is a geometric random variable with parameter p if $\mathbb{P}(X = k) = (1-p)^{k-1}p$ for all $k \geq 1$; then $\mathbb{E}[X] = p^{-1}$. We observe that each T_i in (2.21) is a geometric random variable with parameter

$$p_i = \frac{A - N - i + 1}{A}$$

where $A = N(N+1)/2$ is the number of possible edges and loops on a graph with N vertices. Using this representation, (2.21), and an integral approximation, we compute that

$$\mathbb{E}[T] = A \sum_{i=1}^{S_N - N^{3\alpha\varepsilon}} \frac{1}{A - N - i + 1} \geq A \log \left(\frac{A - N}{A - N - S_N + N^{3\alpha\varepsilon} + 2} \right) \geq S_N - 2N^{3\alpha\varepsilon} \quad (2.22)$$

for sufficiently large N (depending on ε), where we used $\log(1/(1-x)) \geq x$ in (2.22). Defining $T^\circ = T - \mathbb{E}[T]$, we bound

$$\mathbb{P}(T \leq S_N) = \mathbb{P}(T^\circ \leq S_N - \mathbb{E}[T]) \leq \mathbb{P}(T^\circ \leq -2N^{3\alpha\varepsilon}) \leq e^{-2N^{3\alpha\varepsilon}} \mathbb{E}[e^{-T^\circ}]. \quad (2.23)$$

Then to complete the proof, it suffices to bound the right side of (2.23).

Writing $T_i^\circ = T_i^\circ - \mathbb{E}[T_i^\circ]$, we have

$$\mathbb{E}[e^{-T^\circ}] = \prod_{i=1}^{S_N - N^{3\alpha\varepsilon}} \mathbb{E}[e^{-T_i^\circ}] = \prod_{i=1}^{S_N - N^{3\alpha\varepsilon}} \frac{e^{1/p_i}}{1 + p_i^{-1}(e-1)} \leq \exp \sum_{i=1}^{S_N - N^{3\alpha\varepsilon}} (p_i^{-1} - 1). \quad (2.24)$$

We compute

$$p_i^{-1} - 1 = \frac{A}{A - N - i + 1} - 1 = \frac{N + i - 1}{A - N - i + 1} \leq \frac{N + S_N - 1}{A - N - S_N + 1} \leq C_1 N^{-2} S_N \quad (2.25)$$

for sufficiently large N , and a constant $C_1 > 1$. Inserting (2.25) into (2.24) and (2.23), we find

$$\mathbb{P}(T \leq S_N) \leq \exp(-2N^{3\alpha\varepsilon} + C_1 N^{-2} S_N^2) \leq \exp(-2N^{3\alpha\varepsilon} + C_1 N^{-2} S_N^2) \leq \exp(-cN^\varepsilon).$$

This completes the proof. \square

Lemma 2.7. *There exists $c > 0$ such that the following holds for all $\varepsilon \in (0, c)$. We have $\lim_{N \rightarrow \infty} F_{S_N, N, N}^\circ = \lim_{N \rightarrow \infty} \tilde{F}_N$, if the limit on the left exists.*

Proof. Given (2.20), the rest of the proof proceeds similarly to the proof of Lemma 2.5. We indicate only the main points here. Let

$$D = \{a \in \{1, 2, \dots, S\} : c_a = c_b \text{ for some } b < a, \text{ or } c_a = (i, i) \text{ for some } i \in \llbracket 1, N \rrbracket\}$$

denote the set of duplicate edges and loops. As described below (2.15), to sample from the Hamiltonian \tilde{H} , one may begin with $H_{S_N, N, N}^\circ$, remove all edges c_a such that $a \in D$, and restore $|D|$ edges chosen uniformly at random from all size $|D|$ subsets of distinct edges contained in

$$\{(i, j) : 1 \leq i < j \leq N\} \setminus \{c_1, c_2, \dots, c_S\},$$

where $\{c_1, c_2, \dots, c_S\}$ denotes the original set of edges in $H_{S_N, N, N}^\circ$.

As shown in (2.14), the addition or deletion of some edge c_a with weight $d_a(N)$ in the Hamiltonian $H_{S_N, N, N}^\circ$ changes $\mathbb{E}[\log Z_N^\circ(S, N)]$ by at most a constant $C > 1$. Let \mathcal{A} denote the event where $f \leq N^{3\alpha\varepsilon}$ holds. Then, under the coupling given in the previous paragraph, we have

$$\left| \mathbb{E} \left[(\log \tilde{Z} - \log Z_N^\circ(S, N)) \mathbb{1}_{\mathcal{A}} \right] \right| \leq CN^{3\alpha\varepsilon}. \quad (2.26)$$

Further, by a computation nearly identical to (2.17), we have

$$\mathbb{E} \left[|\log \tilde{Z}|^{1+\varepsilon} \right] + \mathbb{E} \left[|\log Z_N^\circ(S, N)|^{1+\varepsilon} \right] \leq CN^5. \quad (2.27)$$

Hölder's inequality shows that

$$\left| \mathbb{E} \left[(\log Z_N^\circ(S, N) - \log \tilde{Z}) \mathbb{1}_{\mathcal{A}^c} \right] \right| \leq \mathbb{P}(\mathcal{A}^c)^{\frac{\varepsilon}{1+\varepsilon}} \mathbb{E} \left[|\log Z_N^\circ(S, N)|^{1+\varepsilon} \right]^{\frac{1}{1+\varepsilon}} + \mathbb{P}(\mathcal{A}^c)^{\frac{\varepsilon}{1+\varepsilon}} \mathbb{E} \left[|\log \tilde{Z}|^{1+\varepsilon} \right]^{\frac{1}{1+\varepsilon}}. \quad (2.28)$$

The conclusion now follows from combining (2.20), (2.27), and (2.28). \square

3. FREE ENERGY FOR SPARSE HAMILTONIAN

In this section we apply the combinatorial interpolation strategy of [2] to show the existence of the limit of the free energy for the multi-edge model.

