EXPLICIT COMPACTIFICATIONS OF MODULI SPACES OF CAMPEDELLI AND BURNIAT SURFACES

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ABSTRACT. We describe explicitly the geometric compactifications, obtained by adding slc surfaces X with ample canonical class, for two connected components in the moduli space of surfaces of general type: Campedelli surfaces with $\pi_1(X) = \mathbb{Z}_2^3$ and Burniat surfaces with $K^2 = 6$.

CONTENTS

1. Introduction

In 1988, Kollár and Shepherd-Barron [KSB88] proposed a way to compactify the moduli space of surfaces of general type by adding *stable surfaces*, i.e. surfaces that have slc (semi log canonical) singularities and ample canonical class K_X, similar to the stable curves in dimension one. This construction was subsequently extended to pairs $(X, B = \sum b_i B_i)$ with divisors and to higher dimensions. The resulting compact moduli spaces are commonly known as KSBA spaces, see [Kol23, Ch. 8], and they will be denoted $\overline{M}^{\mathrm{slc}}$ here.

In the cases with nonzero boundary divisor B there are many papers where these compact moduli spaces are described in detail, e.g. for toric and abelian varieties [Ale02], hyperplane arrangements [HKT06, Ale15], K3 surfaces [AE23, AET23, ABE22, AE22], elliptic surfaces [AB21, Inc20] and many, many more.

However, in the original case of [KSB88], i.e. with $B = 0$, practically no explicit compactifications are known, aside from the moduli of surfaces which are either quotients or special covers of a product of two curves [vO06, Liu12, Rol10], which essentially reduce to moduli of curves. Let us also mention some works on *partial* compactifications: [FPR17, FFP22, FPRR22, CFP$^+$23, GPSZ22].

The goal of this paper is to describe explicitly two complete compactifications in the case with zero boundary divisor, for two irreducible components of the classical moduli space of surfaces of general type, of dimensions 6 and 4:

(1) Campedelli surfaces with $\pi_1(X) = \mathbb{Z}_2^3$. They can be defined as $\mathbb{Z}_2^3$-covers of $\mathbb{P}^2$ ramified in 7 lines B_i. The problem then can be reduced to compactifying the moduli of pairs $(\mathbb{P}^2, \sum_{i=1}^7 \frac{1}{2} B_i)$—which turns out to be quite easy in this case—and applying the theory of singular abelian covers of [AP12].

(2) Burniat surfaces with $K_X^2 = 6$. They can be defined as $\mathbb{Z}_2^2$-covers of the del Pezzo surface $\Sigma = \mathrm{Bl}_3 \mathbb{P}^2$, ramified in 12 curves coming from a particular configuration of 9 lines in $\mathbb{P}^2$. This case, although similar in spirit to the one above, turns out to be much harder.

Our main results are:

Theorem 1. *For Campedelli surfaces, the main irreducible component of the compactification $\overline{M}_{\mathrm{Cam}}^{\mathrm{slc}}$ is*

$$\mathrm{GL}(3, \mathbb{F}_2) \backslash (\mathbb{P}^2)^7 // \mathrm{PGL}(3),$$

a finite $\mathrm{GL}(3, \mathbb{F}_2)$-quotient of a smooth projective GIT quotient $(\mathbb{P}^2)^7 // \mathrm{PGL}(3)$.

Theorem 2. *For Burniat surfaces, the normalization of the compactification $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ is the quotient of a certain moduli space $\overline{M}(\frac{1}{2})$ of labeled stable pairs (Def. 4.5) by the finite group $C_3 \ltimes S_2^4$. The normalization map is a bijection.*

There is a diagram of moduli spaces of labeled stable pairs (see Section 4.10)

$$\overline{M}^{\mathrm{tor}} = \overline{M}(\tfrac{1}{3}) \xleftarrow{\rho_1} \overline{M}(\tfrac{2}{5}) \xleftarrow{\rho_2} \overline{M}(\tfrac{1}{2})$$

in which $\overline{M}^{\mathrm{tor}}$ is a projective toric variety with 8 isolated singularities corresponding to an explicit fan $\mathfrak{F}$ (Def. 4.10), ρ_1 is the blowup at one smooth point, and ρ_2 is the blowup at six disjoint smooth rational curves avoiding the singular locus.

Remark 1.1. The moduli space of smooth surfaces of general type with fixed numerical invariants is open in the corresponding moduli space of stable surfaces, but possibly not dense: here by "compactification" of a class of surfaces we mean its closure in the moduli space of stable surfaces. In fact for Burniat surfaces there are additional components meeting the compactification $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ of Theorem 2 (cf. Remark 4.24). We don't know if the main component of $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ is normal.

On the other hand, in the Campedelli case we conjecture that the compactification $\overline{M}_{\mathrm{Cam}}^{\mathrm{slc}}$ of Theorem 1 is a connected component of the moduli space of stable surfaces.

The boundary $\overline{M}_{\mathrm{Cam}}^{\mathrm{slc}} \setminus M_{\mathrm{Cam}}$ of M_{Cam} is the union of two irreducible divisors and the boundary of $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ is the union of eight irreducible divisors. Note that, contrary to the case of curves, it is not always the case that the complement $\overline{M}^{\mathrm{slc}} \setminus M$ of a component M of the moduli space of surfaces inside its closure in the moduli of stable surfaces is a divisor (see for instance [FPR17], Table 1).

Here all the surfaces corresponding to boundary points are $\mathbb{Z}_2^k$-covers, induced by the bicanonical system, of Gorenstein surfaces with Cartier total branch divisor, so by the Hurwitz formula they have Cartier index 1 or 2. This is also somewhat unexpected, since although the index is known to be bounded for any fixed value of K^2 ([Ale94]) already for $K^2 = 1$ there are examples of stable surfaces with Cartier index 15 ([CFP$^+$23]) and 21 ([GPSZ22]).

The first version [AP23] of this paper was written in 2009, and by some accounts it served as an introduction to the subject to many students. We haven't finished it until now, however, for two reasons:

The main reason was that it used the moduli space of stable pairs $(X, \sum b_i B_i)$ with coefficients $b_i = \frac{1}{2}$, which did not really exist at the time. The moduli with fixed coefficients $b_i \leq \frac{1}{2}$ present problems on the level of the definition of families, as the divisors B_i may form non-flat families. Various solutions, none completely satisfactory, were proposed: working with subschemes $B_i \subset X$ instead of divisors; working with finite maps $B_i \to X$; restricting to seminormal reduced bases, etc. Recently, a good solution involving the notion of K-flatness was proposed, and a complete theory has been firmly put in place in [Kol23].

The second reason was that the initial computation used an ad hoc generalization of the theory of weighted hyperplane arrangements [Ale15], which itself was not fully worked out at the time.

In the present version we give new, easier proofs and provide much sharper, very explicit descriptions of the compactified moduli spaces. We then sketch the original proofs which use the moduli of stable weighted hyperplane arrangements.

Results of the present paper were used in [AH23] to compute the stable surface compactifications of secondary Burniat surfaces and in [Ale23] to investigate kappa classes on KSBA spaces.

We work over $\mathbb{C}$ since the general results of [Kol23] about the existence of the stable pair compactifications are known only over $\mathbb{C}$. But in fact most of the constructions work, or can be modified to work, over any field k of characteristic different from 2. The source of this paper on arXiv includes a sagemath [Sag22] file verifying computations with fans and polytopes.

2. Preliminaries

2.1. **Compact moduli of stable surfaces.** We briefly recall the main definitions and the existence theorem, referring the reader to [Kol23] for more details.

We say that a variety has double crossings if every point is either smooth or has a neighborhood formally isomorphic to $xy = 0$. It is *deminormal* if it is S_2 and has double crossings outside a closed subset of codimension ≥ 2.

Let X be a variety, let B_j, $j = 1, \ldots n$, be effective Weil divisors on X, possibly reducible and with components in common, and let b_j be rational numbers with $0 < b_j \leq 1$. Set $B = \sum_j b_j B_j$.

Definition 2.1. Assume that X is a normal variety. Then X has a canonical Weil divisor K_X defined up to linear equivalence. The pair (X, B) is called *log canonical* (lc) if

    (1) $K_X + B$ is $\mathbb{Q}$-Cartier, i.e. some positive multiple is a Cartier divisor, and

    (2) every prime divisor D of X has multiplicity ≤ 1 in B and for every proper birational morphism $h \colon X' \to X$ with normal X', in the natural formula $K_{X'} + h_*^{-1} B = h^*(K_X + B) + \sum a_i E_i$ one has $a_i \geq -1$. Here, E_i are the irreducible exceptional divisors of π and the pullback h^* is defined by extending $\mathbb{Q}$-linearly the pullback on Cartier divisors. $h_*^{-1} B$ is the strict preimage of B.

The pair (X, B) is called *Kawamata log terminal* (klt) if all $a_i > -1$ and $\operatorname{mult}_D B := \sum b_j \operatorname{mult}_D(B_j) < 1$.

Remark 2.2. If $(X, \sum b_i B_i)$ is a pair such that X is a smooth surface and the B_i are smooth prime divisors that intersect transversally, then Definition 2.1 can be made more explicit. Since the blow-up of X at the points where at least 3 of the B_i meet is a log resolution of (X, B), [Kol13, Cor. 2.23] applies and $(X, \sum b_i B_i))$ is log canonical iff the following conditions hold:

    (1) every prime divisor appears in $\sum b_i B_i$ with multiplicity ≤ 1,

    (2) ever point has multiplicity ≤ 2 for $\sum b_i B_i$.

Definition 2.3. A pair (X, B) is called *semi log canonical* (slc) if

    (1) X is deminormal,

    (2) no divisor B_j contains any component of the double locus of X,

    (3) some multiple of the Weil $\mathbb{Q}$-divisor $K_X + B$, well defined thanks to the previous condition, is Cartier, and

    (4) denoting by $\nu \colon X^\nu \to X$ the normalization, the pair $(X^\nu, (\text{double locus}) + \nu_*^{-1} B)$ is log canonical.

Definition 2.4. A pair (X, B) is a KSBA stable pair, or simply a *stable pair*, if

    (1) (X, B) has slc singularities, and

    (2) $K_X + B$ is ample.

A family of stable pairs over a normal scheme S is a flat morphism $f \colon \mathcal{X} \to S$ together with Weil divisors $\mathcal{B}_i \subset \mathcal{X}$ such that $K_{\mathcal{X}} + \sum_i b_i \mathcal{B}$ is a relative ample $\mathbb{Q}$-Cartier divisor and every geometric fiber is a stable pair. For general (e.g., not

reduced or reduced but not seminormal) base schemes the definition of families of divisors $\mathcal{B}_i$ is much more delicate, and we refer to [Kol23, Ch. 7] for more details. The families we construct will be over reduced normal bases.

Theorem 2.5 ([Kol23], Thm. 8.1). *For fixed $(b_1, \ldots, b_n)$ and fixed $(K_X + B)^{\dim X}$, there exists a coarse moduli space of stable pairs, and it is projective.*

2.2. **Abelian covers.** A G-cover is a finite morphism $X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite

group G. This means that for every component Y_i of Y the G-action on the restricted cover $X \times_Y Y_i \to Y_i$ is faithful. We will restrict ourselves to the case when G is abelian.

When Y is smooth and X is normal, the theory of abelian covers was described in [Par91]. In [AP12] we extended it to the case needed for this paper: when one or both of X and Y are non-normal and deminormal. We briefly review this theory.

The G-action on X with $X/G = Y$ is equivalent to a decomposition:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{F}_\chi, \qquad \mathcal{F}_0 = \mathcal{O}_Y$$

where G acts on $\mathcal{F}_\chi$ via the character χ; each sheaf $\mathcal{F}_\chi$ is generically locally free of rank 1. The variety X is S_2 iff each sheaf $\mathcal{F}_\chi$ is S_2.

Now assume that Y is smooth and that X is S_2. Then $\mathcal{F}_\chi = L_\chi^{-1}$ for some invertible sheaves L_χ on Y. The $\mathcal{O}_Y$-algebra structure on $\pi_* \mathcal{O}_X$ is given by global sections $s_{\chi,\chi'}$ of $L_\chi \otimes L_{\chi'} \otimes L_{\chi+\chi'}^{-1}$. No section $s_{\chi,\chi'}$ is identically zero, since otherwise X would be non reduced, so $D_{\chi,\chi'} = (s_{\chi,\chi'})$ is an effective divisor and $L_\chi \otimes L_{\chi'} \simeq L_{\chi+\chi'}(D_{\chi,\chi'})$.

For the rest of the section we restrict to the case $G = \mathbb{Z}_2^k$, which is especially simple and is enough for our applications. When writing down formulas for this case it is convenient to write G additively and identify the group of characters $G^* = \mathrm{Hom}(G, \mathbb{G}_m)$ with the dual vector space $(\mathbb{Z}_2^k)^\vee$, so that the natural pairing takes place in $\mathbb{Z}_2$.

Then one of the main results of [Par91] is that if X is normal then there exist unique effective divisors D_g labeled by the nonzero elements $0 \neq g \in G$ such that $D_{\chi,\chi'} = \sum_{g:\ \chi(g)=\chi'(g)=1} D_g$. The support of the divisor D_g equals the image of the divisorial part of the set of points $x \in X$ fixed by the automorphism g. and $X \to Y$ is étale outside of $\cup D_g$. We will call $\sum D_g = D_{\mathrm{tot}}$, or simply D, the *total branch divisor*.

We remark (cf. [Par91, § 1]) that for more general abelian groups G the divisors $D_{H,\psi}$ are labeled by the cyclic subgroups $H \subseteq G$ with a choice of a generator ψ of H^*. When $G = \mathbb{Z}_2^k$, the pairs (H, ψ) are obviously in a bijection with the nonzero elements of G. The above description extends to the case Y smooth and X deminormal, see [AP12, Cor. 1.10].

Definition 2.6. The data of the invertible sheaves L_χ for all $\chi \in G^*$ and effective divisors D_g for $0 \neq g \in G$ are called the *building data* of the cover; they satisfy the identities

$$(1) \qquad \text{(written additively)} \quad L_\chi + L_{\chi'} \simeq L_{\chi+\chi'} + \sum_{g:\ \chi(g)=\chi'(g)=1} D_g,$$

called the *fundamental relations*. Note that for $\chi = \chi'$ one gets the relation $2L_\chi \simeq \sum_{g: \chi(g)=1} D_g$. The divisors D_g are called the *branch data*: if $\mathrm{Pic}(Y)$ has no 2-torsion the branch data suffice to determine the cover.

Vice versa, given building data satisfying the fundamental relations, if $H^0(\mathcal{O}_Y^*) = \mathbb{C}^*$ (for example if Y is proper and connected) then there exists a unique cover $\pi \colon X \to Y$ with these building data. Without assuming $H^0(\mathcal{O}_Y^*) = \mathbb{C}^*$, the cover exists and is unique étale-locally.

Example 2.7 ($\mathbb{Z}_2^2$-covers)**.** Consider the group $\mathbb{Z}_2^2$. Denote the nonzero elements of $\mathbb{Z}_2^2$ by R, G, B for the colors red, green and blue, and use the same letters R, G, B to denote the corresponding divisors. Then the fundamental relations imply that there exist line bundles L_1, L_2, L_3 such that:

$$(2) \qquad\qquad 2L_1 = G + B, \quad 2L_2 = B + R, \quad 2L_3 = R + G.$$

Vice versa, assume that one has three effective divisors R, G, B such that the divisors $G + B$, $B + R$, $R + G$ are 2-divisible in $\mathrm{Pic}\, Y$. If $\mathrm{Pic}(Y)$ has no torsion the solutions L_1, L_2, L_3 of (2) are unique and R, G, B together with L_1, L_2, L_3 are the building data of a $\mathbb{Z}_2^2$-cover.

The following is easy:

Lemma 2.8. *Suppose that Y is smooth and connected and $\mathrm{Pic}(Y)$ has no 2-torsion. Then X is connected iff G is generated by the elements g with $D_g \neq 0$.*

Proof. Let $G_0 < G$ be the subgroup generated by the g such that $D_g \neq 0$ and set $X_0 := X/G_0$. Then the induced cover $X_0 \to Y$ is an étale G/G_0-cover and therefore it is trivial since $\mathrm{Pic}(Y)$ has no 2-torsion. So if X is connected then $G = G_0$.

Conversely, if $G = G_0$ then by the fundamental relations for every $\chi \neq 0$ we have $2L_\chi > 0$, hence $h^0(L_\chi^{-1}) = 0$. So $h^0(\mathcal{O}_X) = 1$ by the projection formula, and thus X is connected. $\qquad\square$

The extension of the theory to the case X, Y demi-normal goes as follows (cf. [AP12]). If Y is singular but normal one uses S_2-fication: if $i \colon U \to Y$ is the nonsingular locus, then $\mathrm{codim}(Y \setminus U) \geq 2$, and for any G-cover $\pi \colon X \to Y$ with X demi-normal the restriction $\pi^{-1}(U) =: V \to U$ is a demi-normal G-cover with smooth base, so $\pi_* \mathcal{O}_V = \oplus_{\chi \in G^*} L_\chi^{-1}$ for suitable line bundles L_χ on U. Since X is S_2 by assumption, $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F}_\chi$ with $\mathcal{F}_\chi = i_* L_\chi^{-1}$. The pushforwards of invertible sheaves from U to X are divisorial sheaves corresponding to Weil divisors modulo linear equivalence. Thus, all the same statements about the building data and fundamental relations hold, with the Weil divisors D_g and L_χ taken in the class group $\mathrm{Cl}(Y)$.

By computing in local coordinates (see [Par91, §3]), with the above reduction from the case of a normal Y to the case of a smooth Y, one has for $G = \mathbb{Z}_2^k$:

Lemma 2.9. *Suppose that Y is normal and X is deminormal. Then:*
   (1) the components of $D_{\mathrm{tot}} = \sum_{g \in G} D_g$ have multiplicities ≤ 2;
   (2) X is normal iff D_{tot} is reduced;
   (3) the normalization of X is a G-cover $X' \to Y$ with branch data D_h', defined as follows: a prime divisor E of Y is a component of D_h' iff $h = \sum_g \left(\mathrm{mult}_{D_g} E\right) g$. In particular, E is not a component of $D' := \sum_h D_h'$ iff $sum_g \left(\mathrm{mult}_{D_g} E\right) g = 0$.

Finally, the case when both X and Y are deminormal is treated by [AP12], Theorems 1.13 and 1.17. The main result is that every G-cover $\pi\colon X \to Y$ is obtained from a G-cover $\pi'\colon X' \to \widetilde{Y}$ of the normalization by a gluing construction.

Finally we recall from [AP12] a Hurwitz type formula for the canonical class of X in terms of the canonical class of Y and of the branch data for a cover $\pi\colon X \to Y$ of deminormal varieties. Let E be a prime divisor of Y and set:

- $a_E = 0$ if π is generically étale over E or if E is contained in the double locus of Y,
- $a_E = 1$ if Y is generically smooth along E but X is singular along $\pi^{-1}(E)$,
- $a_E = \frac{1}{2}$ otherwise.

The divisor $D_{\mathrm{Hur}} = \sum_F a_F F$ is called the Hurwitz divisor of π. Note that if Y is normal $D_{\mathrm{Hur}} = \frac{1}{2}D_{\mathrm{tot}}$ and this equality holds more generally if the irreducible components of Y are smooth in codimension 1. We have the following ([AP12, Lem. 2.3, Prop. 2.5]):

Lemma 2.10. *Let $\pi\colon X \to Y$ be a $\mathbb{Z}_2^k$-cover of deminormal varieties. Then:*

(1) $2K_X = \pi^*(2K_Y + 2D_{\mathrm{Hur}})$ in $\mathrm{Cl}(X)$;

(2) K_X is $\mathbb{Q}$-Cartier iff so is $K_Y + D_{\mathrm{Hur}}$, and X is slc iff so is the pair (Y, D_{Hur}).

