

TORIC MULTISECTIONS AND CURVES IN RATIONAL SURFACES

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ABSTRACT. We study multisections of embedded surfaces in 4-manifolds admitting effective torus actions. We show that a simply-connected 4-manifold admits a genus one multisection if and only if it admits an effective torus action. Orlik and Raymond showed that these 4-manifolds are precisely the connected sums of copies of \mathbb{CP}^2 , $\overline{\mathbb{CP}^2}$, and $S^2 \times S^2$. Therefore, embedded surfaces in these 4-manifolds can be encoded diagrammatically on a genus one surface. Our main result is that every smooth, complex curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be put in efficient bridge position with respect to a genus one 4-section. We also analyze the algebraic topology of genus one multisections.

1. INTRODUCTION

Trisections were introduced by Gay and Kirby in 2016 as a novel approach to studying smooth 4-manifolds [GK16]. Soon after, the notion of a bridge trisection was introduced by the fourth author and Zupan as an extension of trisections to the study of smoothly embedded surfaces in 4-manifolds [MZ17, MZ18]. Recent work indicates an elegant interplay between the theory of (bridge) trisections and the study of complex curves and surfaces [LC20, LCM20, LCMS21]. For example, complex curves in \mathbb{CP}^2 happen to admit bridge trisections that are as simple as possible in that they can be decomposed into three trivial disks with respect to the standard genus one trisection of \mathbb{CP}^2 .

The main goal of the present paper is to prove an analogous result for complex curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ by making use of multisections, a generalization of trisections introduced recently by the first author and Naylor [IN20].

Theorem 3.4. *Every smooth, complex curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be isotoped to lie in efficient bridge position with respect to a genus one 4-section.*

Here, *efficient* means that the surface intersects each of the four sectors of the 4-section in a single, trivial disk. For example, if $\mathcal{C}_{p,q}$ denotes the isotopy class of the complex curve of bidegree (q, p) , then by Corollary 3.6, $\mathcal{C}_{p,q}$ admits a $(b, 1)$ -bridge 4-section with $b = pq - p - q + 2$. The proof of Theorem 3.4 is contained in Section 3, where a careful analysis of the genus one 4-section of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given.

As an application of Theorem 3.4, we obtain efficient 4-sections of the complex surfaces that occur as branched covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ along complex curves; see Theorem 4.4 for the detailed statement. For example, the elliptic surface $E(q)$ admits a $(6q - 1, 0)$ 4-section. Diagrams for the 4-sections of $E(1)$ and $K3 = E(2)$ are shown in Section 4, where other connections to branched coverings are explored.

Our analysis of the curves $\mathcal{C}_{p,q}$ makes use of the fact that the genus one 4-section of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is compatible with an effective torus action. In fact, a more general connection exists between 4-manifolds admitting effective torus actions and those admitting genus one multisections, which we henceforth refer to as *toric multisections*. The following can be viewed as a 4-dimensional analogue of the fact that a closed 3-manifold admits an effective torus action if and only if it admits a genus one Heegaard splitting [OR70, Section 2].

Theorem 5.1. *Let X be a closed, simply-connected 4-manifold. Then the following are equivalent.*

- (1) X admits an effective torus action.
- (2) X admits a toric multisection.
- (3) X is diffeomorphic to a connected sum of copies of \mathbb{CP}^2 , $\overline{\mathbb{CP}}^2$, and $S^2 \times S^2$.

Moreover, the following sets of objects are in bijection.

- (4) toric multisections of simply-connected 4-manifolds, up to diffeomorphism
- (5) effective torus actions on simply-connected 4-manifolds, up to equivalence
- (6) loops in the Farey graph, up to conjugation.

Note that $S^1 \times S^3$ is the only non-simply-connected 4-manifold admitting a genus one multisection; see Remark 2.3, but there are infinitely many non-simply-connected 4-manifolds admitting effective torus actions. So, the hypothesis of simple-connectivity is necessary. Theorem 5.1 holds when $X \cong S^4$ (which we think of as $\#^0 X$ for any 4-manifold X), since S^4 admits a toric 2-section, see Remark 2.2.

The first part of Theorem 5.1 is a consequence of the classification of simply-connected 4-manifolds admitting effective torus actions given by Orlik and Raymond [OR70], while the second part makes use of the connection between such 4-manifolds and loops in the Farey graph given by Melvin [Mel82]. Theorem 5.1 is proved in Section 5, where a number of consequences are discussed. For example, we describe how to give a simple computation of the intersection form of a 4-manifold admitting a toric multisection by locating a circular plumbing of disk-bundles over spheres generating the second homology group. We also remark on the following consequence of the second part of Theorem 5.1 and work of Melvin.

Corollary 5.11. *A 4-manifold X admits finitely many efficient toric $(n+2)$ -sections if and only if either $X \cong \#^n \mathbb{CP}^2$ or $X \cong \#^n \overline{\mathbb{CP}}^2$ – i.e., if and only if X is a definite manifold admitting a toric multisection.*

In fact, by work of Melvin, the number of non-diffeomorphic toric $(n+2)$ -sections of $\#^n \mathbb{CP}^2$ is the number t_{n+2} of triangulations of a regular $(n+2)$ -gon (with no added vertices), up to rotations and reflections. For example, $\#^4 \mathbb{CP}^2$ admits 3 distinct 6-sections, which are shown as circuits in the Farey graph in Figure 12.

This paper is motivated in large part by the following question.

Question 1.1. *If \mathcal{C} is a complex curve in a rational surface X , then does \mathcal{C} admit an efficient bridge multisection with respect to the toric multisection of X ?*

With this question in mind, we include in Section 6 an analysis of the algebraic topology of toric multisections. We also discuss gluing of bridge multisections with boundary in Section 5. For a more general discussion of the algebraic topology of multisections, see also [MS21].

If X is a simply-connected 4-manifold with a $(g, 0)$ n -section, then $b_2(X) = (n-2)g$. For the elliptic surface, $E(q)$, we have $b_2(E(q)) = 12q - 2$. If $E(q)$ were to admit an efficient, genus g n -section, then we must have that $(n-2)g = 12q - 2$; in particular g must divide $12q - 2$. Theorem 4.4 shows that $E(q)$ admits an efficient, $(6q-1, 0)$ 4-section, and [LCM20, Theorem 7.7] shows that $E(q)$ admits a $(12q-2, 0)$ trisection. These results can be seen as the boundary cases of the following geography problem.

Question 1.2. *For which values of g does $E(q)$ admit a $(g, 0)$ n -section?*

In particular, the results of this paper produce or rule out all efficient multisections of $E(2)$, except for perhaps a $(2, 0)$ 11-section. Not much is known about the classification of genus two

multisections, which, under a branched covering construction, is equivalent to the classification of 3-bridge multisections. In Section 2, after giving preliminary definitions related to multisections and bridge multisections, we give an infinite family of non-diffeomorphic 3-bridge 4-sections of the unknotted 2-sphere in S^4 , the 2-fold branched covers of which comprise an infinite family of non-diffeomorphic $(2, 1)$ 4-sections of S^4 .

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2. MULTISECTIONS AND BRIDGE MULTISECTIONS

Throughout this section, X will denote a smooth, orientable, closed, connected 4-manifold. Multisections, as defined here, were first studied in [IN20], where they were introduced as a generalization of the trisections introduced by Gay and Kirby [GK16].

Definition 2.1. Let $g \geq 0$, and let $\mathbf{k} = (k_1, \dots, k_n)$, with $n \geq 3$ and $k_i \geq 0$. A (g, \mathbf{k}) -multisection, \mathfrak{M} , of X is a decomposition

$$X = Z_1 \cup \dots \cup Z_n,$$

where, for each $i \in \mathbb{Z}_n$,

- (1) $Z_i \cong \natural^{k_i}(S^1 \times B^3)$,
- (2) $H_i = Z_i \cap Z_{i-1} \cong \#^g(S^1 \times D^2)$, and
- (3) $\Sigma = \bigcap_{i=1}^n Z_i \cong \#^g T^2$.

We adopt the convention that, as oriented manifolds, $\partial H_i = \Sigma$ and $\partial Z_i = H_i \cup_{\Sigma} \overline{H}_{i+1}$. We variously refer to \mathfrak{M} as a (g, \mathbf{k}) n -section, a *genus g n -section*, or an n -section, depending on the context. If $k = k_i$ for all i , then \mathfrak{M} is a (*balanced*) (g, k) -multisection. We call \mathfrak{M} *efficient* if $k_i = 0$ for all i .

Remark 2.2. Technically, it makes sense to consider the degenerate case of 2-sections, and even 1-sections, where a 2-section is a decomposition of a manifold into two 4-dimensional 1-handlebodies, glued along a Heegaard splitting, and a 1-section is a 4-dimensional 1-handlebody with the two halves of a Heegaard splitting of its boundary identified. However, it is easy to see that a $(g; k_1, k_2)$ 2-section can be expanded to a $(g; k_1, k_2, g)$ -trisection, while a $(g; k)$ 1-section can be expanded to a $(g; k, g, g)$ -trisection. It is easy to see that this implies $k_1 = k_2$ and $k = g$ and that these decompositions describe $\#^{k_1}(S^1 \times S^3)$ and $\#^g(S^1 \times S^3)$, respectively, since they are built without 2-handles [MSZ16, LP72].

For these degenerate cases, when $g = 1$, there are two possibilities: S^4 and $S^1 \times S^3$. The toric 2-section of S^4 fits cleanly into the analysis in this paper; see also Remark 5.2. The toric 1-section and 2-section of $S^1 \times S^3$ can be expanded to a toric n -section for any $n \in \mathbb{N}$; cf. Remark 2.3.

By a theorem of Laudenbach and Poénaru [LP72], the *spine* $H_1 \cup \dots \cup H_n$ of an n -section of X determines X up to diffeomorphism. In light of this, two multisections \mathfrak{M} and \mathfrak{M}' are *diffeomorphic* if there is a diffeomorphism $\varphi: X \rightarrow X'$ such that $\varphi(H_i) = H'_i$, up to cyclic reordering.

Since each handlebody H_i is determined by a cut-system of curves α_i , it follows that X is determined by the n -tuple $(\alpha_1, \dots, \alpha_n)$ of g -tuples of curves on Σ , called a *multisection diagram*. Two multisections of a fixed smooth orientable closed 4-manifold are known to be related by a finite sequence of moves [Isl21]. We note that the notion of a multisection studied here differs

from that of Rubinstein and Tillmann, who introduced related structures called multisections for studying PL manifolds in arbitrary dimension [RT18].

The main objects of study in this paper are efficient multisections of genus one, which we refer to as *toric*. Since we are interested in simply-connected 4-manifolds, all toric multisections in this paper will be $(1, 0)$ -multisections (and hence efficient).

Remark 2.3. In a genus g multisection, if $k_i = g$, then the sector Z_i is simply a product cobordism between H_i and H_{i+1} . Usually, this sector can be removed to give a multisection with one fewer sector. The exception to this rule is the degenerate case that $k_i = g$ for all i , in which case the number of sectors can be reduced to one. In this case, the multisection is an open-book decomposition, with binding Σ , page H_1 , and trivial monodromy. It follows that the 4-manifold is diffeomorphic to $\#^g(S^1 \times S^3)$ [MSZ16, Theorem 1.2].

Thus, $S^1 \times S^3$ is the only non-simply-connected 4-manifold admitting a toric multisection, and its toric multisection is unique up to diffeomorphism and collapsing of sectors. See Remark 5.2 for a discussion of this degenerate case in the context of effective torus actions.

Bridge trisections were introduced in [MZ17, MZ18] as an extension of the theory of trisections to the study of embedded surfaces in 4-manifolds. Here, we generalize the notion of a bridge trisection to the setting of 4-manifolds with multisections. A *trivial b -strand tangle* is a pair (H, \mathcal{T}) where H is a handlebody and \mathcal{T} is a collection of properly embedded arcs in H which are simultaneously isotopic into the boundary of H ; a *trivial c -patch disk-tangle* is a pair (Z, \mathcal{D}) where X is a 4-dimensional 1-handlebody and \mathcal{D} is a collection of properly embedded disks in X which are simultaneously isotopic into the boundary of X .

Definition 2.4. Let X be a 4-manifold with an n -section \mathfrak{M} . Let $b \geq 1$ and let $\mathbf{c} = (c_1, \dots, c_n)$ with $c_i \geq 1$. An embedded surface (X, \mathcal{K}) is in (b, \mathbf{c}) -*bridge position* with respect to \mathfrak{M} if

- (1) $(Z_i, \mathcal{D}_i) = (Z_i, Z_i \cap \mathcal{K})$ is a trivial c_i -patch disk-tangle, and
- (2) $(H_i, \mathcal{T}_i) = (H_i, H_i \cap \mathcal{K})$ is a trivial b -strand tangle.

The induced decomposition $\mathfrak{M}_{\mathcal{K}}$

$$(X, \mathcal{K}) = (Z_1, \mathcal{D}_1) \cup \dots \cup (Z_n, \mathcal{D}_n),$$

is called a (b, \mathbf{c}) -*bridge n -section* for \mathcal{K} (relative to \mathfrak{M}). If \mathcal{K} is oriented, we adopt the convention that, as oriented manifolds, $\partial \mathcal{D}_i = \mathcal{T}_i \cap \overline{\mathcal{T}}_{i+1}$. We variously refer to $\mathfrak{M}_{\mathcal{K}}$ as a b -*bridge n -section* or a *bridge n -section*, depending on the context. If \mathfrak{M} is a (g, \mathbf{k}) -multisection, then $\mathfrak{M}_{\mathcal{K}}$ is a $(g, \mathbf{k}; b, \mathbf{c})$ -bridge multisection. If $c = c_i$ for all i , then \mathfrak{M} is a *(balanced) (b, c) -bridge multisection*. We call $\mathfrak{M}_{\mathcal{K}}$ *efficient* or *1-patch* if $c_i = 1$ for all i .

Any two disk-tangles in a handlebody with the same boundary are isotopic rel-boundary [Liv82, MZ18], so the *spine* $(H_1, \mathcal{T}_1) \cup \dots \cup (H_n, \mathcal{T}_n)$ of a bridge n -section of the pair (X, \mathcal{K}) determines the pair (X, \mathcal{K}) , as well as the multisection, up to diffeomorphism. In light of this, two bridge multisections $\mathfrak{M}_{\mathcal{K}}$ and $\mathfrak{M}'_{\mathcal{K}}$ are *diffeomorphic* if there is a diffeomorphism $\varphi: (X, \mathcal{K}) \rightarrow (X', \mathcal{K})$ such that $\varphi((H_1, \mathcal{T}_1) \cup \dots \cup (H_n, \mathcal{T}_n)) = (H'_1, \mathcal{T}'_1) \cup \dots \cup (H'_n, \mathcal{T}'_n)$.

Since each tangle \mathcal{T}_i is trivial, it can be isotoped to lie on Σ as a collection of arcs \mathbf{a}_i , called *shadow arcs* for \mathcal{T}_i . It follows that \mathcal{K} is determined by the n -tuple $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ of b -tuples of arcs on Σ , called a *shadow diagram*.

We conclude this section with a simple example that illustrates complexities that arise when one moves from the consideration of toric multisection to higher genus multisections.

Theorem 2.5. *The 4-sphere admits infinitely many non-diffeomorphic $(2, 1)$ 4-sections. The unknotted 2-sphere in S^4 admits infinitely many non-diffeomorphic $(3, 2)$ -bridge 4-sections with respect to the $(0, 0)$ 4-section of S^4 .*

Proof. Consider the 4-tuple of 3-bridge tangles (H_i, \mathcal{T}_i) with $H_i \cong B^3$ described diagrammatically in Figure 1. It is straightforward to check that, for each $i \in \mathbb{Z}_4$, the union $(H_i, \mathcal{T}_i) \cup (\overline{H_{i+1}}, \overline{\mathcal{T}_{i+1}})$ is an unlink of 2-components. It follows that the union $\bigcup_{i=1}^4 (H_i, \mathcal{T}_i)$ is the spine of a $(3, 2)$ -bridge 4-section of a knotted surface \mathcal{K} , relative to the $(0, 0)$ 4-section of S^4 .

