

AN ATLAS FOR THE PINHOLE CAMERA

SAMEER AGARWAL, TIMOTHY DUFF, MAX LIEBLICH, AND REKHA R. THOMAS

Abstract. We introduce an atlas of algebro-geometric objects associated with image formation in pinhole cameras. The nodes of the atlas are algebraic varieties or their vanishing ideals related to each other by projection or elimination and restriction or specialization respectively. This atlas offers a unifying framework for the study of problems in 3D computer vision. We initiate the study of the atlas by completely characterizing a part of the atlas stemming from the triangulation problem. We conclude with several open problems and generalizations of the atlas.

1. Introduction

The standard model of a pinhole camera (also known as a *projective camera*) in computer vision is a surjective linear projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Such a map is specified by a 3×4 -matrix A of rank 3 up to scaling, which can be realized as a point in \mathbb{P}^{11} . This map sends a *world point* q in \mathbb{P}^3 , which can be realized as a 4-vector up to scale, to its image point $p \in \mathbb{P}^2$, which can be realized as a 3-vector up to scale. The center of the projection map, also called the *camera center*, can be identified with the kernel of the representing matrix A . If we wish to express that a point q is sent to the image p by the camera A , we will write $Aq \sim p$ to indicate equality up to scale.

Much work has been done on the algebraic varieties obtained from a *fixed* arrangement of cameras $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$ ¹ by taking the closed image of the rational imaging map

$$\begin{aligned} \varphi_{\bar{\mathbf{A}}} : \mathbb{P}^3 &\dashrightarrow (\mathbb{P}^2)^m \\ q &\mapsto (\bar{A}_1 q, \dots, \bar{A}_m q). \end{aligned} \tag{1.1}$$

Such a variety is known as a *multiview variety* [2, 5]. Several closely-related sets have been studied in computer vision: for example, the names *joint image* and *natural descriptor* for the constructible set $\text{im } \varphi_{\bar{\mathbf{A}}}$ were coined, respectively, by Triggs [61] and Heyden and Åström [27]. Much previous work has also gone into studying various polynomials that vanish on the multiview variety, eg. [20, 27, 38, 59, 61]. The associated *multiview ideals* have been calculated in [2, 5], and their moduli have been considered in [5, 34].

¹To distinguish between known and unknown quantities, we use a bar over an object to indicate specialization. For instance, A stands for a symbolic 3×4 matrix denoting a camera while \bar{A} is a 3×4 scalar matrix realizing a camera. We also use bold face letters to indicate collections. For instance, we use \mathbf{A} and $\bar{\mathbf{A}}$ to specify a collection of symbolic and scalar cameras respectively.

Consider now the universal incarnation of this problem where we treat the cameras and the world points as unknowns. In particular, one can think about the universal imaging map

$$\begin{aligned} \mathbb{P}^{11} \times \mathbb{P}^3 &\dashrightarrow \mathbb{P}^2 \\ (A, q) &\mapsto Aq \end{aligned} \tag{1.2}$$

that sends a pair (A, q) to Aq . This map describes the image of one unknown point by one unknown camera. More generally, fixing integers $m, n \geq 1$ there is a map

$$(\mathbb{P}^{11})^m \times (\mathbb{P}^3)^n \dashrightarrow (\mathbb{P}^2)^{mn} \tag{1.3}$$

which, where defined, associates m unknown cameras $A_1, \dots, A_m \in \mathbb{P}^{11}$ and n unknown world points $q_1, \dots, q_n \in \mathbb{P}^3$ to their mn images $A_i q_j \sim p_{ij} \in \mathbb{P}^2$.²

For any nonempty Zariski-open $U \subset (\mathbb{P}^{11})^m \times (\mathbb{P}^3)^n$ where the imaging map (1.3) is defined, we can consider the Zariski closure of the graph of the imaging map restricted³ to U :

$$\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n} = \text{cl} \{ ((\bar{\mathbf{A}}, \bar{\mathbf{q}}), \bar{\mathbf{p}}) \in U \times (\mathbb{P}^2)^{mn} \mid \bar{A}_i \bar{q}_j \sim \bar{p}_{ij} \forall i = 1, \dots, m, j = 1, \dots, n \}. \tag{1.4}$$

The set of polynomials vanishing on all points of $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ can be understood as the set of all constraints that must be satisfied for any valid geometry $(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$. This set forms an ideal in the polynomial ring $\mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}]$, generated by polynomials which are homogeneous in each of the $m + n + mn$ groups of variables $A_1, \dots, A_m, q_1, \dots, q_n, p_{11}, \dots, p_{mn}$.

Definition 1.1. *The image formation correspondence* $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ *is the algebraic variety defined by (1.4). We denote its vanishing ideal (Definition 2.1) by* $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$.

In this paper, we will be concerned with the structure of $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ and the vanishing ideals of varieties which may be defined in terms of three natural geometric operations on $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$.

(1) *Coordinate projection.* For example, let $\pi_{\mathbf{q}} : \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n} \rightarrow (\mathbb{P}^{11})^m \times (\mathbb{P}^2)^{mn}$ denote the coordinate projection $\pi_{\mathbf{q}}(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) = (\bar{\mathbf{A}}, \bar{\mathbf{p}})$. We denote its image by

$$\Gamma_{\mathbf{A}, \mathbf{p}}^{m, n} := \pi_{\mathbf{q}}(\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}) = \{(\bar{\mathbf{A}}, \bar{\mathbf{p}}) \mid \exists \bar{\mathbf{q}} \text{ s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}\}. \tag{1.5}$$

We note that, by the projective elimination theorem (see eg. [40, Theorem 4.22]), $\Gamma_{\mathbf{A}, \mathbf{p}}^{m, n}$ is closed in the Zariski topology on $(\mathbb{P}^{11})^m \times (\mathbb{P}^2)^{mn}$.

(2) *Coordinate specialization.* For example, let $\bar{\mathbf{A}}$ be a particular arrangement of cameras. We can form the intersection of $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ with the coordinate planes where $A_i = \bar{A}_i$ for $i = 1, \dots, m$, and then coordinate project away from $(\mathbb{P}^{11})^m$ to obtain

$$\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, n} := \{(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \mid (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}\}. \tag{1.6}$$

²The imaging map of (1.3) models a scenario in which all points are visible in every image. One may consider other scenarios of interest in computer vision, eg. when each point is visible in only some of the images (eg. [19, 29]).

³We note that the definition of $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ is independent of the choice of U . As such, it is insensitive to certain physical assumptions about the camera matrices (eg. that they have full rank, or that their centers do not coincide.) In particular, although a *generic* point $(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ will be such that each A_i has full rank and all $A_i q_j$ are defined, these conditions do not hold for an *arbitrary* point $(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$.

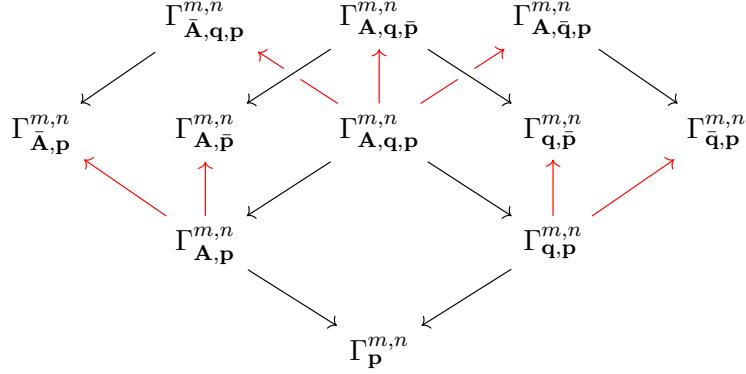


Figure 1. **Atlas of the Pinhole Camera.** Red arrows indicate variable specialization and black arrows indicate projection. An analogous diagram can be constructed where the arrows remain the same but the varieties are replaced by their vanishing ideals.

(3) *Projection + specialization.* Combining 1 and 2, the relevant variety is

$$\Gamma_{\bar{A},p}^{m,n} = \{\bar{p} \mid \exists \bar{q} \text{ s.t. } (\bar{A}, \bar{q}, \bar{p}) \in \Gamma_{A,q,p}^{m,n}\}. \quad (1.7)$$

All the non-trivial varieties (and their vanishing ideals) obtained in this way can be organized as shown in Figure 1⁴. Red arrows indicate variable specialization and black arrows indicate variable elimination.

The aim of this paper is to motivate and initiate the study of this diagram which we will call the *Atlas of the Pinhole Camera*. We call it an atlas because it is a systematic collection of algebro-geometric objects associated with the pinhole camera that also captures the relationships between them, telling us how one can travel from one object to another. For example, knowing that a variety is obtained by specializing a group of variables (red arrow) can be used to easily compute its dimension as the difference of the dimensions of two varieties. See Appendix A for dimension counts for each of the varieties in the atlas.

1.1. Computer Vision and the Atlas. As the ideal $\Gamma_{A,q,p}^{m,n}$ describes *all* the algebraic relationships that hold between cameras, world, and image points in 3D reconstruction, many of the problems studied in 3D computer vision (also known as multiview geometry) can be described in terms of the varieties and ideals that occur in the atlas. So we begin our study of the atlas by summarizing what is known about each node and how it relates to problems in multiview geometry.

⁴Not every result of successively applying projection and specialization operations to $\Gamma_{A,q,p}^{m,n}$ is included here. For example $\Gamma_{A,p}^{m,n}$ and $\Gamma_{q,p}^{m,n}$ are trivial. Similarly $\Gamma_{A,q,\bar{p}}^{m,n}$ and $\Gamma_{A,\bar{q},\bar{p}}^{m,n}$ are defined by linear equations and not interesting for projective cameras. However this can change as the model for the camera is varied. For example, $\Gamma_{A,\bar{q},\bar{p}}^{m,n}$ is an interesting nonlinear variety for Euclidean cameras. See Section 8 for more.

The image formation correspondence, $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ is the starting point of our study. Given noisy image observations $\tilde{\mathbf{p}}$, the problem of finding \mathbf{A} and \mathbf{q} such that they best explain these image observations is the *reconstruction problem*. Other names for this problem are *Structure from Motion* (SfM) [23] and *Simultaneous Localization And Mapping* (SLAM) [58].