3. Campedelli surfaces with $\pi_1(X) = \mathbb{Z}_2^3$

3.1. **Definitions.** We will work with canonical models of surfaces of general type. Thus, our normal surfaces of general type will have canonical (i.e. Du Val) singularities and ample canonical class. As usual in surface theory, we write $p_g(X) = h^2(\mathcal{O}_X)$ and $q(X) = h^1(\mathcal{O}_X)$.

The term *(numerical) Campedelli surface* normally refers to a surface of general type with $K_X^2 = 2$ and $p_g = q = 0$. The first examples of such surfaces were constructed by Campedelli [Cam32] in 1932.

The Campedelli surfaces with fundamental group of order 8 are usually described as free quotients of the intersection of 4 quadrics in $\mathbb{P}^6$ by a group of order 8 (cf. [Miy77] for the case $\pi_1 = \mathbb{Z}_2^3$ and [MLPR09] for the general case). When $G = \mathbb{Z}_2^3$, the quadrics can be taken to be diagonal and it is easy to check that the bicanonical system gives a $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched on 7 lines. These are the surfaces we consider. Over $\mathbb{C}$ at least, they form a connected component in the moduli space of surfaces of general type with canonical singularities.

Definition 3.1. For brevity, a *Campedelli surface* in this paper will denote a $\mathbb{Z}_2^3$-cover $\pi\colon X \to \mathbb{P}^2$ whose building data are 7 lines D_g $(g \in \mathbb{Z}_2^3 \setminus 0)$ for which the cover has canonical singularities.

This means that either the lines are in general position (and then the cover is smooth) or three distinct lines D_{g_1}, D_{g_2}, D_{g_3} intersect at a point and the three elements g_1, g_2, g_3 generate G (in which case the cover has an A_1 singularity, see [AP12, Table 1]).

We will denote the moduli space of Campedelli surfaces with canonical singularities by $\mathrm{M_{Cam}}$, and the open subset of smooth surfaces by $\mathrm{M_{Cam}^0}$.

The fundamental relations (1) have the solution $L_\chi = \mathcal{O}_{\mathbb{P}^2}(2)$ for every $0 \neq \chi \in G^*$. Thus, $h^i(\mathcal{O}_X) = \sum h^i(L_\chi^{-1}) = 0$ for $i = 1, 2$. By Lemma 2.10 one has $K_X = \pi^*(K_{\mathbb{P}^2} + \frac{1}{2}\sum D_g) = \pi^*(\frac{1}{2}h)$, h being the class of a line in $\mathbb{P}^2$, so indeed $K_X^2 = 8 \cdot (\frac{1}{2})^2 = 2$. Using the standard projection formulas for abelian covers, one

can show $|2K_X| = \pi^*|\mathcal{O}_{\mathbb{P}^2}(1)|$, so the covering map π coincides with the bicanonical map. In particular, every automorphism of X descends to an automorphism of $\mathbb{P}^2$ that permutes the 7 branch lines of π. So for a general choice of these lines we have $\mathrm{Aut}(X) = G$.

Let $\mathrm{M}^0(3,7)$ denote the moduli space of arrangements of 7 lines in $\mathbb{P}^2$ in general position, i.e., such that no three of them are concurrent: it is the free $\mathrm{PGL}(3)$-quotient of an open subset of $(\mathbb{P}^{2\vee})^7$. The moduli space $\mathrm{M}^0_{\mathrm{Cam}}$ of smooth Campedelli surfaces is obtained by dividing $\mathrm{M}^0(3,7)$ by the choice of a basis in $\mathbb{Z}_2^3$. Thus, the coarse moduli space is $\mathrm{M}^0_{\mathrm{Cam}} = \mathrm{M}^0(3,7)/\mathrm{GL}(3,\mathbb{F}_2)$.

3.2. Compact moduli spaces.
To describe the compactified moduli space, we need to understand two things:

(1) what are the degenerations of arrangements of 7 lines, and
(2) what happens to the abelian covers.

Theorem 3.2. *Let $F = (\mathbb{P}^2)^7 /\!\!/ \mathrm{PGL}(3)$ be the GIT quotient for the "democratic" polarization $(1,\dots,1)$. Then there exists a family $(\mathcal{Y}, \sum_{i=1}^7 \frac{1}{2}\mathcal{B}_i) \to F$ whose fibers are log canonical hyperplane arrangements $(\mathbb{P}^2, \sum_{i=1}^7 \frac{1}{2}B_i)$. The GIT quotient is also the geometric quotient for the set of stable points, the group action is free, and the quotient is smooth and projective.*

Proof. The space $(\mathbb{P}^2)^7$ parameterizes the set of ordered 7-tuples of hyperplanes in the dual projective plane, and $\mathrm{PGL}(3) = \mathrm{Aut}\,\mathbb{P}^2$. By [MFK94, Prop. 4.3] (see also [DO88, Thm. II.2.1]) the pair $(\mathbb{P}^2; B_1,\dots,B_7)$ is GIT stable (resp. semistable) iff the following two conditions hold:

(1) the number of lines coinciding with a given line is $< \frac{7}{3}$ (resp. $\leq \frac{7}{3}$),
(2) the number of lines passing through a point is $< \frac{14}{3}$ (resp. $\leq \frac{14}{3}$).

On the other hand, by Remark 2.2 the pair $(\mathbb{P}^2, \sum \frac{1}{2}B_i)$ is log canonical iff:

(1') the number of lines that coincide is ≤ 2,
(2') the number of lines passing through a point is ≤ 4.

Since $\lfloor \frac{7}{3} \rfloor = 2$ and $\lfloor \frac{14}{3} \rfloor = 4$, these three pairs of conditions are equivalent, so the sets of stable, semistable and log canonical pairs are all the same. The GIT quotient of the semistable locus is projective. On the other hand, the quotient of the stable locus is a geometric quotient and the stabilizers are finite. By enumeration, one checks that every stable configuration contains 4 lines in general position. Thus, the stabilizers are trivial, the group action is free, and the quotient is smooth.

The universal family $\mathcal{Y}$ is the quotient of the universal family of line arrangements over the semistable = stable locus $\big((\mathbb{P}^2)^7\big)^s$ by $\mathrm{PGL}(3)$. $\square$

3.3. Proof of Theorem 1.

Proof. For any point in $F = (\mathbb{P}^2)^7 /\!\!/ \mathrm{PGL}(3)$, choosing a sufficiently small open neighborhood $U \subset F$, one can identify $\mathcal{Y} \times_Y U$ with $\mathbb{P}^2 \times U$, and the sheaves L_χ and $\mathcal{O}_{\mathcal{Y}}(D_g)$, $g \in \mathbb{Z}_2^3 \setminus 0$, are pullbacks from $\mathbb{P}^2$. Then

$$\mathcal{X} := \mathrm{Spec}_{\mathcal{Y}}\big(\oplus_{\chi \in G^*} L_\chi^{-1}\big)$$

gives a family of semi log canonical surfaces. For each surface X in this family, $\mathcal{O}(2K_X)$ is an ample invertible sheaf defining a $\mathbb{Z}_2^3$-cover $X \to \mathbb{P}^2$ (see section 2.2, and in particular Lemma 2.10), and the local deformations of X are given by deforming the 7 lines D_g. The lines D_g are labeled by $0 \neq g \in \mathbb{Z}_2^3$, and two choices

differ by a choice of a basis, i.e. by an element of $G = \mathrm{GL}(3, \mathbb{F}_2)$. Thus, there is a bijective map from $\overline{M}_{\mathrm{Cam}} = F/G$ to coarse moduli space $\overline{M}_{\mathrm{Cam}}^{\mathrm{slc}}$. We claim that this bijection is an isomorphism to an irreducible component of $\overline{M}_{\mathrm{Cam}}^{\mathrm{slc}}$.

The deformations of X with G-action are equivalent to the deformations of the pair of $\mathbb{P}^2$ with 7 lines. So the space of G-deformations is smooth. The space of G-deformations is immersed into the space of all deformations, as the the tangent space to the former is the linear subspace of the tangent space to the latter on which G acts trivially. $\square$

Remark 3.3. For the "labeled" stable Campedelli surfaces, with the branch divisors labeled by $0 \neq g \in \mathbb{Z}_2^3$, the moduli stack is a gerbe over F banded by $\mathbb{Z}_2^3$. It would have been the quotient stack $[F : \mathbb{Z}_2^3]$ if the sheaves L_χ had global square roots over F. Similarly, for the unlabeled stable Campedelli surfaces, the moduli stack is a gerbe over the stack $[F : \mathrm{GL}(3, \mathbb{F}_2)]$ banded by $\mathbb{Z}_2^3$. We thank Angelo Vistoli for explaining this point to us.

3.4. **Degenerate Campedelli surfaces and their singularities.** The singularities occurring on the degenerate Campedelli surfaces were considered in detail in Tables 1,2,3 of [AP12]. Enumerating the possibilities for the lines D_g, $g \in \mathbb{Z}_2^3 \setminus 0$ gives the following:

Lemma 3.4. *In the notation of* [AP12]:
- (1) *the singularities occurring on degenerate Campedelli surfaces are* 3.1, 3.3, 4.3, 4.4, 2′.1, 3′.1, 3′.4, 4′.5, 4′.6, 4′.7, 4″.4, 4″.5.
- (2) $\overline{M}_{\mathrm{Cam}}$ *contains two boundary divisors, one consisting of surfaces with an A_1 singularity (type* 3.1*) and the other one consisting of surfaces with two $\frac{1}{4}(1,1)$ singularities (type* 3.3*).*

Here, the notation $k.n$ means that k lines pass through a common point on the base surface $\mathbb{P}^2$; n is a case number from [AP12]. Similarly, $k'.n$ means that two of them coincide to form a double line, and $4''.n$ means that there are two pairs of double lines. The integer n refers to the n-th case in the list of possible relations between the k lines.

Proof. The proof of (1) is a direct enumeration of cases. Part (2) is a consequence of the fact that three branch lines D_{g_1}, D_{g_2}, D_{g_3} going through the same point is the only codimension one degeneration. When g_1, g_2, g_3 are linearly independent in $\mathbb{Z}_2^3$, the cover has an A_1 singularity. When $g_1 + g_2 + g_3 = 0$, it has two $\frac{1}{4}(1,1)$ singularities. $\square$

Remark 3.5. It is a straightforward but tedious exercise to list the boundary data of higher codimensions. Indeed, over an infinite field of char $k \neq 2$, modulo S_7 there are 36 configurations of 7 lines in $\mathbb{P}^2$ such that ≤ 2 lines coincide at a time and ≤ 4 lines pass through a common point. Modulo our relabeling group $\mathrm{GL}(3, \mathbb{F}_2) \subset S_7$ there are 175 orbits. It is not very practical to list them all here.

The cases of codimension 2 are as follows:
- (1) The lines D_1, D_2, D_3 pass through a common point, and the lines D_4, D_5, D_6 pass through a common point. There are two cases: $g_1 + g_2 + g_3 = 0$ and $\{g_4, g_5, g_6\}$ is a basis of $\mathbb{Z}_2^3$, or both $\{g_1, g_2, g_3\}$ and $\{g_4, g_5, g_6\}$ are bases.
- (2) The lines D_1, D_2, D_5 pass through a common point, and the lines D_3, D_4, D_5 pass through a common point. There are four orbits depending on the

triples $ijk \in \{125, 345, 567\}$ for which $g_i + g_j + g_k = 0$: all; 125; 567; none. In the last two cases the singularities are the same: two A_1.

(3) Two lines $D_1 = D_2$ coincide. There is only one case, with the singularities $2'.1$, $3'.1$, $3'.4$ of Table 2 in [AP12].

(4) Four lines pass through a common point. Again, there are two cases mod $\mathrm{GL}(3, \mathbb{F}_2)$. This gives the normal, log canonical but not log terminal singularities, cases 4.3 and 4.4 of Table 2 in [AP12].

Lemma 3.6. *A degenerate Campedelli surface may have 1, 2, or 4 (but not 8) irreducible components.*

Proof. By Lemmas 2.9, 2.8, a $\mathbb{Z}_2^3$-cover is not normal iff the branch divisor $\sum_g D_g$ is not reduced and the normalization is also a $\mathbb{Z}_2^3$-cover branched on a divisor contained in $\sum_g D_g$. In addition, the slc condition implies that no three of the D_g can coincide. If $D_g = D_{g'} = h$, the line h occurs in the branch locus of the normalization with label $g + g' \neq 0$. So the normalization is branched on at least four lines and the case of 8 components never occurs.

Up to the action of $\mathrm{GL}(3, \mathbb{F}_2)$, the case of four irreducible components occurs when $D_{100} = D_{011}$, $D_{010} = D_{101}$, $D_{001} = D_{110}$, where we use the natural labels for the nonzero elements of $\mathbb{Z}_2^3$. In this case the normalization has 4 components, each of them a double cover of $\mathbb{P}^2$ ramified in 4 lines corresponding to $g = 111$. Each component is a del Pezzo surface of degree 2 with six A_1 singularities. It is easy to see that up to the action of $\mathrm{GL}(3, \mathbb{F}_2)$ this is the only case with 4 irreducible components.

If we split one of the double lines then the cover has 2 components. Each of them is a del Pezzo of degree 1 with six A_1 singularities. $\square$

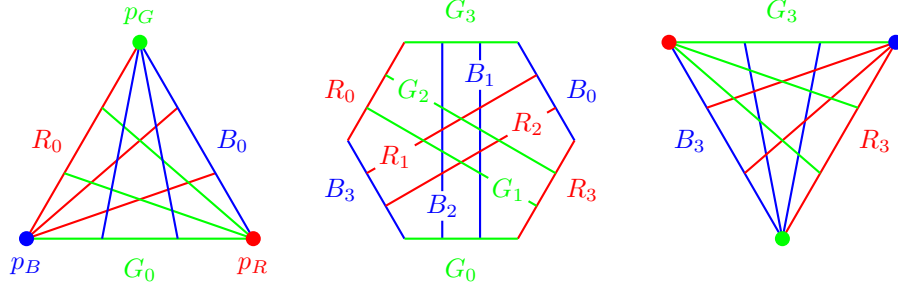## 4. Burniat surfaces with $K_X^2 = 6$

In this paper, we consider only Burniat surfaces with $K_X^2 = 6$. These are the so called "primary" Burniat surfaces. There exist also "secondary" and "tertiary" Burniat surfaces with $K_X^2 = 5$ and 4. They were considered in [Hu14] and [AH23].

4.1. **Set-up and notation.** Consider the arrangement of 9 lines on $\mathbb{P}^2$ shown in the first panel in Fig. 1. Using the RGB color scheme (R for red, G for green, B for blue; in the black and white version of this paper they are shown as solid, dashed and dotted) we denote the sides of the triangle R_0, G_0, B_0 and the vertices p_R, p_G, p_B. The point p_R is the point of intersection of G_0 and B_0, etc. There are additional lines R_1, R_2 through p_B, lines G_1, G_2 through p_R, and lines B_1, B_2 through p_G. We assume that the lines are in general position otherwise.

We note the cyclic RGB symmetry of this picture.

Now blow up the points p_R, p_G, p_B and denote the resulting exceptional divisors on the surface $\Sigma = \mathrm{Bl}_3\, \mathbb{P}^2$ by R_3, G_3, B_3. Note that the arrangement on Σ can be presented as the blowup of $\mathbb{P}^2$ in a different way by contracting R_0, G_0, B_0. The two line arrangements differ by a Cremona transformation.

Definition 4.1. Set $R = \sum_{i=0}^3 R_i$, $G = \sum_{i=0}^3 G_i$, $B = \sum_{i=0}^3 B_i$, corresponding to the 3 nonzero elements of $\mathbb{Z}_2^2$. The divisors $R + G$, $G + B$ and $B + R$ are 2-divisible in $\mathrm{Pic}(\Sigma)$, the fundamental relations (1) have a unique solution (cf. Example 2.7), and there is a (unique) $\mathbb{Z}_2^2$-cover $\pi \colon X \to \Sigma$ with branch data R, G, B. The surface X is called a Burniat surface.

FIGURE 1. Burniat arrangements on $\mathbb{P}^2$ and $\Sigma = \mathrm{Bl}_3 \mathbb{P}^2$

When the lines are chosen generically, so that on Σ only two divisors at a time intersect (and they belong to different elements of $G = \mathbb{Z}_2^2$, which is always true for Burniat arrangements), the Galois cover is smooth. In the notation of section 2.2, we have $D_{\mathrm{Hur}} = \frac{1}{2} D_{\mathrm{tot}} = \frac{1}{2}(R + G + B)$, so $K_\Sigma + D_{\mathrm{Hur}} = -\frac{1}{2} K_\Sigma$ is ample, and by Lemma 2.10 $K_X = \pi^*(K_\Sigma + D_{\mathrm{Hur}})$, and so $K_X^2 = 4 \cdot \frac{6}{4} = 6$. To compute $p_g(X) = q(X) = 0$ one solves equations (1) for L_χ and uses the projection formula $\pi_* \mathcal{O}_X = \mathcal{O}_\Sigma \oplus_{G^* \setminus 0} L_\chi^{-1}$.

By [MLP01] Burniat surfaces form a connected component $\mathrm{M}_{\mathrm{Bur}}$ in the moduli space of canonical surfaces of general type. The dimension of $\mathrm{M}_{\mathrm{Bur}}$ is 4. The map to Σ is the bicanonical map, so it is intrinsic to the surface X.

Definition 4.2. We define the *relabeling group* for the tuple (Σ, R_i, G_i, B_i) to be $\Gamma = C_3 \ltimes S_2^4 \subset S_{12}$, a group of order 48, acting as follows:

(1) S_2^3 acts by exchanging $R_1 \leftrightarrow R_2$, $G_1 \leftrightarrow G_2$, $B_1 \leftrightarrow B_2$ independently.
(2) The remaining copy of S_2 acts by an involution exchanging the six (-1)-curves, the sides of the hexagon $R_0 \leftrightarrow R_3$, $G_0 \leftrightarrow G_3$, $B_0 \leftrightarrow B_3$ at the same time.
(3) The cyclic group C_3 rotates the RGB colors.

This is the subgroup of the automorphism group of the dual graph of $\cup R_i \cup G_i \cup B_i$ that preserves the partition into three parts, arbitrarily colored.

Lemma 4.3. *The automorphism group of a smooth Burniat surface X with the labeled curves R_i, G_i, B_i is the covering group $\mathbb{Z}_2^2$ of $\pi \colon X \to \Sigma$. The smooth Burniat surfaces X, X' defined by two Burniat configurations (Σ, R_i, G_i, B_i) and $(\Sigma, R_i', G_i', B_i')$ are isomorphic iff there exists an automorphism $\alpha \colon \Sigma \to \Sigma$ sending the 12 RGB curves to the 12 $R'G'B'$ curves and such that the induced permutation of the 12 labels is an element of Γ.*

Proof. The automorphism group of Σ preserving the 6 boundary curves is the torus $\mathbb{G}_m^2$ and the subgroup of this torus acting trivially on two non-parallel boundary curves is trivial. The automorphism group of $\mathbb{P}^1$ fixing 3 points is trivial. For a smooth Burniat surface there are even 4 distinct points on each boundary curve. Thus an automorphism of a labeled Burniat surface acts trivially on Σ, so it is an element of the covering group.

Since the bicanonical map $\varphi_{|2K_X|} \colon X \to \Sigma$ is intrinsic, any isomorphism $X \to X'$ induces an isomorphism $\alpha \colon \Sigma \to \Sigma$ permuting the 12 branch curves. It must send the 4 curves in each branch divisor D_g to curves in some $D_{g'}$, so it should permute

the three colors. With the colors fixed, obviously $\alpha(R_1) = R_1$ or R_2, etc. and $\alpha(R_0) = R_0$ or R_3. And if $\alpha(R_0) = R_3$ then $\alpha(G_0) = G_3$ and $\alpha(B_0) = B_3$ since the boundary hexagon is distinguished. So $\alpha \in \Gamma$. $\square$

We note the following obvious observation (cf. Remark 2.2). To simplify the notation, we use R_i, G_i, B_i to denote both the lines in $\mathbb{P}^2$ and their strict transforms in Σ.