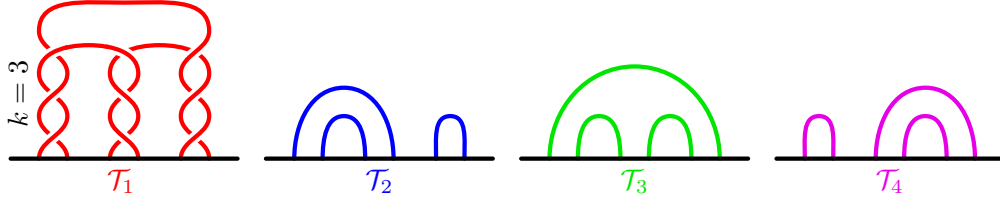


FIGURE 1. The four tangles comprising a $(3, 2)$ -bridge 4-section of the unknotted 2-sphere in S^4 .

Note that $K = \mathcal{T}_1 \cup \overline{\mathcal{T}_3}$ is the pretzel link $P(3, -3, 3)$, while $U = \mathcal{T}_2 \cup \overline{\mathcal{T}_4}$ is the unknot. This multisection induces the standard Morse function $h: S^4 \rightarrow \mathbb{R}$ so that the restriction $f|_{\mathcal{K}}$ satisfies $f^{-1}(\{0\}) = U$ and has 2 minima and a band below U and a band and 2 maxima above U ; see [IN20, Proposition 3.2] and [MZ17, Remark 3.4]. By [Sch85, Main Theorem], \mathcal{K} is unknotted.

Now, let \mathfrak{M}_k denote the bridge multisection obtained by replacing each of the three 3-twist regions of (H_1, \mathcal{T}_1) with k -twist regions (preserving the sign in each case). The above discussion shows these are all $(3, 2)$ -bridge 4-sections of the unknotted 2-sphere, since the cross-section $U_k = \mathcal{T}_2 \cup \overline{\mathcal{T}_4}$ is still unknotted; however, as 4-sections they are non-diffeomorphic, since the cross-sections $K_k = \mathcal{T}_1 \cup \overline{\mathcal{T}_3}$ are the non-equivalent pretzel links $P(k, k, -k)$. This proves the second claim of the theorem.

For the final claim, let $\widetilde{\mathfrak{M}}_k$ denote the multisection obtained as the 2-fold branched cover of \mathfrak{M} . These are $(2, 1)$ 4-sections of S^4 that are non-diffeomorphic, since the cross-sections $\widetilde{H}_1 \cup \widetilde{H}_3$ are the (non-diffeomorphic) Seifert fibered spaces $S^2(\frac{1}{k}, \frac{-1}{k}, \frac{1}{k})$, which are distinguished by the first homology groups [Ori72]. \square

3. BRIDGE POSITION FOR A FAMILY OF CURVES IN $\mathbb{CP}^1 \times \mathbb{CP}^1$

In this section, we prove that there is a family of smooth, complex curves $\mathcal{V}_{p,q}$ in $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ that can be isotoped to lie in 1-patch bridge position with respect to the toric 4-section. The curves $\mathcal{V}_{p,q}$ have homogeneous bidegree (q, p) , so every possible bidegree is represented. Since the moduli space of curves of fixed bidegree is connected, and since smooth curves are generic, this yields Theorem 3.4.

Our analysis proceeds as follows. First, we study in detail the 4-section \mathfrak{M} of $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, which we view through the lens of the symplectic toric structure on X . Next, we introduce a family of singular, reducible complex curves whose smoothings are the curves $\mathcal{V}_{p,q}$, and we determine how they sit relative to \mathfrak{M} . Finally, we study the smoothing $\mathcal{V}_{p,q}$, showing that it is isotopic to a surface in efficient bridge position with respect to \mathfrak{M} .

3.1. The toric 4-section of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Our study of curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ makes use of the structure this manifold inherits as a (symplectic) toric manifold – i.e., a compact, connected (symplectic) manifold equipped with an effective, half-dimensional torus action (and a choice of moment map). In fact, the symplectic structure is not necessary for our analysis, but the fact that our analysis is compatible with the symplectic structure may be of independent interest and useful in future, more geometric considerations. We refer the reader to [CdS03] for an introduction and complete details. In Section 5, we generalize the discussion immediately below to the class of simply-connected 4-manifolds admitting effective torus actions, in which case symplectic structures are not always present. In what follows we write $S^2 \times S^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ interchangeably.

As a warm up example, consider the S^1 (1-torus) action on \mathbb{CP}^1 . We adopt angular coordinates on our tori and homogeneous coordinates on our projective spaces. Then, the action of S^1 on \mathbb{CP}^1 is given by

$$\theta \cdot [x_1 : x_2] = [x_1 : e^{i\theta} x_2],$$

where $\theta \in [0, 2\pi]$. If we equip \mathbb{CP}^1 with the Fubini-Study symplectic form ω_{FS} and moment map $\mu: \mathbb{CP}^1 \rightarrow \mathbb{R}$ given by

$$\mu([x_1 : x_2]) = \frac{|x_1|^2}{|x_1|^2 + |x_2|^2},$$

then $(\mathbb{CP}^1, \omega_{FS}, S^1, \mu)$ is a symplectic toric manifold. The moment polytope (i.e., the image of μ) is the closed interval $[0, 1] \subset \mathbb{R}$.

In what follows, we consider $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ to be equipped with the product Fubini-Study form, with the effective torus action given by

$$(\theta, \phi) \cdot ([x_1 : x_2], [y_1 : y_2]) = ([x_1 : e^{i\theta} x_2], [y_1 : e^{i\phi} y_2]),$$

and with the corresponding moment map $\mu: X \rightarrow \mathbb{R}^2$ given by

$$\mu([x_1 : x_2], [y_1 : y_2]) = \left(\frac{|x_1|^2}{|x_1|^2 + |x_2|^2}, \frac{|y_1|^2}{|y_1|^2 + |y_2|^2} \right).$$

Thus, we find that the moment polytope for X is the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$. In Figure 2, the corners and two edges of the unit square are labeled with their preimages in X under μ .

We can construct a 4-section \mathfrak{M} on X by lifting via the moment map a decomposition of the moment polytope into four squares; compare the following descriptions with Figure 2.

$$\begin{aligned} Z_1 &= \{|x_1| \leq |x_2|, |y_1| \leq |y_2|\}, & H_1 &= \{|x_1| \leq |x_2|, |y_1| = |y_2|\}, \\ Z_2 &= \{|x_1| \geq |x_2|, |y_1| \leq |y_2|\}, & H_2 &= \{|x_1| = |x_2|, |y_1| \leq |y_2|\}, \\ Z_3 &= \{|x_1| \geq |x_2|, |y_1| \geq |y_2|\}, & H_3 &= \{|x_1| \geq |x_2|, |y_1| = |y_2|\}, \\ Z_4 &= \{|x_1| \leq |x_2|, |y_1| \geq |y_2|\}, & H_4 &= \{|x_1| = |x_2|, |y_1| \geq |y_2|\}, \\ \Sigma &= \{|x_1| = |x_2|, |y_1| = |y_2|\}. \end{aligned}$$

Each Z_i is a 4-ball, while the H_i are solid tori, and Σ is a torus. Each of the H_i is determined by a curve α_i on Σ that bounds a disk in H_i . We adopt the following coloring convention: H_1 is red, H_2 is blue, H_3 is green, and H_4 is purple. When we later consider tangles \mathcal{T}_i inside the H_i , we will depict the shadows of the tangles using a lighter shade of the corresponding color. See Figure 2, where we have represented Σ as a square with opposite edges identified via reflection.

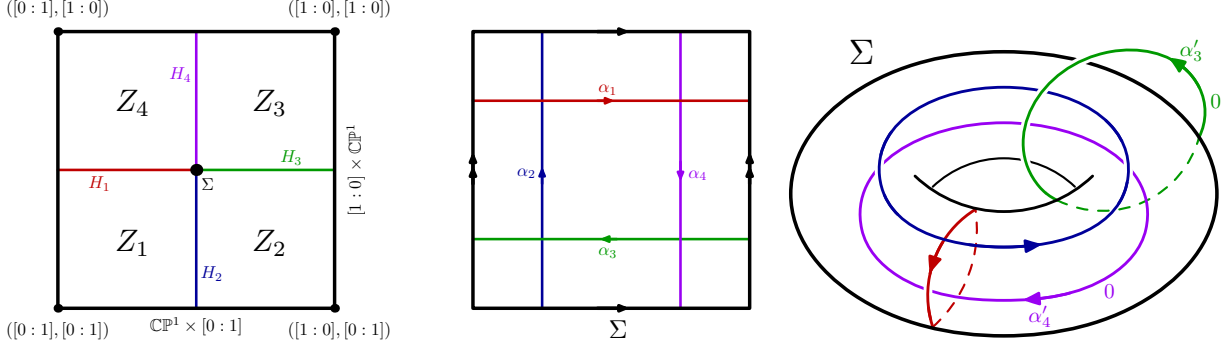


FIGURE 2. **(Left)** A decomposition of the moment polytope for $\mathbb{CP}^1 \times \mathbb{CP}^1$ lifting to a 4-section. **(Middle)** A 4-section diagram for $S^2 \times S^2$. **(Right)** Obtaining a handle decomposition of $S^2 \times S^2$ from the 4-section.

The Z_i inherit orientations from X , and we orient Σ and the H_i by declaring that $\partial Z_i = H_i \cup \overline{H}_{i+1}$ and $\Sigma = \partial H_i$ as oriented submanifolds. Here, and henceforth, we adopt cyclical indexing $i \in \mathbb{Z}_4$. Note that $H_i \cup \overline{H}_{i+1} \cong S^3$ for all i , but $H_i \cup \overline{H}_{i+2} \cong S^1 \times S^2$. More precisely, we have

$$H_1 \cup \overline{H}_3 = \partial\nu(\mathbb{CP}^1 \times [0:1]), \quad \text{and} \quad H_2 \cup \overline{H}_4 = \partial\nu([1:0] \times \mathbb{CP}^1),$$

as oriented manifolds.

The standard handle-decomposition of $S^2 \times S^2$ is evident in the 4-section. Consider Σ as the genus one Heegaard surface in $S^3 = \partial Z_1$. Let α'_3 and α'_4 denote copies of α_3 and α_4 (respectively) that have been isotoped off Σ to lie in H_2 and H_1 (respectively). Let \mathfrak{h}_3 and \mathfrak{h}_4 be 0-framed 2-handles attached along the respective components of the link $\alpha'_3 \cup \alpha'_4$. The effect of attaching \mathfrak{h}_3 is to perform 0-framed Dehn surgery on H_2 along α'_3 ; the resulting handlebody is H_3 . Similarly, H_4 is the result of performing 0-framed Dehn surgery on H_1 along α'_4 . The result of performing 0-framed Dehn surgery on $S^3 = H_1 \cup \overline{H}_2$ along $\alpha'_3 \cup \alpha'_4$ is $S^3 = \overline{H}_4 \cup H_3$, which is ∂Z_4 . In this way, we find that Z_1 corresponds to the 0-handle; Z_2 and Z_3 correspond to 2-handles (\mathfrak{h}_3 and \mathfrak{h}_4 , respectively); and Z_4 corresponds to the 4-handle. The toric 4-section gives precisely the standard handle-decomposition. See Figure 2.

The pieces of the 4-section \mathfrak{M} are preserved set-wise by the action of the torus on X . In particular, we can identify the surface Σ with the torus that is acting upon it. Representing Σ as a square with opposite sides identified, let θ represent the horizontal direction, and let ϕ represent the vertical direction. These coordinates are consistent with the conventions we have established thus far. For example, fixing ϕ and varying θ gives a circle action that amounts to rotation of $\mathbb{CP}^1 \times [1:1]$ about its poles, which corresponds to rotation of Σ in the α_1 -direction. Similarly, ϕ acts on Σ as rotation in the α_2 -direction. Henceforth, we adopt (θ, ϕ) -coordinates on Σ , which allow us to identify the representation square with $[0, 2\pi] \times [0, 2\pi]$ in \mathbb{R}^2 .

We will refer to a curve γ on Σ as an (a, b) -curve if it represents $a[\alpha_1] + b[\alpha_2]$ in

$$H_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z} = \langle \alpha_1, \alpha_2 \rangle.$$

Equivalently, γ can be represented in the identification square by a collection of arcs, each of which has slope $b/a \in \mathbb{Q} \cup \{\infty\}$. For example, α_1 is a $(1, 0)$ -curve, while α_2 is a $(0, 1)$ -curve.

We note for future reference that certain representatives for the generators of the second homology are also evident in the diagram. Let D_i denote a copy of the meridional disk for H_i ; assume $\partial D_i = \partial \overline{D}_{i+2}$. Then,

$$D_1 \cup \overline{D}_3 = \mathbb{CP}^1 \times [1:1], \quad \text{and} \quad D_2 \cup \overline{D}_4 = [1:1] \times \mathbb{CP}^1.$$

Note the connection to the handle-decomposition: D_1 is isotopic to the Seifert disk for α'_3 , and D_3 is the core of \mathfrak{h}_3 . Similarly, D_2 is isotopic to the Seifert disk for α'_4 , and D_4 is the core of \mathfrak{h}_4 . So, in the handlebody diagram in Figure 2, α'_3 is the equator of $\{pt\} \times S^2$, while α'_4 is the equator of $S^2 \times \{pt\}$. Let β_1 and β_2 denote the homology classes of $\mathbb{CP}^1 \times [1 : 1]$ and $[1 : 1] \times \mathbb{CP}^1$, respectively, so,

$$H_2(X) = \mathbb{Z} \oplus \mathbb{Z} = \langle \beta_1, \beta_2 \rangle.$$

Note that, given this set-up, the map from $H_1(\Sigma)$ to $H_2(X)$ given by $\alpha_i \mapsto \beta_i$ is an isomorphism.

3.2. The complex curves \mathcal{V} . Consider the bihomogeneous polynomial $f: \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{C}$ given by

$$f = \left(x_1^{q-1} y_1 + x_2^{q-1} y_2 \right) \left(x_1 y_2^{p-1} + x_2 y_1^{p-1} \right).$$

Let \mathcal{V} be the variety cut out by f . Note that \mathcal{V} has homogeneous bidegree (q, p) and that it is reducible. Let \mathcal{V}_1 be the irreducible variety of homogeneous bidegree $(q-1, 1)$ cut out by

$$f_1 = x_1^{q-1} y_1 + x_2^{q-1} y_2,$$

and let \mathcal{V}_2 be the irreducible variety of homogeneous bidegree $(1, p-1)$ cut out by

$$f_2 = x_1 y_2^{p-1} + x_2 y_1^{p-1}.$$

So, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.

Remark 3.1. Note that \mathcal{V} represents $p\beta_1 + q\beta_2$ in $H_2(X)$.

We now determine how these varieties intersect the 4-section of X .