We begin by stating two preliminary lemmas. The first is our main interpolation result. We prove it at the end of Section 4, below. In its statement, the notation $\text{Bi}(n, p)$ denotes a binomial random variable with n trials and success probability p , and we recall S_N was defined in (2.11). We also recall $Z_v^\circ(u, m)$ is the partition function for the Hamiltonian $H_{u, v, m}^\circ$ defined in Section 2.3.

Lemma 3.1. *For every $1 \leq N_1, N_2 \leq N$ such that $N_1 + N_2 = N$, we have*

$$\mathbb{E}[\log Z_N^\circ(S, N)] \geq \mathbb{E}[\log Z_{N_1}^\circ(\mathcal{M}_1, N)] + \mathbb{E}[\log Z_{N_2}^\circ(\mathcal{M}_2, N)], \quad (3.1)$$

where \mathcal{M}_1 is distributed as $\text{Bi}(S_N, N_1/N)$ and $\mathcal{M}_2 = S_N - \mathcal{M}_1$ is distributed as $\text{Bi}(S_N, N_2/N)$. Here \mathcal{M}_1 and \mathcal{M}_2 are independent of the random variables in the definition of H° .

We also require the following sub-additivity lemma.

Lemma 3.2 ([10, Theorem 23]). *Suppose that the sequence $(a_N)_{N=1}^\infty$ satisfies $a_N \leq a_{N_1} + a_{N-N_1} + \varphi(N)$ for all N, N_1 such that $N/3 \leq N_1 \leq 2N/3$, for some positive increasing function φ with $\int_1^\infty \varphi(t)t^{-2} dt < \infty$. Then $\lim_{N \rightarrow \infty} a_N/N = L$ for some L such that $-\infty \leq L < \infty$.*

3.1. Proof of Theorem 1.1. Let $1 \leq N_1, N_2 \leq N$ be parameters. We define

$$S^{(1)} = \frac{S_N N_1}{N}, \quad S^{(2)} = \frac{S_N N_2}{N}.$$

Lemma 3.3. Fix $\varepsilon > 0$. Then there exists $C(\varepsilon) > 0$ such that the following holds. For every $1 \leq N_1, N_2 \leq N$ such that $N/3 \leq N_1 \leq 2N/3$ and $N_1 + N_2 = N$, we have

$$\left| \mathbb{E}[\log Z_{N_1}^\circ(\mathcal{M}_1, N)] - \mathbb{E}[\log Z_{N_1}^\circ(S^{(1)}, N)] \right| \leq CN^{2/3} \quad (3.2)$$

$$\left| \mathbb{E}[\log Z_{N_2}^\circ(\mathcal{M}_2, N)] - \mathbb{E}[\log Z_{N_2}^\circ(S^{(2)}, N)] \right| \leq CN^{2/3}. \quad (3.3)$$

Proof. We prove only (3.2), since the proof of (3.3) is similar. Recall the definition of \mathcal{M}_1 from (3.1), and note that $S^{(1)} = \mathbb{E}[\mathcal{M}_1]$. By the Chernoff bound, there exists a constant $c(\varepsilon) > 0$ such that

$$\mathbb{P}(\mathcal{A}_1^c) \leq \exp(-cN^\varepsilon), \quad \mathcal{A}_1 = \{|\mathcal{M}_1 - S^{(1)}| \leq c^{-1}N_1^{1/2}N^{\alpha\varepsilon/2+\varepsilon}\}. \quad (3.4)$$

Similarly to the coupling given below (2.20), we may couple the Hamiltonians $H_{\mathcal{M}_1, N_1, N}^\circ$ and $H_{S^{(1)}, N_1, N}^\circ$ by deleting or adding $|\mathcal{M}_1 - S^{(1)}|$ edges c_a from $H_{\mathcal{M}_1, N_1, N}^\circ$, where each member of the set of modified edges is chosen uniformly at random from the set of all possible edges. By (2.14) and the definition of \mathcal{A}_1 , we have

$$\left| \mathbb{E}[(\log Z_{N_1}^\circ(\mathcal{M}_1, N) - \log Z_{N_1}^\circ(S^{(1)}, N)) \mathbb{1}_{\mathcal{A}_1}] \right| \leq CN^{1/2+\alpha\varepsilon/2+\varepsilon}. \quad (3.5)$$

Further, arguing similarly to (2.17), we find

$$\mathbb{E}[|\log Z_{N_1}^\circ(\mathcal{M}_1, N)|^{1+\varepsilon}] + \mathbb{E}[|\log Z_{N_1}^\circ(S^{(1)}, N)|^{1+\varepsilon}] \leq CN^5. \quad (3.6)$$

Then applying Hölder's inequality and using (3.4) (as in (2.28)) shows

$$\left| \mathbb{E}[(\log Z_{N_1}^\circ(\mathcal{M}_1, N) - \log Z_{N_1}^\circ(S^{(1)}, N)) \mathbb{1}_{\mathcal{A}_1^c}] \right| \leq CN^5 \exp(-cN^\varepsilon).$$

Combining the previous line with (3.5) completes the proof. \square

Recall that $\varepsilon > 0$ is a parameter. We write $H_{S^{(1)}, N_1, N}^\circ = \widehat{H}^\circ + \widehat{p}^\circ$, where we define

$$\widehat{H}^\circ = \frac{1}{N^{1/\alpha}} \sum_{1 \leq a \leq S^{(1)}} d_a(N) \sigma_{c_a(1)} \sigma_{c_a(2)} \mathbb{1}_{|N^{-1/\alpha} d_a(N)| \geq N_1^{-\varepsilon}}, \quad (3.7)$$

$$\widehat{p}^\circ = \frac{1}{N^{1/\alpha}} \sum_{1 \leq a \leq S^{(1)}} d_a(N) \sigma_{c_a(1)} \sigma_{c_a(2)} \mathbb{1}_{|N^{-1/\alpha} d_a(N)| < N_1^{-\varepsilon}}. \quad (3.8)$$