Lemma 4.4. *Let* $\left(\mathbb{P}^2, \sum_{i=0}^2 (r_i R_i + g_i G_i + b_i B_i)\right)$ *and* $\left(\Sigma, \sum_{i=0}^3 (r_i R_i + g_i G_i + b_i B_i)\right)$ *be pairs as in Fig. 1, $f\colon \Sigma \to \mathbb{P}^2$ be the blowup, and suppose that*

$$f^*\left(K_{\mathbb{P}^2} + \sum_{i=0}^2 (r_i R_i + g_i G_i + b_i B_i)\right) = K_\Sigma + \sum_{i=0}^3 (r_i R_i + g_i G_i + b_i B_i).$$

Then

$$r_3 = g_0 + g_1 + g_2 + b_0 - 1, \ g_3 = b_0 + b_1 + b_2 + r_0 - 1, \ b_3 = r_0 + r_1 + r_2 + g_0 - 1$$

and the first pair is log canonical iff so is the second one.

4.2. **Variation of weights.** The curves on Σ are split into two groups: *boundary* and *interior*:

$$D_{\mathrm{bry}} = R_0 + R_3 + G_0 + G_3 + B_0 + B_3, \quad D_{\mathrm{int}} = R_1 + R_2 + G_1 + G_2 + B_1 + B_2.$$

We have $D_{\mathrm{bry}} \equiv -K_\Sigma$ and $D_{\mathrm{int}} \equiv -2K_\Sigma$. Thus, the $\mathbb{Q}$-divisor $K_\Sigma + \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}}$ is ample for any $c > \frac{1}{4}$.

Definition 4.5. For $c > \frac{1}{4}$, we denote by $\overline{M}(c)$ the normalization of the compactification for the moduli space of pairs $(\Sigma, \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}})$, which exists by Theorem 2.5, with the labeled and ordered curves R_i, G_i, B_i.

We are ultimately interested in $\overline{M}(\frac{1}{2})$ but we will proceed in stages. By Remark 2.2 the non log canonical singularities of the pair $(\Sigma, \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}})$ occur in the interior $\Sigma \setminus D_{\mathrm{bry}}$ when

(1) $c > \frac{1}{3}$ and 6 of the interior lines meet at a point of $\Sigma \setminus D_{\mathrm{bry}}$, or
(2) $c > \frac{2}{5}$ and 5 of the interior lines meet at a point of $\Sigma \setminus D_{\mathrm{bry}}$.

Since two lines of the same color meet at an interior point iff they coincide, condition (1) above is equivalent to $R_1 = R_2$, $G_1 = G_2$, $B_1 = B_2$ and R_1, G_1, B_1 meet at an interior point, and condition (2) can be rephrased in a similar way.

For $\frac{1}{4} < c \le \frac{1}{3}$ the moduli space $\overline{M}(c)$ is the same. For these weights the singularities are possibly not log canonical only if some of the curves R_i, G_i, B_i with $i = 1, 2$ go to the boundary. These degenerations are purely toric.

4.3. **The toric setup.** We fix the torus embedding $\mathbb{G}_m^2 \hookrightarrow \Sigma$. The coordinates on $\mathbb{G}_m^2$ can be chosen symmetrically to be x, y, z with $xyz = 1$. Then the divisor D_{int} on $\mathbb{G}_m^2$ is given by the equation

$$(3) \qquad\qquad F = (x + r_1)(x + r_2)(y + g_1)(y + g_2)(z + b_1)(z + b_2).$$

The Newton polytope of F is a side-2 hexagon, and F defines a section of $\mathcal{O}_\Sigma(2)$.

We choose the orientation in such a way that $r_i \to 0$ for $i = 1, 2$ means $R_i \to R_0 + G_3$, $g_i \to 0$ means $G_i \to G_0 + B_3$, $b_i \to 0$ means $B_i \to B_0 + R_3$.

There is a natural action of $\mathbb{G}_m^6$ on the equation F, rescaling r_i, g_i, b_i, and the torus action on Σ gives an embedding $\mathbb{G}_m^2 \to \mathbb{G}_m^6$. It gives an exact sequence of tori

$$1 \to N_\Sigma \otimes \mathbb{G}_m \to N_{\mathcal{Y}} \otimes \mathbb{G}_m \to N \otimes \mathbb{G}_m \to 1, \quad \text{where}$$

(1) $N_{\mathcal{Y}} = \mathbb{Z}^6 = \{v = (\rho_1, \rho_2, \gamma_1, \gamma_2, \beta_1, \beta_2)\}$,
(2) the sublattice $N_\Sigma \simeq \mathbb{Z}^2 \subset N_{\mathcal{Y}}$ is the set of vectors $(\rho, \rho, \gamma, \gamma, \beta, \beta)$ with $\rho + \gamma + \beta = 0$,
(3) We identify N with the subset of $\mathbb{Z}^4$ of quadruples $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta})$ such that $\delta + \bar{\rho} + \bar{\gamma} + \bar{\beta} \equiv 0 \pmod 2$ and the map $N_{\mathcal{Y}} \to N$ with

$$v \mapsto (\rho_1 + \rho_2 + \gamma_1 + \gamma_2 + \beta_1 + \beta_2, \rho_1 - \rho_2, \gamma_1 - \gamma_2, \beta_1 - \beta_2).$$

We will define two toric varieties and a toric morphism $\mathcal{Y} \to \overline{M}(\frac{1}{2})$ by explicit fans $\mathfrak{F}_{\mathcal{Y}}$ in $N_{\mathcal{Y}}$ and $\mathfrak{F}$ in N and a map of fans $\mathfrak{F}_{\mathcal{Y}} \to \mathfrak{F}$.

Lemma 4.6. *The relabeling group $\Gamma = C_3 \ltimes S_2^4$ of Definition 4.2 acts as follows:*

(1) *S_2^3 acts on $N_{\mathcal{Y}}$ by switching the order in each of the pairs (ρ_1, ρ_2), (γ_1, γ_2), (β_1, β_2), and on N by sending $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta})$ to $(\delta, \pm\bar{\rho}, \pm\bar{\gamma}, \pm\bar{\beta})$.*
(2) *another S_2 acts by sending $v \to -v$ and $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta}) \to (-\delta, -\bar{\rho}, -\bar{\gamma}, -\bar{\beta})$.*
(3) *C_3 acts by cyclically permuting the groups (ρ_1, ρ_2), (γ_1, γ_2), (β_1, β_2), and by permuting the three coordinates $\bar{\rho}, \bar{\gamma}, \bar{\beta}$ in $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta})$.*

4.4. **Minimal (codimension 1) toric degenerations.** Consider a DVR R with quotient field K and a valuation $\nu \colon K^* \to \mathbb{Z}$ with a generator t of the maximal ideal $\mathfrak{m} = (t) \subset R$, $\nu(t) = 1$. Without loss of generality we can as well take $R = \mathbb{C}[t]_{(t)}$ or $\mathbb{C}[[t]]$. The computations are the same but notation is easier. Instead of a family over $\operatorname{Spec} K \subset \operatorname{Spec} R$ one can equivalently work with a family over the germ $(\Delta, 0)$ of a smooth curve.

Notations 4.7. Any one parameter degeneration of the Burniat configuration is described by a 6-tuple $(r_1, r_2, g_1, g_2, b_1, b_2) \in (K^*)^6$. We can write $r_i = r_i' \cdot t^{\rho_i}$, $g_i = g_i' \cdot t^{\gamma_i}$, $b_i = b_i' \cdot t^{\beta_i}$ with $\rho_i = \nu(r_i)$, $\gamma_i = \nu(g_i)$, $\beta_i = \nu(b_i)$ and with r_i', g_i', b_i' invertible in R. Thus, any one-parameter degeneration defines a vector $(\rho_1, \rho_2, \gamma_1, \gamma_2, \beta_1, \beta_2) \in N_{\mathcal{Y}}$ and its image $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta}) \in N$. Denote the residues of r_i', b_i', g_i' in $R/\mathfrak{m} = \mathbb{C}$ by $\bar{r}_i', \bar{g}_i', \bar{b}_i' \in \mathbb{C}^*$.

We begin by describing the four "minimal" one-parameter degenerations shown in Fig. 2, which we call A, B, C and D. In the figures the thicker lines denote the curves in the double locus of the limit surface, and, as usual in toric geometry, a triangle represents $\mathbb{P}^2$, a rhombus represents $\mathbb{P}^1 \times \mathbb{P}^1$, a trapezoid represents $\mathbb{F}_1$ and a hexagon the del Pezzo surface of degree 6. So in Fig. 2 B is irreducible, A and C consist of three copies of $\mathbb{P}^1 \times \mathbb{P}^1$ and D is the union of two copies of $\mathbb{F}_1$.

Each of these four degenerations produces a 3-dimensional family of limit pairs. The cases are ordered by the slope $\mu = \frac{|\delta|}{|\bar{\rho}| + |\bar{\gamma}| + |\bar{\beta}|}$. The orbits of these one-parameter degenerations under the relabeling group Γ will then define the rays of the fans $\mathfrak{F}_{\mathcal{Y}}$ and $\mathfrak{F}$.

Case A. $(1,1,0,0,0,0), (0,0,1,1,0,0), (0,0,0,0,1,1) \mapsto (2,0,0,0)$.

In the limit, $R_i \to R_0 + G_3$ for $i = 1, 2$. The limits of the divisors G_i, B_i, $i = 1, 2$ remain in the interior of Σ. We have a constant family of varieties $Y = \Sigma \times \operatorname{Spec} R$ and 12 divisors R_i, G_i, B_i on it.
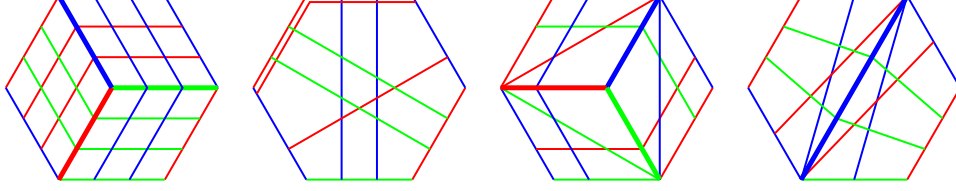
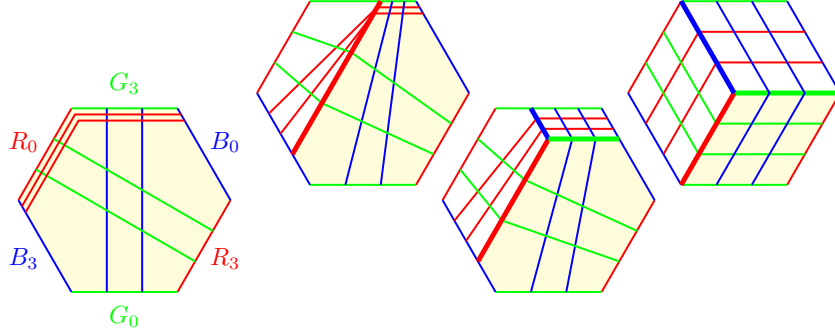FIGURE 2. Minimal toric degenerations of types A, B, C, D.

FIGURE 3. Degeneration for case A

Blow up the line R_0 in the central fiber Y_0. Then the central fiber becomes $\mathrm{Bl}_3\,\mathbb{P}^2 \cup \mathbb{F}_1$. Blowing up the strict preimage of G_3 changes $\mathbb{F}_1$ into $\mathrm{Bl}_2\,\mathbb{P}^2$ and inserts $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. To make such computations, we use the well-known *triple point formula:* Let $Y = \cup Y_j$ be the central fiber in a smooth one-parameter family, and assume that Y is reduced and simple normal crossing. Let C be the intersection $Y_1 \cap Y_2$, suppose it is smooth. Then

$$(C|_{Y_1})^2 + (C|_{Y_2})^2 + (\text{the number of the triple points of } Y \text{ contained in } C) = 0.$$

For the central fiber, the divisor $K_Y + \frac{1}{2}D_{\mathrm{tot}}$ restricted to an irreducible component Y_j is $K_{Y_j} + \frac{1}{2}D_{\mathrm{tot}}|_{Y_j} + (\text{the double locus})|_{Y_j}$. The curves R_i, G_i, B_i appear in the last sum with coefficient $\frac{1}{2}$, and the curves in the double locus with coefficient 1.

A simple computation shows that after the last step on the central fiber $K_Y + \frac{1}{2}D_{\mathrm{tot}}$ is big, nef and zero on 3 curves. The 3-fold pair $(\mathcal{Y}, \frac{1}{2}\mathcal{D}_{\mathrm{tot}})$ is simple normal crossing. The Basepoint-Free Theorem [KM98, Thm. 3.24] immediately implies that a big positive multiple $N(K_{\mathcal{Y}} + \frac{1}{2}\mathcal{D}_{\mathrm{tot}})$ gives a birational morphism contracting the three zero curves.

After the contraction the new central fiber is a union of three $\mathbb{P}^1 \times \mathbb{P}^1$ together with 8 curves on each. The equations of D_{int} on these components are

$$\prod_{i=1,2}(y + \bar{g}'_i)(z + \bar{b}'_i), \quad \prod_{i=1,2}(z + \bar{b}'_i)(x + \bar{r}'_i), \quad \prod_{i=1,2}(x + \bar{r}'_i)(y + \bar{g}'_i).$$

There is a 3-dimensional family of such pairs, parameterized by $(\mathbb{C}^*)^3$, and all of them appear as limits. For example, we can take $\bar{r}'_2 = \bar{g}'_2 = \bar{b}'_2 = 1$ and vary $\bar{r}'_1$, $\bar{g}'_1$, $\bar{b}'_1$. A natural compactification of $(\mathbb{C}^*)^3$ in $\overline{M}(\frac{1}{2})$ is $(\mathbb{P}^1)^3$, sending either R_1 or R_2 to the boundary of the hexagon, and similarly for G_i and B_i.

Remark 4.8. The labeling of the double locus is done in such a way that the fundamental relations (1) of Definition 2.6 hold on each irreducible component, so one has a well defined $\mathbb{Z}_2^2$-cover. Namely, on each irreducible component the divisors $R+G$, $G+B$ and $B+R$ must be divisible by 2. Formally, the new color is obtained by multiplying in the group $\mathbb{Z}_2^2$ the colors intersecting at the blown-up locus. We will justify this choice in Lemma 4.33.

Case B. $(1, 0, 0, 0, 0, 0) \mapsto (1, 1, 0, 0)$. In this case the limit of the curve R_1 coincides with $R_0 + G_3$. The resulting configuration is still log canonical and there is obviously a 3-dimensional family of limits parameterized by $(\mathbb{C}^*)^3$.

The remaining two cases are very similar to case A. We let the pictures do the explanations for the blowups and blowdowns.

Case C. $(0, -1, 1, 0, 1, 0), (1, 0, 0, -1, 1, 0), (1, 0, 1, 0, 0, -1) \mapsto (1, 1, 1, 1)$.

FIGURE 4. Degeneration for case C

Again, the central fiber Y is a union of three $\mathbb{P}^1 \times \mathbb{P}^1$. The equations of D_{int} restricted to these irreducible components are (remember that $xyz = 1$):

$$(x + \bar{r}_1')(y + \bar{g}_2')(z + \bar{b}_2'), \quad (y + \bar{g}_1')(z + \bar{b}_2')(x + \bar{r}_2'), \quad (z + \bar{b}_1')(x + \bar{r}_2')(y + \bar{g}_2').$$

For each component the moduli space is $\mathbb{C}^*$, giving $(\mathbb{C}^*)^3$ as the parameter space. Again, we can take $\bar{r}_2' = \bar{g}_2' = \bar{b}_2' = 1$ and vary $\bar{r}_1'$, $\bar{g}_1'$, $\bar{b}_1'$ to realize all of them.

The natural compactification of each $\mathbb{C}^*$ is $\mathbb{P}^1$. There are two stable degenerations shown in Fig. 5. The degenerations of the three components are independent since the gluings are unique. So the compactification of this family in the stable pair moduli space is $(\mathbb{P}^1)^3$.

FIGURE 5. Further case C degenerations over 0 and $\infty \in \mathbb{P}^1$

FIGURE 6. Degeneration for case D

Case D. $(1, 0, 0, 0, 0, -1), (0, -1, 0, 0, 1, 0) \mapsto (0, 1, 0, 1)$

For each of the two irreducible components, isomorphic to $\mathbb{F}_1$, there is a 2-dimensional moduli space, but they should reduce to the same 4 points in $\mathbb{P}^1$ on the intersection, giving a moduli space of dimension $2 + 2 - 1 = 3$ of such surfaces.
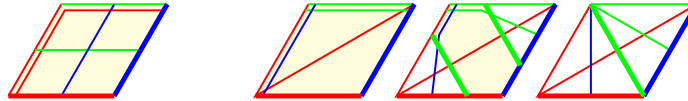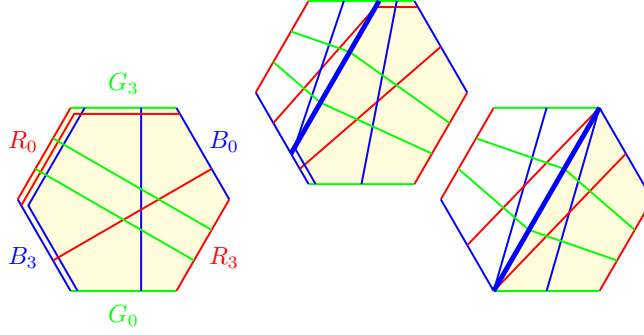
Indeed, the equations of D_{int} on the components are (remember that $xyz = 1$):

$$(x + \bar{r}_2')(z + \bar{b}_1')(y + \bar{g}_1')(y + \bar{g}_2'), \quad (x + \bar{r}_1')(z + \bar{b}_2')(y + \bar{g}_1')(y + \bar{g}_2').$$

Taking $\bar{r}_2' = \bar{g}_2' = \bar{b}_2' = 1$ and varying $\bar{r}_1'$, $\bar{g}_1'$, $\bar{b}_1'$ realizes the parameter space $(\mathbb{C}^*)^3$.

Remark 4.9. It is not hard to give explicit equations of a degeneration of the del Pezzo surface Σ of degree 6 to the underlying surface Y_0 of the limit pair for the minimal degenerations of type A,C and D (Figure 2).

For type A and C consider the threefold $\mathcal{Y} \subset (\mathbb{P}^1)^3 \times \mathbb{A}^1_t$ defined by $x_1 x_2 x_3 + t y_1 y_2 y_3 = 0$, where x_i, y_i are homogeneous coordinates on the i-th copy of $\mathbb{P}^1$, $i = 1, 2, 3$. Denote by Y_t the fiber over $t \in \mathbb{A}^1$ of the projection $\mathcal{Y} \to \mathbb{A}^1$; if $t \neq 0$ the surface Y_t is isomorphic to Σ, while the special fiber Y_0 is the union of three copies of $\mathbb{P}^1 \times \mathbb{P}^1$ intersecting along rulings as shown in Figure 2. The singularities of $\mathcal{Y}$ are three ordinary double points, one on each component of the double curve of Y_0.

For type D consider the $\mathbb{P}^2$-bundle $\mathbb{P} := \text{Proj}\,(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ with relative homogeneous coordinates x, y, z. Denote by u, v the homogeneous coordinates of $\mathbb{P}^1$ and let $\mathcal{Y} \subset \mathbb{P} \times \mathbb{A}^1_t$ be the threefold defined by $xy + t(uvz^2) = 0$. Standard computations show that for $t \neq 0$ the fiber Y_t over $t \in \mathbb{A}^1$ of the projection $\mathcal{Y} \to \mathbb{A}^1$ is a smooth del Pezzo surface of degree 6, hence it is isomorphic to Σ, while the special fiber Y_0 is a union of two copies of $\mathbb{F}_1$ meeting transversally along a section of self-intersection 1. The singularities of $\mathcal{Y}$ are two ordinary double points, both on the double curve of Y_0.