Lemma 3.2. *The intersections of $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is as follows.*

- (1) $\mathcal{V}_1 \cap \text{Int}(Z_1) = \mathcal{V}_1 \cap \text{Int}(Z_3) = \emptyset$.
- (2) $\mathcal{V}_2 \cap \text{Int}(Z_2) = \mathcal{V}_2 \cap \text{Int}(Z_4) = \emptyset$.
- (3) $\mathcal{V}_1 \cap Z_2$ and $\mathcal{V}_1 \cap Z_4$ are each a neatly embedded¹, trivial disk.
- (4) $\mathcal{V}_2 \cap Z_1$ and $\mathcal{V}_2 \cap Z_3$ are each a neatly embedded, trivial disk.
- (5) $\mathcal{V}_1 \cap \text{Int}(H_i) = \mathcal{V}_2 \cap \text{Int}(H_i) = \emptyset$.
- (6) $\mathcal{V}_1 \cap \Sigma$ is a $(-1, q-1)$ -curve passing through the point $(0, \pi)$.
- (7) $\mathcal{V}_2 \cap \Sigma$ is a $(p-1, 1)$ -curve passing through the point $(\pi, 0)$.

Proof. First, consider \mathcal{V}_1 , which is cut out by $x_1^{q-1} y_1 + x_2^{q-1} y_2 = 0$. If $x_1 = 0$ or $y_1 = 0$, then $y_2 = 0$ or $x_2 = 0$, respectively, so we land in Z_4 or Z_2 , respectively. If $x_1 = y_1 = 1$, then we get $1 + x_2^{q-1} y_2 = 0$. It follows that $|x_2| \leq 1$ if and only if $|y_2| \geq 1$, so we land in Z_4 or Z_2 . This completes the proof of (1); the analysis of \mathcal{V}_2 is similar, yielding (2).

Next, it can be checked that $|x_1| = |x_2|$ if and only if $|y_1| = |y_2|$ for points on \mathcal{V} . This proves (5). If we assume $|x_1| = |x_2|$ and $|y_1| = |y_2|$, we can set $x_2 = y_2 = 1$ and $x_1 = e^{i\theta}$ and $y_1 = e^{i\phi}$. Then $f = 0$ reduces to

$$\left(e^{i(q-1)\theta} e^{i\phi} + 1 \right) \left(e^{i\theta} + e^{i(p-1)\phi} \right) = 0.$$

Switching to (θ, ϕ) -coordinates, this becomes

$$(\phi + (q-1)\theta - \pi)(\theta - (p-1)\phi - \pi) = 0.$$

So, $\mathcal{V}_1 \cap \Sigma$ is given by the equation $\phi = -(q-1)\theta + \pi$, while $\mathcal{V}_2 \cap \Sigma$ is given by the equation $\theta = (p-1)\phi + \pi$, proving (6) and (7).

¹Recall that an embedding $\varphi: M \rightarrow N$ is *neat* if $f(\partial M) \subset \partial N$ and $f(\partial M) \pitchfork \partial N$.

It remains to show (3) and (4); first, we consider $Z_1 \cap \mathcal{V}_2$. Since x_2 and y_2 are nonzero on the interior of Z_1 , we can set $x_2 = y_2 = 1$ and adopt affine (x_1, y_1) -coordinates so that $Z_1 = \{(x_1, y_1) : |x_1| \leq 1, |y_1| \leq 1\}$. In this way, we identify Z_1 with the bi-disk $D^2 \times D^2$. In doing so, the polynomial f_2 reduces to $x_1 + y_1^{p-1}$, which cuts out a neatly embedded, complex disk in Z_1 . This is a disk in the 4-ball bounded with a single minimum – it is a smoothing of a cone – hence, it is trivial and isotopic rel-boundary to the Seifert disk spanning its boundary, which is the (unknotted) torus knot $T_{1,p-1}$. A similar argument for the other three sectors completes the proof. \square

3.3. Smoothing the complex curves \mathcal{V} . A consequence of Lemma 3.2 is that \mathcal{V}_1 and \mathcal{V}_2 are 2-spheres that intersect in $(p-1)(q-1)+1$ points, all of which are contained in Σ . The purpose of this section is to show that these singular points can be smoothed to obtain a smooth, complex curve $\mathcal{V}_{p,q}$ that is isotopic to a smooth surface $\mathcal{C}_{p,q}$ that lies in bridge position with respect to \mathfrak{M} . First, we will describe a local, topological modification that will replace a neighborhood of a singular point with an annulus, thus smoothing that singularity. We will then describe an ambient isotopy of the resulting smooth surface that will make it transverse to Σ . At this point, the surface will be in bridge position, and it will be isotopic to the smoothing $\mathcal{V}_{p,q}$, as desired.

Let x be a point in $\mathcal{V}_1 \cap \mathcal{V}_2$, so x lies on Σ . Let B^4 be a small neighborhood of x , which inherits a 4-section structure from \mathfrak{M} . In abuse of notation, let $Z_i = B^4 \cap Z_i$, $H_i = B^4 \cap H_i$, and $\Sigma = B^4 \cap \Sigma$. Note that the Z_i are 4-balls, the H_i are 3-balls, and Σ is a disk. Let $Y_i = S^3 \cap Z_i$, let $P_i = S^3 \cap H_i$, and let $U = S^3 \cap \Sigma$. Note that $U = \partial\Sigma$ is the unknot, the P_i are each a page of the open-book decomposition of S^3 with binding U , and the Y_i are each a spread of pages co-bounded by the P_i and P_{i+1} .

Let $E_x = B^4 \cap \mathcal{V}$. Since \mathcal{V}_1 and \mathcal{V}_2 intersect positively and transversely, E_x consists of a pair of disks that intersect in a positive node singularity. Let $L_i = S^3 \cap \mathcal{V}_i$, so $L = L_1 \cup L_2$ is a positive Hopf link. Note that L_i is contained in $Y_{i-1} \cup Y_{i+1}$, with indices taken in \mathbb{Z}_4 ; cf. Lemma 3.2, parts (1) and (2). Also, $L \cap U$ consists of 4 points. Label these x_1, x_2, x_3 , and x_4 so that $L_1 \cap U = \{x_2, x_4\}$. Let $\tau_i \subset P_i$ be a neatly embedded arc with $\partial\tau_i = \{x_i, x_{i+1}\}$. See Figure 3.

Let Δ_i be a triangle contained in Y_i such that $\partial\Delta_i = \tau_i \cup \tau_{i+1} \cup \lambda_i$, where $\lambda_i \subset L$. Let $A = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$. It is immediate that A is the annular Seifert surface for L . Let A_x denote the result of perturbing the interior of A into the interior of B^4 in such a way as to respect the 4-section structure on B^4 . (For example, $A \cap P_i$ gives rise to $A_x \cap H_i$.)

Let \mathcal{C} denote the surface obtained from \mathcal{V} by replacing node singularity E_x with smooth annulus A_x at each point x in $\mathcal{V}_1 \cap \mathcal{V}_2$. It is immediate that \mathcal{C} is isotopic to the smoothing $\mathcal{V}_{p,q}$ of \mathcal{V} . However, although \mathcal{C} is transverse to Σ near the points x , it is not transverse to Σ everywhere; see Figure 4. Instead, $\mathcal{C} \cap \Sigma$ is a collection of arcs. There is an ambient isotopy of \mathcal{C} that is supported in a neighborhood of such an arc that transforms the arc of intersection into a single point of intersection; see Figure 5.

Having arranged that \mathcal{C} is smoothly embedded and transverse to Σ , we now claim that \mathcal{C} is in bridge position with respect to \mathfrak{M} .

Lemma 3.3. *The smooth surface \mathcal{C} is in 1-patch bridge position with respect to the 4-section \mathfrak{M} .*

Proof. Let $\mathcal{D}_1 = Z_1 \cap \mathcal{C}$. We will describe how \mathcal{D}_1 was obtained from $\mathcal{D}_1'' = Z_1 \cap \mathcal{V}_2$ and conclude that \mathcal{D}_1 is a trivial disk. By Lemma 3.2, we know that \mathcal{D}_1'' is a trivial disk. The first modification we made to \mathcal{D}_1'' was to smooth its intersections with \mathcal{V}_1 . This modification was achieved by removing from \mathcal{D}_1'' the disk $D_1 = B^4 \cap \mathcal{D}_1''$, whose boundary was split as $\partial D_1 = \lambda_1 \cup \delta_1$, where $\delta_1 \subset \partial D_1$ and λ_1 is neatly embedded in D_1 . The disk D_1 was removed and replaced by the triangle Δ_1 ; see Figure 3. However, D_1 and Δ_1 are neatly, ambiently isotopic; the only difference is that Δ_1 is

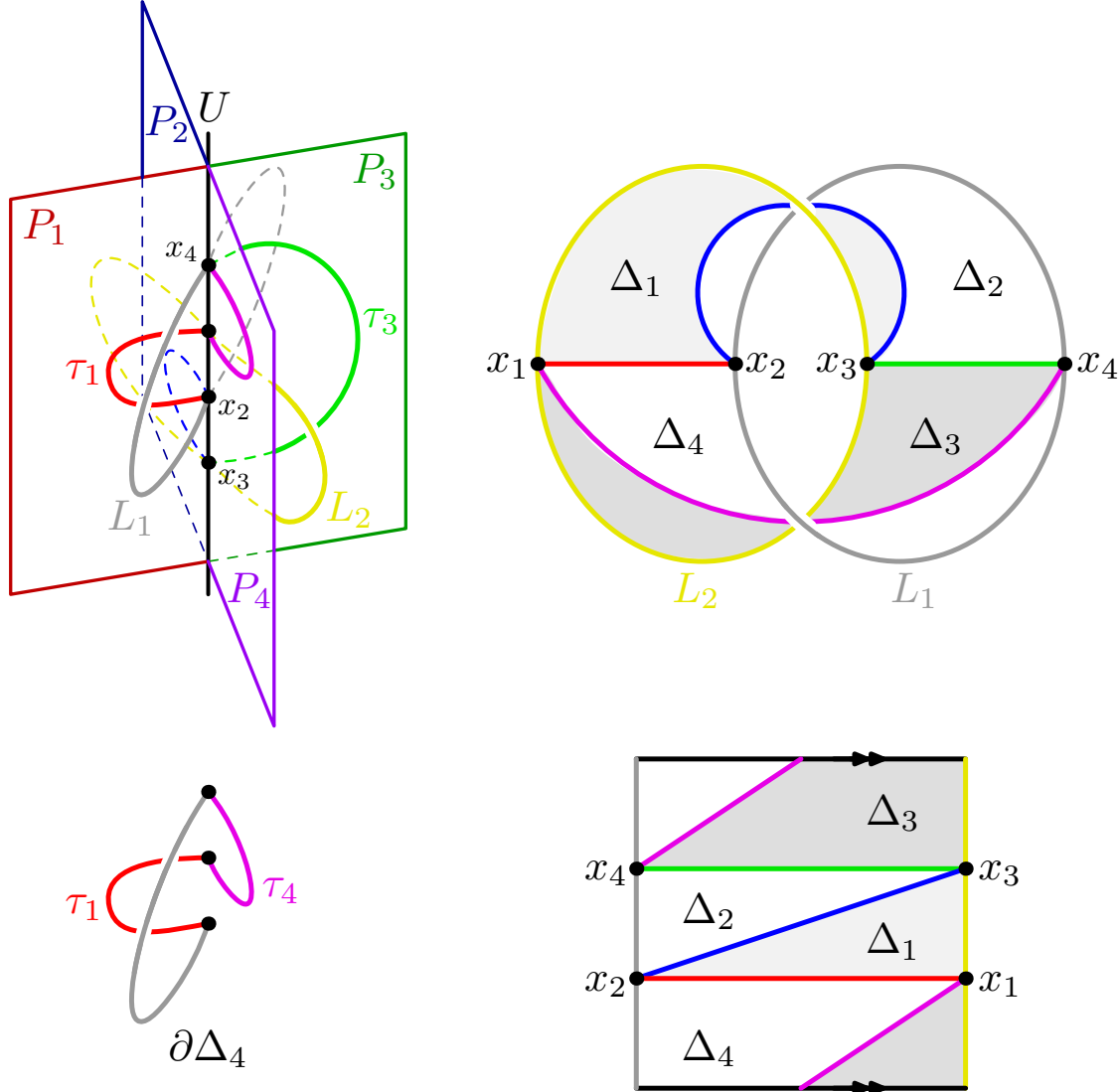


FIGURE 3. The configuration of arcs in S^3 that cut the annular Seifert surface in four triangles.

obtained from D_1 by perturbing δ_1 , which is not transverse to Σ , to the union $\tau_1 \cup \tau_2$, which is transverse to Σ . It follows that the result \mathcal{D}'_1 remains a trivial, neatly embedded disk, even after \mathcal{V} is smoothed to obtain \mathcal{C} .

Besides the smoothing accounted for above, the only other modification made was the ambient isotopy that was used to make \mathcal{C} transverse to Σ . The effect of this isotopy on \mathcal{D}'_1 can be seen in the bottom line of Figure 5. In the left-most union of “tangles,” we see that \mathcal{D}'_1 fails to be transverse to Σ along a collection of arcs, one of which is shown in the local picture. The effect of this ambient isotopy is to straighten out the portion of $\partial\mathcal{D}'_1$ where the failure of transversality occurs. It is clear that the result of these ambient isotopies, which is now the desired disk \mathcal{D}_1 , is indeed a trivial disk, as desired.

A similar argument suffices to prove that $\mathcal{D}_i = Z_i \cap \mathcal{C}$ is a trivial disk for all $i \in \mathbb{Z}_4$. It remains to see that $\mathcal{T}_i = H_i \cap \mathcal{C}$ is a trivial tangle for each $i \in \mathbb{Z}_4$. However, this is clear from the discussion

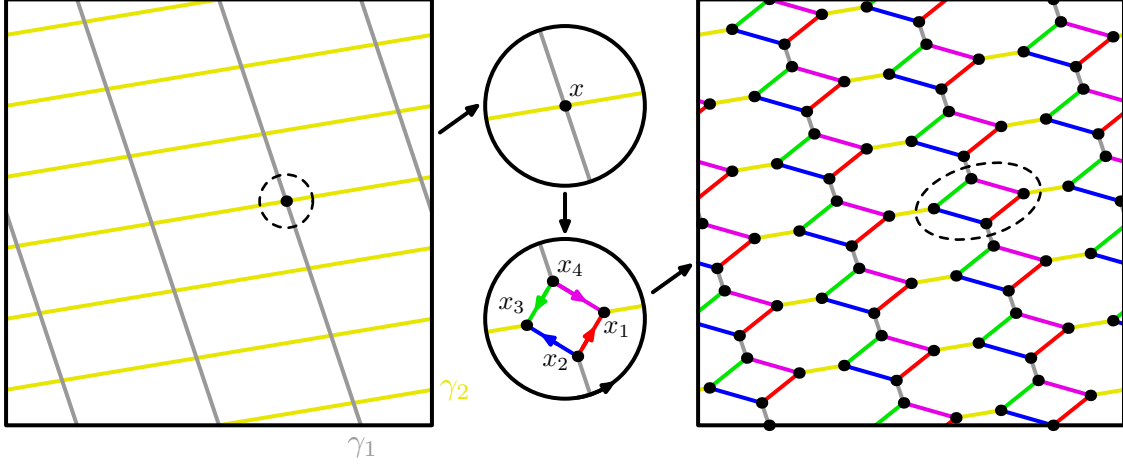


FIGURE 4. Here, $(p, q) = (7, 4)$. **(Left)** The curves of intersection of $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with Σ . **(Middle)** A diagrammatic representation of the local modification used to smooth \mathcal{V} . **(Right)** A diagrammatic representation of the smoothing \mathcal{C} , before it has been made transverse to Σ .