We also define $\widehat{Z}_N^\circ = \sum_{\sigma \in \Sigma_N} \exp(\widehat{H}^\circ(\sigma))$. Set

$$p^{(1)} = p_N^{(1)} = \mathbb{P}\left(|N^{-1/\alpha} d_1(N)| \geq N_1^{-\varepsilon}\right) = \frac{N_1^{\alpha\varepsilon}}{N^{\alpha\varepsilon}}, \quad (3.9)$$

where the last equality follows from the definition of d_a and (1.1), and we supposed that N is sufficiently large (depending on C_0). We let $\{x_a, y_a\}_{1 \leq a \leq S^{(1)}}$ be a collection of mutually independent random variables such that for every interval $I \subset \mathbb{R}$, we have

$$\mathbb{P}(x_a \in I) = (1 - p^{(1)})^{-1} \mathbb{P}(N^{-1/\alpha} d_1(N) \in I \cap (-N_1^{-\varepsilon}, N_1^{-\varepsilon})) \quad (3.10)$$

$$\mathbb{P}(y_a \in I) = (p^{(1)})^{-1} \mathbb{P}(N^{-1/\alpha} d_1(N) \in I \cap ((-\infty, -N_1^{-\varepsilon}] \cup [N_1^{-\varepsilon}, \infty))). \quad (3.11)$$

Let $L^\circ = \{L_a^\circ\}_{1 \leq a \leq S^{(1)}}$ be a collection of independent, identically distributed random variables such that $\mathbb{P}(L_a = 1) = p^{(1)}$ and $\mathbb{P}(L_a = 0) = 1 - p^{(1)}$. We further impose the condition that the collection $\{L_a^\circ, x_a, y_a\}_{1 \leq a \leq S^{(1)}}$ is mutually independent.

With these definitions, we have the distributional equalities

$$\{N^{-1/\alpha} d_a(N)\}_{1 \leq a \leq S^{(1)}} \stackrel{(d)}{=} \{(1 - L_a^\circ)x_a + L_a^\circ y_a\}_{1 \leq a \leq S^{(1)}}, \quad (3.12)$$

$$\widehat{H}^\circ(\sigma) \stackrel{(d)}{=} \sum_{1 \leq a \leq S^{(1)}} L_a^\circ y_a \sigma_{c_a(1)} \sigma_{c_a(2)}, \quad \widehat{p}^\circ(\sigma) \stackrel{(d)}{=} \sum_{1 \leq a \leq S^{(1)}} (1 - L_a^\circ)x_a \sigma_{c_a(1)} \sigma_{c_a(2)}, \quad (3.13)$$

with the dependence between $\widehat{H}^\circ(\sigma)$ and $\widehat{p}^\circ(\sigma)$ expressed through the L_a° . We observe that after conditioning on L° , the sums $\widehat{H}^\circ(\sigma)$ and $\widehat{p}^\circ(\sigma)$ are independent.

Lemma 3.4. *Fix $\varepsilon > 0$. Then there exists $C(\varepsilon) > 0$ such that the following holds. For every $1 \leq N_1 \leq N$ such that $N/3 \leq N_1 \leq 2N/3$, we have*

$$\mathbb{E}[\exp((1 - L_a)x_a)] \leq \exp(CN^{-2\varepsilon}).$$

Proof. By (2.6),

$$\mathbb{E}[\exp((1 - L_a)x_a)] \leq C \exp(C \cdot \mathbb{E}[(1 - L_a)^2 x_a^2]). \quad (3.14)$$

Next, using the definition of d_a from (2.19), we have

$$\mathbb{P}(|N^{-1/\alpha} d_a(N)| > t) = \frac{1}{N^{\alpha\varepsilon} t^\alpha} \quad (3.15)$$

for $t \geq N^{-\varepsilon}$. Then using $|L_{ij}^\circ| \leq 1$, $(1 - p^{(1)})^{-1} \leq 3$ (from (1.1) and the assumption on N_1), and the definition (3.10), we have

$$\mathbb{E}[(1 - L_a^\circ)^2 x_a^2] \leq 3\mathbb{E}\left[\left|N^{-1/\alpha} d_a(N)\right|^2 \mathbb{1}_{|N^{-1/\alpha} d_a(N)| < N_1^{-\varepsilon}}\right] \leq N^{-\alpha\varepsilon} \int_{N^{-\varepsilon}}^{N_1^{-\varepsilon}} t^{1-\alpha} dt \leq CN^{-2\varepsilon}. \quad (3.16)$$

This completes the proof. \square

Lemma 3.5. *Fix $\varepsilon > 0$. Then there exists $C(\varepsilon) > 0$ such that the following holds. For every $1 \leq N_1, N_2 \leq N$ such that $N/3 \leq N_1 \leq 2N/3$ and $N_1 + N_2 = N$, we have*

$$\left| \mathbb{E}[\log Z_{N_1}^\circ(S_{N_1}, N_1)] - \mathbb{E}[\log Z_{N_1}^\circ(S^{(1)}, N)] \right| \leq CN^{1+\varepsilon(\alpha-2)} \quad (3.17)$$

$$\left| \mathbb{E}[\log Z_{N_2}^\circ(S_{N_2}, N_2)] - \mathbb{E}[\log Z_{N_2}^\circ(S^{(2)}, N)] \right| \leq CN^{1+\varepsilon(\alpha-2)}. \quad (3.18)$$