4.5. **Maximal (codimension** 4) **toric degenerations.** Next, we describe 6 maximal degenerations, obtained as combinations of the minimal degenerations of Fig. 2. They are shown in Fig. 7. The last surface is a cone over the cycle of rational curves.

Each of these degenerations naturally defines a maximal-dimensional cone in the 4-dimensional lattice N, and we list the integral generators of its rays.

(AB^3) $(2, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$.

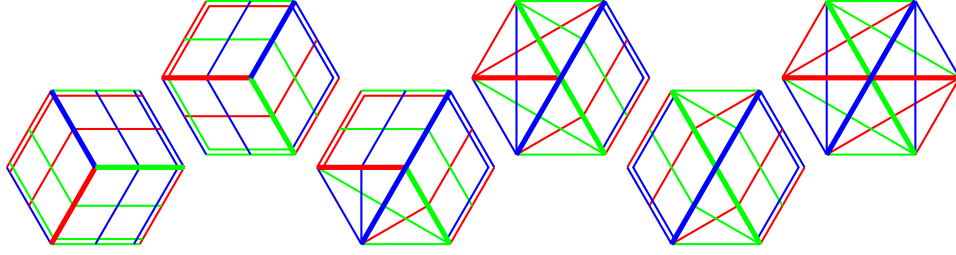(B^{3}C) $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(1, 1, 1, 1)$.

FIGURE 7. Maximal toric degenerations of the Burniat arrangement

(B^2CD) $(1,1,0,0)$, $(1,0,0,1)$, $(1,1,1,1)$, $(0,1,0,1)$.
(BCD^2) $(1,0,0,1)$, $(1,1,1,1)$, $(0,1,0,1)$, $(0,0,1,1)$.
(B^2D^2) $(1,0,0,1)$, $(-1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1)$.
(C^2D^3) $(1,1,1,1)$, $(-1,1,1,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,0)$.

Note that the first five cones are nonsingular, and the last one is non-simplicial.

4.6. **The ad hoc fans $\mathfrak{F}$ and $\mathfrak{F}_\mathcal{Y}$.** We will define the toric family $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ and the corresponding map of fans $\mathfrak{F}_\mathcal{Y} \to \mathfrak{F}$ more intrinsically in Section 4.8, but that theory has a high entry point. However, it is very easy and instructive to define the fans $\mathfrak{F}$, $\mathfrak{F}_\mathcal{Y}$ directly from the minimal and maximal degenerations already found. That is what we do here, calling them the "ad hoc fans". We will check in Proposition 4.17 that they are indeed the same as the fans of Section 4.8.

Definition 4.10. We define $\mathfrak{F}$ to be the fan in N with

(1) The rays $\mathbb{R}_{\geq 0}(2,0,0,0)$, $\mathbb{R}_{\geq 0}(1,1,0,0)$, $\mathbb{R}_{\geq 0}(1,1,1,1)$, $\mathbb{R}_{\geq 0}(0,1,0,1)$ for the minimal degenerations of Section 4.4, and their images under the relabeling group $\Gamma = C_3 \ltimes S_2^4$. We call them the rays of types A, B, C, D respectively.
(2) Maximal cones of the six types listed in Section 4.5 + their images under Γ.

We define $\overline{M}^{\mathrm{tor}}$ to be the toric variety with the fan $\mathfrak{F}$.

We checked the following by hand and also in sage:

Lemma 4.11. $\mathfrak{F}$ *is a complete fan with* $(1, 42, 210, 328, 160)$ *cones of dimensions* 0–4. *It is the normal fan of the polytope* $\Pi(a,b,c)$ *with vertices* $(a, -a+b+4c, -a+b+4c, -a+b+4c)$, $(3c, b+c, b+c, b+c)$, $(2c, b, b+2c, b+2c)$, $(c, b+c, b+c, b+3c)$, $(0, b, b, b+4c)$, $(0, b+2c, b+2c, b+2c)$ *and their images under the action of* Γ, *for any* $a, b, c \in \mathbb{R}$ *with* $b, c > 0$ *and* $b + 4c > a > 3c$.

Note: the fact that the fan $\mathfrak{F}$ is complete implies that we got all the degenerations.

Lemma 4.12. *The Γ-orbits of the cones in $\mathfrak{F}$ of dimensions* 1–4 *are as follows:*

(1) A(2), B(12), C(16), D(12).
(2) AB(12), B^2(6), B^2(24), BC(48), BD(24), BD(24), CD(48), D^2(24).
(3) AB^2(24), B^3(16), B^2C(48), B^2D(24), B^2D(12), B^2D(12), BCD(48), BCD(48), BD^2(48), CD^2(48).
(4) AB^3(16), B^3C(16), B^2CD(48), BCD^2(48), B^2D^2(24), C^2D^3(8).

Corollary 4.13. *The toric variety $\overline{M}^{\mathrm{tor}}$ for the fan $\mathfrak{F}$ is projective. It has* 8 *isolated singularities corresponding to the C^2D^3 cones. Each singularity is isomorphic to the cone over* $\left(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1,1)\right)$ *and admits two small resolutions.*

We now define the fan $\mathfrak{F}_{\mathcal{Y}}$ for the presumed family $f\colon \mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ from general principles of toric geometry: The rays of $\mathfrak{F}_{\mathcal{Y}}$ should correspond to the torus-invariant divisors in $\mathcal{Y}$. The maximal cones of $\mathfrak{F}_{\mathcal{Y}}$ should correspond to the torus-fixed points in $\mathcal{Y}$, mapped by f to the torus-fixed points in $\overline{M}^{\mathrm{tor}}$. So under the projection $N_{\mathcal{Y}} \to N$ the maximal cones of $\mathfrak{F}_{\mathcal{Y}}$ must map to the maximal cones of $\mathfrak{F}$.

Definition 4.14. The torus-invariant divisors in $\mathcal{Y}$ are of two kinds:

(1) Those that correspond to the irreducible components of the minimal degenerations. Under f they map to the divisors of types A, B, C, D in $\overline{M}^{\mathrm{tor}}$. The corresponding rays of $\mathfrak{F}_{\mathcal{Y}}$ are listed in Section 4.4. For example, there are the A-type rays $\mathbb{R}_{\geq 0}(1,1,0,0,0,0)$, $\mathbb{R}_{\geq 0}(0,0,1,1,0,0)$, $\mathbb{R}_{\geq 0}(0,0,0,0,1,1)$ mapping to the A-type ray $\mathbb{R}_{\geq 0}(2,0,0,0)$ of $\mathfrak{F}$. Thus, for each type-A ray of $\mathfrak{F}$ there are three A-type rays of $\mathfrak{F}_{\mathcal{Y}}$ mapping to it, for B-type there is one, for C-type there are three, and for D-type there are two.

(2) The divisors and their rays corresponding to the 6 boundary curves which do not vary in the family:

$$R_0\,(-1,-1,0,0,1,1), \quad G_0\,(1,1,-1,-1,0,0), \quad B_0\,(0,0,1,1,-1,-1),$$
$$R_3\,(1,1,0,0,-1,-1), \quad G_3\,(-1,-1,1,1,0,0), \quad B_3\,(0,0,-1,-1,1,1).$$

These divisors dominate $\overline{M}^{\mathrm{tor}}$, and the rays map to $(0,0,0,0) \in N$. We will call these the divisors and rays of type Δ.

We define the rays of $\mathfrak{F}_{\mathcal{Y}}$ to be these rays and their images under $\Gamma = C_3 \ltimes S_2^4$, for a total of $2 \cdot 3 + 12 + 16 \cdot 3 + 12 \cdot 2 + 6 = 96$ rays.

For each surface in Fig. 7 there are 7 torus-fixed points, corresponding to the 7 vertices of the polytopes. For each of them we list the irreducible components of the minimal degenerations whose closures contain this point, and the boundary divisors containing it. The corresponding rays define a maximal-dimensional cone of $\mathfrak{F}_{\mathcal{Y}}$. We define the maximal cones of $\mathfrak{F}_{\mathcal{Y}}$ to be their images under Γ.

Lemma 4.15. *For the maximal cones of $\mathfrak{F}$ of the six types listed in Section 4.5, the following are the types of maximal cones of $\mathfrak{F}_{\mathcal{Y}}$ mapping to them:*

    (AB³) A^3B^3, three $A^2B^3\Delta^2$, three $AB^3\Delta^2$.
    (B³C) B^3C^3, three $B^3C^2\Delta^2$, three $B^3C\Delta^2$.
    (B²CD) $B^2C^3D^2$, two $B^2CD\Delta^2$, two $B^2C^2D\Delta^2$, $B^2C^2D^2\Delta^2$, $B^2CD^2\Delta^2$.
    (BCD²) BC^3D^4, $BCD^2\Delta^2$, $BC^2D^2\Delta^2$, two $BC^2D^3\Delta^2$, two $BCD^3\Delta^2$.
    (B²D²) B^2D^4, two $B^2D^2\Delta^2$, four $B^2D^3\Delta^2$.
    (C²D³) C^6D^6, six $C^3D^4\Delta^2$.

One can check with sagemath that the fan $\mathfrak{F}_{\mathcal{Y}}$ is complete. This also follows by observing that for each maximal cone $\tau \in \mathfrak{F}$ the cones defined above cover its preimage in $N_{\mathcal{Y}} \otimes \mathbb{R}$. We give a more direct proof in Proposition 4.17 by showing that $\mathfrak{F}_{\mathcal{Y}}$ is the normal fan of a convex polytope.

4.7. Moduli interpretation of a fiber fan. In Section 4.8 we will define the fans $\mathfrak{F}$, $\mathfrak{F}_{\mathcal{Y}}$ and the toric family $f\colon \mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ more intrinsically. We will show that the ad hoc fan $\mathfrak{F}$ defined in the previous section is in fact an instance of a fiber fan, which is well known to have a moduli interpretation. One consequence of this fact is an explicit description of the family of "varying" divisors D_{int}, in addition to the "fixed" boundary divisor D_{bry}.

Let $\phi\colon M_P \simeq \mathbb{Z}^{n_P} \to M_Q \simeq \mathbb{Z}^{n_Q}$ be an affine map and $A \subset M_P$ a finite set. Consider two polytopes $P = \mathrm{Conv}\, A$ and $Q = \mathrm{Conv}\, \phi(A)$, assume maximal-dimensional. Recall that a lattice polytope P defines a toric variety with an ample line bundle as follows: $(V_P, L_P) = (\mathrm{Proj}\, S_P, \mathcal{O}(1))$, where the graded algebra S_P is

$$S_P = \oplus_{d \geq 0} H^0(V_P, \mathcal{O}(d)) = \oplus \mathbb{C} e^{(d,m)} \quad \text{with } (d,m) \in \mathrm{Cone}(1, P) \cap \mathbb{M}_P,$$

$\mathbb{M}_P = \mathbb{Z} \oplus M_P$. The above data defines two projective toric varieties $T_P \curvearrowright (V_P, L_P)$ and $T_Q \curvearrowright (V_Q, L_Q)$ and a finite morphism $j\colon V_Q \to V_P$ such that $j^*(L_P) = L_Q$.

In this situation, Billera and Sturmfels [BS92] defined the fiber polytope $\Sigma(P, Q)$ in the lattice $\ker(M_P \to M_Q)$ as the Minkowski integral of the fibers P_q, $q \in Q$. They also prove that it is a weighted Minkowski sum of finitely many fibers, over the barycenters of an appropriate subdivision of Q. The faces of $\Sigma(P, Q)$ are in a bijection with the coherent tilings $Q = \cup(Q_i, A_i)$, $Q_i = \mathrm{Conv}\, \phi(A_i)$ for some subsets $A_i \subset A$. The vertices of $\Sigma(P, Q)$ correspond to the tight tilings.

The fiber fan is the normal fan of the fiber polytope, defining a toric variety $V_{\Sigma(P,Q)}$. There exist (at least) four different moduli interpretations of $V_{\Sigma(P,Q)}$:

(1) As the "Chow quotient" $V_P /// T_Q$ [KSZ91]. This is the closure in the Chow variety of V_P of the T_P-orbit of the cycle $[j(V_Q)]$. Of course the T_P-action factors through the action of the quotient torus $T_P / j^*(T_Q)$.
(2) As a toric Hilbert scheme of V_P [PS02, HS04].
(3) As a moduli space of stable toric varieties with a finite morphism to V_P [Ale02], [Ale15, Sec. 2.5], [AK10].
(4) As the target of a morphism of toric varieties $V_{P+\Sigma(P,Q)} \to V_{\Sigma(P,Q)}$.

This paper is not the right place to discuss the common parts and the differences between these approaches. (In fact, they are all equivalent in our particular case, the key property being that the sets $(1, \phi(A_i))$ span the semigroups $\mathrm{Cone}(1, Q_i) \cap \mathbb{M}_Q$.) We simply take the fourth approach: it is the easiest and sufficient for our purposes.

The normal fan of the Minkowski sum $P + \Sigma(P, Q)$ comes with two maps to the normal fans of P and $\Sigma(P, Q)$, defining two projections $p_1\colon V_{P+\Sigma(P,Q)} \to V_P$ and $p_2\colon V_{P+\Sigma(P,Q)} \to V_{\Sigma(P,Q)}$ and a finite morphism $V_{P+\Sigma(P,Q)} \to V_P \times V_{\Sigma(P,Q)}$. The second projection gives an equidimensional family over $V_{\Sigma(P,Q)}$ which may have non-reduced fibers in general.

Let Y_t be a fiber in this family over a point $t \in V_{\Sigma(P,Q)}$. The torus orbit $\mathrm{orb}(t) \subset V_{\Sigma(P,Q)}$ containing t corresponds to a face of the polytope $\Sigma(P, Q)$ and, by the above, to a tiling $Q = \cup(Q_i, A_i)$. The irreducible components Y_i of Y_{red} are toric varieties V_{Q_i} for the polytopes Q_i in this tiling.

For each fiber, the first projection $Y_t \to V_P$ is a finite morphism, which may not be an embedding in general, and the images of the irreducible components may be non-normal. Each fiber Y_t comes with two divisors:

(1) The "fixed" Weil boundary divisor D_{bry} corresponding to the boundary ∂Q.
(2) The "varying" divisor D_{var}, the pullback of the Cartier divisor on V_P that is the zero divisor of the section $\sum_{a \in A} e^a \in H^0(V_P, L_P)$.

We mention the following fact which we do not use but which may help the reader understand some of the features of our construction: Under some additional conditions (for example, when all the fibers are reduced and the map $\phi|_A\colon A \to M_Q$ is injective), the fibers $(Y, D_{\mathrm{bry}} + \epsilon D_{\mathrm{var}})$ are stable pairs for $0 < \epsilon \ll 1$. But we want the pair $(Y, \frac{1}{2} D_{\mathrm{bry}} + (\frac{1}{4} + \epsilon) D_{\mathrm{int}})$ to be stable instead. So what we do is a kind of a "weighted" version of the "standard" construction.

4.8. $\mathfrak{F}$ is a fiber fan, and a family of pairs over $\overline{M}^{\mathrm{tor}}$. We are now ready to state our construction.

Definition 4.16. In Section 4.3 we defined two co-character lattices $N_{\mathcal{Y}} = \mathbb{Z}^6$ and $N_\Sigma = \{\rho, \gamma, \beta \mid \rho + \gamma + \beta = 0\}$ supporting the fans of $\mathcal{Y}$ and Σ, with a natural inclusion $i \colon N_\Sigma \to N_{\mathcal{Y}}$. The dual lattices $M_\Sigma = \mathbb{Z}^6$ and $M_{\mathcal{Y}} = \mathbb{Z}^3/\mathbb{Z}(1,1,1)$ are lattices of monomials, supporting polytopes of projective toric varieties. We define an affine linear map $\phi \colon M_{\mathcal{Y}} \to M_\Sigma$ such that $-\phi^* \colon N_\Sigma \to N_{\mathcal{Y}}$ equals i. (All of our fans and polytopes are centrally symmetric, so $-\phi$ is chosen for convenience.)

For $a = (k_1, \ldots, k_6) \in M_{\mathcal{Y}}$, we set $\phi(a) = (2 - k_1 - k_2, 2 - k_3 - k_4, 2 - k_5 - k_6) = (\ell_1, \ell_2, \ell_3)$. We interpret the monomial e^a as $r_1^{k_1} r_2^{k_2} g_1^{k_3} g_2^{k_4} b_1^{k_5} b_2^{k_6} \cdot x^{\ell_1} y^{\ell_2} z^{\ell_3}$ and the monomial $e^{\phi(a)}$ as $x^{\ell_1} y^{\ell_2} z^{\ell_3}$ with $xyz = 1$, so that the above $rgb\text{-}xyz$ monomial is projected to its xyz-part.

Let $A(F) \subset M_{\mathcal{Y}}$ be the set of the $2^6 = 64$ monomials appearing in the polynomial $F(r_i, g_i, b_i; x, y, z)$ of Equation (3). Its projection $\phi(A(F))$ is the set of the integral point of a hexagon of side 2, which we call $2Q$. We list the rgb- and xyz-parts of e^a for $a \in A(F)$ in Fig. 8. For example, $b_1 x^2 y^2 z$ gives b_1 and $x^2 y^2 z$. As a shortcut, we denote $r = r_1 r_2$, $g = g_1 g_2$, $b = b_1 b_2$. The notation r_i means that one has to repeat the monomial for r_1 and r_2, etc.

Let $A \subset A(F)$ be the subset of 46 points mapping to the 7 lattice points of the small, side-1 hexagon Q. Let $P = \mathrm{Conv}\, A$. By definition, we have $Q = \mathrm{Conv}\, \phi(A)$.

FIGURE 8. Monomials in $F(r_i, g_i, b_i; x, y, z)$

Proposition 4.17. *The normal fans of the polytopes $\Sigma(P, Q)$ and $P + \Sigma(P, Q)$ coincide with the ad hoc fans $\mathfrak{F}$, $\mathfrak{F}_{\mathcal{Y}}$, and the morphism $V_{P+\Sigma(P,Q)} \to V_{\Sigma(P,Q)}$ is identified with the toric family $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ of Section 4.6.*

Proof. We computed $\Sigma(P, Q)$ using the description given in [BS92, Cor. 2.6] as the convex hull of a set of explicit vectors Φ_Δ, as Δ go over the triangulations of Q with vertices in the multiset $\phi(A)$. Then we confirmed that $18\Sigma(P, Q)$ is a translate of the polytope $\Pi(6, 5, 1)$ defined in Lemma 4.11, whose normal fan is $\mathfrak{F}$. (In fact, the Minkowski sum of certain 6 fibers P_q already gives $\mathfrak{F}$, but we don't need this.)