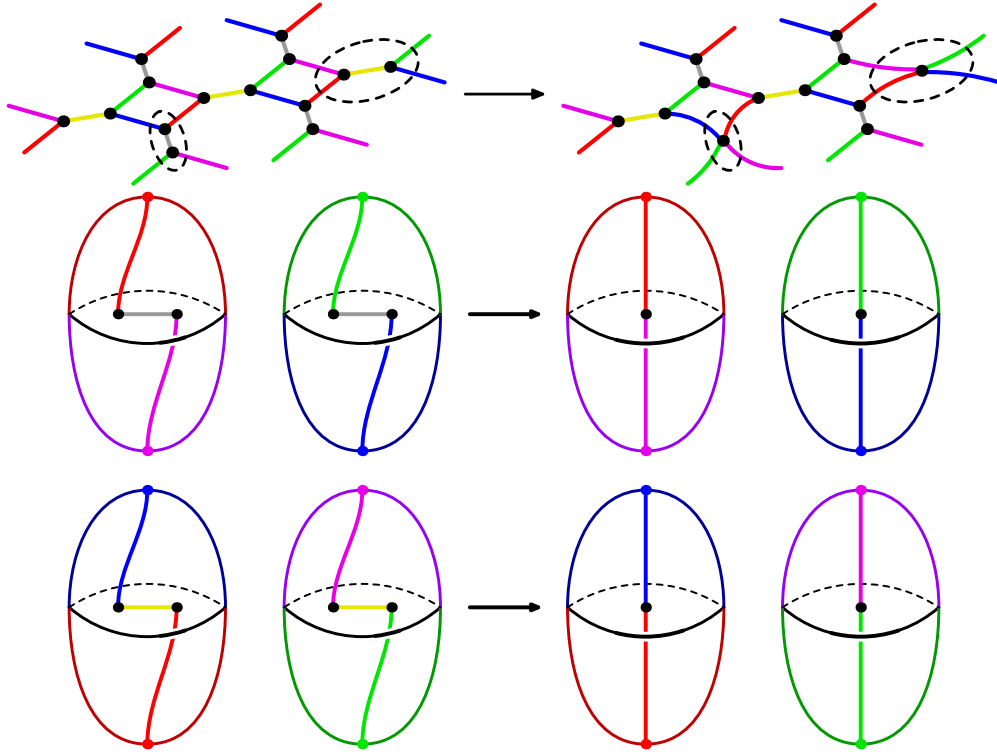


FIGURE 5. The local ambient isotopy used to make \mathcal{C} transverse to Σ , shown diagrammatically **(Top line)** and ambiently within the spine of \mathfrak{M} **(Bottom lines)**.

of \mathcal{D}_1 just given; to see this, we focus on \mathcal{T}_1 , as it lies in $\partial\mathcal{D}_1$. By Lemma 3.2, $\partial\mathcal{D}_1'' \cap \text{Int}(H_1) = \emptyset$ for all $i \in \mathbb{Z}_4$. As discussed above, the smoothing modification amounted to taking portions of \mathcal{D}_1'' that were lying flat in Σ and perturbing them via ambient isotopy to be transverse to Σ ; see

Figure 3. Therefore, we have that \mathcal{D}'_1 intersects H_1 in a collection of trivial strands, plus some flat arcs that come from the residual non-transversality of \mathcal{C} ; see Figure 5. The ambient isotopy that eliminated these flat arcs affected the trivial strands only via an isotopy of their boundary points. It follows that the result, which is precisely \mathcal{T}_1 , is a trivial tangle, as desired. A similar argument for the other \mathcal{T}_i completes the proof. \square

Let $\mathcal{V}_{p,q}$ denote a smooth, complex curve obtained from \mathcal{V} by a small, analytic perturbation. The curve $\mathcal{V}_{p,q}$ is smoothly isotopic to the smooth surface \mathcal{C} , since the former is obtained from the singular curve by an arbitrarily small analytic perturbation, while the latter is obtained by a local transformation near the nodes that matches the effect the perturbation there. We are now ready to prove our first theorem from the introduction.

Theorem 3.4. *Every smooth, complex curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be isotoped to lie in efficient bridge position with respect to a genus one 4-section.*

Proof. Let \mathcal{S} be a smooth, complex curve of homogeneous bidegree (p, q) . Then, \mathcal{S} is isotopic to $\mathcal{V}_{p,q}$, since the moduli space of curves of fixed bidegree is connected and since smooth curves are generic in this moduli space. On the other hand, $\mathcal{V}_{p,q}$ is isotopic to \mathcal{C} , which lies in efficient bridge position with respect to \mathfrak{M} . \square

Note that there is no reason to expect that \mathcal{C} (as constructed) is algebraic. For this reason, we will denote by $\mathcal{C}_{p,q}$ the smooth isotopy class of $\mathcal{V}_{p,q}$, and we will work henceforth with the former, rather than the latter.

3.4. A shadow diagram for $\mathcal{C}_{p,q}$. We now describe how a shadow diagram for the efficient bridge 4-section of $\mathcal{C}_{p,q}$ can be obtained in practice; see Figure 6, where the case of $(p, q) = (7, 4)$ is shown. First, draw the curves $\gamma_1 = \mathcal{V}_1 \cap \Sigma$ and $\gamma_2 = \mathcal{V}_2 \cap \Sigma$ on Σ as $(-1, q-1)$ and $(p-1, 1)$ slopes, respectively. Note that these curves intersect in $b = (p-1)(q-1) + 1$ points, and these points of intersection divide each of the curves into the b segments. Let \mathbf{x} be a collection of $2b$ points on $\gamma_1 \cup \gamma_2$ so that each point of \mathbf{x} lies at the midpoint of one of the $2b$ segments. For each $i \in \mathbb{Z}_4$, let \mathbf{a}_i be a collection of b straight arcs in Σ with $\partial \mathbf{a}_i = \mathbf{x}$. These should be drawn so that their union is embedded (i.e., they don't intersect in their interior), so that, at each point of \mathbf{x} , they are arranged cyclically counterclockwise, and so that the union $\mathbf{a}_1 \cup \mathbf{a}_2$ is isotopic to γ_2 . (This will uniquely determine their placement; cf. Figure 6.)

The following is a useful characterization of the shadow arcs just described.

Proposition 3.5. *The union $\mathbf{a}_1 \cup \mathbf{a}_3$ is a $(p-2, q)$ -curve, and the union $\mathbf{a}_2 \cup \mathbf{a}_4$ is a $(-p, q-2)$ -curve. The shadow arcs give a tiling of Σ by congruent parallelograms.*

Proof. Recall the identification of Σ with the Euclidean square $[0, 2\pi] \times [0, 2\pi]$, and arrange γ_1 and γ_2 to be geodesic. Then, for each i , every arc of \mathbf{a}_i can be drawn as a geodesic segment meeting γ_1 (respectively, γ_2) at the same angle. Moreover, since the \mathbf{a}_i connect midpoints of the segments $\gamma_1 \setminus \gamma_2$ to midpoints of the segments $\gamma_2 \setminus \gamma_1$, we find that the arcs of \mathbf{a}_i are parallel to those of \mathbf{a}_{i+2} . \square

In light of the proposition, there is a slightly more streamlined way to draw the shadow diagram: Instead of starting with γ_1 and γ_2 , one can first draw the curves $\mathbf{a}_1 \cup \mathbf{a}_3$ and $\mathbf{a}_2 \cup \mathbf{a}_4$. Then, the bridge points are the points of intersection of these curves. We have illustrated this approach in Figures 8 and 9.

A second implication of this proposition is that \mathcal{C} intersects $\mathbb{CP}^1 \times [1 : 1] = D_1 \cup \overline{D}_3$ in q points and intersects $[1 : 1] \times \mathbb{CP}^1 = D_2 \cup \overline{D}_4$ in p points; since $\partial D_1 = \alpha_1 = \alpha_3$ is a $(1, 0)$ -curve, while

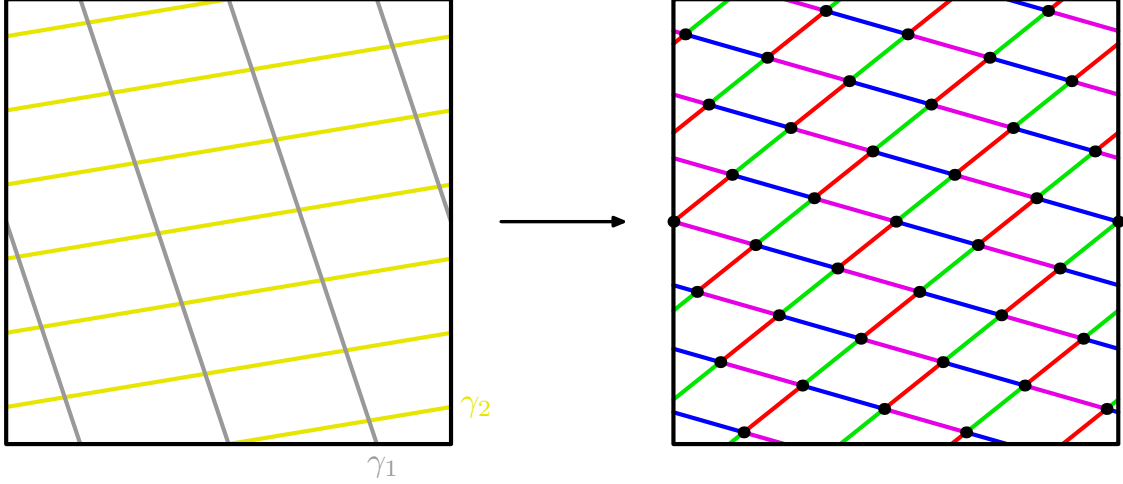


FIGURE 6. **(Left)** The intersection of \mathcal{V} with Σ . **(Right)** A shadow diagram corresponding to the 4-section of $(X, \mathcal{V}_{p,q})$. Here, $(p, q) = (7, 4)$, and $\mathcal{V}_{p,q}$ is a curve of genus 18.

$\partial D_2 = \alpha_2 = \alpha_4$ is a $(0, 1)$ -curve, these intersections can be seen explicitly in the diagram by considering $(\mathbf{a}_1 \cup \mathbf{a}_3) \cap \alpha_1$ and $(\mathbf{a}_2 \cap \mathbf{a}_4) \cap \alpha_2$. This shows that $[\mathcal{C}_{p,q}] = p\beta_1 + q\beta_2$ in $H_2(S^2 \times S^2)$; cf. Remark 3.1.

A consequence of the above discussion is a calculation of the complexity of the efficient bridge multisection. Let $\mathcal{C}_{p,q}$ denote the smooth isotopy class of the complex curve $\mathcal{V}_{p,q}$. Note $\chi(\mathcal{K}) = \sum_i c_i - b(n-2)$ if \mathcal{K} admits a (b, \mathbf{c}) -bridge n -section.

Corollary 3.6. *The surface $\mathcal{C}_{p,q}$ in $S^2 \times S^2$ admits a $(b, 1)$ -bridge 4-section with $b = pq - p - q + 2$, has genus $(p-1)(q-1)$, and represents $p\beta_1 + q\beta_2$ in $H_2(S^2 \times S^2)$.*

4. BRANCHED COVERS

Branched covers of trivial tangles are handlebodies, therefore, bridge multisections naturally give rise to multisections of their branched covers. (This fact has been extensively explored in the literature [BCKM19, CK17, LCM20, LCMS21, MZ17, MZ18].) Moreover, efficient bridge multisections give rise to efficient multisections, since the branched cover of a trivial disk-tangle with one patch is a 4-ball.

Many interesting 4-manifolds are obtained through branched coverings of complex curves in complex surfaces. In [LCM20], the authors gave examples of branched covers over complex curves in trisected complex surfaces. Many of these constructions generalize to the case of multisections and yield lower genus representations of these branched covers. In the present setting, we will obtain multisections of the manifolds $\mathcal{X}_{p,q,n}$ which are the n -fold cyclic branched covers of $(S^2 \times S^2, \mathcal{C}_{p,q})$, with n dividing $\gcd(p, q)$.

4.1. Branched covers along $\mathcal{C}_{p,q}$. To get started, we need to calculate the fundamental group $\pi(\mathcal{C}_{p,q})$ of the complement of $\mathcal{C}_{p,q}$.

Proposition 4.1. *Let $d = \gcd(p, q)$, then $\pi(\mathcal{C}_{p,q}) \cong \langle a \mid a^d \rangle$.*

Proof. The exterior $(S^2 \times S^2) \setminus \nu(\mathcal{C}_{p,q})$ can be built by starting with 4-dimensional thickening of $\Sigma \setminus \nu(\mathbf{x})$ and attaching 4-dimensional 2-handles of two types. First, attach a 2-handle along the

boundary of a tubular neighborhood of each shadow arc, then, attach four 2–handles, one along each of the curves α_i that define the handlebodies of the 4–section \mathfrak{M} . (In the shadow diagram, the α_i should be drawn disjoint from the shadow arcs corresponding to \mathcal{T}_i .) This builds the restriction of the exterior to (a thickening of) the spine of the 4–section; the construction is completed by the addition of 4–dimensional 3–handles and 4–handles to build the exteriors of the trivial disks in the sectors. These final handles don’t affect the fundamental group calculation, so $\pi(\mathcal{C}_{pq})$ can be calculated as the fundamental group of the 3–dimensional thickening, which deformation retracts onto the 2–complex given by $\Sigma \setminus \nu(\mathbf{x})$ and the cores of the 2–handles.

The orientation on $\mathcal{C}_{p,q}$ induces an orientation on the bridge points $\mathbf{x} = \Sigma \cap \mathcal{C}_{p,q}$ such that any two bridge points connected by a shadow arc have opposite orientation. The group is generated by the (oriented) meridional curves to these bridge points, of which there are $2b$ and the curves α_1 and α_2 on Σ . The 2–cells that are attached along the boundaries of regular neighborhoods of the shadow arcs have the effect of (coherently) identifying all the oriented meridional curves to the bridge points. Let a be the class of a meridional curve to a positive bridge point. This leaves three generators: a , $[\alpha_1]$, and $[\alpha_2]$.

We claim that α_1 and α_3 can be chosen to induce the relation a^q and kill $[\alpha_1]$. Draw a curve ω on Σ that (i) intersects the midpoints of the left and right edges of the identification square, (ii) is isotopic to α_1 , and (iii) is the union of diagonal arcs to the parallelograms cut out by the shadow arcs; cf. Figure 7, where ω is not drawn, but indicated by bridge points labeled with $+$. The number of positive bridge points intersected by ω will be the number of intersections of γ_1 with the bottom of the square, plus one; this is q . Now, let α_1 and α_3 be pushoffs (with opposite orientation to each other) of ω that are disjoint from the shadow arcs corresponding to \mathcal{T}_1 and \mathcal{T}_3 , respectively, so α_1 and α_3 co-bound an annulus containing q positive bridge points. Then, $\alpha_1 \cup \alpha_3$ is homotopic to a curve α'_3 that encloses q positive bridge points. This implies the relation a^q in $\pi(\mathcal{C}_{p,q})$, and $[\alpha_1]$ is killed by the cut disk D_1 .

A similar argument for α_2 and α_4 gives the relation a^p and kills $[\alpha_2]$, so we have $\pi(\mathcal{C}_{p,q}) \cong \langle a \mid a^p, a^q \rangle$, as desired. \square

Since $\pi(\mathcal{C}_{p,q})$ is cyclic of order $\gcd(p, q)$, if n divides $\gcd(p, q)$, the n –fold cyclic branched covering of $S^2 \times S^2$ over $\mathcal{C}_{p,q}$ exists. The following proposition shows how this construction can be used to obtain a multisection of the resulting branched cover, and gives the parameters for the resulting multisection.