Assuming Gaussian noise, the maximum likelihood estimate of \mathbf{A} and \mathbf{q} can be found by solving the following optimization problem:

$$\arg \min_{(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}})} \sum_i^n \sum_j^m \|\bar{p}_{ij} - \tilde{p}_{ij}\|^2 \text{ s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}. \quad (1.8)$$

The quantity $\|\bar{p}_{ij} - \tilde{p}_{ij}\|^2$, known as the *reprojection error*, is understood to be the squared Euclidean norm between dehomogenized p and \tilde{p} . That is, we assume that the \bar{p} and \tilde{p} are finite points. The optimization problem (1.8) is also known as the *bundle adjustment* problem [63]. *Projective factorization* is another method for solving the reconstruction problem [41, 62]. In this method, one tries to find a nearby feasible point on $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ by solving a matrix factorization problem.

Eliminating \mathbf{q} (also known as *structure*) from the constraint set in (1.8) we obtain $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n}$ and the *bundle adjustment without structure* problem:

$$\arg \min_{\bar{\mathbf{A}}, \bar{\mathbf{p}}} \sum_i^n \sum_j^m \|\bar{p}_{ij} - \tilde{p}_{ij}\|^2 \text{ s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{p}}^{m,n}. \quad (1.9)$$

Bundle adjustment without structure is of interest because eliminating the world points \mathbf{q} significantly reduces the dimensionality of the problem [44, 47, 48, 50, 52]. The two-view ($m = 2$) version of this problem is particularly important, as it is usually the first step in incremental 3D reconstruction algorithms.

Similarly $\Gamma_{\mathbf{q}, \mathbf{p}}^{m,n}$ is obtained from $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ by eliminating the cameras (also known as *motion*) and $\Gamma_{\mathbf{p}}^{m,n}$ is obtained from $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ by eliminating both cameras and points. One can formulate the reconstruction problem as a nearness problem on each of these varieties, and each formulation leads to its own version of the bundle adjustment problem. As a result we will refer to all four ideals $I_{\mathbf{A}, \mathbf{p}}^{m,n}, I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}, I_{\mathbf{q}, \mathbf{p}}^{m,n}, I_{\mathbf{p}}^{m,n}$ as *bundle adjustment ideals*.

Of these four ideals, only $I_{\mathbf{A}, \mathbf{p}}^{m,n}$ has been studied to a certain extent [59]. Before our work, very little was known about the structure of the other three ideals. This may come as a surprise to the reader given the importance of the reconstruction problem in 3D computer vision. The lack of study can be explained by computational reasons. Substituting $p_{ij} \sim A_i q_j$ into the objective and dehomogenizing it gives us the more commonly occurring form of (1.8) as an unconstrained rational optimization problem. This form is preferred in practice because it can be solved much more efficiently using simpler algorithms than the form in (1.8) [63]. Similarly, the bundle adjustment without structure problem is usually solved by approximating (1.9) by an unconstrained optimization problem where the objective is the sum of squares of certain polynomials (the so-called epipolar and trifocal constraints [23]).

These transformations have the disadvantage that they obscure the geometric structure of the problem. For example, by formulating the reconstruction problem as an optimization problem

over the variety $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$, we can study it using the tools of complex and real algebraic geometry. It also reveals two other versions of the problem not considered before. This perspective has been useful in constructing effective algorithms for solving the *triangulation problem*.

In *triangulation* [26], the cameras $\mathbf{A} = \bar{\mathbf{A}}$ are known and fixed, we are given the noisy images $\tilde{\mathbf{p}}$ of an unknown world point, and we seek a solution to

$$\arg \min_{\bar{\mathbf{q}}, \bar{\mathbf{p}}} \sum_i^m \|\bar{p}_i - \tilde{p}_i\|^2 \text{ s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, 1}, \quad (1.10)$$

i.e., find the world point q that best explains the image observations. The variety $\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, n}$ is the slice of $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ that is obtained by fixing the cameras. So the above optimization problem is better re-written in terms of $\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, 1}$ as:

$$\arg \min_{\bar{\mathbf{q}}, \bar{\mathbf{p}}} \sum_i^m \|\bar{p}_i - \tilde{p}_i\|^2 \text{ s.t. } (\bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, 1}. \quad (1.11)$$

The *multiview variety* $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, 1}$ is obtained from $\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, 1}$ by eliminating the world point from it. As we noted earlier, the multiview variety and sets related to it are perhaps the most well-studied objects in computer vision [5, 23, 27, 38, 59, 61]. Using $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, 1}$ we can re-write the triangulation problem as:

$$\arg \min_{\bar{\mathbf{p}}} \sum_i^m \|\bar{p}_i - \tilde{p}_i\|^2 \text{ s.t. } \bar{\mathbf{p}} \in \Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, 1}. \quad (1.12)$$

This form of the triangulation problem has been used effectively in practice [3, 26, 35]. We will refer to $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, n}$ and $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m, n}$ as the *triangulation ideals*.

The optimization problem in (1.12) finds the nearest point on the variety $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, 1}$. One measure of complexity of problems of this form is the concept of the *Euclidean distance degree* of a variety [18]. Recently the affine Euclidean distance degree of $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, 1}$ was computed [39], settling a conjecture of Stévenen et al [54].

In resectioning [23, Chapter 7], the world points $\mathbf{q} = \bar{\mathbf{q}}$ are known and fixed, we are given their noisy images $\tilde{\mathbf{p}}$ by an unknown camera A and we seek a solution to:

$$\arg \min_{\bar{\mathbf{A}}, \bar{\mathbf{p}}} \sum_j^n \|\bar{p}_j - \tilde{p}_j\|^2 \text{ s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{1, n}. \quad (1.13)$$

i.e., find the camera matrix A that best explains the image observations. Resectioning is also known as the *Perspective-n-Point Problem*. Like triangulation, it is a fundamental problem in 3D computer vision and a considerable effort has been devoted to solving it [33]. The variety $\Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{1, n}$ is the slice of $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{1, n}$ that is obtained by fixing the world points, so like triangulation we can re-write (1.13) as an optimization problem over $\Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{1, n}$. Also like triangulation, we can eliminate

\mathbf{A} and formulate resectioning as a nearest point to the variety problem on $\Gamma_{\bar{\mathbf{q}}, \mathbf{p}}^{1,n}$. As a result we will refer to $I_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{m,n}, I_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n}$ as *resectioning ideals*.

The resectioning and triangulation problems are in a sense duals of each other. So one might be inclined to think that the structure of $I_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n}$ would be as well-studied as that of $I_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$. This however is not the case, very little is known about the structure of the resectioning ideals $I_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{m,n}$ and $I_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n}$ and their varieties [45, 49].

Let us now talk about the three varieties that are obtained by fixing the image points $\mathbf{p} = \bar{\mathbf{p}}$. From $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n}$ we obtain $\Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$. The vanishing ideal of $\Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$ is the set of polynomials on camera matrices that satisfy given image observations. Among these is a distinguished set of polynomials which are multilinear forms over the image coordinates. Longuet-Higgins was the first to study the matrix defining the bilinear form for a pair of Euclidean cameras – the *essential matrix* [17, 24, 36]. The generalization of the essential matrix to projective cameras is known as the *fundamental matrix* [37]. For three and four cameras we get the trifocal tensor [4, 23, 25] and the quadrifocal tensor [43, 51] respectively. We note that the varieties of these tensors and their vanishing ideals have been studied [4, 17, 43]. However, they do not appear in our atlas.

When we are given the minimal number of image observations such that $\Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$ is zero dimensional (modulo PGL_4), finding cameras $\bar{\mathbf{A}}$ that lie on $\Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$ is a fundamental problem in computer vision and the subject of intensive study. These problems are known as *minimal problems*. Starting with Nistér’s work [42] on estimating the essential matrix using five point correspondences, there has been an explosion of work on using methods of computational algebraic geometry for solving these problems [31, 32, 53].

From $\Gamma_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n}$ we get $\Gamma_{\mathbf{q}, \bar{\mathbf{p}}}^{m,n}$, by fixing the image points $\mathbf{p} = \bar{\mathbf{p}}$. The vanishing ideal of this variety is the set of polynomials that a set of 3D points must satisfy to explain a set of given image observations. Finally, we have the intriguing variety $\Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \bar{\mathbf{p}}}^{m,n}$. Points on this variety correspond to 3D reconstructions determined by a set of image observations. A particularly interesting case is the two view, seven point problem, which is closely related to the *seven point algorithm* for estimating the fundamental matrix [23].

The above is a necessarily brief presentation. However, we hope that we have convinced the reader that the nodes/objects of the atlas are related to key problems in 3D computer vision. Some of these objects have been studied before, but many remain completely unexplored. We hope that by systematically constructing these objects as slices and projections of a single variety, and organizing them in the atlas will reveal more of their shared structure and propel their study.

In the remainder of this paper, we initiate the study of the atlas by exploring a part of it in detail. The next section introduces our main results.

1.2. A Square in the Atlas. Polynomial constraints, particularly those which are implied by rank constraints, pervade the multiview geometry literature. An extensive catalogue of these constraints may be found, for instance, in [38, Chapter 8]. In the context of the atlas, many of these constraints are sufficient to describe various varieties locally. For instance, we have

$$Aq \sim p \quad \Rightarrow \quad \exists \lambda \text{ s.t. } Aq = \lambda p \quad \Rightarrow \quad \text{rank}(Aq - p) \leq 1. \quad (1.14)$$

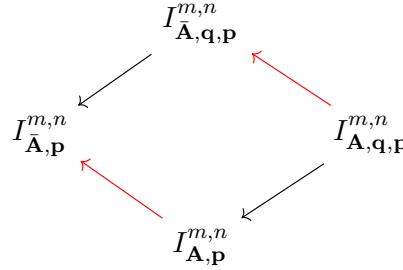


Figure 2. The four ideals studied in this paper.

Thus, the condition $Aq \sim p$ may be expressed as the vanishing of the 2×2 minors of $(Aq \ p)$, giving equations defining $\Gamma_{A,q,p}^{m,n}$ near a generic point. However, the question of whether these equations *globally* define the full closed correspondence $\Gamma_{A,q,p}^{m,n}$ is more subtle. In fact, already for 2 symbolic cameras A_1, A_2 and image points p_1, p_2 , the 2-focal polynomial (cf. Definition 3.1)

$$\det \begin{pmatrix} A_1 & p_1 & 0 \\ A_2 & 0 & p_2 \end{pmatrix}$$

is contained in $I_{A,q,p}^{2,1}$, and yet this polynomial is not in the ideal generated by all 2×2 minors of the matrices $(A_1 q \ p_1)$ and $(A_2 q \ p_2)$. Thus, in a precise sense, the rank constraints of (1.14) are not complete.