We computed $P + \Sigma(P, Q)$ and its normal fan in sage and confirmed that the normal fan coincides with $\mathfrak{F}_{\mathcal{Y}}$. Its $160 \cdot 7$ maximal cones are exactly the same as

those described in Lemma 4.15. A sage file with these computations is included in the source of this paper on arXiv. $\qquad\square$

Lemma 4.18. *The morphism* $f\colon \mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ *is flat with reduced fibers.*

Proof. From the definition, the images of maximal cones in $\mathfrak{F}_{\mathcal{Y}}$ are maximal cones in $\mathfrak{F}$, and it is easy to check that the image $p(\sigma)$ of any cone $\sigma \in \mathfrak{F}_{\mathcal{Y}}$ is a cone of $\mathfrak{F}$. The integral generators of σ map to the integral generators of $p(\sigma)$, and it is easy to check that for any cone $p(\sigma) \in \mathfrak{F}$ the integral generators of the rays generate $p(\sigma) \cap N$. This implies that $p(\sigma \cap N_{\mathcal{Y}}) = p(\sigma) \cap N$. By [Mol21, Thm. 2.1.4], which is an extension of Kato's flatness criterion [Kat89, Prop. 4.1], this implies that the morphism f is flat with reduced fibers. $\qquad\square$

Lemma 4.19. *The morphism* $\mathcal{Y} \to V_P \times \overline{M}^{\mathrm{tor}}$ *is a closed embedding, and for every fiber* Y *of* $f\colon \mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ *the restriction of the line bundle* $p_1^*(L_P)$*, thereafter denoted by* $\mathcal{O}_Y(1)$*, is very ample. The boundary divisor* $D_{\mathrm{bry}} \sim -K_Y$ *on each fiber is given by a section of* $\mathcal{O}_Y(1)$*, and so is Cartier.*

Proof. Using the torus action and the openness of very ampleness, it is enough to check the statement for the fibers over the torus-fixed points of $\overline{M}^{\mathrm{tor}}$, which are listed in Fig. 7. The statements follows because for each irreducible component Y_i in these fibers the linear system generated by the sections s_a corresponding to the vertices of the polytopes Q_i is very ample. Indeed, there are only two types of components in Fig. 7: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ and $(\mathbb{P}^2, \mathcal{O}(1))$ corresponding to a square and a triangle. The boundary divisor D_{bry} is given by the section corresponding to the central point of the hexagon Q. And it is easy to check that $K_Y + D_{\mathrm{bry}} \sim 0$, same as for an ordinary toric variety. $\qquad\square$

As we recalled at the end of Section 4.7, the varying divisor D_{var} given by the "standard" construction is not the internal divisor D_{int} that we are after. Indeed, D_{var} corresponds to the polytope Q and is defined by a section of $L_P = \mathcal{O}(1)$, but D_{int} should correspond to $2Q$ and a section of $\mathcal{O}(2)$.

Definition 4.20. Let $A_0 = \{a_0 \in A \mid \phi(a_0) = \text{the center of } Q\}$. The 10 monomials e^{a_0}, $a_0 \in A_0$ are $\{x^2 y^2 z^2, r_i g_j b_k \cdot xyz, r_1 r_2 g_1 g_2 b_1 b_2\}$. We have $A_0 \subset A \subset A(F)$. The convex hull of $A(F)$ is the Newton polytope of F.

It is easy to check that for any $a_0 \in A_0$, one has $a_0 + A(F) \subset A + A$ and $a_0 + \mathrm{Newton}(F) \subset 2P$. Thus, $e^{a_0} F$ defines a global section of $\mathcal{O}_{V_P}(2)$. For any variety $Y_t \to V_P$ in our family, the restrictions of the 10 sections $e^{a_0} F$ are sections of $\mathcal{O}_{Y_t}(2)$ which are multiples of each other. If one of them is not identically zero on each irreducible component of Y_t then it defines a Cartier divisor D_{int} on Y_t and also a relative Cartier divisor $\mathcal{D}_{\mathrm{int}}$ on the family $\mathcal{Y} \times_{\overline{M}^{\mathrm{tor}}} U \to U$ for some open neighborhood $U \ni t$.

Proposition 4.21. *For any variety* Y *in the family* $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$*, the Cartier divisor* D_{int} *of Definition 4.20 is well defined and thus there is a relative Cartier divisor* $\mathcal{D}_{\mathrm{int}}$*. With this divisor included, the pairs* $(Y, D_{\mathrm{bry}}, D_{\mathrm{int}})$ *in the family* $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ *are the same as the pairs described in Sections 4.4, 4.5.*

Proof. We will freely use Notation 4.7 and the notation introduced in Section 4.7 and Definition 4.16. Let $Y_t := Y = \cup Y_i$ be a variety in our fiber fan family, and let $Q = \cup(Q_i, A_i)$ be the tiling corresponding to the orbit $\mathrm{orb}(t) \subset V_{\Sigma(P,Q)}$, as

explained in Section 4.7, so that $Y_i \simeq V_{Q_i}$. We claim that there exists $a_0 \in A_0$ such that the section $e^{(1,a_0)} \in H^0(V_P, \mathcal{O}(1))$ is not identically zero on each irreducible component Y_i. This is equivalent to the statement that for any face of the fiber polytope and the corresponding tiling $Q = \cup(Q_i, A_i)$, each set $\phi(A_i)$ contains the central point of Q. It is sufficient to check this for maximal degenerations, shown in Fig. 7, which follows because the central point $0 \in Q$ is a vertex of each Q_i and one always has $\mathrm{Vert}\, Q_i \subset \phi(A_i)$.

Pick $a_0 \in A_0$ as above. The height function $h|_{A_0}$ achieves a minimum at this point. F contains $e^{(1,a_0)}$, so the restriction of $e^{(1,a_0)}F$ to an irreducible component Y_i contains the monomial corresponding to central point of Q with a coefficient which is a nontrivial polynomial in $\bar{r}'_j, \bar{g}'_j, \bar{b}'_j$ (see Notation 4.7). On the other hand, it is a product of linear terms $x + \bar{r}'_j$, $y + \bar{g}'_j$, $z + \bar{b}'_j$, x, y, z. So it is not identically zero for any $\bar{r}'_j, \bar{g}'_j, \bar{b}'_j$. Below, we compute these restrictions explicitly.

Now consider a one-parameter family of pairs $(\mathcal{Y}^K, \mathcal{D}^K_{\mathrm{bry}}, \mathcal{D}^K_{\mathrm{int}}) \to \mathrm{Spec}\, K$. We identify $(\mathcal{Y}^K, \mathcal{D}^K_{\mathrm{bry}})$ with the constant family of toric varieties $\Sigma^K := \Sigma \times \mathrm{Spec}\, K$. We get a map $p_1 \colon \Sigma^K \to V_P^K$ and the pullback $p_1^* \colon H^0(V_P^K, \mathcal{O}(1)) \to H^0(\Sigma^K, \mathcal{O}(1))$:

$$\oplus_{a \in A} K e^{(1,a)} \to \oplus_{m \in Q \cap M_\Sigma} K e^{(1,m)}, \quad e^{(1,k_i)} \to r_1^{k_1} r_2^{k_2} g_1^{k_3} g_2^{k_4} b_1^{k_5} b_2^{k_6} \cdot x^{\ell_1} y^{\ell_2} z^{\ell_3}$$

with $r_i, g_i, b_i \in K$ and $x^{\ell_1} y^{\ell_2} z^{\ell_3} = e^{(1,\phi(a))}$. We call the valuation

$$\nu(r_1^{k_1} r_2^{k_2} g_1^{k_3} g_2^{k_4} b_1^{k_5} b_2^{k_6}) = k_1 \rho_1 + k_2 \rho_2 + k_3 \gamma_1 + k_4 \gamma_2 + k_5 \beta_1 + k_6 \beta_6 \in \mathbb{Z}$$

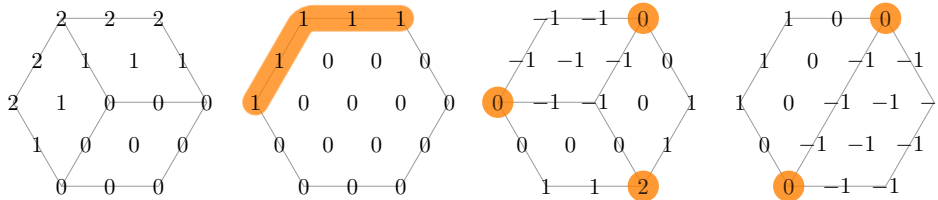*the height $h(a)$ of the monomial* $a = (k_i)$.

Thus, every point $a \in A$ defines a point $(a, h(a))$ in $M_\Sigma \oplus \mathbb{Z}$. The convex hull of the set of these points for $a \in A$ projects to the hexagon $Q \subset M_\Sigma \otimes \mathbb{R}$, and projections of the facets of the lower envelope are the polytopes Q_i corresponding to the limit surface by the fiber fan construction. The completed family $\mathcal{Y} \to \mathrm{Spec}\, R$ is $\mathrm{Proj}\, S$, where S is the R-subalgebra of $S_Q \otimes K$ generated by the monomials $t^{h(a)} e^{(1,\phi(a))}$, and the central fiber is $\mathrm{Proj}\, S/tS$.

The limit of divisor $\mathcal{D}^K_{\mathrm{int}}$ is obtained by restricting the section $e^{a_0} F$ to $\mathcal{Y}$ and reducing it mod t. This is computed as follows. We extend the tiling $Q = \cup Q_i$ by dilation to the tiling $2Q = \cup 2Q_i$. The polytopes $2Q_i$ are projections of the facets of the lower envelope defined above, extended by linearity away from the center.

For each monomial $a = (k_i) \in F(A)$ we compute its height $h(a)$. If the point $(a, h(a))$ lies above the extended lower envelope, then $e^{(1,a)}$ reduces to 0 mod t. If it lies on the extended lower envelope then it reduces to the monomial $\bar{r}'^{k_1}_1 \bar{r}'^{k_2}_2 \bar{g}'^{k_3}_1 \bar{g}'^{k_4}_2 \bar{b}'^{k_5}_1 \bar{b}'^{k_6}_2 \cdot x^{\ell_1} y^{\ell_2} z^{\ell_3}$, with $\bar{r}'_i, \bar{g}'_i, \bar{b}'_i$ introduced in Notation 4.7.

We now make concrete computations for the minimal degenerations. In each case we lift a vector $(\delta, \bar{\rho}, \bar{\gamma}, \bar{\beta}) \in N$ to a vector $(\rho_1, \rho_2, \gamma_1, \gamma_2, \beta_1, \beta_2) \in N_{\mathcal{Y}}$. Two lifts differ by an element of N_Σ, i.e. a linear function on M_Σ, and lead to the same result.

Concretely, we choose the lifts of the rays to be the first vectors in the cases A, B, C, D of Section 4.4: $(1,1,0,0,0,0)$, $(1,0,0,0,0,0)$, $(0,-1,1,0,1,0)$, $(1,0,0,0,0,-1)$. The answers for the minimal heights $h(a)$ over each point $\phi(a) \in 2Q$ are given in Fig. 9, for all the monomials in F. The minimal heights for $a \in A$ are in the central 7 points, for the hexagon Q of side 1. The picture shows the dilated domains of linearity, the polytopes $2Q_i$. For the highlighted places, *all* monomials of F mapping to it vanish. For the non-highlighted places, some monomials survive.

FIGURE 9. The minimal heights for the rays A, B, C, D

In more detail, for each polytope $2Q_i$ we compute the sum of all monomials in F whose heights lie on the lower envelope, and get the following answers:

(A) $x^2 \cdot (y + g_1)(y + g_2)(z + b_1)(z + b_2)$ and two more by symmetry.
(B) $x(x + r_2)(y + g_1)(y + g_2)(z + b_1)(z + b_2)$
(C) $(x + r_1)(y + g_2)(z + b_2) \cdot yzr_2$ and two more by symmetry.
(D) $(x + r_2)(y + g_1)(y + g_2)(z + b_1) \cdot xb_2$ and one more by symmetry.

After plugging in $r_i = r_i' t^{\rho_i}$, $g_i = g_i' t^{\gamma_i}$, $b_i = b_i' t^{\beta_i}$, and recalling the generators of the algebra S above, we see that the equations of D_{int} restricted to the irreducible components of $\mathrm{Proj}\, S/tS$ are

(A) $(y + \bar{g}_1')(y + \bar{g}_2')(z + \bar{b}_1')(z + \bar{b}_2')$ and two more by symmetry.
(B) $x(x + \bar{r}_2')(y + \bar{g}_1')(y + \bar{g}_2')(z + \bar{b}_1')(z + \bar{b}_2')$
(C) $(x + \bar{r}_1')(y + \bar{g}_2')(z + \bar{b}_2')$ and two more by symmetry.
(D) $(x + \bar{r}_2')(y + \bar{g}_1')(y + \bar{g}_2')(z + \bar{b}_1')$ and one more by symmetry.

So they are exactly the same as in Section 4.4. Thus, the limits computed by the fiber polytope technology, with our "weighted" twist, are exactly the same as those that we obtained in Section 4.4 by doing the Minimal Model Program steps. This proves the statement for the rays of $\mathfrak{F}$.

For a vector h in the interior of arbitrary cone $\tau \in \mathfrak{F}$ with rays r_k and $a \in A(F)$, the height $h(a)$ is a positive combination of the heights $h_k(a)$. This implies that the tiling for τ is the intersection of the tilings for r_k, and the set of the nonvanishing monomials is the intersection of such sets for r_k. One checks that the restriction of F to each component Y_i is a product of several linear terms $x + \bar{r}_j'$, $y + \bar{g}_j'$, $z + \bar{b}_j'$, x, y, z and that the degenerations given by the fiber polytope technology are the same as in Sections 4.4, 4.5.   □

Warning 4.22. The interior divisor $\mathcal{D}_{\mathrm{int}}$ is the zero set of F, so it is a Cartier divisor. Over the open subset $(\mathbb{C}^*)^4 \subset \overline{M}^{\mathrm{tor}}$ it splits into the sum of six divisors $R_1 + R_2 + G_1 + G_2 + B_1 + B_2$. Since $\mathcal{D}_{\mathrm{int}}$ is the closure of this set, it naturally splits into the sum of six Weil divisors. However, one can see from the pictures for the minimal degenerations C and D that sometimes these Weil divisors are not $\mathbb{Q}$-Cartier.

4.9. Nontoric degenerations.

Case E. In the toric family $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ this degeneration occurs over the unit point $1 \in \mathbb{G}_m^4$. For the pair $(\Sigma, \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}})$ this configuration is log canonical only when $c \le \frac{1}{3}$, and is not log canonical when $c > \frac{1}{3}$. Consider the latter case.
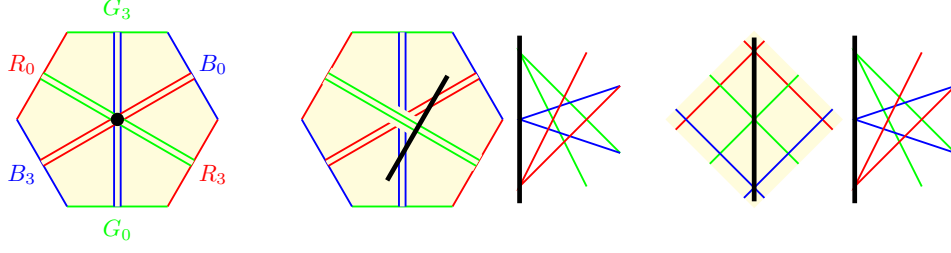
FIGURE 10. Degeneration for case E

For a one-parameter degeneration $Y \to (\Delta, 0)$ with Δ immersed into $\mathbb{G}_m^4$, the resolution is obtained by blowing up a point in the central fiber. Then the central fiber becomes $Y_0 = \mathrm{Bl}_p \Sigma \cup \mathbb{P}^2$ with a configuration of lines on $\mathbb{P}^2$ which is shown in the central panel of Fig. 10. The intersection number for the strict preimages of the curves R_i, G_i, B_i, $i = 1, 2$ with $K_Y + \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}}$ on the 3-fold Y is $1 - 2c$. It follows that the divisor $K_{Y_0} + \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}}$ is ample for $\frac{1}{3} < c < \frac{1}{2}$.

For $c = \frac{1}{2}$ it is big, nef and semiample by the standard Basepoint-Free Theorem [KM98, Thm. 3.24]. It then defines a contraction of three curves, the strict transforms on the blow up of Σ of $R_1 = R_2$, $B_1 = B_2$, $G_1 = G_2$. So on the resulting stable model the central fiber is $Y_0 = (\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2$, as shown in the right most panel of Fig. 10.

Now consider the configuration of lines on the $\mathbb{P}^2$ in the right panel with the condition that not all of the lines $R_1, R_2, G_1, G_2, B_1, B_2$ pass through the same point. Such a configuration is described by 6 lines with the equations $x + r_1 z$, $x + r_2 z$, $y + g_1 z$, $y + g_2 z$, $-x - y + b_1 z$, $-x - y + b_2 z$ modulo a free action by the matrix group

$$G = \left\{ \begin{pmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{with } e_1, e_2 \in \mathbb{C}, \ \lambda \in \mathbb{C}^*. \right\}$$

It is easy to see that the quotient is $(\mathbb{A}^6/\mathbb{A}^2)/\mathbb{G}_m \simeq \mathbb{P}^3$ and that it is naturally identified with the projectivization of the tangent space of $1 \in \mathbb{G}_m^4$. Indeed, an easy computation shows that the limits of the families with the immersed bases $(\Delta, 0) \to \mathbb{G}_m^4$ for different tangent vectors are exactly the surfaces described above.

Case F. The configuration is shown in Fig. 11. There are five curves passing through the same point. It is not log canonical if $c > \frac{2}{5}$. A one-parameter degeneration is resolved by a single blowup of a point in the central fiber, which becomes $Y_0 = \mathrm{Bl}_p \Sigma \cup \mathbb{P}^2$. A similar computation as above shows that the moduli space of the lines in the right panel of Fig. 11, not all of which pass through the same point, is $(\mathbb{A}^5/\mathbb{A}^2)/\mathbb{G}_m \simeq \mathbb{P}^2$.

Case G. Two curves in the same pencil coincide, e.g. $R_1 = R_2$. The pair $(\Sigma, \frac{1}{2}D)$ is still log canonical.

Case H. Three curves from the three pencils, e.g. R_1, G_1, B_1 pass through the same point. The pair $(\Sigma, \frac{1}{2}D)$ is still log canonical, and even stronger: it is klt.
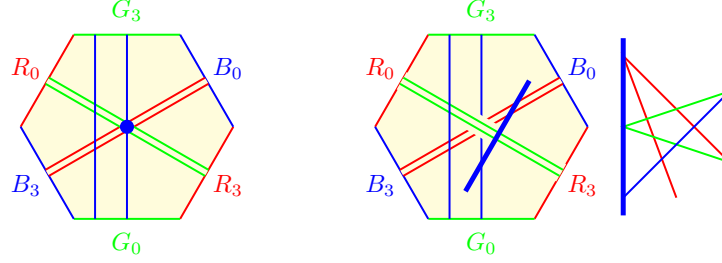
FIGURE 11. Degeneration for case F

We also include the following codimension 2 degeneration in which a new type of irreducible components of Y appears.

Mixed case EF. In case E, suppose that 5 of the 6 lines in the $\mathbb{P}^2$ pass through the same point q. Again, this is not a log canonical configuration if $c > \frac{2}{5}$, and a generic one-parameter degeneration with this limit is resolved by one additional blowup of the 3-fold at q. For $\frac{1}{3} < c < \frac{1}{2}$ this produces the central fiber $Y_0 = \mathrm{Bl}_p \Sigma \cup \mathbb{F}_1 \cup \mathbb{P}^2$. For $c = \frac{1}{2}$ it becomes $Y_0 = (\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{F}_1 \cup \mathbb{P}^2$.
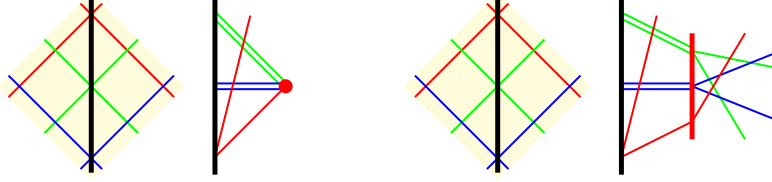

FIGURE 12. Degeneration for case EF

Remark 4.23. One may ask why we do not consider the configurations in the cases E and EF when 6 lines in the $\mathbb{P}^2$ on the right pass through a common point. The answer is that they do not appear in the degenerations $Y \xrightarrow{f} (\Delta, 0) \xrightarrow{g} \overline{M}(\frac{1}{2})$ where the classifying map g is unramified over $0 \in \Delta$.

They *do* appear if g if ramified to order d over 0. (Note that $g(0)$ is a smooth point of $\overline{M}(\frac{1}{2})$). In that case one has to blow up the central fiber d times, after which the intermediate ruled surfaces are contracted one by one when passing to the canonical model. The picture is quite similar to Miles Reid's "pagoda" [Rei83]. For the same reason, in case F we do not consider the case when 5 lines pass through a common point.