Proposition 4.2. *Suppose that (X, \mathcal{K}) admits a $(g, \mathbf{k}; b, \mathbf{c})$ bridge m –section. Then, the n –fold cyclic branched cover $(\tilde{X}, \tilde{\mathcal{K}})$ admits a $(g', \mathbf{k}'; b, \mathbf{c})$ bridge m –section where*

$$g' = ng + (n - 1)(b - 1), \quad \mathbf{k}' = n\mathbf{k} + (n - 1)(\mathbf{c} - \mathbf{1}).$$

Proof. This theorem essentially follows from the results in Section 2.7 of [MZ17]. The branched covering \tilde{X} over X along \mathcal{K} restricts to a branched covering of \tilde{Z}_i over Z_i along \mathcal{D}_i and \tilde{H}_i over H_i along \mathcal{T}_i for each $i \in \mathbb{Z}_m$. Since \mathcal{T}_i is a trivial b –strand tangle in the genus g handlebody H_i , the cover \tilde{H}_i is a genus $ng + (n - 1)(b - 1)$ handlebody. The branched cover of the genus k_i 4–dimensional 1–handlebody Z_i along the c_i –component trivial disk–tangle \mathcal{D}_i , is again a 4–dimensional 1–handlebody \tilde{Z}_i , with $k'_i = nk_i + (n - 1)(c_i - 1)$. (Simply notice that a c –component trivial disk–tangle is a trivial c –strand tangle cross an interval.) \square

Applying the previous proposition to a multisection with each $c_i = 1$ and $k_i = 0$, we obtain the following corollary.

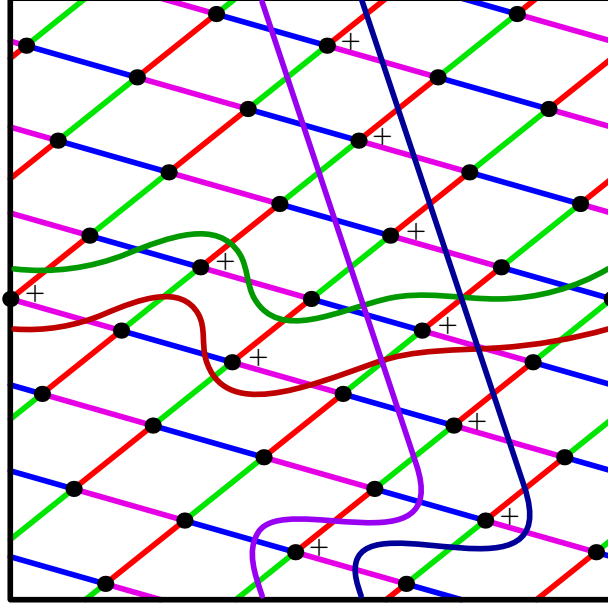


FIGURE 7. $\pi(\mathcal{C}_{7,4}) \cong \langle a \mid a^7, a^4 \rangle \cong 1$. The curves α_1 and α_3 are shown running left to right in red and green, respectively. The curve ω would connect the four bridge points between these curves labeled with $+$.

Corollary 4.3. *If (X, \mathcal{K}) admits an efficient $(g; b)$ bridge m -section, then (\tilde{X}, \tilde{K}) admits an efficient $(g'; b)$ bridge m -section, with $g' = ng + (n - 1)(b - 1)$.*

Combining Corollaries 3.6 and 4.3, we immediately obtain the main theorem of this section.

Theorem 4.4. *The complex surface $\mathcal{X}_{p,q,n}$ admits an efficient $(g, 0)$ 4-section where*

$$g = n + (n - 1)(p - 1)(q - 1).$$

In particular:

- (1) *the rational surface $\mathcal{X}_{2,2q,2} \cong \mathbb{CP}^2 \# (4q + 1)\overline{\mathbb{CP}}^2$ admits an efficient $(g, 0)$ 4-section with*

$$g = 2q + 1;$$
- (2) *the elliptic surface $\mathcal{X}_{4,2q,2} \cong E(q)$ admits an efficient $(g, 0)$ 4-section with*

$$g = 6q - 1;$$
- (3) *the Horikawa surface $\mathcal{X}_{6,2q,2} \cong H(q)$ admits an efficient $(g, 0)$ 4-section with*

$$g = 10q - 3.$$

Given a shadow diagram for (X, \mathcal{K}) , there is a straightforward procedure to draw a multisection diagram for \tilde{X} ; see [LCM20] for details, presented in the setting of trisections. Multisection diagrams for the 4-sections of $K3 = \mathcal{X}_{4,4,2}$ and $E(1) = \mathcal{X}_{4,2,2}$ obtained as branched coverings of the surfaces $\mathcal{C}_{p,q}$ (as described in Subsection 3.4) are shown in Figures 8 and 9, respectively.

4.2. Branching over the central surface. Given a multisectioned 4-manifold, a natural operation is to consider the 4-manifold obtained as the cyclic branched cover over the central surface. This will always produce a multisectioned 4-manifold.

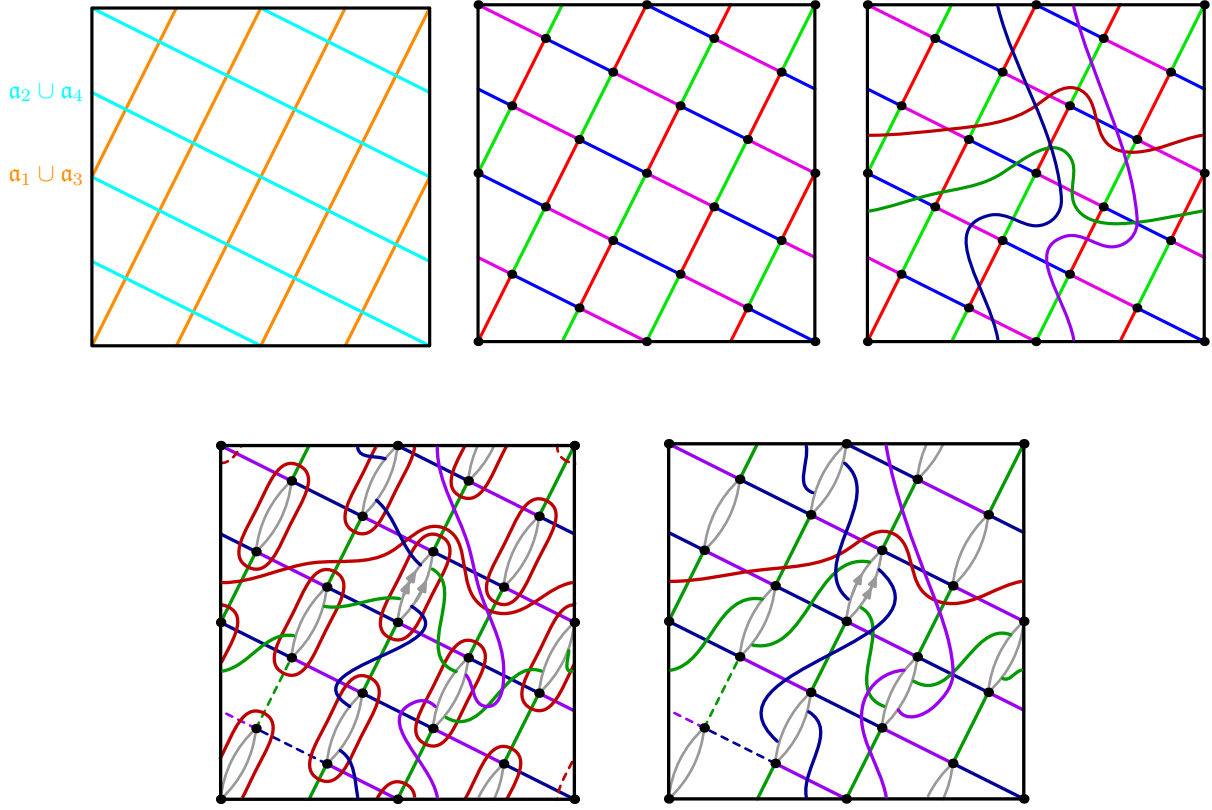


FIGURE 8. **(Top)** The shadow diagram corresponding to $(p, q) = (4, 4)$. **(Bottom)** An $(11, 0)$ 4-section diagram for the $K3$ surface.

Let X be a closed, smooth oriented 4-manifold with \mathfrak{M} a genus g n -section of X with central surface Σ and diagram $(\Sigma; \alpha_1, \dots, \alpha_n)$. Let $X(\mathfrak{M}, r)$ denote the r -fold cyclic branched cover of X over Σ .

Lemma 4.5. *The n -section \mathfrak{M} lifts to a genus g (rn) -section $\widetilde{\mathfrak{M}}$ of $X(\mathfrak{M}, r)$ with diagram $(\Sigma; \alpha_1, \dots, \alpha_n, \dots, \alpha_1, \dots, \alpha_n)$.*

Furthermore, if \mathfrak{M} is efficient, then $\widetilde{\mathfrak{M}}$ is efficient as well.

Proof. The central surface Σ bounds the handlebody H_1 determined by the cut-system α_1 . Therefore, to construct the cyclic branched cover, we cut along H_1 , take r copies of $X \setminus H_1$, and glue them cyclically. Each 4-dimensional sector Z_i has r lifts $\tilde{Z}_{i,1}, \dots, \tilde{Z}_{i,r}$. After reindexing the sectors by setting $\tilde{Z}_{i,k} = \tilde{Z}_{n(k-1)+i}$, it is clear that the decomposition $X(\mathfrak{M}, r) = \tilde{Z}_1 \cup \dots \cup \tilde{Z}_{rn}$ is a multisection.

A multisection is efficient if and only if each sector Z_i is diffeomorphic to a 4-ball, a property which clearly lifts to the cyclic cover. \square

In some simple cases, we can completely determine the diffeomorphism type of the cyclic branched cover over the central surface.

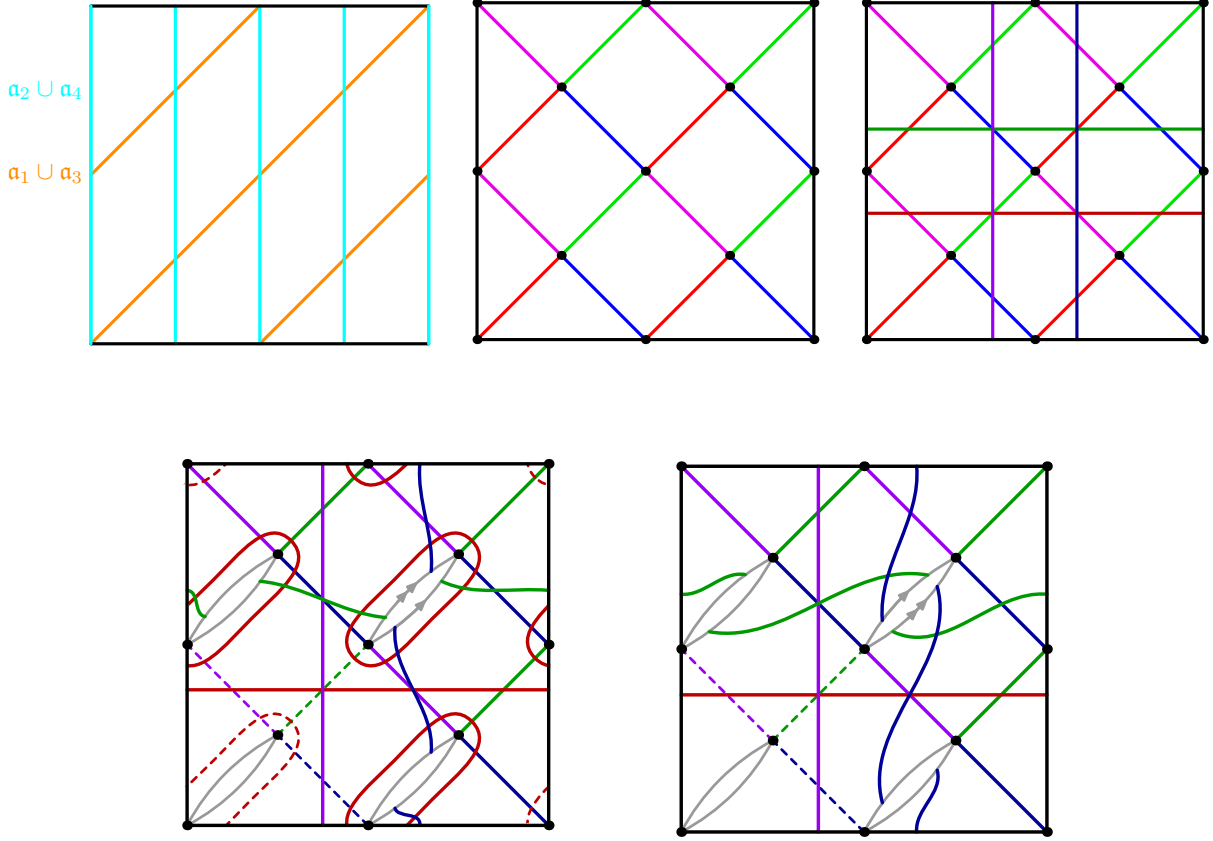


FIGURE 9. **(Top)** The shadow diagram corresponding to $(p, q) = (4, 2)$. **(Bottom)** A $(5, 0)$ 4-section diagram for the elliptic surface $E(1)$.

Proposition 4.6. *Suppose that \mathfrak{T} is a $(g, 0)$ -trisection of X . Then*

$$X(\mathfrak{T}, r) \cong (\#^r X) \# \left(\#^{g(r-1)} (S^2 \times S^2) \right).$$

To prove this, we need the following lemma.

Lemma 4.7. *Let α, β be a pair of geometrically dual cut-systems on a genus g surface. Then the tuple $(\alpha, \beta, \alpha, \beta)$ gives a 4-section of $\#^g (S^2 \times S^2)$.*

Proof. Since α, β are geometrically dual, we can completely decompose the genus g 4-section into the connected sum of g toric 4-sections.

Now assume that $g = 1$. Since α, β are geometrically dual, we can immediately identify this 4-section with a 4-section of $S^2 \times S^2$. \square

Proof of Proposition 4.6. Consider the union $Z_i \cup Z_{i+1}$ of two successive sectors of the multisection $X(\mathfrak{T}, r)$, which has diagram

$$(\Sigma; \alpha, \beta, \gamma, \dots, \alpha, \beta, \gamma).$$

This is a bisection of a compact 4-manifold with boundary and is determined by a triple of cut-systems (α, β, γ) . Since \mathfrak{T} was a $(g, 0)$ -trisection, this is a bisection of $X \setminus B^4$. We can remove a connected summand a copy of X from $X(\mathfrak{T}, r)$ by replacing $Z_i \cup Z_{i+1}$ with a 4-ball bounded by $H_\alpha \cup \overline{H}_\gamma \cong S^3$. Note that this is equivalent to decreasing the number of sectors of the multisection

$X(\mathfrak{T}, r)$ by one; moreover we have replaced the subsequence (α, β, γ) in the multisection diagram by the subsequence (α, γ) .

Inductively, we can remove each β from the multisection diagram. The result is a connected sum decomposition

$$X(\mathfrak{T}, r) \cong (\#^r X) \# Y,$$

where Y has a $2r$ -section with diagram $(\Sigma; \alpha, \gamma, \dots, \alpha, \gamma)$.

Now consider the subsequence $(\alpha, \gamma, \alpha, \gamma)$, which is a multisection diagram for the union of three successive sectors of the multisection. By the previous lemma, it is a diagram for $\#^g (S^2 \times S^2) \setminus B^4$. We can remove these three sectors and replace them with a B^4 . Consequently, we have replaced the subsequence $(\alpha, \gamma, \alpha, \gamma)$ with the subsequence (α, γ) .

Inductively, we can repeat this $r - 2$ times, until we are left with a 4-section with diagram $(\Sigma; \alpha, \gamma, \alpha, \gamma)$, which specifies $\#^g (S^2 \times S^2)$. \square

5. TORIC MULTISECTIONS

5.1. Toric multisections and effective torus actions. The purpose of this section is to prove the following theorem, which shows how toric multisections fit into the classical picture of the classification of simply-connected 4-manifolds admitting effective torus-actions, as illuminated by Orlik and Raymond [OR70] and Melvin [Mel82]. We refer the reader to [Mel82] for definitions and complete details.