The **main results** of our paper characterize the four ideals appearing in Figure 2 for all m and n . This square, although not comprising the full atlas, is a natural point of departure since it includes both $I_{A,q,p}^{m,n}$ and the multiview ideal $I_{A,p}^{m,1}$.

In the case of the triangulation ideals $I_{\bar{A},q,p}^{m,n}$ and $I_{\bar{A},p}^{m,n}$, Theorem 7.1 and Theorem 7.2, identify simple, explicit generators and Gröbner bases under suitable genericity assumptions. For the bundle adjustment ideals $I_{A,q,p}^{m,n}$ and $I_{A,p}^{m,n}$, our characterization is more subtle. Theorem 7.3 determines these ideals up to saturation by certain minors of the matrix of indeterminates $(A_1^\top \mid \dots \mid A_m^\top)$.

Studying the bundle adjustment ideals with their corresponding triangulation ideals is revealing in interesting ways. Despite being closely related, many facts about the triangulation ideals do not transfer to their bundle adjustment ideals—see Examples 4.1, 7.1, and 7.2. The seemingly simple act of making the cameras symbolic gives the bundle adjustment ideals a more elaborate structure. It also offers further insight into the structure of the triangulation ideals. For instance, it explains the origin of the quadrifocal tensor.

The quadrifocal tensor and the associated 4-view constraints [51] are an oddity in multiview geometry, where they are known to be redundant – i.e. they are not needed to cut out the multiview variety. However, they are needed to form a Gröbner basis for the multiview ideal [2, 5]. We show in Example 4.1 that, unlike the case of $\Gamma_{\bar{A},p}^{m,1}$, four-view constraints are necessary to cut out the variety $\Gamma_{A,p}^{m,1}$, and that these constraints only become redundant after specialization to a particular arrangement of cameras.

The structure of the ideals $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ and $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ plays an important role in our study. In Section 3, we determine explicit generators for $I_{\mathbf{A}, \mathbf{p}}^{m,1}$, which form a Gröbner basis for a large class of term orders. Using this result, we then recover the strongest known results about the multiview ideal $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ as simple corollaries using specialization arguments. Section 5 and Section 6 follow the same pattern, first giving an explicit Gröbner basis for $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$, and then specializing to obtain $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$. We then establish our main results in Section 7 as consequences of the $n = 1$ cases treated in Sections 3–6. Closing the paper, we list several open problems and further research directions centered around the atlas from Figure 1.

1.3. Notation. The notation $p \sim p'$ is used to denote equality of p and p' in projective space.

In sections 3–6, we restrict attention to the case of a single world point $q \in \mathbb{P}^3$ whose image in the i th camera A_i , for $i = 1, \dots, m$, is $p_i \in \mathbb{P}^2$. We will use MATLAB notation to refer to subentities of A, q and p : $A_i[j, k]$ for the entry of matrix A_i in row j and column k , $q[j]$ for the j th entry of q and $p_i[j]$ for the j th entry of p_i . When multiple world points are involved we let p_{ij} be the image of q_j in A_i for each $j = 1, \dots, n$.

The reader has already encountered our use of a bar over an object to indicate specialization. For instance, A stands for a symbolic 3×4 matrix denoting a camera while \bar{A} is a 3×4 scalar matrix realizing a camera. Boldface font will be reserved for sets of variables or ordered row indices of submatrices. For example, if $\emptyset \neq \mathbf{r} \subseteq \{1, 2, 3\}$ is a collection of row indices of A_i , we let $A_i[\mathbf{r}, :]$ denote the $|\mathbf{r}| \times 4$ submatrix of A consisting of the rows indexed by \mathbf{r} in the usual order. The notation $\mathbf{A}, \mathbf{q}, \mathbf{p}$ will denote collections of m cameras, n world points and the corresponding mn image points and $\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}$ will refer to their specializations.

Acknowledgements. We thank Jessie Loucks Tavitas, Erin Connelly & Craig Citro for helpful discussions. Timothy Duff acknowledges support from the National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship (DMS-2103310). Max Lieblich was partially supported by a National Science Foundation Grant (DMS - 1902251). Rekha Thomas was partially supported by a National Science Foundation grant (DMS - 1719538).

2. Tools

2.1. Vanishing Ideals. The homogeneous coordinate ring of the product of projective spaces $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ is the polynomial ring $\mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$ in the k sets of indeterminates $\mathbf{x}_i = \{x_{i1}, \dots, x_{i(n_i+1)}\}$, which we equip with the multigrading $\deg(x_{ij}) = e_i \in \mathbb{Z}^k$. A set of polynomials $\{f_1, \dots, f_s\} \subset \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$, where each f_i is homogeneous with respect to the multigrading, has a well-defined vanishing locus:

$$V(f_1, \dots, f_s) = \{x \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \text{ s.t. } f_1(x) = \dots = f_s(x) = 0\}.$$

These algebraic varieties are the closed sets in the Zariski topology on $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. The polynomials f_1, \dots, f_s are said to vanish on a set X if $X \subset V(f_1, \dots, f_s)$.

Definition 2.1. *The vanishing ideal $I(X)$ of a set $X \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ is the ideal generated by all $f \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$ vanishing on X .*

For polynomials $f_1, \dots, f_s \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$, we use the standard notation $\langle f_1, \dots, f_s \rangle$ for the ideal they generate. The ideal $\langle f_1, \dots, f_s \rangle$ need not be a vanishing ideal in the sense of Definition 2.1, although it will always be contained in the vanishing ideal $I(V(f_1, \dots, f_s))$. A necessary condition for the equality of these ideals is that $\langle f_1, \dots, f_s \rangle$ is radical. We will rely on the following Gröbner basis result to certify that an ideal is radical. For the basics of Gröbner basis theory, see [15].

Proposition 2.2. *Let $<$ be a monomial order on the polynomial ring $R[\mathbf{x}] = R[x_1, \dots, x_k]$ where R is an integral domain. Suppose $g_1, \dots, g_s \in R[\mathbf{x}]$ form a Gröbner basis with respect to $<$, and their leading terms $in_<(g_1), \dots, in_<(g_s)$ are squarefree monomials in the variables x_1, \dots, x_k . Then*

- (1) $\langle g_1, \dots, g_s \rangle$ is radical, and
- (2) when $R = \mathbb{R}$ is the field of real numbers, $\langle g_1, \dots, g_s \rangle$ is real radical.

For a proof of the first statement, see Appendix B, and for the second statement, see [64, Proposition 1.2].

Gröbner bases satisfying the conditions of Proposition 2.2 establish much more than the radicality of the ideal they generate. For particular monomial orders, they serve the extra purpose of deducing results about specializations and eliminations of these ideals. They also imply a nice bonus for ideals defined over \mathbb{R} , namely, Proposition 2.2 (2) establishes that the ideal generated by the Gröbner basis is *real radical*. In other words, this ideal is also the vanishing ideal of all real points in its variety.

In our applications of Proposition 2.2, R is almost always a field, where many standard results about Gröbner bases [15, Ch 2] are at our disposal. The sole exception is in the proof of Theorem 7.3, where R is obtained by localizing a polynomial ring at the powers of certain polynomials in $\mathbb{C}[\mathbf{A}]$.

As a reminder, a main contribution of this paper is to explicitly describe the following four vanishing ideals and their corresponding varieties (see Definition 1.1, (1.4)–(1.7)):

$$I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n} = I\left(\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}\right), \quad I_{\mathbf{A}, \mathbf{p}}^{m, n} = I\left(\Gamma_{\mathbf{A}, \mathbf{p}}^{m, n}\right), \quad I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, n} = I\left(\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m, n}\right), \quad I_{\bar{\mathbf{A}}, \mathbf{p}}^{m, n} = I\left(\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m, n}\right).$$

These ideals live in the polynomial rings $\mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}]$, $\mathbb{C}[\mathbf{A}, \mathbf{p}]$, $\mathbb{C}[\mathbf{q}, \mathbf{p}]$ and $\mathbb{C}[\mathbf{p}]$ respectively.

Throughout the paper, we employ the following simple Nullstellensatz-based *recognition criterion* to determine vanishing ideals.

Proposition 2.3 (Recognition Criterion). *Given a closed subvariety $X \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, its vanishing ideal $I(X)$ is generated by the \mathbb{Z}^k -homogeneous polynomials $f_1, \dots, f_s \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$ if and only if the following three conditions hold:*

- (1) f_1, \dots, f_s cut out X set-theoretically, i.e., $V(f_1, \dots, f_s) = X$.
- (2) $\langle f_1, \dots, f_s \rangle$ is saturated with respect to the irrelevant ideal:

$$\langle f_1, \dots, f_s \rangle = \langle f_1, \dots, f_s \rangle : (\mathfrak{m}_{\mathbf{x}_1} \cap \dots \cap \mathfrak{m}_{\mathbf{x}_k})^\infty$$

where $\mathfrak{m}_{\mathbf{x}_i} = \langle x_{i1}, \dots, x_{i(n_i+1)} \rangle$.

- (3) $\langle f_1, \dots, f_s \rangle$ is radical.

This recognition criterion is a standard consequence of Hilbert's Nullstellensatz, but we provide a proof in Appendix B for completeness. In all of our results, we establish Condition 3 in Proposition 2.3 via explicit Gröbner bases as in Proposition 2.2.

If M is a $m \times n$ matrix over a polynomial ring, then for $\ell \leq \min(m, n)$ we let $\text{minors}(\ell, M)$ denote the ideal generated by all $\ell \times \ell$ minors of M . We will consider matrices M in which all entries are polynomials, some of which may be variables or scalars. Here is a small example illustrating some of the above tools.