Remark 4.24. Degenerations of type E can be deformed by smoothing the three pairs of branch lines on the $\mathbb{P}^2$ component to conics tangent to the double locus (represented by the black line). These deformations depend on 6 moduli: 3 parameters for each conic tangent to the black line at an assigned point minus the dimension of the group of automorphisms of $\mathbb{P}^2$ inducing the identity on the black line, which is 3. Similarly, the pairs of fibers of the same color on the $\mathbb{P}^1 \times \mathbb{P}^1$ component can be smoothed to curves of type $(1, 1)$ tangent to the black line. This gives

3 additional moduli (one for each color), so we have an irreducible 9-dimensional family of deformations.

By [AP12, Ex. 1.17], for each such deformation Y there is a $\mathbb{Z}_2^2$-cover $X \to Y$ with X a stable surface that is a deformation of a limit of smooth Burniat surfaces. By dimension reasons all these surfaces are contained in a component of the moduli space of stable surfaces different from $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$.

The situations is similar for minimal degenerations of type C and D (cf. Figure 2): we get three additional parameters for type C and two for type D..

4.10. The moduli spaces $\overline{M}(\frac{1}{3}) = \overline{M}^{\mathrm{tor}}$, $\overline{M}(\frac{2}{5})$ and $\overline{M}(\frac{1}{2})$.

Theorem 4.25. *For any $\frac{1}{4} < c \leq \frac{1}{3}$, the toric family $f \colon \mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ is a family of stable pairs $\left(Y, \sum_{i=0,3} \frac{1}{2}(R_i + G_i + B_i) + \sum_{i=1,2} c(R_i + G_i + B_i)\right)$. The fibers are distinct. $\overline{M}(\frac{2}{5})$ is projective and the subset for which $Y = \Sigma$ is open and dense.*

Proof. By Lemma 4.19 and Proposition 4.21, the divisors D_{bry} and D_{int} are relative Cartier divisors, $K + \frac{1}{2}D_{\mathrm{bry}} + cD_{\mathrm{int}}$ is $\mathbb{Q}$-linearly equivalent to $(-1 + \frac{1}{2} + 2c)K$, hence it is ample for $c > \frac{1}{4}$. By looking at the degenerations one observes that for $c \leq \frac{1}{3}$ the pair is log canonical away from the toric boundary, i.e. no nontoric degenerations occur.

We now check that the fibers in the family $\mathcal{Y} \to \overline{M}^{\mathrm{tor}}$ of Section 4.8 are pairwise non-isomorphic. If $Y = \Sigma$ then this amounts to showing that the pair (Y, R_i, G_i, B_i) uniquely defines the 6-tuple $(r_1, r_2, g_1, g_2, b_1, b_2)$ in the Equation (3), up to the action of $\mathbb{G}_m^2$. This 6-tuple defines a unique embedding $Y \subset V_P$ since the projective coordinates of V_p are the monomials in r_i, g_i, b_i with the boundary curves. But this is obvious, since these six coefficients are the coordinates of the intersections of the curves R_i, G_i, B_i with the boundary curves.

For a general $Y = \cup Y_k$ one goes through a similar argument for the irreducible components Y_k. All the types of irreducible components already appear in Figs. 2 and 7. $\qquad\square$

Definition 4.26. Motivated by this theorem, we set $\overline{M}(\frac{1}{3}) := \overline{M}^{\mathrm{tor}}$.

The locus in $\overline{M}(\frac{1}{3})$ of the points for which the pairs $(Y, \frac{1}{2}D_{\mathrm{bry}} + \frac{1}{2}D_{\mathrm{int}})$ are not semi log canonical is six curves $\mathcal{C}_k$ corresponding to the nontoric degeneration of type F, when 5 of the curves pass through a single point. Each of these curves is $\mathcal{C}_k \simeq \mathbb{P}^1$: we include the cases when the remaining sixth curve goes to the boundary. These six curves intersect at the origin of the torus $1 \in \mathbb{G}_m^4 \subset \overline{M}(\frac{1}{3})$, corresponding to case E.

Definition 4.27. Let $\rho_1 \colon \overline{M}(\frac{2}{5}) \to \overline{M}(\frac{1}{3})$ be the blowup at the origin $1 \in \mathbb{G}_m^4 \subset \overline{M}(\frac{1}{3})$ with exceptional divisor $E \simeq \mathbb{P}^3$. Let $\mathcal{Y}(\frac{2}{5}) \to \overline{M}(\frac{2}{5})$ be the blowup in

$$\mathcal{Y} \times_{\overline{M}(\frac{1}{3})} \overline{M}(\tfrac{2}{5})$$

of $Z \simeq E$, the preimage of the point of the point $p = R_1 \cap R_2 \cap G_1 \cap G_2 \cap B_1 \cap B_2$.

Thus, the fibers of $\mathcal{Y}(\frac{2}{5}) \to \overline{M}(\frac{2}{5})$ are the same as the fibers of $\mathcal{Y}(\frac{1}{3}) \to \overline{M}(\frac{1}{3})$ outside of E, and they are isomorphic to $\mathrm{Bl}_p \Sigma \cup \mathbb{P}^2$ over E, as described in case E.

Theorem 4.28. *For any $\frac{1}{3} < c \leq \frac{2}{5}$, the family $\mathcal{Y}(\frac{2}{5}) \to \overline{M}(\frac{2}{5})$ is a family of stable pairs $\left(Y, \sum_{i=0,3} \frac{1}{2}(R_i + G_i + B_i) + \sum_{i=1,2} c(R_i + G_i + B_i)\right)$. The fibers are distinct. $\overline{M}(\frac{2}{5})$ is projective and the subset for which $Y \cong \Sigma$ is open and dense.*

Proof. This follows immediately from the computations in the case E above. The variety $\overline{M}(\frac{2}{5})$ is a blowup of the projective variety $\overline{M}(\frac{1}{3})$, so it is projective. According to the computation in the case E, the additional fibers are parameterized by $E = \mathbb{P}^3$, so all the fibers in this family are distinct. $\square$

On $\overline{M}(\frac{2}{5})$ the preimages of the six curves are disjoint and lie in the smooth locus. Let us denote them by $\mathcal{C}_k$ again.

Definition 4.29. Let $\rho_2 \colon \overline{M}(\frac{1}{2}) \to \overline{M}(\frac{2}{5})$ be the blowup along $\cup \mathcal{C}_k$ with exceptional divisors E_k, each of them a $\mathbb{P}^2$-bundle over $\mathcal{C}_k$. Let $\mathcal{Y}(\frac{1}{2}^-) \to \overline{M}(\frac{1}{2})$ be the blowup in

$$\mathcal{Y}(\tfrac{2}{5}) \times_{\overline{M}(\frac{2}{5})} \overline{M}(\tfrac{1}{2})$$

of the preimages of the points of intersection of 5 lines, isomorphic to $\cup E_k$.

The fibers are described in the cases F and EF above: The fibers of $\mathcal{Y}(\frac{1}{2}^-) \to \overline{M}(\frac{1}{2})$ are the same as the fibers of $\mathcal{Y}(\frac{2}{5}) \to \overline{M}(\frac{2}{5})$ outside of $\cup E_k$. They are isomorphic to $\mathrm{Bl}_p \Sigma \cup \mathbb{P}^2$ over $\cup E_k$ outside of the strict preimage of E. And the fibers are isomorphic to $\mathrm{Bl}_p \Sigma \cup \mathrm{Bl}_q \mathbb{P}^2 \cup \mathbb{P}^2$ over $\cup E_k$ intersected with the strict preimage of E. Further, let $\mathcal{Y}(\frac{1}{2}^-) \to \mathcal{Y}(\frac{1}{2})$ be the contraction defined by $K_{\mathcal{Y}(\frac{1}{2}^-)} + \frac{1}{2}\mathcal{D}_{\mathrm{bry}} + \frac{1}{2}\mathcal{D}_{\mathrm{int}}$, relative over $\overline{M}(\frac{1}{2})$. This replaces $\mathrm{Bl}_p \Sigma$ by $\mathbb{P}^1 \times \mathbb{P}^1$ as described in the cases E and EF.

Theorem 4.30. *For any $\frac{2}{5} < c < \frac{1}{2}$ (resp. for $c = \frac{1}{2}$) the family $\mathcal{Y}(\frac{1}{2}^-) \to \overline{M}(\frac{1}{2})$ (resp. $\mathcal{Y}(\frac{1}{2}) \to \overline{M}(\frac{1}{2})$) is a family of stable pairs $\left(Y, \sum_{i=0,3} \frac{1}{2}(R_i + G_i + B_i) + \sum_{i=1,2} c(R_i + G_i + B_i)\right)$. The fibers are distinct. $\overline{M}(\frac{1}{2})$ is projective and the subset for which $Y = \Sigma$ is open and dense.*

Proof. The proof is immediate from the description in the cases E, F, EF. The contraction $\mathcal{Y}(\frac{1}{2}^-) \to \mathcal{Y}(\frac{1}{2})$ exists by applying the Basepoint-Free Theorem [KM98, Thm. 3.24] over the base $\overline{M}(\frac{1}{2})$. $\square$

4.11. **All degenerations of pairs, summarized.** As a summary, in Table 1 we list the types or irreducible components of the stable pairs $(Y, \sum_{i=0}^{3} \frac{1}{2}(R_i + G_i + B_i))$ that appear in the family over $\overline{M}(\frac{1}{2})$. For completeness, we also include the minimal degenerations B, G, H in which $Y = \Sigma$ but the $\mathbb{Z}_2^2$-cover X is not smooth.

The volume of a component Y_k is

$$4\left((K_Y + \tfrac{1}{2}D_{\mathrm{bry}} + \tfrac{1}{2}D_{\mathrm{int}})|_{Y_k}\right)^2 = \left(K_X|_{X_k}\right)^2, \quad \text{where } X_k = X \times_Y Y_k.$$

Note that in a degeneration $X = \cup X_k$ the volumes add up to $K_X^2 = 6$.

The last column lists the conditions for the pair $\left(\mathbb{P}^2, \sum_{i=0}^{2}(r_i R_i + g_i G_i + b_i B_i)\right)$, resp. $\left(\Sigma, \sum_{i=0}^{3}(r_i R_i + g_i G_i + b_i B_i)\right)$, from which this component of Y originates (as can be seen by the methods of Section 5) to be log canonical, using the notation of Lemma 4.4. The main inequalities, in black, are the ones that fail when all $r_i = g_i = b_i = \frac{1}{2}$, i.e. lead to non log canonical singularities, and the ones in gray lead to log canonical singularities. It is clear that the former inequalities correspond

#	Cases	Surface	Vol	Log canonical conditions
0		Σ	6	
0	B	Σ	6	$(r_0 + r_1 \leq 1,\ g_3 + r_1 \leq 1)$
0	G	Σ	6	$(r_1 + r_2 \leq 1)$
0	H	Σ	6	$(r_1 + g_1 + b_1 \leq 2)$
1	A	$\mathbb{P}^1 \times \mathbb{P}^1$	2	$(r_0 + r_1 + r_2 \leq 1, g_3 + r_1 + r_2 \leq 1)$
2	C	$\mathbb{P}^1 \times \mathbb{P}^1$	2	$(r_3 + r_2 + b_1 \leq 1,\ g_0 + g_1 + r_2 \leq 1,$
				$b_0 + b_1 \leq 1,\ b_3 + g_1 \leq 1)$
3	D	$\mathbb{F}_1$	3	$(r_0 + r_1 + b_2 \leq 1,\ g_3 + r_1 \leq 1,\ b_3 + b_2 \leq 1)$
4	CD, D^2	$\mathbb{P}^2$	1	$(r_0 + r_1 + b_2 \leq 1,\ b_0 + b_1 + g_2 \leq 1,$
				$b_3 + b_2 \leq 1,\ r_3 + b_1 \leq 1)$
5	E, EF	$\mathbb{P}^1 \times \mathbb{P}^1$	2	$(r_1 + r_2 + g_1 + g_2 + b_1 + b_2 \leq 2)$
6	E	$\mathbb{P}^2$	4	$(r_0 + g_0 + b_0 \leq 1)$
7	F	$\mathrm{Bl}_1 \Sigma$	5	$(r_1 + r_2 + g_1 + g_2 + b_1 \leq 2)$
8	F, EF	$\mathbb{P}^2$	1	$(r_0 + g_0 + b_0 + b_2 \leq 1)$
9	EF	$\mathbb{F}_1$	3	$(r_0 + g_0 + b_0 \leq 1, r_1 + r_2 + g_1 + g_2 + b_1 \leq 2,$
				$r_1 + r_2 \leq 1,\ g_1 + g_2 \leq 1)$

TABLE 1. Irreducible components of stable pairs in $\overline{M}(\frac{1}{2})$

to the double curves of Y. By observation, the latter ones correspond to the log centers of the pair which are contained in D_{int}.

Lemma 4.31. *Up to an isomorphism, there are* 10 *surfaces Y underlying the stable pairs in $\overline{M}(\frac{1}{2})$, with the irreducible components: Σ, $3(\mathbb{P}^1 \times \mathbb{P}^1)$, $2\mathbb{F}_1$, two types of $2(\mathbb{P}^1 \times \mathbb{P}^1) \cup 2\mathbb{P}^2$, $(\mathbb{P}^1 \times \mathbb{P}^1) \cup 4\mathbb{P}^2$, $6\mathbb{P}^2$, $(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2$, $(\mathrm{Bl}_1 \Sigma) \cup \mathbb{P}^2$ and $(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{F}_1 \cup \mathbb{P}^2$. For each of them, the divisors K_Y and $D = D_{\mathrm{tot}} = 2D_{\mathrm{Hur}}$ are Cartier, D is ample, and $2K_Y + D$ is very ample.*

Proof. In the toric cases these surfaces appear in the minimal cases A, C, D and the maximal cases of Fig. 7. For the nontoric cases there are additionally cases E, F and EF. It is easy to see that the gluings of the irreducible components $Y = \cup Y_k$ are unique.

For the toric fibers, we checked that D is ample and $2K_Y + D$ is very ample in Lemma 4.19 and Proposition 4.21: both are sections of $\mathcal{O}(1)$. Thus, it is enough to look at the surfaces appearing in the nontoric cases E and EF.

In case E, $2K_Y + D$ restricted to $\mathbb{P}^1 \times \mathbb{P}^1$ and to $\mathbb{P}^2$ is $\mathcal{O}(1,1)$ and $\mathcal{O}(2)$ respectively. Consider $\mathbb{P}^3 \cup \mathbb{P}^5 \subset \mathbb{P}^6$ with $\mathbb{P}^3 \cap \mathbb{P}^5 = \mathbb{P}^2$. The join of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and the Veronese embedding $v_2 \colon \mathbb{P}^2 \to \mathbb{P}^5$ intersecting along a conic in $\mathbb{P}^2$ is our surface Y embedded by $|2K_Y + D|$. Case EF is obtained by a further toric degeneration of $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ into a union $\mathbb{F}_1 \cup \mathbb{P}^2$ with $(2K_Y + D)|_{\mathbb{F}_1} = \mathcal{O}(s_1 + 2f)$ and $(2K_Y + D)|_{\mathbb{P}^2} = \mathcal{O}(1)$. Thus, $|2K_Y + D|$ embeds $\mathbb{F}_1 \cup \mathbb{P}^2$ into $\mathbb{P}^4 \cup \mathbb{P}^2 \subset \mathbb{P}^5$ intersecting along a line $\mathbb{P}^4 \cap \mathbb{P}^2 = \mathbb{P}^1$. $\qquad\square$

4.12. **Proof of Theorem 2.** We start with an instructive example.

Example 4.32. Consider the degeneration described in case A shown in Fig. 3. Let $\mathcal{Y}_{\mathrm{ini}} = \Sigma \times \Delta \to (\Delta, 0)$ be the initial family, and let $\sigma \colon \mathcal{Y} \to \mathcal{Y}_{\mathrm{ini}}$ be the composition of two blowups with exceptional divisors V_1, V_2, so that the central fiber of the

family $\mathcal{Y} \to (\Delta, 0)$ is

$$Y = V_0 \cup V_1 \cup V_2 = \Sigma \cup \mathrm{Bl}_1\, \mathbb{F}_1 \cup \mathbb{F}_0.$$

On $\mathcal{Y}_{\mathrm{ini}}$ the divisors $\mathcal{R}_{\mathrm{ini}}$, $\mathcal{G}_{\mathrm{ini}}$, $\mathcal{B}_{\mathrm{ini}}$ are pullbacks from Σ, so there exist the sheaves L_χ pulled back from Σ which together with the branch divisors $\mathcal{R}_{\mathrm{ini}}$, $\mathcal{G}_{\mathrm{ini}}$, $\mathcal{B}_{\mathrm{ini}}$ provide the building data for a $\mathbb{Z}_2^2$-cover $\mathcal{X}_{\mathrm{ini}} \to \mathcal{Y}_{\mathrm{ini}}$, and $\mathcal{X}_{\mathrm{ini}}$ is a normal variety.

Let $\mathcal{X} = \mathcal{X}_{\mathrm{ini}} \times_{\mathcal{Y}_{\mathrm{ini}}} \mathcal{Y}$ and let $\widetilde{\mathcal{X}}$ be its normalization. Further, let $(\Delta', 0) \to (\Delta, 0)$ be the base change of degree 2 ramified at 0,

$$\mathcal{Y}' = \mathcal{Y} \times_\Delta \Delta', \quad \mathcal{X}' = \widetilde{\mathcal{X}} \times_\Delta \Delta',$$

and let $\widetilde{\mathcal{X}}'$ be the normalization of $\mathcal{X}'$. We claim that

(1) $\mathcal{Y}'$ is a normal variety with central fiber isomorphic to $Y = V_0 \cup V_1 \cup V_2$,
(2) $\pi' \colon \widetilde{\mathcal{X}}' \to \mathcal{Y}'$ is a $\mathbb{Z}_2^2$-cover whose branch divisors do not contain any components of the fibers of $\mathcal{Y}' \to \Delta'$, and
(3) the restrictions of π' to V_0, V_1, V_2 are $\mathbb{Z}_2^2$-covers such that the curves $V_i \cap V_j$ appear in R, G, B exactly as shown in Fig. 3.

Indeed, the branch divisors for $\mathcal{X} \to \mathcal{Y}$ are $\sigma^*(\mathcal{R}_{\mathrm{ini}})$, $\sigma^*(\mathcal{G}_{\mathrm{ini}})$, $\sigma^*(\mathcal{B}_{\mathrm{ini}})$, and it is clear that $\sigma^*(\mathcal{R}_{\mathrm{ini}})$ contains $3V_1 + 2V_2$ and $\sigma^*(\mathcal{G}_{\mathrm{ini}})$ contains V_2. By Lemma 2.9, the branch divisors for the normalized cover $\widetilde{\mathcal{X}} \to \mathcal{Y}$ contain V_1 and V_2 instead, so that the total branch divisor is reduced.

Now make the degree 2 base change. Part (1) is obvious. The branch divisors for $\mathcal{X}' \to \mathcal{Y}'$ now contain $2V_1$ and $2V_2$ respectively. By Lemma 2.9 again, the branch divisors for $\widetilde{\mathcal{X}}' \to \mathcal{Y}'$ do not contain either, proving part (2). Part (3) follows by Lemma 4.33.

The variety $\widetilde{\mathcal{X}}'$ is normal and the family $\widetilde{\mathcal{X}}' \to \mathbb{A}_s^1$ is flat. By Lemma 2.10 $K_{\widetilde{\mathcal{X}}'} = \pi^*(K_{\mathcal{Y}'} + \frac{1}{2} D_{\mathrm{tot}})$ is relatively big and nef. Moreover, by the Hurwitz formula

$$K_{\widetilde{\mathcal{X}}'} + \widetilde{\mathcal{X}}_0' = \pi^* \left(K_{\mathcal{Y}'} + \tfrac{1}{2} D + \mathcal{Y}_0' \right).$$

Since the central fiber $(\mathcal{Y}_0', \frac{1}{2} D)$ is slc–in fact it has simple normal crossings–the pair $\left(\mathcal{Y}', \frac{1}{2} D + \mathcal{Y}_0' \right)$ is slc by Inversion of Adjunction. By Lemma 2.10 the pair $(\widetilde{\mathcal{X}}', \widetilde{\mathcal{X}}_0')$ is slc. By Adjunction, implies that $\widetilde{\mathcal{X}}'$ is slc and that $\widetilde{\mathcal{X}}'$ has canonical singularities along $\widetilde{\mathcal{X}}_0'$. Then by the standard construction, already contained in [KSB88], its relative canonical model over Δ', obtained by contraction by the linear system $|N K_{\widetilde{\mathcal{X}}'}|$ for some $N \gg 0$, provides a family of stable pairs. So we have described the stable limit of Burniat surfaces in this example.