Recall that the *Farey graph* F is the graph where

- (1) vertices are rational numbers p/q ,
- (2) an edge connects a/b to p/q if $aq - bp = \pm 1$.

We regard F as embedded in the unit disk; see Figure 12.

If we consider a point in the Farey graph as an ordered pair of coprime integers using the obvious isomorphism $p/q \rightarrow (p, q)$, then any two points in the Farey graph which are connected by an edge form a basis for $\mathbb{Z} \oplus \mathbb{Z}$. As the group $SL(2, \mathbb{Z})$ acts transitively on such bases, $SL(2, \mathbb{Z})$ acts transitively on the vertices and edges of the Farey graph (See also [Mel82]). Following Melvin, we call two loops $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha' = (\alpha'_1, \dots, \alpha'_n)$ in the Farey graph *conjugate* if there is some $A \in SL(2, \mathbb{Z})$ such that $\alpha'_i = A\alpha_i$ for all $i = 1, \dots, n$.

The main theorem of this section is a strengthening of [IN20, Proposition 5.5].

Theorem 5.1. *Let X be a closed, simply-connected 4-manifold. Then the following are equivalent.*

- (1) X admits an effective torus action.
- (2) X admits a toric multisection.
- (3) X is diffeomorphic to a connected sum of copies of \mathbb{CP}^2 , $\overline{\mathbb{CP}}^2$, and $S^2 \times S^2$.

Moreover, the following sets of objects are in bijection.

- (4) toric multisections of simply-connected 4-manifolds, up to diffeomorphism
- (5) effective torus actions on simply-connected 4-manifolds, up to equivalence
- (6) loops in the Farey graph, up to conjugation.

Proof. (1) and (3) are equivalent by [OR70]. We will first show how (1) implies (2).

If X has an effective T^2 -action we may consider the orbit space X^* together with the orbit map $\pi: X \rightarrow X^*$. Orlik-Raymond showed that X^* is a 2-disk with boundary consisting of singular orbits and isolated fixed points and interior consisting of principal orbits. In particular, we can

think of X^* as an n -gon, where the vertices are the fixed points, and points interior to the edges of X^* are points with isotropy group isomorphic to S^1 . Melvin describes how the boundary of X^* can be identified with a loop $\alpha = (\alpha_1, \dots, \alpha_n)$ in the Farey graph [Mel82] – i.e., $\alpha_i = b_i/a_i \in \mathbb{Q}$ and $|b_i a_{i+1} - b_{i+1} a_i| = 1$, with indices taken in \mathbb{Z}_n . Precisely, the isotropy subgroup of a point projecting to the interior of the i^{th} edge of the n -gon is the subgroup $G(a_i, b_i) \triangleleft T^2$ isomorphic to S^1 and determined by flowing along the slope b_i/a_i on the torus.

Let Γ be a tree in X^* , with a single vertex v_0 in the interior of X^* , a vertex v_i in the interior of each edge of X^* and an edge e_i connecting v_i to v_0 , for each $i = 1, \dots, n$. Then, $\Sigma = \pi^{-1}(v_0)$ is a torus, $H_i = \pi^{-1}(e_i)$ is a solid torus, with boundary Σ and core $\pi^{-1}(v_i)$. H_i is determined by the fact that the slope $\alpha_i = b_i/a_i$ on Σ bounds a disk in H_i .

Now $\pi^{-1}(X^* \setminus \nu(\Gamma))$ is a collection Z_1, \dots, Z_n . We claim that each Z_i is a 4-ball. To see this, note that the two handlebodies at the boundary of Z_i are solid tori forming a Heegaard splitting of S^3 . The sector of X^* corresponding to Z_i can be filled by smaller parallel copies of the two edges of Γ at the boundary until the edges reach the unique corner point of X^* . The preimage of each pair of handlebodies under π of these edges is a copy of S^3 and the preimage of the corner point is a point. This presents Z_i as $S^3 \times I/S^3 \times 0 = \{pt\} \cong B^4$.

It follows that Γ lifts to give a toric n -section of X . Thus, (1) implies (2); note that n is the number of fixed points of the action. Moreover, the multisection diagram is $(\Sigma; \alpha_1, \dots, \alpha_n)$.

We now describe how (2) implies (1). From the definition of a toric multisection, we have that there is a multisection of X such that

- (1) the central surface Σ is T^2 ,
- (2) each handlebody H_i is a solid torus $D^2 \times S^1$,
- (3) each 4-dimensional sector Z_i is a 4-ball $D^2 \times D^2$.

We will show that the (effective) T^2 -action of the central surface on itself can be extended to an action on H_i and Z_i , and therefore, on all of X . In particular, the T^2 -action respects the multisection decomposition.

If we take coordinates (θ_1, θ_2) on Σ , then the action of T^2 (with coordinates (ψ_1, ψ_2)) on Σ is given by

$$(\psi_1, \psi_2) \cdot (\theta_1, \theta_2) = (\theta_1 + \psi_1, \theta_2 + \psi_2).$$

Taking the radial coordinates (r, θ_1, θ_2) on H_i , the action extends trivially:

$$(\psi_1, \psi_2) \cdot (r, \theta_1, \theta_2) = (r, \theta_1 + \psi_1, \theta_2 + \psi_2).$$

Extending thusly over H_i and H_{i+1} gives an extension of the action to $S^3 = H_i \cup \overline{H}_{i+1} = \partial Z_i$. Finally, we can parametrize $Z_i \cong B^4$ as the bi-disk $D^2 \times D^2$ and extend the action in the obvious way:

$$(\psi_1, \psi_2) \cdot (r_1, \theta_1, r_2, \theta_2) = (r_1, \theta_1 + \psi_1, r_2, \theta_2 + \psi_2).$$

In this way, X admits an effective torus action.

We now extract a bit more information from the circumstances of an effective torus action on a toric multisection.

By definition of a toric multisection, for any pair of adjacent handlebodies H_i and H_{i+1} , the slopes $\alpha_i = b_i/a_i$ and $\alpha_{i+1} = b_{i+1}/a_{i+1}$ determining the handlebodies satisfy $|b_i a_{i+1} - b_{i+1} a_i| = 1$. It follows that the curve α_i is isotopic to the core of H_{i+1} , and vice versa. This shows that the isotropy subgroup of the action for points on the core of H_{i+1} (say) is $G(a_i, b_i)$, since $G(a_i, b_i)$ acts by rotation on α_i and the disk it bounds in H_i , thus fixing the core point-wise. From this, it follows that the origin $(0, \theta_1, 0, \theta_2)$ of Z_i is a fixed point of the action.

The orbit space $\pi(\Sigma) = \Sigma^*$ is a point, since Σ is an orbit, and $\pi(H_i) = H_i^*$ is a closed interval, with $\partial H_i^* = \Sigma^* \sqcup C_i^*$, where C_i is the core of H_i , and C_i^* is decorated with the orbit data b_i/a_i . Therefore, the quotient of the spine by the T^2 -action is a tree T^* consisting of a central, n -valent vertex Σ^* and the n decorated leaves C_i^* . Finally, the orbit space $Z_i^* = \pi(Z_i)$ is a square, with

$$\partial Z_i^* = (H_i^* \cup_{\Sigma^*} H_{i+1}^*) \cup_{C_i^* \sqcup C_{i+1}^*} (E_i^* \cup_{O_i^*} E_{i+1}^*),$$

where O_i^* is the orbit space of the origin of Z_i (a fixed point), and $E_i^* = \pi(E_i)$, where E_i is the cone in Z_i on the core C_i of H_i .

Therefore, $X^* = \pi(X)$ is a union of n -squares, glued cyclically to the edges of the tree T^* . Figure 2 (left) shows this arrangement for $n = 4$ and $X = S^2 \times S^2$. Walking along the boundary of X^* , we meet the leaves of T^* cyclically, and recording their weights, we get the sequence $(b_1/a_1, \dots, b_n/a_n)$, which is the loop in the Farey graph described by Melvin [Mel82]. A multisection \mathfrak{M} is uniquely determined by its diagram $(\Sigma; \alpha_1, \dots, \alpha_n)$, and the diagram of \mathfrak{M} is uniquely specified up to diffeomorphism and cyclic re-indexing, which correspond precisely to conjugacy of loops in the Farey graph. This shows that the sets (4) and (6) are in bijective correspondence. That (5) and (6) are in bijective correspondence is the main result of [Mel82]. \square

Remark 5.2. As discussed in Remarks 2.2 and 2.3, there is a (balanced) toric 2-section for S^4 and (balanced) toric n -sections for $S^1 \times S^3$ for any $n \geq 2$. The development in the proof above apply equally well to these degenerate cases, and we find effective torus actions on S^4 and $S^1 \times S^3$ coming from the multisection structure.

For $X = S^1 \times S^3$, the main difference is that, since all the slopes in the diagram for \mathfrak{M} are the same, the disks E_i get (collectively) replaced by a torus E that intersects each handlebody in its core. It follows that X^* is a 2-disk, with the entire boundary circle E^* labeled with the same slope. Note that $S^1 \times S^3$ also admits effective torus actions that restrict to effective circle actions on S^3 whose ordinary orbits are (p, q) -torus links (which gives S^3 the structure of a Seifert fibered space). The orbit space in this case is the orbifold $S^2(p, q)$, so these actions are not equivalent to the one coming from the $(1, 1)$ multisection.

For $X = S^4$, we simply have that T^* is linear, having only two vertices on ∂X^* .

For these reasons, $S^1 \times S^3$ and S^4 fit into the scheme of this paper (with minor caveat) as the unique manifolds admitting a toric 1-section and 2-section, respectively.

5.2. Toric multisections with boundary. The definition of an n -section naturally extends to manifolds with boundary by simply dropping the requirement in Definition 2.1 that (Σ, H_1, H_n) be a Heegaard splitting of $\#^k(S^1 \times S^2)$. Instead, $\partial X = \overline{H}_1 \cup_{\Sigma} H_n$ will instead form a Heegaard splitting of the boundary of the manifold. Note that in the case of a toric multisection with boundary, the boundary 3-manifold will admit a genus one Heegaard splitting, and so is a lens space, S^3 , or $S^1 \times S^2$. We begin with some examples.

Example 5.3. (*Disk-bundles over S^2*)

The toric multisection diagram

$$(0/1, 1/0, n/1),$$

encodes a 2-section of the disk-bundle over S^2 with Euler number n . See Figure 10 (left).

Example 5.4. (*Dual spheres*)

The toric multisection diagram

$$(0/1, 1/0, p/1, (pq-1)/q),$$

encodes a 3-section of the neighborhood of a dual pair of 2-spheres, with Euler numbers p and q , respectively. See Figure 10 (right).

Both of the previous examples could be interpreted as a linear plumbing of 2-spheres and in fact, all toric multisections with boundary are of this form.

Proposition 5.5. *Let $(\alpha_1, \dots, \alpha_{n+1})$ be a diagram for a toric n -section of a manifold with boundary X , where $\langle \alpha_i, \alpha_{i+1} \rangle = 1$ for all $i = 1, \dots, n$. (In particular, we view all α_i as oriented). Then X is diffeomorphic to the linear plumbing of $(n - 1)$ 2-spheres, where the Euler number e_i of the i^{th} 2-sphere is given by the formula*

$$e_i = \langle \alpha_{i+2}, \alpha_i \rangle.$$

Proof. First, we will check that the union of two consecutive sectors $Z_i \cup Z_{i+1}$ is a disk bundle over S^2 . Both Z_i and Z_{i+1} are diffeomorphic to B^4 and $H_{i+1} = Z_i \cap Z_{i+1}$ is a tubular neighborhood of the unknot in $S^3 = \partial Z_i$. Therefore, gluing Z_{i+1} to Z_i is equivalent to attaching a 4-dimensional 2-handle to Z_i along an unknot. The framing here is the surface framing of the curve α_{i+2} and in the Heegaard splitting of Z_i given by the curves α_i and α_{i+1} is the algebraic intersection number $\langle \alpha_{i+2}, \alpha_i \rangle$. Consequently, the result is a disk bundle over S^2 with Euler number determined by the framing of the handle attachment.

The union $Z_{i-1} \cup Z_i \cup Z_{i+1}$ is obtained by gluing $Z_i \cup Z_{i+1}$ to $Z_{i-1} \cup Z_i$, where both components are both disk bundles over S^2 . We can identify $Z_i \cong D^2 \times D^2$ so that $\partial Z_i = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$ is the Heegaard splitting induced by the multisection. Consequently, this identification of the two disk bundles is by definition their plumbing. Inducting over the rest of the sectors, we see that this multisection corresponds to a linear plumbing graph. \square

Proposition 5.5 leads quickly to the following characterization of the intersection form for closed toric multisections.

Proposition 5.6. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection diagram for a closed 4-manifold X . Assume that $\langle \alpha_i, \alpha_{i+1} \rangle = 1$ for all $i = 1, \dots, n - 1$. Then the intersection form Q_X is given by the matrix*

$$\begin{bmatrix} e_1 & 1 & 0 & & \dots & \dots & 0 \\ 1 & e_2 & 1 & & & & \\ 0 & 1 & e_3 & & & & \vdots \\ & & & \ddots & & & \\ \vdots & & & & e_{n-4} & 1 & 0 \\ & & & & 1 & e_{n-3} & 1 \\ 0 & \dots & & 0 & 1 & e_{n-2} \end{bmatrix},$$

where

$$e_i = \langle \alpha_{i+2}, \alpha_i \rangle.$$

Proof. Given a multisection $X = Z_1 \cup \dots \cup Z_n$ of a closed manifold, we get a multisection with boundary of $X \setminus B^4$ given by $Z_1 \cup \dots \cup Z_{n-1}$. By Proposition 5.5, this is a linear plumbing of $(n - 2)$ spheres. The condition $\langle \alpha_i, \alpha_{i+1} \rangle = 1$ determines the orientation on the spheres, so that they intersect sequentially at a positive point. And the integer $e_i = \langle \alpha_{i+1}, \alpha_i \rangle$ is the self-intersection number of the i^{th} sphere. \square

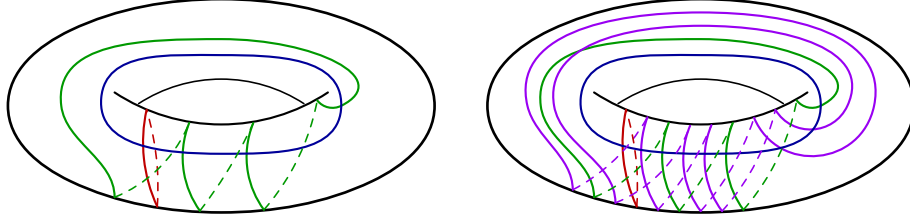


FIGURE 10. (left) A disk bundle over S^2 with Euler number 3. (Right) Plumbing the disk bundle of Euler number 3 with the disk bundle of Euler number 2.

5.3. Blowing up. Two fundamental operations in 4-manifold topology are blowing up and taking connected sums with $S^2 \times S^2$. Under these operations, most of the complexity of smooth simply connected 4-manifolds dissolves. In this subsection we describe how to modify a toric multisection to a multisection of its blow-ups or connected sum with $S^2 \times S^2$ and outline a procedure for the proper transform of a toric bridge multisection.

Lemma 5.7. *In a toric multisection diagram, replacing the subsequence (α_i, α_{i+1}) (viewed as oriented classes in $H_1(T^2)$) with $(\alpha_i, \alpha_i + \alpha_{i+1}, \alpha_{i+1})$ is equivalent to connect summing with*

- (1) \mathbb{CP}^2 if $\langle \alpha_{i+1}, \alpha_i \rangle = 1$,
- (2) $\overline{\mathbb{CP}}^2$ if $\langle \alpha_{i+1}, \alpha_i \rangle = -1$.