Proof. We prove only (3.17), since the proof of (3.18) is similar. As a first step towards (3.17), we claim

$$\left| \mathbb{E}[\log Z_{N_1}(S^{(1)}, N)] - \mathbb{E}[\log \widehat{Z}_N^\circ] \right| \leq CN^{1+\varepsilon(\alpha-2)}, \quad (3.19)$$

where we recall that \widehat{Z}_N° was defined after (3.8). To this end, note that $\mathbb{E}\langle \widehat{p}^\circ \rangle = 0$ by integrating first in the x_a variables, and that

$$\mathbb{E}_x[e^{\widehat{p}^\circ(\sigma)}] = \mathbb{E}_x \left[\prod_{1 \leq a \leq S^{(1)}} \exp((1 - L_a^\circ)x_a) \right] \leq \exp(S^{(1)}N^{-2\varepsilon}) \leq \exp(3N^{-2\varepsilon}S_N) \leq \exp(CN^{1+\varepsilon(\alpha-2)}),$$

where we used the independence of the $(1 - L_a)x_a$ variables for the equality, and Lemma 3.4 for the first inequality. Thus by Lemma 2.2 with $x = \widehat{H}^\circ$ and $y = \widehat{p}^\circ$ as in (3.7), we obtain (3.19).

Next, we claim that

$$\left| \mathbb{E}[\log Z_{N_1}^\circ(S_{N_1}, N_1)] - \mathbb{E}[\log \widehat{Z}_N^\circ] \right| \leq CN^{1+\varepsilon(\alpha-2)}. \quad (3.20)$$

Together with (3.19), the previous equation implies the desired conclusion (3.17).

To prove (3.20), we begin by identifying the distribution of $N_1^{-1/\alpha} d_a(N_1)$, the (rescaled) coupling distribution for the Hamiltonian $H_{S_{N_1}, N_1, N_1}^\circ$. Using the definition of d_a from (2.19), we have

$$\mathbb{P}\left(\left|N_1^{-1/\alpha} d_a(N_1)\right| > t\right) = \frac{1}{N_1^{\alpha\varepsilon} t^\alpha},$$

for $t \geq N_1^{-\varepsilon}$. Similarly, we obtain $\mathbb{P}(|y_a| > t) = 1/(N_1^{\alpha\varepsilon} t^\alpha)$ for $t \geq N_1^{-\varepsilon}$. Hence, the variables $N_1^{-1/\alpha} d_a(N_1)$ and y_a are identically distributed, and we have the distributional equality

$$H_{S_{N_1}, N_1, N_1}^\circ \stackrel{(d)}{=} \sum_{1 \leq a \leq S_{N_1}} y_a \sigma_{c_a(1)} \sigma_{c_a(2)}, \quad (3.21)$$

where we recall that the c_a variables are sampled uniformly from the set $\{(i, j)\}_{1 \leq i \leq j \leq N_1}$.

Now note that the definition \widehat{H}° in (3.13) differs from (3.21) only in the number of nonzero couplings y_a (given by the indices a such that $L_a = 1$). There are $S^{(1)}$ nonzero couplings in $H_{S^{(1)}, N_1, N}^\circ$, and the number nonzero couplings in \widehat{H}° is binomial with $S^{(1)}$ trials and success probability $p^{(1)}$. The expectation of this distribution is

$$\frac{N_1^{\alpha\varepsilon}}{N^{\alpha\varepsilon}} S^{(1)} = C_0 \cdot \frac{N_1^{1+\alpha\varepsilon}}{2} + O(1) = S_{N_1} + O(1). \quad (3.22)$$

Then an argument nearly identical to the one that proved (3.2) shows (3.20). This completes the proof. \square

Proposition 3.6. *There exists $c > 0$ such that the following holds for all $\varepsilon \in (0, c)$. We have $\lim_{N \rightarrow \infty} F_{S, N, N}^\circ = L$ for some L satisfying $-\infty < L \leq \infty$.*

Proof. Let $1 \leq N_1, N_2 \leq N$ be integers such that $N/3 \leq N_1 \leq 2N/3$. Lemma 3.1 and Lemma 3.3 together imply that

$$\mathbb{E}[\log Z_N^\circ(S, N)] + CN^{2/3} \geq \mathbb{E}[\log Z_{N_1}^\circ(S^{(1)}, N)] + \mathbb{E}[\log Z_{N_2}^\circ(S^{(2)}, N)],$$

if ε is chosen small enough (relative to α). Then Lemma 3.5 implies that

$$\mathbb{E}[\log Z_N^\circ(S, N)] + CN^{1+\varepsilon(\alpha-2)} \geq \mathbb{E}[\log Z_{N_1}^\circ(S_{N_1}, N_1)] + \mathbb{E}[\log Z_{N_2}^\circ(S_{N_2}, N_2)].$$

Now set $a_N = -\mathbb{E}[\log Z_N^\circ(S, N)]$ and $\varphi(t) = Ct^{1+\varepsilon(\alpha-2)}$, and observe that $\varphi(t)/t^2$ is integrable on $[1, \infty)$ since $1 + \varepsilon(\alpha - 2) < 1$. We then apply Lemma 3.2 to conclude. \square

Proof of Theorem 1.1. By combining Lemma 2.1, Lemma 2.5, Lemma 2.7, and Proposition 3.6, we find that $\lim_{N \rightarrow \infty} F_N$ exists and $\lim_{N \rightarrow \infty} F_N > -\infty$. It remains to show that this limit does not equal $+\infty$. To accomplish this, we will show that F_N is uniformly bounded. We consider the Hamiltonian $\widehat{H}_\star(\sigma)$ defined by

$$\widehat{H}_\star(\sigma) = \frac{1}{N^{1/\alpha}} \sum J_{ij} \mathbb{1}_{|J_{ij}| \geq R_\star} \sigma_i \sigma_j, \quad \widehat{p}_\star(\sigma) = \frac{1}{N^{1/\alpha}} \sum J_{ij} \mathbb{1}_{|J_{ij}| < R_\star} \sigma_i \sigma_j, \quad R_\star = N^{1/\alpha}.$$

We note that $H(\sigma) = \widehat{H}_\star(\sigma) + \widehat{p}_\star(\sigma)$, and define $Z_{N,\star}$ and $F_{N,\star}$ by analogy with (2.1).