Lemma 4.33 (Colors for the double crossing locus add up)**.** *Let $\mathcal{Y} \to (\Delta, 0)$ be a flat family over a smooth curve with central fiber $V_1 \cup V_2$ such that $\mathcal{Y}$ is normal and it is generically smooth along $V_1 \cap V_2$.*

Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a $\mathbb{Z}_2^k$-cover with normal $\mathcal{X}$ such that V_i appear in branch divisors D_{g_i}, $i = 1, 2$. (Here, $g_i = 0$ means that V_i is not in D_{tot}.) Let $(\Delta', 0) \to (\Delta, 0)$ be a degree 2 cover ramified over 0, $\mathcal{Y}' = \mathcal{Y} \times_\Delta \Delta'$, $\mathcal{X}' = \mathcal{X} \times_\Delta \Delta'$, and let $\widetilde{\mathcal{X}}'$ be the normalization of $\mathcal{X}'$.

Then the total branch divisor of the $\mathbb{Z}_2^k$-cover $\pi' \colon \widetilde{\mathcal{X}}' \to \mathcal{Y}'$ does not contain the V_i, and $V_1 \cap V_2$ is contained in the branch divisor $D_{g_1 + g_2}$ of π'.

Proof. For the cover $\mathcal{X}' \to \mathcal{Y}'$ either $g_i = 0$ and V_i is not in the branch locus or $g_i \neq 0$ and V_i is contained in D_{g_i} with multiplicity 2. So by Lemma 2.9, for the normalized cover $\widetilde{\mathcal{X}}' \to \mathcal{Y}'$ the branch divisors contain neither V_1 nor V_2.

Generically along $V_1 \cap V_2$, the variety $\mathcal{Y}'$ has an A_1-singularity, and the components V_1, V_2 of the central fiber of $\mathcal{Y}' \to \Delta'$ are not Cartier along $V_1 \cap V_2$. Let U be a sufficiently small open neighborhood of $V_1 \cap V_2$ in $\mathcal{Y}'$ and $\widetilde{U} \to U$ be the resolution, with exceptional divisor $E \subset \widetilde{U}$.

For the cover $\mathcal{Z}' = \mathcal{X}' \times_{\mathcal{Y}'} \widetilde{U} \to \widetilde{U}$ the branch divisors are the pullbacks from U. So, E appears in D_{g_1} and D_{g_2} with coefficient 1 if $g_1 \neq g_2$ or with coefficient 2 if $g_1 = g_2$. By Lemma 2.9 for the normalized cover $\widetilde{\mathcal{Z}}' \to \widetilde{U}$, E appears in the branch divisor $D_{g_1+g_2}$. So in the restriction of $\widetilde{\mathcal{Z}}' \to \widetilde{U}$ to V_i the divisor $V_i \cap E$ appears in $D_{g_1+g_2}$. The double locus $V_i \cap E$ equals $V_i \cap V_j \cap U$, with the same neighborhoods in V_i, and the normalized covers $\widetilde{\mathcal{X}}'$, $\widetilde{\mathcal{Z}}'$ agree. This proves the statement. $\square$

We are now ready to prove Theorem 2. We will give two proofs: (1) by analyzing the stable limits of one-parameter families, and (2) by constructing families of stable Burniat surfaces over an open cover of the moduli space $\overline{M}(\frac{1}{2})$.

First proof of Theorem 2. Consider a one-parameter family $\mathcal{X} \to \Delta \setminus 0$ of Burniat surfaces. After a finite base change, it can be realized as a $\mathbb{Z}_2^2$-cover $\mathcal{X} \to \mathcal{Y}$ of a family of stable pairs $(\mathcal{Y}, \frac{1}{2}\mathcal{R} + \frac{1}{2}\mathcal{G} + \frac{1}{2}\mathcal{B})$ over a punctured one-dimensional base $\Delta \setminus 0$. To simplify the notation, let us assume that we have made this reduction.

Since $\overline{M}(\frac{1}{2})$ is complete, we have an extension $\Delta \to \overline{M}(\frac{1}{2})$ and the pullback of the family of Theorem 4.30 over $\overline{M}(\frac{1}{2})$ gives a family over Δ whose central fiber $(Y, \frac{1}{2}R + \frac{1}{2}G + \frac{1}{2}B)$ is stable. We claim that the stable limit of the Burniat family is the $\mathbb{Z}_2^2$-cover of Y. In particular, the stable limit of Burniat surfaces is uniquely defined by the image of $0 \in \Delta$ in $\overline{M}(\frac{1}{2})$. Since $\overline{M}(\frac{1}{2})$ is normal and the moduli space of smooth Burniat surfaces is M^0/Γ, this implies that the normalization of the closure of the moduli space of Burniat surfaces in the moduli space of stable surfaces is $\overline{M}(\frac{1}{2})/\Gamma$, thus proving Theorem 2.

The proof of the above statement is the same as in Example 4.32, with minor changes. We start with the normal variety $\mathcal{Y} \to \Delta$. The solution to the fundamental relations exists over $\Delta \setminus 0$. Then, denoting by $\mathcal{R}, \mathcal{G}, \mathcal{B}$ the closures of the divisors over $\Delta \setminus 0$, there exist divisorial sheaves L_1, L_2, L_3 on $\mathcal{Y}$ such that in the class group $\mathrm{Cl}(\mathcal{Y})$ one has

$$2L_1 = \mathcal{G} + \mathcal{B} + \sum V_{1,k}$$

for some components $V_{1,k}$ of the central fiber Y, and similarly for L_2 and L_3. We make a $2:1$ base change $(\Delta', 0) \to (\Delta, 0)$, so that the pullbacks of $V_{1,k}$ are $2V_{1,k}$, and find the building data with reduced total branch divisor. As in Example 4.32, this removes the vertical components from the total branch divisor. If $\widetilde{\mathcal{X}}' \to \mathcal{Y}'$ is the $\mathbb{Z}_2^2$-cover for these data then by Lemma 2.10 the pair $(\widetilde{\mathcal{X}}', \widetilde{\mathcal{X}}'_0)$ has slc singularities and $K_{\widetilde{\mathcal{X}}'}$ is relatively ample. By adjunction, it follows that

(1) $\widetilde{\mathcal{X}}'$ has canonical singularities near $\widetilde{\mathcal{X}}'_0$, thus both are Cohen-Macaulay,
(2) the central fiber $\widetilde{\mathcal{X}}'_0$ has slc singularities, and
(3) $K_{\widetilde{\mathcal{X}}'}$ is relatively ample.

Therefore, $\widetilde{\mathcal{X}}'_0$ is the stable limit of $\mathcal{X} \to \Delta \setminus 0$. The induced map $\widetilde{\mathcal{X}}'_0 \to \mathcal{Y}$ is a $\mathbb{Z}_2^2$-cover between deminormal varieties with slc singularities. The restrictions of this

map to the irreducible components Y_i of Y are also $\mathbb{Z}_2^2$-covers. In all cases, excluding case 4 of Table 1, "colors" of curves in the double crossing locus are uniquely determined by the fact that the fundamental relations (1) on the irreducible components Y_i have a solution. When some component Y_i is of type 4, we look at a two-step degeneration, from a simpler limits where the components are $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. This implies that the colors are uniquely determined as well.

So the stable limit of Burniat surfaces is uniquely determined by the stable limit $(Y, \frac{1}{2}R + \frac{1}{2}G + \frac{1}{2}B)$, as claimed. The normalization map is a bijection by Lemma 4.35. $\qquad\square$

For another proof, we construct a family of stable Burniat surfaces over open sets covering $\overline{M}(\frac{1}{2})$, after appropriate $2^n : 1$ base changes.

Second proof of Theorem 2. Consider the family $f\colon \mathcal{Y}(\frac{1}{2}) \to \overline{M}(\frac{1}{2})$ of Theorem 4.30. Over the open subset

$$U = \overline{M}(\tfrac{1}{2}) \setminus \big[(\text{toric boundary}) \cup (\cup_{k=1}^6 E_k)\big] = \mathbb{G}_m^4 \setminus \cup_{k=1}^6 \mathcal{C}_k$$

the morphism f is smooth, every fiber is isomorphic to Σ and we have a family of effective divisors R_i, G_i, B_i, $i = 0, 1, 2, 3$. The variety $\mathcal{Y}(\frac{1}{2})$ is normal and the complement $\overline{M}(\frac{1}{2}) \setminus U$ is a union of divisors.

Define the Weyl divisors $\mathcal{R}_i, \mathcal{G}_i, \mathcal{B}_i$ on $\mathcal{Y}(\frac{1}{2})$ to be the closures of the divisors over $f^{-1}(U)$. Consider the divisors $\mathcal{R} + \mathcal{G}, \mathcal{G} + \mathcal{B}, \mathcal{B} + \mathcal{R}$. (As before, $\mathcal{R} = \sum \mathcal{R}_i$, $\mathcal{G} = \sum \mathcal{G}_i$, $\mathcal{B} = \sum \mathcal{B}_i$.) Their restrictions to every fiber over a point in U are 2-divisible. Since $\mathrm{Cl}(U) = 0$, there exist divisorial sheaves L_1, L_2, L_3 on $\overline{M}(\frac{1}{2})$ with

$$2L_1 = \mathcal{G} + \mathcal{B} + \sum \mathcal{V}_{1,k} \text{ in } \mathrm{Cl}\big(\mathcal{Y}(\tfrac{1}{2})\big)$$

for some Weil divisors $\mathcal{V}_{1,k}$ supported on the boundary $f^{-1}\big(\overline{M}(\frac{1}{2}) \setminus U\big)$, and similarly for L_2 and L_3.

Let $s \in \overline{M}(\frac{1}{2}) \setminus U$ be a point on the boundary. Suppose that it is an intersection of the (finitely many) boundary divisors H_n. Pick a sufficiently small affine neighborhood $s \in W \subset \overline{M}(\frac{1}{2})$. There exists a finite cover $\mu\colon V \to W$ with normal V such that

    (1) μ is branched along each H_n with multiplicity divisible by 2,
    (2) the base changed family $\mathcal{Y}_V = \mathcal{Y}(\frac{1}{2}) \times_{\overline{M}(\frac{1}{2})} V \to V$ is normal.

Indeed, outside of the 8 singular torus-fixed points (see Corollary 4.13) the boundary of U is normal-crossing, and near the toric boundary we can take the toric base change corresponding to the embedding $2N \to N$ of the cocharacter lattices.

Since $\mathrm{Cl}(U) = 0$, it follows that $\mathrm{Cl}(\overline{M}(\frac{1}{2}))$ is generated by H_n. On the cover the multiplicities of H_n double. So the pullbacks of $\mathcal{R} + \mathcal{G}, \mathcal{G} + \mathcal{B}, \mathcal{B} + \mathcal{R}$ to $\mathcal{Y}_V$ are divisible by 2 and thus satisfy the fundamental relations (1) for a $\mathbb{Z}_2^2$-cover $\pi_V\colon \mathcal{X}_V \to \mathcal{Y}_V$. The restriction of this family to a generic one-parameter subfamily is the same as described in the first proof. So π_V is a family of stable surfaces and it induces a classifying morphism $V \to \overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$. Clearly, it descends to $W \subset \overline{M}(\frac{1}{2})$ and for different open neighborhoods W_1, W_2 the map is the same on $W_1 \cap W_2$. So we get a morphism $\overline{M}(\frac{1}{2}) \to \overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$.

Finally, we have to divide by the relabeling group for different choices of labeling for smooth Burniat surfaces, giving a well-defined morphism $\overline{M}(\frac{1}{2})/\Gamma \to \overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$. We note here that the natural Γ-action on the fan $\mathfrak{F}$ gives the action on $\overline{M}(\frac{1}{3})$, and that

the unions of centers of the blowups ρ_1, ρ_2 in Section 4.10 are invariant, giving a natural Γ-action on $\overline{M}(\frac{1}{2})$.

The morphism $\overline{M}(\frac{1}{2})/\Gamma \to \overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ is finite, it is a bijection on a dense open subset, and the source is a normal variety. It follows that the normalization of an irreducible component of $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ is indeed $\overline{M}(\frac{1}{2})/\Gamma$. □

Lemma 4.34. *Let X be a degenerate Burniat surface and let $\pi\colon X \to Y$ be the corresponding $\mathbb{Z}_2^2$-cover. Then, denoting $D = D_{\mathrm{tot}} = 2D_{\mathrm{Hur}}$, the linear system $|2K_X|$ coincides with $\pi^*|2K_Y + D|$, it is base point free and maps X to $Y \subset \mathbb{P}^6$.*

Proof. The surfaces X and Y are slc. By Lemma 4.31, Y is Gorenstein and D is Cartier. By Lemma 2.10 we have the equality of line bundles $2K_X = \pi^*(2K_Y + D)$.

For $i = 1, 2$ one has $h^i(2K_Y + D) = h^{2-i}(-K_Y - D)$, and the latter is zero by [LR14, Prop. 3.1] since $K_Y + D$ is ample by Lemma 4.31. Similarly, $h^i(2K_X) = 0$ for $i = 1, 2$. Thus, $h^0(2K_X) = \chi(2K_X)$ and $h^0(2K_Y + D) = \chi(2K_Y + D)$.

Both X and $(Y, \frac{1}{2}D)$ are flat limits of surfaces and the Euler characteristic is locally constant in flat families. Thus, $\chi(2K_X) = \chi(2K_Y + D) = 7$ since this holds generically. Thus, $h^0(2K_X) = h^0(2K_Y + D)$, which implies $|2K_X| = \pi^*|2K_Y + D|$. The last part follows by Lemma 4.31. □

The following Lemma implies the last claim of Theorem 2.

Lemma 4.35. *The degenerate Burniat surfaces corresponding to different points of $\overline{M}(\frac{1}{2})/\Gamma$ are non-isomorphic. In other words, the map from $\overline{M}(\frac{1}{2})/\Gamma$ to the moduli space of stable surfaces is a bijection to the closure of M_{Bur}.*

Proof. By Theorem 4.30, the fibers $(Y, \sum_{i=0}^3 \frac{1}{2}(R_i + G_i + B_i))$ over different points of $\overline{M}(\frac{1}{2})$ are non-isomorphic. By the previous Lemma 4.34, if $-K_Y$ is very ample then we one can recover the pair $(Y, \sum_{i=0}^3 \frac{1}{2}(R_i + G_i + B_i))$ from X, since the $\mathbb{Z}_2^2$-cover is intrinsic: it is the bicanonical map. By Lemma 4.31 this is the case in all cases except when $Y = (\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{F}_1 \cup \mathbb{P}^2$. In the latter case, $|2K_X|$ and $|-K_Y|$ map X and Y to $(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2$. But we observe that the middle component $\mathbb{F}_1$ in the pair $(Y, \sum_{i=0}^3 \frac{1}{2}(R_i + G_i + B_i))$ is unique and does not vary in moduli. □

4.13. Degenerate Burniat surfaces.

Theorem 4.36. *The boundary of the moduli space M_{Bur} of smooth Burniat surfaces in the compactification $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ consists of 8 divisors corresponding to the degenerations A, B, C, D, E, F, G, H of Sections 4.4 and 4.9.*

Proof. These are all the minimal degenerations modulo Γ and for each of them we found an irreducible 3-dimensional family of pairs in $\overline{M}(\frac{1}{2})$ and of their $\mathbb{Z}_2^2$-covers. Thus, the closure of each set is a divisor in $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}} = \overline{M}(\frac{1}{2})/\Gamma$. Outside of the union of these divisors one has $Y = \Sigma$ and the curves R_i, G_i, B_i are in general position. So the covers are smooth Burniat surfaces. □

We now describe the degenerate Burniat surfaces for a general point in each of these divisors.

Remark 4.37. All the surfaces in $\overline{M}_{\mathrm{Bur}}^{\mathrm{slc}}$ satisfy $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$, since they are stable limits of smooth surfaces and slc singularities are Du Bois ([KK10]). It is also possible to double check this vanishing directly in all cases, as is done in

Examples 3.5 and 3.6 of [AP12] for the last degeneration to the right in Figure 7 and for the degeneration of type E.

Warning 4.38. It frequently happens that the components X_k of a stable Burniat surface X are not S_2, even though the surface $X = \cup X_k$ is. This happens when for some point $p \in X_k$ the stabilizer group for the G-action in X is bigger than the stabilizer group for the corresponding S_2 surface X_k^σ. The stabilizer group G_p is generated by the elements $R, G, B \in \mathbb{Z}_2^2$ for the preimages of the curves R_i, G_i, B_i (in X, resp. in X_k only) containing p. When the two stabilizers are different, X_k is obtained from X_k^σ by gluing several points.

For each irreducible component Y_k of a fiber Y of the universal family $\mathcal{Y} \to \overline{M}(\frac{1}{2})$, the S_2-fication X_k^σ of the $\mathbb{Z}_2^2$-cover $X_k \to Y_k$ is given by the procedure described in [AP12], which we reviewed in Section 2.2. The geometric characters of X_k^σ can be all recovered from the branch data: the canonical class, and therefore K^2 and the Kodaira dimension, can be computed using Lemma 2.10, and the cohomology of the structure sheaf can be computed using the decomposition $\mathcal{O}_Y \bigoplus (\oplus_{i=1}^3 L_i^{-1})$ of its direct image. (Note that since all the Y_k are simply connected and therefore, a fortiori, have torsion free Picard group, their decomposition is uniquely determined by the branch data). Finally the slc singularities that can occur in our situation have been analyzed in [AP12].

Case A. (Fig. 3). In the general case, namely when all the lines are distinct, each component is a smooth bielliptic surface (so $K^2 = p_g = 0$, $q = 1$) and the Albanese pencil is the pull back of the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ that contains 2 pairs of branch lines in different branch divisors R, G or B. Two components are glued transversally along a smooth elliptic curve. All three components meet at one point, which is a degenerate cusp of X.

Another description, useful in understanding the degenerations, is as follows. For the general case, consider three elliptic curves E_1, E_2 and E_3, and on each E_k a translation τ_k by a point of order 2 and a rational involution σ_k. Let σ_k' be the involution induced by σ_k on $E_k' := E_k/\tau_k$. Take $X_k := (E_{k+1} \times E_{k+2}')/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on E_{k+1} via τ_{k+1} and on E_{k+2}' via σ_{k+2}' (the index k varies in $\mathbb{Z}_3$). The surfaces X_k and X_{k+1} are glued along a curve isomorphic to E_{k+2}', which on X_k is a fiber of the Albanese pencil $X_k \to E_{k+1}'$ and on X_{k+1} is half of a fiber of the rational pencil $X_{k+1} \to E_k'/\sigma_k' = \mathbb{P}^1$.

Letting two lines in the same branch divisor coincide corresponds to degenerating one of the E_k to a cycle of two rational curves. Letting two lines that are in different branch divisors on one component coincide corresponds to degenerating one of the E_k to a nodal rational curve. At most three degenerations of this type can occur at the same time.

This surface appears very nicely as a degeneration of Burniat surface in the form given by Inoue [Ino94], with the parameter $\lambda \to 0$ or ∞.

Case B. The surface X is non-normal, with singularities of types $2'.1$, $2'.2$, $3'.2$, $3'.4$, and $4''.6$ in Tables 2 and 3 of [AP12]. The normalization is a (non-minimal) properly elliptic surface with 2 A_1 singularities.