Example 2.1. [14, Example 1.2] Let f_1, f_2, f_3 be the 2×2 minors of the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1^2 & b_1 b_2 & b_2^2 \end{pmatrix}$$

in the polynomial ring $\mathbb{C}[a_1, a_2, a_3, b_1, b_2]$. Explicitly,

$$f_1 = b_1(a_1 b_2 - a_2 b_1), \quad f_2 = a_1 b_2^2 - a_3 b_1^2, \quad f_3 = b_2(a_2 b_2 - a_3 b_1).$$

The ideal $J = \langle f_1, f_2, f_3 \rangle = \text{minors}(2, M)$ is homogeneous with respect to the \mathbb{Z}^2 -grading $\deg(a_i) = e_1$, $\deg(b_i) = e_2$. However it is not the vanishing ideal of $V(f_1, f_2, f_3)$. Indeed, it does not satisfy Condition 2 in Proposition 2.3; for instance the polynomial $a_1 b_2 - a_2 b_1$ is an element of $J : (\langle a_1, a_2, a_3 \rangle \cap \langle b_1, b_2 \rangle)^\infty$, but not J . Condition 3 also fails; $f = b_2(a_2^2 - a_1 a_3) \notin J$, while $f^3 \in J$.

The vanishing ideal of $V(J) \subset \mathbb{P}^2 \times \mathbb{P}^1$ turns out to be

$$I = \langle a_3 b_1 - a_2 b_2, a_1 b_2 - a_2 b_1, a_1 a_3 - a_2^2 \rangle.$$

The three generators of I form a Gröbner basis for both Lex and GRevLex orders in the variable ordering $b_2 < b_1 < a_3 < a_2 < a_1$. The GRevLex initial ideal $\langle a_3 b_1, a_2 b_1, a_2^2 \rangle$ is not squarefree. However, the Lex initial ideal $\langle a_2 b_2, a_1 b_2, a_1 a_3 \rangle$ is squarefree, so Proposition 2.2 lets us verify that I is radical. \square

To prove that Condition 2 in Proposition 2.3 holds, we make frequent use of the following lemma about GRevLex Gröbner bases.

Lemma 2.4 ([55, Lemma 12.1]). *Let G be a Gröbner basis for an ideal $I \subset \mathbb{C}[x_1, \dots, x_l]$ with respect to the GRevLex order where x_l is the cheapest variable:*

$$x_l < \dots < x_1.$$

Then

$$G' = \{g \in G \text{ s.t. } x_l \text{ does not divide } g\} \cup \{(g/x_l) \text{ s.t. } g \in G \text{ and } x_l \text{ divides } g\}$$

is a GRevLex Gröbner basis for the ideal quotient $I : x_l$.

2.2. Generic Cameras. Broadly speaking, the main results of this paper may be divided into three progressively stronger categories: *geometric* results giving set-theoretic equations for the varieties of interest, *ideal-theoretic* results which characterize the vanishing ideals of these varieties, and, strongest of all, results about the *Gröbner bases* of these vanishing ideals. For the geometric and ideal-theoretic results, our genericity hypotheses will be that the camera arrangement $\bar{\mathbf{A}}$ has pairwise distinct centers. For the Gröbner basis results we will need stronger notions of genericity which we discuss in this section.

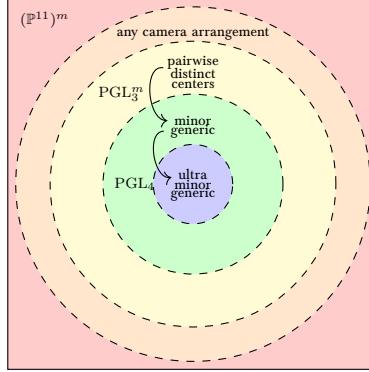


Figure 3. Classes of camera arrangements related by inclusion and equivalence up to projective change-of-coordinates.

Definition 2.5 (Minor Generic). *A camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$ is minor generic if all 4×4 minors of the $4 \times 3m$ matrix $(\bar{A}_1^\top \mid \dots \mid \bar{A}_m^\top)$ are nonzero (cf. [5, Sec. 2], [2, Sec. 3].)*

Definition 2.6 (Ultra Minor Generic). *A camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$ is ultra minor generic if all $k \times k$ minors of the $4 \times 3m$ matrix $(\bar{A}_1^\top \mid \dots \mid \bar{A}_m^\top)$ are nonzero for any $k \in [4]$.*

We now have three notions of genericity for camera arrangements: pairwise distinct centers, minor generic, and ultra minor generic. They are progressively stronger in the sense established by the following proposition, whose proof may be found in Appendix B.

Proposition 2.7. *If a camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$ is ultra minor generic then it is minor generic, and if $\bar{\mathbf{A}}$ is minor generic, then it has pairwise distinct centers.*

We next show that a converse statement of sorts is possible under suitable group actions. This is made precise in Theorem 2.8 and summarized pictorially in Figure 3.

Consider the following group actions on a camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$:

$$\begin{aligned} (\mathrm{PGL}_3)^m \times (\mathbb{P}^{11})^m &\rightarrow (\mathbb{P}^{11})^m \\ (H_1, \dots, H_m) \bar{\mathbf{A}} &= (H_1 \bar{A}_1, \dots, H_m \bar{A}_m), \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathrm{PGL}_4 \times (\mathbb{P}^{11})^m &\rightarrow (\mathbb{P}^{11})^m \\ H \bar{\mathbf{A}} &= (\bar{A}_1 H^{-1}, \dots, \bar{A}_m H^{-1}), \end{aligned} \tag{2.2}$$

and their combined action:

$$\begin{aligned} \mathrm{PGL}_4 \times (\mathrm{PGL}_3)^m \times (\mathbb{P}^{11})^m &\rightarrow (\mathbb{P}^{11})^m \\ (H, (H_1, \dots, H_m)) \bar{\mathbf{A}} &= (H_1 \bar{A}_1 H^{-1}, \dots, H_m \bar{A}_m H^{-1}). \end{aligned} \tag{2.3}$$

The combined group action may be interpreted as a simultaneous change of coordinates in both the world \mathbb{P}^3 and in the set of images $(\mathbb{P}^2)^m$. We will say that two camera arrangements are equivalent if they lie in the same orbit under one of these group actions.

Theorem 2.8. (1) *A camera arrangement $\bar{\mathbf{A}}$ has pairwise distinct centers if and only if it is equivalent to a minor generic camera arrangement under the group action in (2.1).*
 (2) *A camera arrangement $\bar{\mathbf{A}}$ is minor generic if and only if it is equivalent to an ultra minor generic arrangement under the group action in (2.2).*
 (3) *A camera arrangement $\bar{\mathbf{A}}$ has pairwise distinct centers if and only if it is equivalent to an ultra minor generic camera arrangement under the group action in (2.3).*

Unlike the condition of pairwise distinct centers, minor genericity and ultra minor genericity are algebraic conditions which are not preserved under coordinate change. These algebraic genericity assumptions play a central role in deducing results about specializations of the bundle adjustment ideals that we study in this paper.

3. $I_{\mathbf{A}, \mathbf{p}}^{m,1}$: Constraints on cameras and image points for $n = 1$

In this section we study the ideal $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ consisting of all polynomials that constrain a tuple of m symbolic cameras and the images of a single world point in them. Given one camera A and a point $p \in \mathbb{P}^2$, there is always a world point $q \in \mathbb{P}^3$ such that $Aq \sim p$. Therefore, the ideal $I_{\mathbf{A}, \mathbf{p}}^{1,1}$ is trivial, and we will assume throughout this section that we have $m \geq 2$ cameras.

Let $\mathbf{A} = (A_1, \dots, A_m)$ and $\mathbf{p} = (p_1, \dots, p_m)$ represent m symbolic cameras and image points in them.

Definition 3.1. *A k -focal of the pair (\mathbf{A}, \mathbf{p}) is any polynomial in $\mathbb{C}[\mathbf{A}, \mathbf{p}]$ that arises as a maximal minor of the $3k \times (4 + k)$ matrix:*

$$\begin{pmatrix} A_{\sigma_1} & p_{\sigma_1} & 0 & \dots & 0 \\ A_{\sigma_2} & 0 & p_{\sigma_2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_k} & 0 & \dots & 0 & p_{\sigma_k} \end{pmatrix}, \quad (3.1)$$

where $\sigma = \{\sigma_1, \dots, \sigma_k\}$ is any k -element subset of $[m] = \{1, \dots, m\}$ with $k \geq 2$.

Since $4 + k \leq 3k$ for all $k \geq 2$, a k -focal is specified by a choice of $4 + k$ rows of the matrix (3.1). Note that all k -focals of (\mathbf{A}, \mathbf{p}) lie in the ideal $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ since if there exists a world point q such that $A_i q \sim p_i$ for all $i \in [m]$, then the matrix (3.1) is rank deficient and hence all maximal minors of (3.1) are zero.

We now state the main result of this section, which establishes the plausible, but by no means obvious, fact that the k -focals generate $I_{\mathbf{A}, \mathbf{p}}^{m,1}$. Moreover, a distinguished subset of these k -focals form a Gröbner basis with respect to a large class of monomial orders.

Theorem 3.2. *For any number of cameras m , and $n = 1$ world point, the 2-, 3-, and 4-focals form a Gröbner basis for $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ with respect to any product order $<$ with $\mathbf{A} < \mathbf{p}$.*

We will recall the notion of a product order before proving Theorem 3.2. First, we explain why the 2-, 3-, and 4-focals are the only “interesting” k -focals.

For the sake of more compact notation, we let

$$(\mathbf{A} | \mathbf{p}) [\mathbf{r}]_\sigma = \begin{pmatrix} A_{\sigma_1}[\mathbf{r}_1, :] & p_{\sigma_1}[\mathbf{r}_1] & 0 & \dots & 0 \\ A_{\sigma_2}[\mathbf{r}_2, :] & 0 & p_{\sigma_2}[\mathbf{r}_2] & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_k}[\mathbf{r}_k, :] & 0 & \dots & 0 & p_{\sigma_k}[\mathbf{r}_k] \end{pmatrix}, \quad (3.2)$$

denote the k -focal submatrix of (3.1) determined by σ and the sets $\mathbf{r}_i \subseteq \{1, 2, 3\}$ indexing the chosen rows of A_{σ_i} and p_{σ_i} . Thus, every k -focal is the determinant of a matrix of the form in (3.2).

The following “bumping up/bumping down” lemma establishes that all k -focals are elementary consequences of 2-focals, 3-focals, and 4-focals. For $I_{\mathbf{A}, \mathbf{p}}^{m,1}$, this observation is a classical result, dating back at least to [61, Sec. 5] (see also, eg., [59, Proposition 2], or [2, Lemma 2.2].)