Proof. Removing the subsequence (α_i, α_{i+1}) from the multisection diagram, removes a B^4 from the toric multisection. By Proposition 5.5, the toric bisection with boundary given by the curves $(\alpha_i, \alpha_i + \alpha_{i+1}, \alpha_{i+1})$ is a D^2 -bundle over S^2 with Euler number $\langle \alpha_{i+1}, \alpha_i \rangle$. Thus the total process removes a ball and glues in, either $\mathbb{CP}^2 \setminus B^4$ or $\overline{\mathbb{CP}}^2 \setminus B^4$ depending on the given intersection number. \square

Lemma 5.8. *In a toric multisection, replacing the subsequence (α_i, α_{i+1}) with $(\alpha_i, \alpha_{i+1}, \alpha_i, \alpha_{i+1})$ is equivalent to connect summing with $S^2 \times S^2$.*

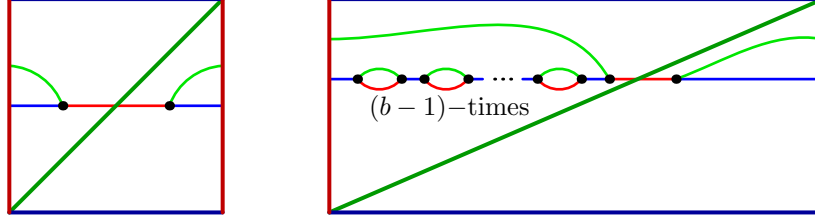
Proof. Following along the lines of Lemma 5.7, this operation removes a B^4 and replaces it with a toric 3-section with boundary of a neighborhood of a plumbing of two 2-spheres, with Euler numbers

$$\langle \alpha_i, \alpha_i \rangle = \langle \alpha_{i+1}, \alpha_{i+1} \rangle = 0.$$

Therefore, the replacement manifold is $(S^2 \times S^2) \setminus B^4$. \square

Recall that, topologically, the proper transform of a 4-manifold/surface pair (X^4, \mathcal{K}) is the 4-manifold/surface pair $(X^4 \# \overline{\mathbb{CP}}^2, \mathcal{K} \# \mathbb{CP}^1)$. On a multisection diagram, this can be accomplished by a relative connected sum operation. Recall that $\mathbb{CP}^1 \subset \overline{\mathbb{CP}}^2$ has a $(1, 1)$ bridge trisection relative to a toric trisection $\overline{\mathbb{CP}}^2 = W_1 \cup W_2 \cup W_3$. This can be seen in Figure 11 (left). We can perturb this bridge trisection to a $(b; 1, c_2, c_3)$ -bridge trisection with $c_2 + c_3 = b + 1$ (shown in the Figure 11 (right) for $c_2 = 1$ and $c_3 = b$). We can also stabilize the $(1, 0)$ trisection to a $(g; 0, k_1, k_2)$ -trisection, where $k_1 + k_2 = g - 1$. Despite these changes, (W_1, \mathcal{S}_1) is still a (B^4, D^2) .

Now let $X = Z_1 \cup \dots \cup Z_n$ be a multisection of the 4-manifold X and let $\mathcal{K} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_n$ be a b -bridge multisection of the surface S with $\mathcal{D}_i \subset Z_i$. Suppose that some pair (Z_i, \mathcal{D}_i) is diffeomorphic as a pair to (W_1, \mathcal{S}_1) , so both pairs give the standard pair (B^4, D^2) . Since S^3 has a unique Heegaard splitting in each genus [Wal68], and since the unknot has a unique b -bridge

FIGURE 11. A b -bridge trisection diagram of $\mathbb{CP}^1 \subset \overline{\mathbb{CP}}^2$.

splitting with respect to these Heegaard splittings [HS98], there is a diffeomorphism

$$\phi : (\partial W_1, \partial \mathcal{S}_1) \rightarrow (\partial Z_i, \partial \mathcal{D}_i),$$

respecting these decompositions.

The blow-up is then given by $X \# \overline{\mathbb{CP}}^2 = (X \setminus Z_i) \cup_\phi (\overline{\mathbb{CP}}^2 \setminus W_1)$. Moreover, when forming the proper transform in this fashion we naturally obtain a bridge multisection given by

$$(X \# \overline{\mathbb{CP}}^2, \mathcal{K} \# \mathbb{CP}^1) = (Z_1 \cup \dots \cup Z_{i-1} \cup W_2 \cup W_3 \cup Z_{i+1} \cup \dots \cup Z_n, \\ \mathcal{D}_1 \cup \dots \cup \mathcal{D}_{i-1} \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{D}_{i+1} \cup \dots \cup \mathcal{D}_n).$$

This technique might be used to describe bridge trisections of complex curves in rational surfaces that are obtained as proper transforms of complex curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ using the bridge trisections described in Sections 3 and 4.

5.4. Classification of toric n -sections. Our next result gives a classification of the diffeomorphism types of 4-manifolds which admit a toric multisection. We will later show in Section 6 that we can precisely determine the smooth 4-manifold underlying a toric multisection using algebraic information.

Theorem 5.9. *If X admits a toric n -section, then there is a diffeomorphism*

$$X \cong (\#^a \mathbb{CP}^2) \# (\#^b \overline{\mathbb{CP}}^2) \# (\#^c S^2 \times S^2),$$

for some integers a, b, c satisfying $a + b + 2c = n - 2$. Moreover, the connected sum decomposition respects the multisection and the corresponding torus-action.

Proof. Let \mathfrak{M} be a toric n -section of a 4-manifold X , with $n \geq 2$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be the corresponding loop in the Farey graph, described in the proof of Theorem 5.1. We will prove the theorem by induction on n . If $n = 2$, then $X \cong S^4$ and the theorem holds. If $n = 3$, then X is diffeomorphic to \mathbb{CP}^2 or $\overline{\mathbb{CP}}^2$. Assume the theorem is true for any $3 \leq k \leq n - 1$.

First, assume that α backtracks at some point: $\alpha_i = \alpha_{i+2}$. Without loss of generality (re-indexing and applying an automorphism if necessary), we can assume α has the form

$$\alpha = (0/1, 1/0, 0/1, 1/m, \alpha_5, \dots, \alpha_n),$$

where $m \in \mathbb{Z}$. Note that $H_1 \cup \overline{H}_4$ is a Heegaard splitting of S^3 , so that it may be capped off with B^4 to form a closed 4-section which is a connected summand of our original toric multisection. By Proposition 5.1 of [IN20] this manifold is $S^2 \times S^2$ if m is even and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ if m is odd.

Let $X' = (X \setminus (Z_1 \cup Z_2 \cup Z_3)) \cup Z'$, where $Z' \cong B^4$ and let \mathfrak{M}' be the $(n-2)$ -section of X' obtained by replacing $Z_1 \cup Z_2 \cup Z_3$ with Z' . By the inductive hypothesis, the theorem is true for X' and \mathfrak{M}' , so $X' = (\#^a \mathbb{CP}^2) \# (\#^b \overline{\mathbb{CP}}^2) \# (\#^{c'} S^2 \times S^2)$, with the connected sum decomposition and the (induced) T^2 -action respecting \mathfrak{M}' . From this it follows that $X \cong (\#^a \mathbb{CP}^2) \# (\#^b \overline{\mathbb{CP}}^2) \# (\#^c S^2 \times S^2)$, with

$c = c' + 1$, and the theorem holds for X . Note that this connected sum decomposition respects the multisection in the sense that the $S^2 \times S^2 \setminus B^4$ is a union of sectors, giving it a 4-section. Moreover, the torus action restricts to a torus action on the spine, so it respects this decomposition.

Next, assume that α doesn't backtrack. In this case, α contains a circuit (an embedded loop of edges). (The following elegant argument is due to Melvin.) Any circuit in the Farey graph bounds a triangulated disk. Consider the interior edge ε of this triangulation whose distance is shortest in terms of the Euclidean metric (applied to the disk on which the Farey graph lives). Then, ε co-bounds a triangle with two edges of the circuit. Without loss of generality, assume the first two vertices of the triangle are $\alpha_1 = 0/1$ and $\alpha_2 = 1/0$. Then, $\alpha_3 = \pm 1/1$, and $Z_1 \cup Z_2$ is diffeomorphic to $(\mathbb{CP}^2) \setminus B^4$ or $(\overline{\mathbb{CP}}^2) \setminus B^4$, respectively, by Lemma 5.7. In either case, we can remove $Z_1 \cup Z_2$ from the multisection, replacing them with $Z' \cong B^4$, as in the first part of the proof. The proof is completed by the inductive hypothesis just as before, with $a = a' + 1$ or $b = b' + 1$, depending on the case. Just as before the decomposition respects the multisection (in the sense that punctured projective plane is the union of two sectors, giving it a 2-section). And again, the torus action restricts to a torus action on the spine, so it respects this decomposition. \square

The above proof also gives the following proposition, which is useful in its own right.

Proposition 5.10. *Every loop in the Farey graph with $n \geq 3$ edges is conjugate to a loop $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $(\alpha_1, \alpha_2, \alpha_3)$ has one of the following forms:*

- (1) $(0/1, 1/0, 1/1)$,
- (2) $(0/1, 1/0, -1/1)$, or
- (3) $(0/1, 1/0, 0/1)$.

In light of the techniques of the previous two proofs, we can apply [Mel82, Theorem 2] to the class of toric multisections.

Corollary 5.11. *A 4-manifold X admits finitely many toric $(n+2)$ -sections if and only if either $X \cong \#^n \mathbb{CP}^2$ or $X \cong \#^n \overline{\mathbb{CP}}^2$ – i.e., if and only if X is definite.*

Proof. This follows from [Mel82, Theorem 2], combined with Theorem 5.1 above. The equivalence with the statement that X is definite is Donaldson's Diagonalization Theorem [Don83]. \square

Example 5.12. The 4-manifold $\#^n \mathbb{CP}^2$ (or its mirror) admits a unique $(n+2)$ -section if and only if $n \leq 3$. In contrast, if $n = 4, 5$ or 6 , then $\#^4 \mathbb{CP}^2$, $\#^5 \mathbb{CP}^2$, and $\#^6 \mathbb{CP}^2$ admit 3, 4, and 12 non-diffeomorphic $(n+2)$ -sections, respectively. For $n = 4$, diagrams for the multisections are given by the tuples

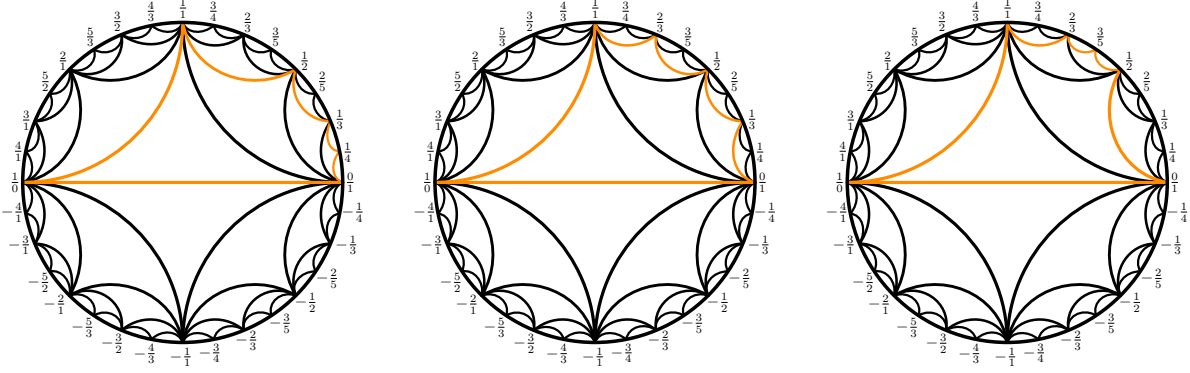
- $(0/1, 1/0, 1/1, 1/2, 1/3, 1/4)$,
- $(0/1, 1/0, 1/1, 2/3, 1/2, 1/3)$, and
- $(0/1, 1/0, 1/1, 2/3, 3/5, 1/2)$,

which are shown as circuits in the Farey graph in Figure 12.

The 4-manifolds $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ each admit infinitely many distinct 4-sections, the diagrams of which are given by the 4-tuples

$$(0/1, 1/0, 0/1, 1/m),$$

with $m \in \mathbb{Z}$ and even values of m give $\mathbb{CP}^1 \times \mathbb{CP}^1$ (See [IN20] Proposition 5.1). Algebraically, these correspond to the infinite family of Hirzebruch surfaces and the corresponding T^2 -actions come from the Kähler toric structures on these manifolds.

FIGURE 12. Diagrams for the three toric multisections of $\#^4\mathbb{CP}^2$.

6. ALGEBRAIC TOPOLOGY OF TORIC MULTISECTIONS

In this section we will give formulas to calculate the algebraic topology of toric multisections. In light of Theorem 5.1, these invariants are sufficient to determine the diffeomorphism type of the underlying 4-manifold. Namely, the manifolds admitting toric multisections are determined by their Euler characteristic, whether or not they are spin, and their signature. The Euler characteristic of a toric n -section can easily be computed to be $n + 2$. We are also able to quickly determine if the manifold underlying a toric multisection is spin.

Proposition 6.1. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection diagram for X . Suppose that*

$$\alpha_1 = (1, 0), \quad \alpha_2 = (0, 1), \quad \alpha_i = (a_i, b_i).$$

Then X admits a spin structure if and only if $a_i b_i \equiv 0 \pmod{2}$ for all $i = 3, \dots, n$.

Proof. Recall that a spin structure can be interpreted as a trivialization of TX along the 1-skeleton that extends across the 2-skeleton. Given a multisection of X , we can construct a handle decomposition such that the 2-skeleton consists of T^2 plus 2-handles attached along each α_i .

There is a unique spin structure on T^2 that extends across the 2-handles attached along α_1 and α_2 . Extentability is measured by a quadratic enhancement $q: H_1(T^2; \mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ which satisfies the formula

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle \pmod{2}.$$

For our chosen spin structure, we have that

$$q(\alpha_1) = q(a_i \alpha_1) = q(\alpha_2) = q(b_i \alpha_2) = 0.$$

Therefore,

$$q(a_i \alpha_1 + b_i \alpha_2) = \langle a_i \alpha_1, b_i \alpha_2 \rangle = a_i b_i \pmod{2}.$$

□

Note that the proof of Theorem 5.9 and Proposition 6.1 shows that X is spin if and only if the path α in the Farey graph is contained in a tree, since otherwise, the path would traverse two edges of a triangle, one vertex of which would violate the above proposition.

6.1. Maslov index and the signature. Following along the lines of [GK16] and [IK19], we will determine the signature of the 4-manifold underlying a toric multisection using Wall's nonadditivity of the signature [Wal69]. As this involves a Maslov index of Lagrangians, we begin with a discussion of this invariant. Suppose V is a finite dimensional real vector space, and $\omega: V \times V \rightarrow \mathbb{R}$ is a non-singular symplectic form. A half-dimensional subspace $L \subset V$ is called a *Lagrangian*, if it is maximally isotropic – i.e.,

$$L = \{x \in V \mid \omega(x, y) = 0, \text{ for every } y \in L\}.$$

For any three Lagrangians L_1, L_2 , and L_3 , define a symmetric form

$$\theta: L_1 \oplus L_2 \oplus L_3 \times L_1 \oplus L_2 \oplus L_3 \rightarrow \mathbb{R},$$

by

$$\theta((x_1, x_2, x_3), (y_1, y_2, y_3)) = \omega(x_2, y_1) + \omega(y_2, x_1) + \omega(x_3, y_2) + \omega(y_3, x_2) + \omega(x_1, y_3) + \omega(y_1, x_3)$$

Then the *Maslov index*, $\mu(L_1, L_2, L_3)$, is defined to be the signature of θ .