Case C. (Fig. 4). In the general case, the surfaces X_1^σ, X_2^σ and X_3^σ, are singular Enriques surfaces. The surfaces X_k meet transversally at one point p_0 which is smooth for all of them, so X has a degenerate cusp there. Two components X_k and

X_{k+1} are glued along a rational curve with a node p_{k+2}. At p_{k+2} there is additional gluing and the surface such that p_{k+2} lies on 3 lines in the same branch divisor is not S_2 there.

Case D. (Fig. 6). Each surface X_k^σ is a singular properly elliptic surface with $2D_4$ and one $\frac{1}{4}(1,1)$ singularities and with $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$. The elliptic fibration is given by $|2K|$ and it is the pullback of the ruling of $\mathbb{F}_1$. The two components are glued along a rational curve with two nodes p_1, p_2, where there is an additional gluing. Each component is not S_2 at one of the points p_k (the one with three branch lines of the same color going through the point).

Case E. (Fig. 10). In this case we have $X_i = X_i^\sigma$ for $i = 1, 2$. The surface X_1, the double cover of $Y_1 = \mathbb{P}^1 \times \mathbb{P}^1$ is a singular del Pezzo surface with $K^2 = 2$, with 6 A_1 singularities. It is the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by the diagonal action of $\mathbb{Z}_2^2$. The second component X_2 is a degenerate Enriques surface. The two surfaces are glued along a curve with 6 nodes that is the union of 4 smooth rational curves. If we let two of the pairs of lines in the same branch divisor coincide on Y_2, then X_2 becomes reducible and the normalization is the union of two quadric cones.

Case F. (Fig. 11). Both components are S_2. The component X_1 which is the cover of the blow up Y_1 of Σ at one point has $K^2 = 2$, $h^1(\mathcal{O}) = 1$, $h^2(\mathcal{O}) = 0$. It is not normal along the preimages of the two double lines in the branch locus, where it has double crossings points. The normalization of X_1 is a ruled surface with $h^1(\mathcal{O}) = 1$, whose Albanese pencil is induced by the pencil that has no fiber contained in the branch locus with multiplicity 2 (the blue one in the picture). The second component X_2 is the same degenerate Enriques surface in case E, but in this case X_1 and X_2 are glued along the union of two rational curves meeting transversally at two points.

Case G. The surface becomes non-normal, with double crossings singularities. The normalization X^ν is a non minimal bielliptic surface. In fact if, say, $G_1 = G_2$, then contracting R_0 and R_3 gives $\mathbb{P}^1 \times \mathbb{P}^1$ with the same configuration of lines as in case A, so the cover is a bielliptic surface $\bar{X}$ and X^ν is the blow up of $\bar{X}$ at two points.

Case H. The surface acquires a $\frac{1}{4}(1,1)$ singularity, with desingularization a Burniat surface with $K^2 = 5$.

The cases above involve all of the irreducible components Y_k of Table 1 except for #4 and #9. We now describe the covers in these two cases.

#4. X_k is a singular del Pezzo surface with $K^2 = 1$ with $2D_4$ singularities over the point where three lines of the same color meet and a $\frac{1}{4}(1,1)$ point over the point where three lines of three distinct colors meet. It is glued to the neighboring components along two rational curves with a node.

#9. X_k is a non-normal surface with $K^2 = -4$. It has two irreducible components that are del Pezzo surfaces with $K^2 = 6$ and $2A_1$ singularities. has double crossing singularities along two disjoint rational curves.

Remark 4.39. In Lemma 4.12 we listed the cones of the fan $\mathfrak{F}$ modulo Γ, i.e. the toric degenerations of the pairs $(Y, \frac{1}{2}D)$. There are 29 of them. Adding non-toric degenerations, there are EF, BF, and then all the possible subcases of these 31

cases obtained by adding some combinations of the G and H degenerations. It does not seem practical to list all of these possibilities here.

Remark 4.40. Although the space $\overline{M}_{\text{Bur}}$ which we constructed is irreducible, in the larger space of stable surfaces there are definitely other irreducible components intersecting $\overline{M}_{\text{Bur}}$. For example, in the degeneration E of Fig. 10 the pairs of lines on $\mathbb{P}^2$ can be deformed to conics tangent to the double locus. Similarly, the three divisors of type $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ can be smoothed, keeping them tangent to the double locus. Since the induced $\mathbb{Z}_2^2$ covers of the double curve $\mathbb{P}^1$ have the same normalization, the covers can be glued. This gives a family of dimension 12. Many of the other degenerations produce other irreducible components in the moduli of stable surfaces.

5. AN ALTERNATIVE DESCRIPTION WITH WEIGHTED LINE ARRANGEMENTS

In the Campedelli case, the compactification of the moduli space of pairs $(\mathbb{P}^2, \sum_{i=1}^{7} \frac{1}{2} D_i)$ is a special case of a more general situation for the weighted hyperplane arrangements $(\mathbb{P}^{r-1}, \sum_{i=1}^{n} d_i D_i)$ for some fixed weights $0 < d_i \leq 1$, considered in [Ale15]. The theory becomes rather trivial in this particular case.

In the Burniat case, the pairs $\left(\text{Bl}_3 \mathbb{P}^2, \sum_{i=0}^{3} \frac{1}{2}(R_i + G_i + B_i) \right)$ arise from a configuration of 9 lines in $\mathbb{P}^2$ (in two ways, related by a Cremona transformation). The construction of the compactified moduli spaces can be done as an application.

We sketch this alternative way here, omitting some details.

5.1. **Compact moduli space for weighted hyperplane arrangements.** Let $\beta = (b_1, \ldots, b_n) \in (0,1]^n$ be a fixed vector and consider the moduli space $M_\beta(r,n) = M_\beta(\mathbb{P}^{r-1}, n)$ of log canonical pairs $(\mathbb{P}^{r-1}, \sum b_i B_i)$ where B_i are some hyperplanes on $\mathbb{P}^{r-1}$, some of which are allowed to coincide. This moduli space can be easily constructed as a free quotient of an open subset of $(\mathbb{P}^{r-1\vee})^n$ by a free action of $\text{PGL}(r)$, since the log canonical pairs as above have trivial automorphism groups.

[Ale15] constructs a certain projective scheme $\overline{M}_\beta(r,n)$ together with a family of stable pairs $(X, \sum b_i B_i)$, called stable weighted hyperplane arrangements over it, containing $M_\beta(r,n)$ as an open subset. The theory is a generalization of [HKT06] to the weighted case, in the same way Hassett's moduli spaces $\overline{M}_{g,\beta}$ of weighted stable curves [Has03] generalize the Deligne-Mumford-Knudsen's spaces $\overline{M}_{g,n}$. The basic tools used in the description are:

(1) The hypersimplex

$$\Delta(r,n) = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1, \ \sum x_i = r\}.$$

(2) Matroid polytopes $P \subset \Delta(r,n)$ associated to hyperplane arrangements $(\mathbb{P}^{r-1}, B_1, \ldots, B_n)$. They are defined as follows: for each subset of indices $I \subset \{1, \ldots, n\}$ one adds the inequality $\sum_{i \in I} x_i \leq \text{codim} \cap_{i \in I} B_i$. It is enough to consider flats, i.e. maximal sets I producing the same linear space $Z = \cap_{i \in I} B_i$. And one can omit the normal crossing intersections since for them the inequalities follow from $x_i \leq 1$. One way to understand this matroid polytope is that this is the set of weights $(x_i) \in [0,1]^n$ for which the pair $(\mathbb{P}^{r-1}, \sum x_i B_i)$ is log canonical.

(3) The weighted hypersimplex, the "window"

$$\Delta_\beta(r,n) = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq b_i, \ \sum x_i = r\}.$$

Then a stable weighted hyperplane arrangement $(X, \sum b_i B_i)$ is described by a *partial* face-fitting cover $\cup P_k$ of $\Delta(r, n)$ by matroid polytopes P_k intersecting the interior of $\Delta_\beta(r, n)$ and such that $\cup P_k$ completely covers the window $\Delta_\beta(r, n)$.

Then the irreducible components X_k of X are in a bijection with the polytopes $P_k \cap \Delta_\beta(r, n)$, and the configurations of the divisors B_i on X_k can be read off this combinatorial gadget as well.

We now explain how this general theory applies to the Campedelli line arrangements and sketch an extension of this theory to the Burniat line arrangements.

5.2. Campedelli line arrangements.

In this case the matroid polytope associated to a line arrangement $(\mathbb{P}^2, \sum_{i-1}^7 \frac{1}{2} D_i)$ is a subset of $\Delta(3, 7) = \{(x_i) \in [0, 1]^7 \mid \sum x_i = 3\}$ satisfying the following additional conditions:

(1) If several lines B_i, $i \in I$, coincide then $\sum_{i \in I} x_i \leq 1$.
(2) If several lines B_i, $i \in I$, pass through a common point then $\sum_{i \in I} x_i \leq 2$.

The weighted "window" is

$$\Delta_{(\frac{1}{2}, \dots, \frac{1}{2})}(3, 7) = \{(x_1, \dots, x_7) \mid 0 \leq x_i \leq \tfrac{1}{2}, \ \sum x_i = 3\}.$$

What makes the situation easy is the fact, easily checked, that none of the equations $\sum_{i \in I} x_i = 1$ and $\sum_{i \in I} x_i = 2$ cuts the interior of $\Delta_{(\frac{1}{2}, \dots, \frac{1}{2})}(3, 7)$. Thus, in this case each partial matroid tiling consists of a unique matroid polytope P. As in Remark 3.5, there are 36 such matroid polytopes modulo S_7, and 175 modulo $\mathrm{GL}(3, \mathbb{F}_2)$.

The geometric consequence of this combinatorial statement is that for every stable weighted hyperplane arrangement $(X, \sum_{i=1}^7 \frac{1}{2} B_i)$, the underlying variety X is irreducible and isomorphic to $\mathbb{P}^2$, which is the same conclusion that we arrived to in Section 3.2 by an easier method.

5.3. Matroid tilings for Burniat arrangements.

Consider the initial generic Burniat line arrangement of the 9 lines R_i, G_i, B_i, $i = 0, 1, 2$, on $\mathbb{P}^2$ in Fig. 1. Associated with it is the matroid polytope in $\Delta(3, 9)$ of the 9-tuples (r_i, g_i, b_i), $i = 0, 1, 2$, satisfying the inequalities $0 \leq r_i, g_i, b_i \leq 1$, $\sum(r_i + g_i + b_i) = 3$, and

$$g_0 + g_1 + g_2 + b_0 \leq 2, \ b_0 + b_1 + b_2 + r_0 \leq 2, \ r_0 + r_1 + r_2 + g_0 \leq 2.$$

We will call it the Burniat polytope and denote by Δ^{Bur}. Recall from Lemma 4.4 that there are three additional coordinates

$$r_3 = g_0 + g_1 + g_2 + b_0 - 1, \ g_3 = b_0 + b_1 + b_2 + r_0 - 1, \ b_3 = r_0 + r_1 + r_2 + g_0 - 1.$$

It is easy to see that $\sum_{i=0}^2 (r_i + g_i + b_i) = 3 \iff \sum_{i=1}^3 (r_i + g_i + b_i) = 3$. Indeed, this is equivalent to $K + \sum(r_i R_i + g_i G_i + b_i B_i) \equiv 0$ on the respective $\mathbb{P}^2$'s, and this condition is preserved by the Cremona transformation.

In terms of these variables the above inequalities become $r_3, g_3, b_3 \leq 1$. The situation is not totally symmetric, however, because it is *not* true that $r_3, g_3, b_3 \geq 0$ on Δ^{Bur}. By analogy with the weighted hyperplane case, we define a "window", the weighted Burniat polytope as the subset

$$\Delta^{\mathrm{Bur}}_{(\frac{1}{2}, \dots, \frac{1}{2})} = \{0 \leq r_i, g_i, b_i \leq \tfrac{1}{2} \text{ for all } i = 0, 1, 2, 3\} \subset \Delta^{\mathrm{Bur}}$$

We now observe that the inequalities in Table 1 defining the irreducible components of the degenerate surfaces Y are in fact the inequalities defining certain matroid subpolytopes $P_k \subset \Delta^{\mathrm{Bur}}$. Moreover, the degenerations A, B, ..., H and

EF naturally correspond to the partial matroid tilings of Δ^{Bur} – completely covering the window $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\dots,\frac{1}{2})}$ – which are listed in Table 2. The inequalities in black are those that cut through the interior of $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\dots,\frac{1}{2})}$. The reason for this will become clear in the next section.

Case	Vol	Matroid polytopes
A	2	$(r_0 + r_1 + r_2 \leq 1,\ g_3 + r_1 + r_2 \leq 1)$,
	+2	$(g_0 + g_1 + g_2 \leq 1,\ b_3 + g_1 + g_2 \leq 1)$,
	+2	$(b_0 + b_1 + b_2 \leq 1,\ r_3 + b_1 + b_2 \leq 1)$.
B	6	$(r_0 + r_1 \leq 1,\ r_1 + g_3 \leq 1)$.
C	2	$(r_3 + r_2 + b_1 \leq 1,\ g_0 + g_1 + r_2 \leq 1,\ b_0 + b_1 \leq 1,\ b_3 + g_1 \leq 1)$,
	+2	$(g_3 + g_2 + r_1 \leq 1,\ b_0 + b_1 + g_2 \leq 1,\ r_0 + r_1 \leq 1,\ r_3 + b_1 \leq 1)$,
	+2	$(b_3 + b_2 + g_1 \leq 1,\ r_0 + r_1 + b_2 \leq 1,\ g_0 + g_1 \leq 1,\ g_3 + r_1 \leq 1)$.
D	3	$(r_0 + r_1 + b_2 \leq 1,\ g_3 + r_1 \leq 1,\ b_3 + b_2 \leq 1)$,
	+3	$(r_3 + r_2 + b_1 \leq 1,\ g_0 + r_2 \leq 1,\ b_0 + b_1 \leq 1)$.
E	2	$(r_1 + r_2 + g_1 + g_2 + b_1 + b_2 \leq 2)$,
	+4	$(r_0 + g_0 + b_0 \leq 1)$.
F	5	$(r_1 + r_2 + g_1 + g_2 + b_1 \leq 2)$,
	+1	$(r_0 + g_0 + b_0 + b_2 \leq 1)$.
G	6	$(r_1 + r_2 \leq 1)$.
H	6	$(r_1 + g_1 + b_1 \leq 2)$.
EF	2	$(r_1 + r_2 + g_1 + g_2 + b_1 + b_2 \leq 2)$,
	+3	$(r_0 + g_0 + b_0 \leq 1,\ r_1 + r_2 + g_1 + g_2 + b_1 \leq 2,\ r_1 + r_2 \leq 1,\ g_1 + g_2 \leq 1)$,
	+1	$(r_0 + g_0 + b_0 + b_2 \leq 1)$.

TABLE 2. Matroid tilings of $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\dots,\frac{1}{2})}$ for some degenerations

5.4. **Compactification for the Burniat arrangements.** We give a sketch of a second construction of the compactification, in addition to the one in Section 4.

As in Section 5.1, let $M_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$ denote the moduli space of log canonical pairs of $\mathbb{P}^2$ with 9 lines, which we label R_i, G_i, B_i, $i = 0, 1, 2$. It comes with the compactification $\overline{M}_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$. Let $Z \subset M_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$ be the closed subset of arrangements for which there are three quadruples of lines passing through a common point, as in Fig. 1:

$$R_0 \cap R_1 \cap R_2 \cap G_0 = p_B,\ G_0 \cap G_1 \cap G_2 \cap B_0 = p_R,\ B_0 \cap B_1 \cap B_2 \cap R_0 = p_G.$$

Let $\overline{Z}$ be its closure in $\overline{M}_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$, with the reduced scheme structure. Over it we have a family $\widehat{\mathcal{Y}} \to \overline{Z}$ of stable pairs $(\widehat{Y}, \sum_{i=0,1,2} \frac{1}{2}(R_i + G_i + B_i))$.

Stable pairs are described by partial matroid tilings of Δ^{Bur} covering the window $\Delta_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$, which can be computed explicitly. Alternatively, one can start by classifying the tilings of the small window $\Delta_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$ itself, a significantly easier task. The latter tilings describe the irreducible components $\widehat{Y}_k$ of $\widehat{Y}$. The possibilities for the incidence relations between the curves R_i, G_i, B_i can then be added in the second step.

The points p_R, p_G, p_B are log centers of the pair $(\mathbb{P}^2, \sum \frac{1}{2}(R_i + G_i + B_i)$: on the blowup the exceptional divisors R_3, G_3, B_3 have discrepancy -1. One checks

that in the family $\widehat{\mathcal{Y}}$ these points give three disjoint sections and that every fiber is smooth at these points. (This is a special case of a general phenomenon.) Let $\widetilde{\mathcal{Y}} \to \widehat{\mathcal{Y}}$ be the blowup at these sections, and consider the divisor

$$K_{\widetilde{\mathcal{Y}}} + \sum_{i=0}^{3} \tfrac{1}{2}(\mathcal{R}_i + \mathcal{G}_i + \mathcal{B}_i) = f^*\big(K_{\widehat{\mathcal{Y}}} + \sum_{i=0}^{2} \tfrac{1}{2}(\mathcal{R}_i + \mathcal{G}_i + \mathcal{B}_i)\big) - \tfrac{1}{2}(\mathcal{R}_3 + \mathcal{G}_3 + \mathcal{B}_3).$$

One checks that this divisor is big and nef on each fiber. By the Basepoint-Free Theorem [KM98, Thm. 3.24] it is relatively semiample and defines a contraction to a family $\mathcal{Y} \to \overline{Z}$ of stable pairs $(Y, \sum_{i=0}^{3} \tfrac{1}{2}(R_i + G_i + B_i))$. Some fibers in this family may be isomorphic. We have a classifying map to the moduli space of stable pairs. Let

$$\overline{Z} \to \overline{Z}' \to \overline{M}^{\mathrm{slc}}$$

be its Stein factorization. Then $\overline{Z}' \to \overline{M}^{\mathrm{slc}}$ is a finite birational map to the closure of Z in $\overline{M}^{\mathrm{slc}}$. The normalization of $\overline{Z}'$ provides the required compactification for the moduli of the pairs $\big(\Sigma, \sum(\tfrac{1}{2}R_i + \tfrac{1}{2}G_i + \tfrac{1}{2}B_i)\big)$.

Remark 5.1. It follows that $\overline{M}(\tfrac{1}{2})$ is the normalization of $\overline{Z}'$.

Remark 5.2. If one is interested only in the irreducible components Y_k of Y then some additional considerations show that it suffices to look only at the tilings of the polytope $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\ldots,\frac{1}{2})}$ itself, instead of computing partial covers of $\Delta(3,9)$ containing $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\ldots,\frac{1}{2})}$. In other words, the grayed out inequalities can be ignored. This makes the computation much easier.

Remark 5.3. The same computations work for the window

$$0 \le r_0, r_3, g_0, g_3, b_0, b_3 \le \tfrac{1}{2}, \quad 0 \le r_1, r_2, g_1, g_2, b_1, b_2 \le c$$

for any $\tfrac{1}{4} < c \le \tfrac{1}{2}$. For $\tfrac{1}{4} < c \le \tfrac{1}{3}$, $\tfrac{1}{3} < c \le \tfrac{2}{5}$ and $\tfrac{2}{5} < c < \tfrac{1}{2}$ respectively they produce the spaces $\overline{M}(\tfrac{1}{3})$, $\overline{M}(\tfrac{2}{5})$, $\overline{M}(\tfrac{1}{2}^-)$ of Section 4.10. For smaller c the polytopes 6 and 8 of Table 1 do not intersect $\Delta^{\mathrm{Bur}}_{(\frac{1}{2},\ldots,\frac{1}{2})}$, so some tilings become simpler. By the general theory [Ale15] there are reduction morphisms between the moduli spaces $\overline{M}(\tfrac{1}{3}) \leftarrow \overline{M}(\tfrac{2}{5}) \leftarrow \overline{M}(\tfrac{1}{2}^-) = \overline{M}(\tfrac{1}{2})$, same as in Section 4.10.

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