For $2 \leq k-1 < m$, a $(k-1)$ -focal $\det(\mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma$ can be “bumped-up” to a k -focal by including one row (say j th) from a new camera A_i with $i \notin \sigma$ as follows:

$$\det \begin{pmatrix} (\mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma & 0 \\ A_i[j, :] & p_i[j] \end{pmatrix}. \quad (3.3)$$

There are $3(m-k)$ choices for the pair (i, j) .

Proposition 3.3. *(Bumping-up) A “bumped-up” k -focal (3.3) can be factored as*

$$\det \begin{pmatrix} (\mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma & 0 \\ A_i[j, :] & p_i[j] \end{pmatrix} = p_i[j] \cdot \det(\mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma, \quad (3.4)$$

and (Bumping-down) every nonzero k -focal for $k > 4$ has the form (3.4) (up to sign.)

Proof. (3.4) is easily verified by Laplace expansion of the determinant. To prove bumping-down, first note that every term in any k -focal has degree 4 in the camera variables since the camera variables occur in exactly four columns of the matrix (3.1). Furthermore, this k -focal is homogeneous of degree $(\#\mathbf{r}_i - 1)$ in the variables of the camera A_i . To see this, scale the k cameras by $c_1, \dots, c_k \in \mathbb{C}$ and use multi-linearity of \det in rows and columns to obtain

$$\begin{aligned} \det((c_1, \dots, c_k) \cdot \mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma &= \left(\prod_{i=1}^k c_i^{\#\mathbf{r}_i} \right) \cdot \det(\mathbf{A} | (c_1^{-1}, \dots, c_k^{-1}) \cdot \mathbf{p})[\mathbf{r}]_\sigma \\ &= \left(\prod_{i=1}^k c_i^{\#\mathbf{r}_i - 1} \right) \cdot \det(\mathbf{A} | \mathbf{p})[\mathbf{r}]_\sigma. \end{aligned}$$

Since $4 = \sum_{i=1}^k (\#\mathbf{r}_i - 1)$, and $\#\mathbf{r}_i - 1 \geq 0$ for all i , at most four cameras can use more than one row. Thus, if $k > 4$, there exists a camera A_i using a single row indexed by $\mathbf{r}_i = \{j\}$, which implies the factorization in (3.4). \square

As explained in [23, Ch. 17], the 2-, 3-, and 4-focals encode many familiar entities from multiview geometry. A 2-focal has the form

$$(\mathbf{A} \mid \mathbf{p}) [\{1, 2, 3\}, \{1, 2, 3\}]_{\{i,j\}} = \det \begin{pmatrix} A_i & p_i & 0 \\ A_j & 0 & p_j \end{pmatrix} = p_j^\top F_{ij}(A) p_i,$$

where $F_{ij}(A)$ is the 3×3 *fundamental matrix* associated to the pair (A_i, A_j) . Similarly, for $\{j_1, j_2\}, \{k_1, k_2\} \subset \{1, 2, 3\}$, the 3-focal

$$\begin{aligned} & \det(\mathbf{A} \mid \mathbf{p}) [\{1, 2, 3\}, \{j_1, j_2\}, \{k_1, k_2\}]_{\{i,j,k\}} \\ &= \det \begin{pmatrix} A_i & p_i & 0 & 0 \\ A_j[\{j_1, j_2\}, :] & 0 & p_j[\{j_1, j_2\}, :] & 0 \\ A_k[\{k_1, k_2\}, :] & 0 & 0 & p_k[\{k_1, k_2\}, :] \end{pmatrix} \end{aligned}$$

may be viewed as a polynomial in $\mathbb{C}[\mathbf{p}]$ whose coefficients in $\mathbb{C}[\mathbf{A}]$ make up a particular *trifocal tensor* for the triple (A_i, A_j, A_k) . Note in the equation above the privileged index i , also known as the covariant index; for each triple (A_i, A_j, A_k) , there are 27 choices of 3-focals where the covariant index may be i, j , or k . Finally, each 4-tuple (A_i, A_j, A_k, A_l) gives rise to 81 4-focals

$$\begin{aligned} & \det(\mathbf{A} \mid \mathbf{p}) [\{i_1, i_2\}, \{j_1, j_2\}, \{k_1, k_2\}, \{l_1, l_2\}]_{\{i,j,k,l\}} \\ &= \det \begin{pmatrix} A_i[\{i_1, i_2\}, :] & p_i[\{i_1, i_2\}, :] & 0 & 0 & 0 \\ A_j[\{j_1, j_2\}, :] & 0 & p_j[\{j_1, j_2\}, :] & 0 & 0 \\ A_k[\{k_1, k_2\}, :] & 0 & 0 & p_k[\{k_1, k_2\}, :] & 0 \\ A_l[\{l_1, l_2\}, :] & 0 & 0 & 0 & p_l[\{l_1, l_2\}, :] \end{pmatrix}, \end{aligned}$$

whose coefficients in $\mathbb{C}[\mathbf{A}]$ form the 81 *quadrifocal tensors*.

We will use Proposition 2.3 to prove Theorem 3.2. This in turn will rely on Proposition 2.2. We begin by proving that both the m -focals and the 2-, 3-, and 4-focals cut out $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$ *set-theoretically*. The latter will establish Condition 1 of the recognition criterion.

Proposition 3.4.

$$\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1} = V(2\text{-}, 3\text{-}, \text{ and } 4\text{-focals}) = V(m\text{-focals}).$$

Proof. We first show that $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1} \subseteq V(2\text{-}, 3\text{-}, \text{ and } 4\text{-focals})$. Suppose $(\bar{\mathbf{A}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$, and fix a representation for this point in homogeneous coordinates. Then there exists a nonzero 4×1 matrix \bar{q} and scalars $\lambda_1, \dots, \lambda_m$ with $\bar{A}_i \bar{q} = \lambda_i \bar{p}_i$. This implies that the kernel of each k -focal matrix (3.1) associated to $(\bar{\mathbf{A}}, \bar{\mathbf{p}})$ contains the nonzero vector

$$(-\bar{q} \quad \lambda_{\sigma_1} \quad \cdots \quad \lambda_{\sigma_k})^\top. \quad (3.5)$$

Thus all k -focals, in particular all 2-, 3-, and 4-focals, vanish at $(\bar{\mathbf{A}}, \bar{\mathbf{p}})$.

The containment $V(2\text{-}, 3\text{-}, \text{ and } 4\text{-focals}) \subseteq V(m\text{-focals})$ follows from Proposition 3.3.

To finish, we argue that $V(m\text{-focals}) \subseteq \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$. Consider a point

$$(\bar{\mathbf{A}}, \bar{\mathbf{p}}) = (\bar{A}_1, \dots, \bar{A}_m, \bar{p}_1, \dots, \bar{p}_m) \in V(m\text{-focals}),$$

with fixed homogeneous coordinates as before, and consider the vector (3.5) for $\sigma = [m]$. To prove this inclusion we will construct a sequence of points

$$(\bar{A}_1^{(n)}, \dots, \bar{A}_m^{(n)}, \bar{p}_1, \dots, \bar{p}_m) \in \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$$

converging to $(\bar{\mathbf{A}}, \bar{\mathbf{p}})$. Since $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$ is closed in the Euclidean topology, this will imply that $(\bar{\mathbf{A}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$, giving the desired inclusion. To construct the sequence, set $\bar{A}_i^{(n)} = \bar{A}_i$ for any i with $\lambda_i \neq 0$. For each index i with $\lambda_i = 0$, choose an arbitrary camera \bar{A}'_i such that $\bar{A}'_i \bar{q} \sim \bar{p}_i$ and set $\bar{A}_i^{(n)} = \bar{A}_i + (1/n)\bar{A}'_i$. This gives a sequence of points

$$(\bar{A}_1^{(n)}, \dots, \bar{A}_m^{(n)}, \bar{A}_1^{(n)} \bar{q}, \dots, \bar{A}_n^{(n)} \bar{q})_{n \geq 1}$$

in $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$. To see that it converges to the given point $(\bar{\mathbf{A}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$, we may consider each camera separately: if $\lambda_i \neq 0$, then $\bar{A}_i^{(n)} = \bar{A}_i$ and

$$\bar{A}_i^{(n)} \bar{q} = \bar{A}_i \bar{q} \sim \bar{p}_i,$$

for all n , whereas if $\lambda_i = 0$, then $\bar{A}_i^{(n)} \rightarrow \bar{A}_i$ as $n \rightarrow \infty$, and

$$\bar{A}_i^{(n)} \bar{q} = \bar{A}_i \bar{q} + \frac{1}{n} \bar{A}'_i \bar{q} = \frac{1}{n} \bar{A}'_i \bar{q} \sim \bar{A}'_i \bar{q} \sim \bar{p}_i.$$

□

Proposition 3.4 shows that the ideals of m -focals and 2-, 3-, and 4-focals both cut out $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$ set-theoretically; therefore, by Proposition 2.3,

$$\begin{aligned} I_{\mathbf{A}, \mathbf{p}}^{m,1} &= \sqrt{\langle 2-, 3-, \text{ and } 4\text{-focals} \rangle : (\mathfrak{m}_{\mathbf{A}} \cap \mathfrak{m}_{\mathbf{p}})^\infty} \\ &= \sqrt{\langle m\text{-focals} \rangle : (\mathfrak{m}_{\mathbf{A}} \cap \mathfrak{m}_{\mathbf{p}})^\infty}. \end{aligned}$$

Theorem 3.2 will establish the following precise relationship between the three ideals:

$$I_{\mathbf{A}, \mathbf{p}}^{m,1} = \langle 2-, 3-, \text{ and } 4\text{-focals} \rangle = \langle m\text{-focals} \rangle : \mathfrak{m}_{\mathbf{p}}^\infty. \quad (3.6)$$

Our next step in the proof of Theorem 3.2 is to note that the m -focals form a universal Gröbner basis for the ideal they generate. As noted in [59][Proposition b3], the m -focal ideal belongs to the class of *sparse determinantal ideals*, studied in algebraic geometry and commutative algebra. Sparse (maximal) determinantal ideals are generated by the maximal minors of matrices whose nonzero entries are distinct indeterminates. These ideals were studied by Giusti and Merle [21], who determined their dimensions and associated primes, and later by Boocher [9], who, in addition to determining their minimal free resolutions and several related invariants, showed that the maximal minors form a universal Gröbner basis. The following is an immediate consequence of [9][Proposition 5.4] and Proposition 2.2.