The second inequality in Lemma 2.2 yields

$$\mathbb{E}[\log Z_{N,\star}] = \mathbb{E}[\log \sum_\sigma e^{\widehat{H}_\star(\sigma) + \widehat{p}_\star(\sigma)}] \leq \mathbb{E}[\log \sum_\sigma e^{\widehat{H}_\star(\sigma)}] + O(N). \quad (3.23)$$

Therefore, it suffices to bound the expectation on the right side of (3.23). Define the Hamiltonian H_0 by $H_0(\sigma) = 0$. Its associated partition function is $Z_0 = 2^N$. Then removing all the nonzero

couplings of H_\star using (2.13) with $\varepsilon = 0$ gives

$$|\log Z_0 - \log Z_{N,\star}| \leq N^{-1/\alpha} \sum_{1 \leq i < j \leq N} |J_{ij}| \mathbb{1}_{|J_{ij}| > R_\star}, \quad (3.24)$$

which implies

$$|\mathbb{E}[\log Z_0] - \mathbb{E}[\log Z_\star]| \leq CN^{2-1/\alpha} \cdot \mathbb{E}[|J_{ij}| \mathbb{1}_{|J_{ij}| > R_\star}] = O(N), \quad (3.25)$$

where we used (1.1) to compute $\mathbb{E}[|J_{ij}| \mathbb{1}_{|J_{ij}| > R_\star}] \leq N^{-1+1/\alpha}$. Since $|\log Z_0| \leq CN$, equation (3.25) implies $N^{-1}\mathbb{E}[\log Z_{\star,N}] \leq C$. Combining this bound with (3.23) completes the proof. \square

4. INTERPOLATION

In this section, we prove Lemma 3.1.

4.1. Proof of Lemma 3.1. Recall the notation of Section 2.3. Given integers $v, u > 0$, define $\mathbb{G}(v, u)$ to be the random multi-graph on the vertex set $\llbracket 1, v \rrbracket$ with edge set $\{c_a\}_{1 \leq a \leq u}$. We will construct a sequence of multi-graphs interpolating between $\mathbb{G}(N, S_N)$ and the disjoint union of $\mathbb{G}(N_1, \mathcal{M}_2)$ and $\mathbb{G}(N_2, \mathcal{M}_2)$.

Given N_1, N_2 such that $N_1 + N_2$ and an integer r such that $0 \leq r \leq N$, we define \mathbb{G}_r as follows. Let χ be a Bernoulli random value that takes the value 1 with probability N_1/N , and is 0 otherwise, and let $\{\chi_a\}_{1 \leq a \leq N}$ be a collection of independent random variables distributed as χ . Let $\{c_a^{(1)}\}_{1 \leq a \leq S}$ be independent edges chosen uniformly at random from the set $\{(i, j)\}_{1 \leq i \leq j \leq N_1}$, and define $\{c_a^{(2)}\}_{1 \leq a \leq S}$ similarly for $\{(i, j)\}_{N_1+1 \leq i \leq j \leq N_2}$. We define the random variables $\{c_a^{(-)}\}_{1 \leq a \leq S}$ by letting $c_a^{(-)} = c_a^{(1)}$ if $\chi = 1$, and $c_a^{(-)} = c_a^{(2)}$ if $\chi = 0$. The graph \mathbb{G}_r is then defined for $0 \leq r \leq S_N$ by the random edge set $\{c_a\}_{1 \leq a \leq r} \cup \{c_a^{(-)}\}_{r+1 \leq a \leq S}$. We see that the graphs \mathbb{G}_r interpolate between $\mathbb{G}(N, S)$ when $r = S_N$ and the disjoint union of $\mathbb{G}(N_1, \mathcal{M}_2)$ and $\mathbb{G}(N_2, \mathcal{M}_2)$ when $r = 0$.

We define a Hamiltonian and partition function corresponding to \mathbb{G}_r by

$$H^{(r)}(\sigma) = N^{-1/\alpha} \sum_{1 \leq a \leq r} d_a(N) \sigma_{c_a(i)} \sigma_{c_a(j)} + N^{-1/\alpha} \sum_{r+1 \leq a \leq S_N} d_a(N) \sigma_{c_a^{(-)}(1)} \sigma_{c_a^{(-)}(2)}$$

and $Z^{(r)} = Z_N^{(r)} = \sum_{\sigma \in \Sigma_N} \exp(H^{(r)}(\sigma))$. We also define the graph $\widehat{\mathbb{G}}_r$ using the random edge set $\{c_a\}_{1 \leq a \leq r-1} \cup \{c_a^{(-)}\}_{r+1 \leq a \leq S}$, which omits the r -th edge. The corresponding Hamiltonian and partition function are defined by

$$H^{(r,-)}(\sigma) = N^{-1/\alpha} \sum_{1 \leq a \leq r-1} d_a(N) \sigma_{c_a(i)} \sigma_{c_a(j)} + N^{-1/\alpha} \sum_{r+1 \leq a \leq S_N} d_a(N) \sigma_{c_a^{(-)}(1)} \sigma_{c_a^{(-)}(2)},$$

$$Z^{(r,-)} = Z_N^{(r,-)} = \sum_{\sigma \in \Sigma_N} \exp(H^{(r,-)}(\sigma)).$$

Lemma 4.1. *For every $1 \leq r \leq S_N$,*

$$\mathbb{E}[\log Z^{(r)}] \geq \mathbb{E}[\log Z^{(r-1)}]. \quad (4.1)$$

Proof. It suffices to show that

$$\mathbb{E}[\log Z^{(r)} \mid \widehat{\mathbb{G}}_r] \geq \mathbb{E}[\log Z^{(r-1)} \mid \widehat{\mathbb{G}}_r], \quad (4.2)$$

where the notation in the previous inequality denotes the conditional expectation over the edges and weights of $\widehat{\mathbb{G}}_r$. The remaining randomness is in the choice of edge c_r (or $c_r^{(-)}$) and the weight d_r . We write $x = c_r(1)$ and $y = c_r(2)$.