In particular, if $V = \mathbb{R}^2$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a triple of basis vectors for L_1, L_2 , and L_3 , then the symmetric form θ is represented by the matrix

$$\begin{bmatrix} 0 & \omega(\mathbf{v}_2, \mathbf{v}_1) & \omega(\mathbf{v}_1, \mathbf{v}_3) \\ \omega(\mathbf{v}_2, \mathbf{v}_1) & 0 & \omega(\mathbf{v}_3, \mathbf{v}_2) \\ \omega(\mathbf{v}_1, \mathbf{v}_3) & \omega(\mathbf{v}_3, \mathbf{v}_2) & 0 \end{bmatrix}.$$

We extend the Maslov index to n -tuples inductively by setting

$$\mu(L_1, \dots, L_n, L_{n+1}) = \mu(L_1, \dots, L_n) + \mu(L_1, L_n, L_{n+1}).$$

Lemma 6.2. *Let L_1, L_2 , and L_3 be a triple of Lagrangians in \mathbb{R}^2 and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a triple of basis vectors for the three Lagrangians. Define*

$$\tau(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \det([\mathbf{v}_1 \ \mathbf{v}_2]) \cdot \det([\mathbf{v}_2 \ \mathbf{v}_3]) \cdot \det([\mathbf{v}_3 \ \mathbf{v}_1]).$$

Then

$$\mu(L_1, L_2, L_3) = \begin{cases} 1 & \text{if } \tau > 0, \\ -1 & \text{if } \tau < 0, \\ 0 & \text{if } \tau = 0. \end{cases}$$

In particular, if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} p \\ q \end{bmatrix},$$

then

$$\mu(L_1, L_2, L_3) = \begin{cases} 1 & \text{if } bq(aq - bp) < 0, \\ -1 & \text{if } bq(aq - bp) > 0, \\ 0 & \text{if } bq = 0. \end{cases}$$

Proof. If $L_1 = L_2$ and therefore, $\det([\mathbf{v}_1 \ \mathbf{v}_2]) = 0$, then $\omega(\mathbf{v}_1, \mathbf{v}_2) = 0$, and the form θ is isomorphic to the form $\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \oplus \langle 0 \rangle$ for some c , which has signature 0. The same holds by any cyclic permutation of the triple $\{L_1, L_2, L_3\}$. This covers all the cases where $\tau = 0$.

Now suppose that no pair of Lagrangians agree. The formula for τ does not change when \mathbf{v}_2 is replaced by $-\mathbf{v}_2$. Therefore, we can assume $\det([\mathbf{v}_1 \ \mathbf{v}_2]) > 0$. The Maslov index is invariant under symplectic equivalence, which are precisely the area-preserving linear transformations on

\mathbb{R}^2 . Consequently, by a rotation and a shear map, followed by scaling $\mathbf{v}_1, \mathbf{v}_2$, we can assume that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $\mathbf{v}_3 = \begin{bmatrix} r \\ s \end{bmatrix}$, then the form θ can be represented by the matrix

$$\begin{bmatrix} 0 & -1 & s \\ -1 & 0 & r \\ s & r & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2rs \end{bmatrix},$$

and $\tau = rs$. The signature of θ is precisely the sign of $2rs$, which is also the sign of τ . \square

We next show how the signature of a toric multisection can be calculated using a Maslov index determined by the curves in a multisection diagram. Namely, each curve in a multisection diagram determines a Lagrangian subspace of symplectic vector space $H_1(T^2; \mathbb{R}) = \mathbb{R}^2$ (equipped with the symplectic form given by the intersection form) given by the span of the curve in homology. The signature will be the Maslov index of these spaces. In the following proposition we denote by α_i both the curve on T^2 as well as the Lagrangian subspace spanned by $[\alpha_i] \in H_1(T^2; \mathbb{R})$.

Proposition 6.3. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection diagram for a smooth 4-manifold X . Then*

$$\sigma(X) = \mu(\alpha_1, \dots, \alpha_n).$$

Proof. If X is closed, then $X \setminus B^4$ has the same signature as X and admits a toric multisection determined by the same α_i . So, it suffices to prove the proposition in the case that X has non-empty boundary.

The proof is by induction on the number of sectors, n . The base case is $n = 2$, in which $\alpha_1 = (1, 0), \alpha_2 = (0, 1), \alpha_3 = (\pm 1, 1)$ representing $X = \mathbb{CP}^2 \setminus B^4$ or $\overline{\mathbb{CP}^2} \setminus B^4$. A quick computation using Lemma 6.2 verifies that in this case $\sigma(X) = \mu(\alpha_1, \alpha_2, \alpha_3)$.

Now assume the result is true for $k < n$, and let $(\alpha_1, \dots, \alpha_n)$ be a diagram for X . Let X' be the result of removing the interior of Z_{n-1} , so $\partial X' = H_1 \cup H_{n-1}$. (Note: there is no Z_n , since X has non-empty boundary.) We have $X = X' \cup_{H_{n-1}} Z_{n-1}$, and by Wall's nonadditivity of signature [Wal69],

$$(1) \quad \sigma(X) = \sigma(X') + \sigma(Z_{n-1}) - \mu(\alpha_n, \alpha_{n-1}, \alpha_1) = \sigma(X') + \mu(\alpha_1, \alpha_{n-1}, \alpha_n).$$

By the inductive hypothesis, we have $\sigma(X') = \mu(\alpha_1, \dots, \alpha_{n-1})$, so the right side is $\mu(\alpha_1, \dots, \alpha_n)$, by the definition of Maslov index. \square

Combining Lemma 6.2 and Proposition 6.3, we can compute the signature directly from the multisection diagram.

Lemma 6.4. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection diagram for X , with $\alpha_1 = (1, 0)$ and $\alpha_i = (a_i, b_i)$ for $2 \leq i \leq n$. Let k_+ be the set of indices i for $2 \leq i \leq n-1$, where b_i and b_{i+1} are nonzero and $\frac{a_i}{b_i} < \frac{a_{i+1}}{b_{i+1}}$; similarly define k_- to be the set of indices i for $2 \leq i \leq n-1$, where b_i and $b_{i+1} + 1$ are nonzero and $\frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1} + 1}$. Then $\sigma(X) = k_+ - k_-$.*

Proof. By Proposition 6.3, we have that

$$\sigma(X) = \mu(\alpha_1, \dots, \alpha_n) = \sum_{i=2}^{n-2} \mu(\alpha_1, \alpha_i, \alpha_{i+1}) = \sum_{i=2}^{n-1} \mu \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} \right).$$

We can assume that $b_i, b_{i+1} \geq 0$ without changing the Maslov index and without changing the quotients $\frac{a_i}{b_i}$ and $\frac{a_{i+1}}{b_{i+1}}$. Also, recall that by assumption, $a_i b_{i+1} - a_{i+1} b_i = \pm 1 \neq 0$. Therefore, we

can apply the formula of Lemma 6.2 to obtain

$$\mu \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} \right) = \begin{cases} 0 & \text{if } b_i \cdot b_{i+1} = 0, \\ 1 & \text{if } b_i \cdot b_{i+1} > 0 \text{ and } a_i b_{i+1} - a_{i+1} b_i < 0, \\ -1 & \text{if } b_i \cdot b_{i+1} > 0 \text{ and } a_i b_{i+1} - a_{i+1} b_i > 0. \end{cases}$$

Summing over all $i = 2, \dots, n-1$ yields the signature formula. \square

Theorem 6.5. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection of X with*

$$\alpha_1 = (1, 0), \quad \alpha_2 = (0, 1), \quad \alpha_i = (a_i, b_i).$$

Then

- (1) *If $a_i b_i = 0 \pmod{2}$ for all $i = 1, \dots, n$, then X is diffeomorphic to $\#^{\frac{n}{2}-1}(S^2 \times S^2)$.*
- (2) *Otherwise, X is diffeomorphic to $(\#^r \mathbb{CP}^2) \# (\#^s \overline{\mathbb{CP}}^2)$, where*

$$\begin{aligned} r &= \frac{n}{2} - 1 + \frac{1}{2}\sigma(X), \\ s &= \frac{n}{2} - 1 - \frac{1}{2}\sigma(X), \end{aligned}$$

where r and s can be calculated by Lemma 6.4.

Proof. By Theorem 5.1, we know that

$$X \cong (\#^a \mathbb{CP}^2) \# (\#^b \overline{\mathbb{CP}}^2) \# (\#^c S^2 \times S^2).$$

From this decomposition, it is clear that X is spin if and only if $a = b = 0$. According to Proposition 6.1, a toric multisection is spin if and only if $a_i b_i = 0 \pmod{2}$ for all $i = 1, \dots, n$. Thus, X is diffeomorphic to several copies of $S^2 \times S^2$ if and only if its multisection diagram satisfies this condition.

Now suppose that $a + b > 0$. Recall that

$$\begin{aligned} \mathbb{CP}^2 \# S^2 \times S^2 &\cong (\#^2 \mathbb{CP}^2) \# \overline{\mathbb{CP}}^2, \\ \overline{\mathbb{CP}}^2 \# S^2 \times S^2 &\cong \mathbb{CP}^2 \# (\#^2 \overline{\mathbb{CP}}^2). \end{aligned}$$

Therefore, if a or b is nonzero, we can replace the parameters (a, b, c) with $(a + c, b + c, 0)$. In this case, $r = a + c = b_2^+(X)$ and $s = b + c = b_2^-(X)$. Therefore, we have

$$\begin{aligned} r + s &= n - 2, \\ r - s &= \sigma(X), \end{aligned}$$

so that calculating the signature via Lemma 6.4 allows us to solve for r and s . \square

6.2. The extended Farey graph and almost complex structures. It follows from the Wu formula that a simply-connected 4-manifold X admits an almost-complex structure J if and only if $b_2^+(X)$ is odd; see Exercise 1.4.16(b) of [GS99]. Here, we express the condition that $b_2^+(X)$ is odd in terms of loops in the Farey graph.

We start by introducing an extension of the Farey graph \tilde{F} . Loops in the extended Farey graph will correspond to almost-complex 4-manifolds admitting toric multisections.

Definition 6.6. The extended Farey graph \tilde{F} is the directed graph where

- (1) vertices consist of primitive elements $(p, q) \in \mathbb{Z}^2$,

(2) there is an oriented edge from (a, b) to (p, q) if

$$\det \begin{bmatrix} a & p \\ b & q \end{bmatrix} = 1.$$

In particular, each vertex p/q of the Farey graph lifts to two vertices (p, q) and $(-p, -q)$ of \tilde{F} and each edge of the Farey graph lifts to four edges.

There is a map $\pi: \tilde{F} \rightarrow F$ that sends edges to edges and vertices to vertices. However, note that π is not a 2-to-1 covering map. Nonetheless, loops in \tilde{F} map to loops in F and the following lemma shows that “half” the loops in F lift to loops in \tilde{F} .

Lemma 6.7. *Every oriented path γ in the Farey graph F starting at $1/0$ lifts to a unique, oriented path $\tilde{\gamma}$ in \tilde{F} starting at $(1, 0)$.*

Proof. To prove the statement, we use induction on the edge-length of γ . Suppose that γ has length 1, consisting of one edge connecting $1/0$ to $p/1$. The rational number $p/1$ has two lifts $(p, 1)$ and $(-p, -1)$ in \tilde{F} . Lift this edge in F to the edge in \tilde{F} connecting $(1, 0)$ to $(p, 1)$.

Now suppose $\gamma = \gamma' * e$, where γ' is a path of length $n - 1$ and e is an edge connecting p/q to r/s . By induction, γ' has a lift to \tilde{F} that ends at either (p, q) or $(-p, -q)$; without loss of generality we can assume it is (p, q) . The two lifts of r/s to \tilde{F} are (r, s) and $(-r, -s)$. If $ps - qr = 1$, there is a directed edge in \tilde{F} from (p, q) to (r, s) , which is the required lift of e . Otherwise, there is a directed edge in \tilde{F} from (p, q) to $(-r, -s)$, which is the required lift of e . \square

Proposition 6.8. *Let $(\alpha_1, \dots, \alpha_n)$ be a toric multisection diagram for X . Then $b_2^+(X)$ is odd if and only if the loop in F lifts to a loop in \tilde{F} .*

Proof. The multisection diagram determines a path from α_1 to α_n in the Farey graph. This lifts to a path in \tilde{F} by Lemma 6.7. This lift corresponds to possibly replacing α_i with $-\alpha_i$ for i in some subset of $\{1, \dots, n\}$. Equivalently, we can assume that the intersection pairing satisfies $\langle \alpha_i, \alpha_{i+1} \rangle = 1$ for $i = 1, \dots, n - 1$. The closed loop lifts to a loop in \tilde{F} if and only if there is a directed edge from α_n to α_1 , which in terms of the intersection pairing is equivalent to $\langle \alpha_n, \alpha_1 \rangle = 1$.

Recall that we can assume $\alpha_1 = (1, 0)$. Let $\alpha_2 = (a_2, b_2)$. Then by the assumption we must have that $\langle \alpha_1, \alpha_2 \rangle = b_2 > 0$. Further, if $\alpha_n = (a_n, b_n)$, then the loop lifts if and only if $\langle \alpha_n, \alpha_1 \rangle = -b_n > 0$. More generally, suppose that $b_i = 0$. Then $a_i = \pm 1$. In the first case, α_i fits into the subsequence:

$$(a_{i-1}, b_{i-1}), (1, 0), (a_{i+1}, b_{i+1}),$$

with $b_{i-1} < 0 < b_{i+1}$. In the latter, α_i fits into the subsequence

$$(a_{i-1}, b_{i-1}), (-1, 0), (a_{i+1}, b_{i+1}),$$

with $b_{i-1} > 0 > b_{i+1}$. Let k_0 denote the number of slopes (including $i = 1$) with $b_i = 0$.

Consider a sequential pair $\alpha_i = (a_i, b_i)$ and $\alpha_{i+1} = (a_{i+1}, b_{i+1})$. The contribution of the pair to $\sigma(X)$ is the Maslov index of the triple $(\alpha_1, \alpha_i, \alpha_{i+1})$. Since $\langle \alpha_i, \alpha_{i+1} \rangle = 1$, it follows from Lemma 6.2 that

$$(2) \quad \mu(\alpha_1, \alpha_i, \alpha_{i+1}) = \begin{cases} 1 & \text{if } b_i b_{i+1} < 0, \\ -1 & \text{if } b_i b_{i+1} > 0, \\ 0 & \text{if } b_i b_{i+1} = 0. \end{cases}$$

Let k_{\pm} denote the number of pairs (α_i, α_{i+1}) such that $\mu(\alpha_1, \alpha_i, \alpha_{i+1}) = \pm 1$ and k_0 be the number of slopes with $b_i = 0$. Then

$$\begin{aligned}\sigma(X) &= k_+ - k_-, \\ b_2(X) &= 2k_0 + k_+ + k_- - 2, \\ b_2^+(X) &= \frac{1}{2}(b_2(X) + \sigma(X)) = k_0 + k_+ - 1, \\ b_2^-(X) &= \frac{1}{2}(b_2(X) - \sigma(X)) = k_0 + k_- - 1.\end{aligned}$$

Now, the coefficient b_i changes sign or becomes zero exactly $(k_0 + k_+)$ -times (once for each $\alpha_i = (\pm 1, 0)$ and once for each edge with $b_i b_{i+1} < 0$). As these slopes can be consistently oriented to ensure that each intersection number is positive, we have that the sign must change an even number of times. Therefore,

$$k_0 + k_+ = b_2^+(X) + 1 = 0 \pmod{2}.$$

□

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