Proposition 3.5. *For $n = 1$ world point and any number of cameras m , the m -focals form a universal Gröbner basis for the ideal they generate, and this ideal is radical.*

Proposition 3.5 does not directly give us any information about $\langle 2-, 3-, \text{ and } 4\text{-focals} \rangle$, since its generators are minors of several different matrices which share variables. However, Theorem 3.2 asserts that the 2-, 3-, and 4-focals form a Gröbner basis for a large class of term orders. To get there, we next develop a result of a similar flavor to Lemma 2.4, about Gröbner bases of multihomogeneous ideals in $\mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}]$ whose elements are linear in each set of variables $\mathbf{x}_1, \dots, \mathbf{x}_k$. This is Lemma 3.8, which deals with a product order instead of a GRevLex order.

Recall that two monomial orders \prec_x on $\mathbb{C}[\mathbf{x}]$ and \prec_y on $\mathbb{C}[\mathbf{y}]$ specify a *product order* on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ with $\mathbf{y} \prec \mathbf{x}$ as follows: $\mathbf{x}^{\alpha_1} \mathbf{y}^{\beta_1} \prec \mathbf{x}^{\alpha_2} \mathbf{y}^{\beta_2}$ when either $\mathbf{x}^{\alpha_1} \prec_x \mathbf{x}^{\alpha_2}$ or $\alpha_1 = \alpha_2$ and $\mathbf{y}^{\beta_1} \prec_y \mathbf{y}^{\beta_2}$. More generally, we may define a product order on $\mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$ with $\mathbf{x}_k \prec \dots \prec \mathbf{x}_1$ by comparing monomials first using a monomial order \prec_{x_1} on $\mathbb{C}[\mathbf{x}_1]$ and breaking any ties successively with $\prec_{x_2}, \dots, \prec_{x_k}$. We also need the following condition on the monomial supports of a (partially) multilinear polynomial g .

Definition 3.6. Suppose $g \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}]$ is multilinear in $\mathbf{x}_1, \dots, \mathbf{x}_k$ and supported in each group of variables on the subsets $\mathbf{x}'_j \subset \mathbf{x}_j$ for $j = 1, \dots, k$. We say that g is *well-supported* on $\mathbf{x}_1, \dots, \mathbf{x}_k$ if every monomial of the form $\prod_{j=1}^k x_j$ with $x_j \in \mathbf{x}'_j$ divides some term of g .

Essentially, g is well-supported if it is multilinear in $\mathbf{x}_1, \dots, \mathbf{x}_k$ and its monomial support in $\mathbf{x}'_1, \dots, \mathbf{x}'_k$ (the \mathbf{x} variables appearing in the support of g) is as large as possible.

Example 3.1. Consider

$$g = x_{12}x_{22} + y^2x_{11}x_{21} + 3x_{11}x_{22} + 2yx_{12}x_{21}$$

in the polynomial ring $\mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, y]$. Then g is well-supported on $\mathbf{x}_1 = \{x_{11}, x_{12}, x_{13}\}$ and $\mathbf{x}_2 = \{x_{21}, x_{22}\}$. The monomial support of g comes from $\mathbf{x}'_1 = \{x_{11}, x_{12}\}$ and $\mathbf{x}'_2 = \mathbf{x}_2$ and indeed, all four products of variables from \mathbf{x}'_1 and \mathbf{x}'_2 divide some term of g .

Suppose now that we remove the first term from g :

$$g' = y^2x_{11}x_{21} + 3x_{11}x_{22} + 2yx_{12}x_{21}.$$

Then g' is not well-supported on \mathbf{x}_1 and \mathbf{x}_2 since no term of g' is divisible by $x_{12}x_{22}$. \square

The following fact is used implicitly in the proof of [5, Theorem 2.1].

Proposition 3.7. Suppose $g \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}]$ is multilinear in $\mathbf{x}_1, \dots, \mathbf{x}_k$ and well-supported on $\mathbf{x}_1, \dots, \mathbf{x}_k$. Let $\mathbf{x} = \mathbf{x}_1 \cup \dots \cup \mathbf{x}_k$, and \prec be a product order with $\mathbf{y} \prec \mathbf{x}$ formed from \prec_x and \prec_y . Then the leading term in $\prec(g)$ depends only on \prec_y and the relative ordering of the variables within each group $\mathbf{x}_1, \dots, \mathbf{x}_k$.

Proof. Let x_1, \dots, x_k be the largest variables with respect to \prec_x in each of the sets $\mathbf{x}'_1, \dots, \mathbf{x}'_k$. Since g is well-supported on $\mathbf{x}_1, \dots, \mathbf{x}_k$, it has a monomial of the form $(\prod_{j=1}^k x_i) \cdot \mathbf{y}^\alpha$. Choosing \mathbf{y}^α maximal with respect to \prec_y , $\text{in}_\prec(g) = c \cdot (\prod_{j=1}^k x_i) \cdot \mathbf{y}^\alpha$ for some $c \in \mathbb{C}$. \square

We note that Proposition 3.7 may fail when g is not well-supported. In Example 3.1, suppose $x_{11} \prec x_{12}$ in \mathbf{x}_1 and $x_{21} \prec x_{22}$ in \mathbf{x}_2 . Then since, $x_{12}x_{22}$ is not in the monomial support of g' , we could have either $\text{in}_\prec(g') = 2yx_{12}x_{21}$ or $\text{in}_\prec(g') = 3x_{11}x_{22}$.

Lemma 3.8. *With notation as in Proposition 3.7, let G be a Gröbner basis for a \mathbb{Z}^{k+1} -homogeneous ideal I in $\mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}]$ with respect to a product order where $\mathbf{y} < \mathbf{x}$. If every element of G is multilinear and well-supported on $\mathbf{x}_1, \dots, \mathbf{x}_k$ and x_i is the cheapest variable in \mathbf{x}_i , then*

$$G' = \{g \in G \text{ s.t. } x_i \text{ does not divide } g\} \cup \{(g/x_i) \text{ s.t. } g \in G \text{ and } x_i \text{ divides } g\}$$

is a Gröbner basis with respect to $<$ for the ideal quotient $I : x_i$.

Proof. The proof is analogous to that given for Lemma 2.4 in [55]. For any $f \in I : x_i$, we have that $x_i f \in I$ and hence $\text{in}_<(g)$ divides $x_i \text{in}_<(f)$ for some $g \in G$. We must show that the leading term of some element of G' divides $\text{in}_<(f)$. We argue by considering two cases. In the first case, if x_i divides g , then the leading term of $g/x_i \in G'$ divides $\text{in}_<(f)$. In the second case, x_i does not divide g and hence $g \in G'$. Proposition 3.7 then implies that x_i also does not divide $\text{in}_<(g)$. Indeed, if x_i did divide $\text{in}_<(g)$, then x_i would be the most expensive variable in \mathbf{x}'_i , and this would force $\mathbf{x}'_i = \{x_i\}$. By the well-supportedness of g , this would imply that x_i divides g , a contradiction. Thus x_i does not divide $\text{in}_<(g)$, which implies $\text{in}_<(g)$ divides $\text{in}_<(f)$. \square

Applying Lemma 3.8 over *all* product orders with $\mathbf{y} < \mathbf{x}$ yields the following.

Corollary 3.9. *With notation as in Lemma 3.8, if G is a Gröbner basis with respect to all product orders with $\mathbf{y} < \mathbf{x}$, then the same is true for G' .*

Thus, if G is a Gröbner basis with respect to all product orders with $\mathbf{y} < \mathbf{x}$, then we obtain a Gröbner basis with respect to all product orders for every successive quotient of $\langle G \rangle$ by variables in \mathbf{x} . With this tool in hand, we are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. We construct an ascending chain of ideals

$$\langle m\text{-focals} \rangle = J_0 \subset J_1 \subset \dots \subset J_m = \langle 2\text{-, 3\text{-, and 4-focals} } \rangle$$

where each $J_k = \langle G_k \rangle$ is given by an explicit set of generators G_k which we now define. Set $G_0 = \{m\text{-focals}\}$, and for $k > 0$,

$$G_k = \{g \in G_{k-1} \text{ s.t. } p_k[i] \text{ doesn't divide } g \ \forall i\} \cup \\ \{(g/p_k[i]) \text{ s.t. } g \in G_{k-1} \text{ and } p_k[i] \text{ divides } g \text{ for some } i\}.$$

Applying Proposition 3.3, one can also fix a $j \in [3]$ and obtain G_k as

$$G_k = \{g \in G_{k-1} \text{ s.t. } p_k[i] \text{ doesn't divide } g \ \forall i\} \cup \\ \{(g/p_k[j]) \text{ s.t. } g \in G_{k-1} \text{ and } p_k[j] \text{ divides } g\}.$$

Let $<$ be any product order with $\mathbf{A} < \mathbf{p}$. We now argue that G_k is a Gröbner basis with respect to $<$ for all k . This is true for $k = 0$ by Proposition 3.5, so suppose G_{k-1} is a Gröbner basis with respect to $<$ for some $k > 0$. Then Proposition 3.3 implies that any element $g \in G_k$ is a ℓ -focal for some integer ℓ . An ℓ -focal is the determinant of a minor of (3.1) for $k = \ell$ involving $4 + \ell$ rows. Every term in this determinant is multilinear in $\mathbf{p}_1, \dots, \mathbf{p}_m$ and thus involves exactly one variable from each subset of supporting variables $\mathbf{p}'_1 \subset \mathbf{p}_1, \dots, \mathbf{p}'_m \subset \mathbf{p}_m$. Conversely, every possible product of ℓ variables, one from each \mathbf{p}'_i appears in a term of the ℓ -focal. Therefore, g is well-supported on $\mathbf{p}_1, \dots, \mathbf{p}_m$. Moreover, each term of the ℓ -focal is the product of a \mathbf{p} -monomial as above and a 4×4 minor of of the symbolic matrix $(A_1^\top \mid \dots \mid A_m^\top)$.