We compute

$$\begin{aligned} & \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r] - \mathbb{E}[\log Z^{(r,-)}] \\ &= \mathbb{E} \left[\log \frac{e^{-d_r} \sum_{\sigma} \mathbb{1}_{\sigma_x \neq \sigma_y} \exp(H^{(r,-)}(\sigma)) + e^{d_r} \sum_{\sigma} \mathbb{1}_{\sigma_x = \sigma_y} \exp(H^{(r,-)}(\sigma))}{\sum_{\sigma} \exp(H^{(r,-)}(\sigma))} \middle| \widehat{\mathbb{G}}_r \right]. \end{aligned} \quad (4.3)$$

The same expression holds with $Z^{(r)}$ replaced by $Z^{(r-1)}$, and x and y replaced by $x^{(-)} = c_r^{(-)}(1)$ and $y^{(-)} = c_r^{(-)}(2)$, respectively. In the following two cases, we will compute both of these expressions, after conditioning on d_r . The computations will differ depending on the sign of d_r .

Case I: $d_r < 0$. Let μ denote the Gibbs measure for the Hamiltonian $H^{(r,-)}(\sigma)$. Using (4.3), we have

$$\begin{aligned} & \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] \\ &= -d_r + \mathbb{E} \left[\log \frac{\sum_{\sigma} \mathbb{1}_{\sigma_x \neq \sigma_y} \exp(H^{(r,-)}(\sigma)) + e^{2d_r} \sum_{\sigma} \mathbb{1}_{\sigma_x = \sigma_y} \exp(H^{(r,-)}(\sigma))}{\sum_{\sigma} \exp(H^{(r,-)}(\sigma))} \middle| \widehat{\mathbb{G}}_r \right] \\ &= -d_r + \mathbb{E} \left[\log \left(1 - (1 - e^{2d_r}) \mu(\sigma_x = \sigma_y) \right) \middle| \widehat{\mathbb{G}}_r, d_r \right]. \end{aligned}$$

Observe that $0 < (1 - e^{2d_r}) \mu(\sigma_x = \sigma_y) < 1$ because $d_r < 0$, so it is permissible to Taylor expand the logarithm. Therefore, introducing replicas σ^ℓ , we have

$$\begin{aligned} & \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r \\ &= - \sum_{k=1}^{\infty} \mathbb{E} \left[\frac{(1 - e^{2d_r})^k \mu(\sigma_x = \sigma_y)^k}{k} \middle| \widehat{\mathbb{G}}_r, d_r \right] \\ &= - \sum_{k=1}^{\infty} \frac{(1 - e^{2d_r})^k}{k} \mathbb{E} \left[\sum_{\sigma^1, \dots, \sigma^k} \frac{\exp \left(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell) \right)}{(Z^{(r,-)})^k} \mathbb{1}_{\{\sigma_x^\ell = \sigma_y^\ell, \forall \ell\}} \middle| \widehat{\mathbb{G}}_r, d_r \right] \\ &= - \sum_{k=1}^{\infty} \frac{(1 - e^{2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp \left(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell) \right)}{(Z^{(r,-)})^k} \cdot \mathbb{E} \left[\mathbb{1}_{\{\sigma_x^\ell = \sigma_y^\ell, \forall \ell\}} \right]. \end{aligned} \quad (4.4)$$

For every set of replicas $\sigma = (\sigma^1, \dots, \sigma^k)$, we introduce the following equivalence relation on $\llbracket 1, N \rrbracket$. For $i, j \in \llbracket 1, N \rrbracket$, we say that $i \sim j$ if $\sigma_i^\ell = \sigma_j^\ell$ for all replicas $\ell = 1, \dots, k$. Denote the number of equivalence classes induced by \sim by J , and let $\{O_s\}_{s=1}^J = \{O_s(\sigma)\}_{s=1}^J$ be the set of these equivalence classes. Recalling the definition of x and y , we compute $\mathbb{E}[\mathbb{1}_{\{\sigma_x^\ell = \sigma_y^\ell, \forall \ell\}}] = \sum_{s=1}^J (|O_s|/N)^2$. Then, recalling (4.4), we have

$$\begin{aligned} & \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r \\ &= - \sum_{k=1}^{\infty} \frac{(1 - e^{2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp \left(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell) \right)}{(Z^{(r,-)})^k} \sum_{s=1}^J \left(\frac{|O_s|}{N} \right)^2. \end{aligned} \quad (4.5)$$

The computation for $\mathbb{E}[\log Z^{(r-1)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r]$ is analogous, and we now outline the main steps. Note that in \mathbb{G}_{r-1} , the r -th edge is added using the two-step sampling procedure described at the beginning of this proof, where first the value χ_r is sampled, and then $c_r^{(-)}$ is sampled

from either $c_r^{(1)}$ or $c_r^{(2)}$, depending on the value of χ_r . Recall that $(x^{(-)}, y^{(-)})$ denotes the random edge $c_r^{(-)}$. We find

$$\mathbb{E} \left[\mathbb{1}_{\{\sigma_{x^{(-)}}^\ell = \sigma_{y^{(-)}}^\ell, \forall \ell\}} \right] = \sum_{s=1}^J \left(\frac{N_1}{N} \left(\frac{|O_s \cap \llbracket 1, N_1 \rrbracket|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_s \cap \llbracket 1, N_2 \rrbracket|}{N_2} \right)^2 \right). \quad (4.6)$$

Then the analogue of (4.4) holds for $\mathbb{E}[\log Z^{(r-1)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r]$, with x and y replaced by $x^{(-)}$ and $y^{(-)}$, respectively, and we conclude that

$$\begin{aligned} \mathbb{E}[\log Z^{(r-1)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r = \\ - \sum_{k=1}^{\infty} \frac{(1 - e^{2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell))}{(Z^{(r,-)})^k} \\ \times \sum_{s=1}^J \left(\frac{N_1}{N} \left(\frac{|O_s \cap \llbracket 1, N_1 \rrbracket|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_s \cap \llbracket 1, N_2 \rrbracket|}{N_2} \right)^2 \right). \end{aligned} \quad (4.7)$$

Case II: $d_r > 0$. We proceed as in the previous case to obtain

$$\mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r = \mathbb{E} \left[\log \left(1 - (1 - e^{-2d_r}) \mu(\sigma_x \neq \sigma_y) \right) | \widehat{\mathbb{G}}_r, d_r \right].$$

Taylor expanding the logarithm, we obtain as before that

$$\begin{aligned} \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r \\ = - \sum_{k=1}^{\infty} \frac{(1 - e^{-2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell))}{(Z^{(r,-)})^k} \mathbb{E}[\mathbb{1}_{\{\sigma_x^\ell \neq \sigma_y^\ell, \forall \ell\}}]. \end{aligned} \quad (4.8)$$

We now compute the term $\mathbb{E}[\mathbb{1}_{\{\sigma_x^\ell \neq \sigma_y^\ell, \forall \ell\}}]$. We recall the equivalence classes O_s defined in the previous case. For every class O_s , there exists an equivalence class O_r , for some $r = r(s)$, of vertices such that $\sigma_i^\ell \neq \sigma_j^\ell$ for all ℓ if $i \in O_s$ and $j \in O_r$. This gives a pairing of equivalence classes. Then we have $\mathbb{E}[\mathbb{1}_{\{\sigma_x^\ell \neq \sigma_y^\ell, \forall \ell\}}] = N^{-2} \sum_{s=1}^J |O_s| |O_r|$ which combined with (4.8) yields

$$\begin{aligned} \mathbb{E}[\log Z^{(r)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r \\ = - \sum_{k=1}^{\infty} \frac{(1 - e^{-2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell))}{(Z^{(r,-)})^k} \sum_{s=1}^J \left(\frac{|O_s|}{N} \right) \left(\frac{|O_r|}{N} \right). \end{aligned} \quad (4.9)$$

Similarly, we compute

$$\mathbb{E}[\mathbb{1}_{\{\sigma_{x^{(-)}}^\ell = \sigma_{y^{(-)}}^\ell, \forall \ell\}}] = \frac{N_1}{N} \sum_{s=1}^J \frac{|O_s \cap \llbracket 1, N_1 \rrbracket|}{N_1} \cdot \frac{|O_r \cap \llbracket 1, N_1 \rrbracket|}{N_1} + \frac{N_2}{N} \sum_{s=1}^J \frac{|O_s \cap \llbracket 1, N_2 \rrbracket|}{N_2} \cdot \frac{|O_r \cap \llbracket 1, N_2 \rrbracket|}{N_2},$$

leading to

$$\begin{aligned} \mathbb{E}[\log Z^{(r-1)} | \widehat{\mathbb{G}}_r, d_r] - \mathbb{E}[\log Z^{(r,-)} | d_r] + d_r \\ = - \sum_{k=1}^{\infty} \frac{(1 - e^{-2d_r})^k}{k} \sum_{\sigma^1, \dots, \sigma^k} \frac{\exp(\sum_{\ell=1}^k H^{(r,-)}(\sigma^\ell))}{(Z^{(r,-)})^k} \\ \times \left(\frac{N_1}{N} \sum_{s=1}^J \frac{|O_s \cap \llbracket 1, N_1 \rrbracket|}{N_1} \cdot \frac{|O_r \cap \llbracket 1, N_1 \rrbracket|}{N_1} + \frac{N_2}{N} \sum_{s=1}^J \frac{|O_s \cap \llbracket 1, N_2 \rrbracket|}{N_2} \cdot \frac{|O_r \cap \llbracket 1, N_2 \rrbracket|}{N_2} \right). \end{aligned} \quad (4.10)$$

Conclusion. Observe that $1 - e^{2(-x)} = 1 - e^{-2x}$, so the powers $(1 - e^{2d_r})^k$ in the Taylor expansions in above two cases are the same if $|d_r| = x$ in each case. Further, observe that the density of d_r is symmetric, by definition. We now subtract (4.7) from (4.5), subtract (4.10) from (4.9), and take expectation over d_r and $\widehat{\mathbb{G}}_r$ in each expression. The upshot of this computation is that to establish (4.2), it suffices to prove for a fixed replica σ that

$$\begin{aligned} & \sum_{s=1}^J \left(\frac{N_1}{N} \left(\frac{|O_s \cap [1, N_1]|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_s \cap [1, N_2]|}{N_2} \right)^2 \right) \\ & + \sum_{s=1}^J \left(\frac{N_1}{N} \left(\frac{|O_s \cap [1, N_1]|}{N_1} \right) \left(\frac{|O_r \cap [1, N_1]|}{N_1} \right) + \frac{N_2}{N} \left(\frac{|O_s \cap [1, N_2]|}{N_2} \right) \left(\frac{|O_r \cap [1, N_2]|}{N_2} \right) \right) \\ & \geq \sum_{s=1}^J \left(\frac{|O_s|}{N} \right)^2 + \sum_{s=1}^J \left(\frac{|O_s|}{N} \right) \left(\frac{|O_r|}{N} \right). \end{aligned}$$