By Corollary 3.9, G_k is a Gröbner basis with respect to $<$ for the ideal

$$J_k = J_{k-1} : p_k[1] = J_{k-1} : p_k[2] = J_{k-1} : p_k[3]. \quad (3.7)$$

When $k = m$, Proposition 3.3 establishes the equality $G_m = \{2\text{-, 3\text{-, and 4-focals}}\}$ and hence $J_m = \langle 2\text{-, 3\text{-, and 4-focals}} \rangle$. Further, G_m is a Gröbner basis with respect to all product orders with $\mathbf{A} < \mathbf{p}$.

To conclude that $J_m = I_{\mathbf{A}, \mathbf{p}}^{m,1}$, we verify that J_m satisfies the three conditions of Proposition 2.3. Proposition 3.4 gives the set-theoretic Condition 1. For Condition 3, we must show that J_m is radical; this follows from Proposition 2.2 since the lead terms of the 2-, 3-, and 4-focals are all squarefree. Finally, for Condition 2, we calculate

$$\begin{aligned} J_m &\subset J_m : \mathfrak{m}_{\mathbf{p}}^{\infty} \\ &= J_m : \mathfrak{m}_{\mathbf{p}} \quad (J_m \text{ is radical}) \\ &\subset J_m : \left(\prod_{k=1}^m (p_k[1] \cdot p_k[2] \cdot p_k[3]) \right) \\ &= ((J_m : p_1[1]) : \cdots) : p_m[3] \\ &= J_m, \quad (\text{Corollary 3.9 w/ } G = G_m) \end{aligned}$$

forcing $J_m = J_m : \mathfrak{m}_{\mathbf{p}}^{\infty}$. A similar calculation, shown below, gives $J_m = J_m : \mathfrak{m}_{\mathbf{A}}^{\infty}$:

$$\begin{aligned} J_m &= J_0 : \mathfrak{m}_{\mathbf{p}} \quad (\text{Proposition 3.3}) \\ &= (J_0 : \mathfrak{m}_{\mathbf{A}}) : \mathfrak{m}_{\mathbf{p}} \quad (\text{Proposition 3.3, Proposition 3.5}) \\ &= (J_0 : \mathfrak{m}_{\mathbf{p}}) : \mathfrak{m}_{\mathbf{A}} \\ &= J_m : \mathfrak{m}_{\mathbf{A}} \\ &= J_m : \mathfrak{m}_{\mathbf{A}}^{\infty}. \quad (J_m \text{ is radical}) \end{aligned}$$

Thus $J_m = J_m : (\mathfrak{m}_{\mathbf{A}} \cap \mathfrak{m}_{\mathbf{p}})^{\infty}$, and we may conclude from Proposition 2.3 that

$$\langle 2\text{-, 3\text{-, and 4-focals}} \rangle = J_m = I_{\mathbf{A}, \mathbf{p}}^{m,1}.$$

□

4. $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$: The multiview ideal for $n = 1$

We now specialize the camera arrangement \mathbf{A} and consider the ideal $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$. Recall that the *multiview variety* is the closed image of the following rational imaging map associated to a camera arrangement $\bar{\mathbf{A}}$:

$$\begin{aligned} \varphi_{\bar{\mathbf{A}}} : \mathbb{P}^3 &\dashrightarrow (\mathbb{P}^2)^m \\ q &\mapsto (\bar{A}_1 q, \dots, \bar{A}_m q). \end{aligned} \quad (4.1)$$

The closure of the image of $\varphi_{\bar{A}}$ is precisely the variety $\Gamma_{\bar{A},p}^{m,1}$ in (1.7). Thus, the vanishing ideal $I_{\bar{A},p}^{m,1}$ is none other than the *multiview ideal* of the camera arrangement \bar{A} [2, 5]. Much previous work has also gone into studying various polynomials that vanish on $\Gamma_{\bar{A},p}^{m,1}$ [20, 27, 38, 59, 61].

In this section, we show how Theorem 3.2 recovers two important previously-known theorems about multiview ideals. Each of these previous results assumes the camera arrangement \bar{A} is generic in one of the senses discussed in Section 2.2. The first theorem characterizes a *universal Gröbner basis* for the multiview ideal of a minor generic camera arrangement; that is, a Gröbner basis with respect to all possible monomial orders on $\mathbb{C}[\mathbf{p}]$.

Theorem 4.1 ([5, Theorem 2.1]). *For minor generic \bar{A} , the specialized 2-, 3-, and 4-focals from $I_{\bar{A},p}^{m,1}$ form a universal Gröbner basis for the multiview ideal $I_{\bar{A},p}^{m,1}$.*

In the work of Aholt, Sturmfels, and Thomas [5], Theorem 4.1 is one of the main results needed to study the Hilbert Scheme parametrizing multiview ideals. We will obtain this result as a direct corollary of Theorem 3.2. Our argument is essentially independent of the original proof, which requires relating the multiview ideal to yet another Hilbert scheme studied in [12]. In our proof, Booher's result in Proposition 3.5 does much of the heavy lifting. Booher's result in turn relies on an older result of Bernstein, Sturmfels, and Zelevinsky [8, 56], the proof of which was later greatly simplified by Conca, De Negri, and Gorla [13]. We also note that $I_{\bar{A},p}^{m,1}$ is *not* a sparse determinantal ideal, since the matrices involved have some nonzero constant entries. Therefore Proposition 3.5 cannot be applied directly to prove Theorem 4.1.

Recall that minor genericity of \bar{A} is stronger than \bar{A} having pairwise distinct centers. For arbitrary \bar{A} with pairwise distinct centers, the Gröbner-theoretic Theorem 4.1 no longer applies. However, we still have the following ideal-theoretic characterization of $I_{\bar{A},p}^{m,1}$.

Theorem 4.2 ([2, Theorem 3.7]). *For \bar{A} with pairwise distinct centers, $I_{\bar{A},p}^{m,1}$ is generated by the specialized 2- and 3-focals.*

The set-theoretic analogue of Theorem 4.2 is a classical result of multiview geometry. Indeed, it can be shown under even stronger genericity assumptions on \bar{A} that just the 2-focals cut out $\Gamma_{\bar{A},p}^{m,1}$ for $m \geq 4$ [27, 59], although they do not generate $I_{\bar{A},p}^{m,1}$.

We observe that the 4-focals, although needed for Theorem 4.1, are not needed for Theorem 4.2. Indeed, it is a well-established principle in the vision literature that the 4-focals "do not add more information" [20] and "are always completely unnecessary" [59]. However, these statements apply only to the multiview ideal $I_{\bar{A},p}^{m,1}$. By contrast, the ideal $I_{\bar{A},p}^{m,1}$ is *not* generated by only the 2- and 3-focals.

Example 4.1. We investigate $I_{\bar{A},p}^{4,1} = \langle 2-, 3-, \text{ and } 4\text{-focals} \rangle$ using the computer algebra system Macaulay2 [22]. The containment $\langle 2- \text{ and } 3\text{-focals} \rangle \subsetneq I_{\bar{A},p}^{4,1}$ is strict, since there is a nonzero remainder upon polynomial division of any 4-focal f by a Gröbner basis for the ideal $\langle 2- \text{ and } 3\text{-focals} \rangle$. However, polynomial division also shows, whenever $i \in [m]$ and $1 \leq i_1 < i_2 < i_3 \leq 4$, that

$$\det A_i[:, \{i_1, i_2, i_3\}] \cdot f \in \langle 2- \text{ and } 3\text{-focals} \rangle. \quad (4.2)$$

(4.2) implies more generally that for any number of cameras m , $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$ may be obtained from successive ideal quotients of $\langle 2\text{- and } 3\text{-focals} \rangle$ by certain subdeterminants of the symbolic camera matrices A_1, \dots, A_m . \square

We reiterate that the 4-focals are not needed to generate $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$, but they are needed to generate $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ where the cameras are symbolic. As Example 4.1 shows, this is because the open condition that each symbolic camera matrix has full rank is not enforced when we consider $I_{\mathbf{A}, \mathbf{p}}^{m,1}$. Moreover, for *any* camera arrangement $\bar{\mathbf{A}}$, Example 4.1 recovers the well-known fact that

$$\langle \text{specialized 2- and 3-focals} \rangle = \langle \text{specialized 2-, 3-, and 4-focals} \rangle.$$

since, for each i , specializing \mathbf{A} to $\bar{\mathbf{A}}$ in each of the four polynomials $\det A_i[:, \{i_1, i_2, i_3\}]$ yields four scalars, at least one of which is nonzero. Bearing this fact in mind, we still work with the 4-focals in the remainder of this section, as they are necessary for the Gröbner-theoretic Theorem 4.1 and in order to apply the results of the previous section. Thus, we state the well-known set-theoretic analogue of Theorem 4.2 as follows.

Proposition 4.3. *If $\bar{\mathbf{A}}$ is a camera arrangement with pairwise distinct centers, then*

$$I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1} = V(\langle \text{specialized 2-, 3-, and 4-focals} \rangle).$$

We provide a complete, self-contained proof of Proposition 4.3 since it is an important step in our proofs of Theorem 4.1 and Theorem 4.2. Also, we appeal to the geometric argument used in the proof when establishing various results later in the text (eg. Theorem 6.1, Theorem 7.5.)

Before proceeding to the proofs, let us consider three simple camera arrangements and their multiview ideals. For the first example, camera centers are distinct, so Proposition 4.3 applies.

Example 4.2. For $m = 2$ cameras with distinct centers, the multiview ideal is a principal ideal which can be obtained by specializing the sole 2-focal generating $I_{\mathbf{A}, \mathbf{p}}^{2,1}$. The \mathbb{Z}^2 -graded Hilbert series of $I_{\mathbf{A}, \mathbf{p}}^{2,1}$ in this case equals

$$\frac{1 - T_{\mathbf{p}_1} T_{\mathbf{p}_2}}{(1 - T_{\mathbf{p}_2})^3 (1 - T_{\mathbf{p}_1})^3} = 1 + 3T_{\mathbf{p}_1} + 3T_{\mathbf{p}_2} + 6T_{\mathbf{p}_1}^2 + 8T_{\mathbf{p}_1}T_{\mathbf{p}_2} + 6T_{\mathbf{p}_2}^2 + \dots$$

Up to coordinate changes in the world and images, such an arrangement takes the form $\bar{\mathbf{A}} = ((I \ 0), (0 \ I))$. Although generic in the sense of having distinct centers, this arrangement is special in the sense that its multiview ideal is *toric* [5, Sec 4]:

$$I_{\bar{\mathbf{A}}, \mathbf{p}}^{2,1} = \langle -p_1[2]p_2[2] + p_1[3]p_2[1] \rangle.$$

\square

The next two examples illustrate why the assumption of distinct centers cannot be relaxed in Proposition 4.3. Namely, when camera centers coincide, the multiview ideal can contain polynomials which are not specializations of polynomials in $I_{\mathbf{A}, \mathbf{p}}^{m,1}$. See [2][Example 3.10] for another instance of this phenomenon.