Fix some replica s and corresponding $r = r(s)$ (as defined in the second case above), and consider just these terms in the sum. It suffices to show that

$$\begin{aligned} & \left(\frac{N_1}{N} \left(\frac{|O_s \cap [1, N_1]|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_s \cap [1, N_2]|}{N_2} \right)^2 \right) \\ & + \left(\frac{N_1}{N} \left(\frac{|O_r \cap [1, N_1]|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_r \cap [1, N_2]|}{N_2} \right)^2 \right) \\ & + 2 \left(\frac{N_1}{N} \left(\frac{|O_s \cap [1, N_1]|}{N_1} \right) \left(\frac{|O_r \cap [1, N_1]|}{N_1} \right) + \frac{N_2}{N} \left(\frac{|O_s \cap [1, N_2]|}{N_2} \right) \left(\frac{|O_r \cap [1, N_2]|}{N_2} \right) \right) \\ & \geq \left(\frac{|O_r|}{N} \right)^2 + \left(\frac{|O_s|}{N} \right)^2 + 2 \left(\frac{|O_s|}{N} \right) \left(\frac{|O_r|}{N} \right). \end{aligned}$$

The right side of the previous inequality factors as $\left(\frac{|O_s|}{N} + \frac{|O_r|}{N} \right)^2$, whereas the left side factors as

$$\frac{N_1}{N} \left(\frac{|O_s \cap [1, N_1]|}{N_1} + \frac{|O_r \cap [1, N_1]|}{N_1} \right)^2 + \frac{N_2}{N} \left(\frac{|O_s \cap [1, N_2]|}{N_2} + \frac{|O_r \cap [1, N_2]|}{N_2} \right)^2.$$

The left side is thus greater than the right side by the convexity of $x \mapsto x^2$. This establishes (4.1) and completes the proof. \square

Proof of Lemma 3.1. We apply Lemma 4.1 in succession for $r = 1, \dots, S$ to obtain

$$\mathbb{E}[\log Z^{(N)}] \geq \mathbb{E}[\log Z^{(0)}]. \quad (4.11)$$

By the definition of $Z^{(r)}$, equation (4.11) is exactly the claim (3.1). \square

5. SELF-AVERAGING

Proof of Theorem 1.2. This follows from the following Proposition together with Markov's inequality. \square

Proposition 5.1. *For all $\delta > 0$,*

$$\mathbb{E} \left[\left| N^{-1} \log Z_N - N^{-1} \mathbb{E}[\log Z_N] \right|^{\alpha-\delta} \right] \leq C N^{1-\alpha+2\delta}.$$

Proof. Let $\mathcal{I} = \{(i, j) : 1 \leq i < j \leq N\}$ and fix an arbitrary bijection $f: \mathcal{I} \rightarrow \{1, 2, \dots, |\mathcal{I}|\}$. We use J_x with $x \in \{1, 2, \dots, |\mathcal{I}|\}$ as shorthand for $J_{f^{-1}(x)}$. Set $\mathcal{F}_x = \sigma(J_y : y \leq x)$ for all $x \in \{1, 2, \dots, |\mathcal{I}|\}$, where this notation denotes the σ -algebra generated by the given couplings J_y . Consider the martingale

$$A_x = \frac{1}{N} \mathbb{E}[\log Z_N \mid \mathcal{F}_x] - \frac{1}{N} \mathbb{E}[\log Z_N], \quad (5.1)$$

with the convention that $A_0 = 0$.

Define the martingale difference sequence $D_x = A_x - A_{x-1}$ for $x \geq 1$, so that $A_x = \sum_{y \leq x} D_y$. Set

$$H^{(x)}(\sigma) = \frac{1}{N^{1/\alpha}} \sum_{i < j} J_{ij} \sigma_i \sigma_j \mathbb{1}_{f(i,j) \neq x}, \quad Z_N^{(x)} = \sum_{\sigma \in \Sigma_N} e^{H^{(x)}(\sigma)},$$

where $H^{(x)}(\sigma)$ is similar to the Hamiltonian $H(\sigma)$, except with the coupling J_x set equal to zero. Let $\langle \cdot \rangle_x$ denote the Gibbs measure with respect to $H^{(x)}$. Then we have (by definition) $Z_N = Z_N^{(x)} \langle e^{N^{-1/\alpha} J_x \sigma_x} \rangle_x$. We write

$$N \cdot D_x = \mathbb{E}[\log \langle e^{N^{-1/\alpha} J_x \sigma_x} \rangle_x \mid \mathcal{F}_x] - \mathbb{E}[\log \langle e^{N^{-1/\alpha} J_x \sigma_x} \rangle_x \mid \mathcal{F}_{x-1}],$$

where we use the equality $\mathbb{E}[Z^{(x)} \mid \mathcal{F}_x] = \mathbb{E}[Z^{(x)} \mid \mathcal{F}_{x-1}]$. Bounding $e^{N^{-1/\alpha} J_x \sigma_x}$ in absolute value in each expectation gives

$$|D_x| \leq N^{-1-1/\alpha} (|J_x| + \mathbb{E}[|J_x|]),$$

which implies $|D_x|^p \leq 2^p N^{-1-1/\alpha} (|J_x|^p + \mathbb{E}[|J_x|]^p)$ for any $p \in (1, 2)$. By Burkholder's inequality with exponent $p \in (1, \alpha)$, and the fact that $p/2 < 1$, we have

$$\mathbb{E}[|A_{|\mathcal{I}|} - A_0|^p] \leq C_p \mathbb{E}[(\sum_x D_x^2)^{p/2}] \leq C_p \mathbb{E}[\sum_x D_x^p] \leq C_p N^{2-p-p/\alpha} \mathbb{E}[|J_x|]^p.$$

Set $g(p) = 2 - p(1 + 1/\alpha)$. By choosing $p(\delta) = \alpha - \delta$, we find

$$\mathbb{E}[|A_{|\mathcal{I}|} - A_0|^{\alpha-\delta}] \leq C N^{1-\alpha+2\delta},$$

where $C = C(\delta) > 1$ depends on δ . This completes the proof. \square

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