Example 4.3. For $\bar{\mathbf{A}} = ((I \ 0), (I \ 0))$, the camera centers coincide. Its multiview ideal is generated by the 2×2 minors of the 3×2 matrix $(p_1 \ p_2)$, with Hilbert series

$$\frac{1 - 3T_{\mathbf{p}_1}T_{\mathbf{p}_2} + T_{\mathbf{p}_1}^2T_{\mathbf{p}_2} + T_{\mathbf{p}_1}T_{\mathbf{p}_2}^2}{(1 - T_{\mathbf{p}_2})^3(1 - T_{\mathbf{p}_1})^3} = 1 + 3T_{\mathbf{p}_1} + 3T_{\mathbf{p}_2} + 6T_{\mathbf{p}_1}^2 + 6T_{\mathbf{p}_1}T_{\mathbf{p}_2} + 6T_{\mathbf{p}_2}^2 + \dots$$

In contrast to the previous example, specializing the 2-focal generating $I_{\bar{\mathbf{A}}, \mathbf{p}}^{2,1}$ to $\bar{\mathbf{A}}$ results in the zero polynomial. Thus, in general, $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$ cannot be obtained from $I_{\mathbf{A}, \mathbf{p}}^{m,n}$ by specialization when the cameras in $\bar{\mathbf{A}}$ do not have distinct centers. \square

Example 4.4. For the camera arrangement

$$\bar{\mathbf{A}} = (\bar{A}_1, \bar{A}_2, \bar{A}_3) = ((I \ 0), (I \ 0), (0 \ I)),$$

the first two camera centers coincide. The multiview ideal is

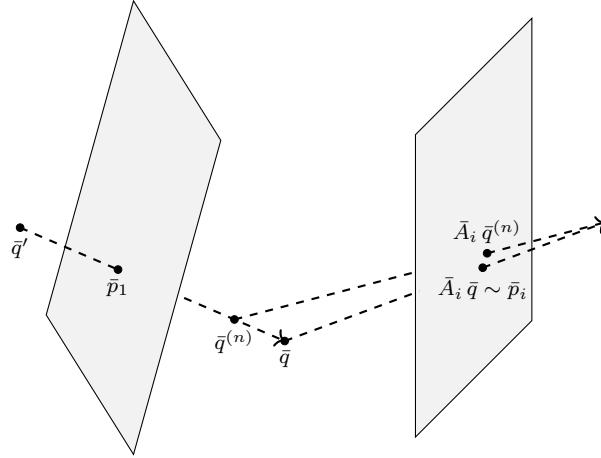
$$\langle \det(\bar{\mathbf{A}} \mid \mathbf{p}) [1 : 3, 1 : 3]_{\{1,3\}}, \det(\bar{\mathbf{A}} \mid \mathbf{p}) [1 : 3, 1 : 3]_{\{2,3\}} \rangle + \text{minors}(2, (p_1 \ p_2)). \quad (4.3)$$

The first summand above is generated by specializations of the 2-focals in $I_{\bar{\mathbf{A}}, \mathbf{p}}^{3,1}$ corresponding to the camera pairs (\bar{A}_1, \bar{A}_3) and (\bar{A}_2, \bar{A}_3) with distinct centers. The second summand is the multiview ideal from the previous example. Upon specializing $I_{\bar{\mathbf{A}}, \mathbf{p}}^{3,1}$ to this particular arrangement $\bar{\mathbf{A}}$, the 2-focal on (A_1, A_2) and all 3-focals in $I_{\bar{\mathbf{A}}, \mathbf{p}}^{3,1}$ become zero, in contrast to the case of a generic camera triple. \square

Theorem 4.2 asserts that the failure to obtain $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$ by specialization from $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ occurs *only* in examples with coincident centers like in Example 4.3 and Example 4.4. We begin by proving the set-theoretic version.

Proof of Proposition 4.3. The inclusion $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1} \subset V(\langle \text{specialized 2-, 3-, and 4-focals} \rangle)$ follows along similar lines as the inclusion $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,1} \subseteq V(2\text{-, 3\text{-, and 4-focals})}$ in the proof of Proposition 3.4. The proof of the reverse inclusion is analogous to the proof of $V(m\text{-focals}) \subseteq \Gamma_{\mathbf{A}, \mathbf{p}}^{m,1}$ in that we construct a sequence in $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$ that limits to any given point in $V(\langle \text{specialized 2-, 3-, and 4-focals} \rangle)$. However, instead of perturbing cameras, we perturb the associated world point, as illustrated in Figure 4. We note that a similar argument appears in the proof of [59, Proposition 1].

Let $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$ be a camera arrangement with distinct centers. Suppose $\bar{\mathbf{p}}$ lies in the vanishing locus of the $\bar{\mathbf{A}}$ -specialized 2-, 3-, and 4-focals. Then $\bar{\mathbf{p}}$ also lies in the vanishing locus of all m -focals and the corresponding specialized m -focal matrix is rank-deficient, so we may consider an element in its nullspace as in (3.5). At most one of $\lambda_1, \dots, \lambda_m$ in that null vector can be zero since if, say, $\lambda_1 = \lambda_2 = 0$, then \bar{q} is the center of both \bar{A}_1 and \bar{A}_2 . If all λ_i are nonzero, then $\bar{\mathbf{p}}$ is in the image of the imaging map (4.1) and thus $\bar{\mathbf{p}} \in \Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$. Otherwise, we may assume without loss of generality that $\lambda_1 = 0$, so that \bar{q} is the center of \bar{A}_1 . We now write $\bar{\mathbf{p}}$ as the limit of a sequence of points in $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$ as $n \rightarrow \infty$, showing that $\bar{\mathbf{p}} \in \Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$. Let \bar{q}' be such that $\bar{A}_1\bar{q}' \sim \bar{p}_1$, and, working affinely, let $\bar{q}^{(n)} = \bar{q} + (1/n)\bar{q}'$. Set $\bar{\mathbf{p}}^{(n)} = (\bar{A}_1\bar{q}^{(n)}, \dots, \bar{A}_m\bar{q}^{(n)})$.

Figure 4. Proof of Proposition 4.3; the $\lambda_1 = 0$ case.

Passing to a subsequence if necessary, we may assume that $\bar{A}_i \bar{q}^{(n)} \neq 0$ for all $i \geq 1$ and all $n \geq 1$. Observe for $i = 1$ and all $n \geq 1$ that

$$\bar{A}_1 \bar{q}^{(n)} \sim \bar{A}_1 \bar{q}' \sim \bar{p}_1$$

For $i \geq 2$, in the limit as $n \rightarrow \infty$,

$$\bar{A}_i \bar{q}^{(n)} = \bar{A}_i \bar{q} + (1/n) \bar{A}_i \bar{q}' \rightarrow \bar{A}_i \bar{q} \sim \bar{p}_i.$$

We have exhibited \bar{p} as the limit of points in $\Gamma_{\bar{A}, \mathbf{p}}^{m,1}$. Since $\Gamma_{\bar{A}, \mathbf{p}}^{m,1}$ is closed in the Euclidean topology, this implies that $\bar{p} \in \Gamma_{\bar{A}, \mathbf{p}}^{m,1}$, proving the desired inclusion. \square

Proof of Theorem 4.1. Let $<$ be any product order on $\mathbb{C}[\mathbf{A}, \mathbf{p}]$ with $\mathbf{A} < \mathbf{p}$, formed from product orders $<_{\mathbf{p}}$ on $\mathbb{C}[\mathbf{p}]$ and $<_{\mathbf{A}}$ on $\mathbb{C}[\mathbf{A}]$. We note the following identity:

$$in_{<} \left(\sum_{\mathbf{p}^{\alpha_1} <_{\mathbf{p}} \dots <_{\mathbf{p}} \mathbf{p}^{\alpha_k}} g_{\alpha_i}(\mathbf{A}) \mathbf{p}^{\alpha_i} \right) = in_{<_{\mathbf{A}}} (g_{\alpha_k}(\mathbf{A})) \mathbf{p}^{\alpha_k}. \quad (4.4)$$

Using this identity and Theorem 3.2, application of Buchberger's S-pair criterion [15, §2.6] shows that the \bar{A} -specialized 2-, 3-, and 4-focals form a Gröbner basis with respect to $<_{\mathbf{p}}$ as long as none of their leading coefficients $g_{\alpha_k}(\bar{A})$ vanish. Each of these leading coefficients is a 4×4 minor of $(\bar{A}_1^\top \mid \dots \mid \bar{A}_m^\top)$ and by minor genericity, they do not vanish. Since $<_{\mathbf{p}}$ is an arbitrary monomial order on $\mathbb{C}[\mathbf{p}]$, the specialized 2-, 3-, and 4-focals form a universal Gröbner basis. To show they generate $I_{\bar{A}, \mathbf{p}}^{m,1}$, apply the recognition criterion of Proposition 2.3. The set-theoretic statement is just Proposition 4.3, and the remaining conditions follow as in Theorem 3.2. \square

Combining the last example with our previous results allows us to recover the main ideal-theoretic result of [2].

Proof of Theorem 4.2. When $\bar{\mathbf{A}}$ is minor generic, this follows by applying Theorem 4.1 and specializing (4.2) for every 4-focal f . For $\bar{\mathbf{A}}$ with pairwise distinct centers, Theorem 2.8 shows that there exists $h = (H_1, \dots, H_m) \in (\mathrm{PGL}_3)^m$ such that

$$h\bar{\mathbf{A}} = (H_1\bar{A}_1, \dots, H_m\bar{A}_m)$$

is minor generic. Consider the automorphism of \mathbb{Z}^m -graded rings defined as

$$\begin{aligned} L_h : \mathbb{C}[\mathbf{p}] &\rightarrow \mathbb{C}[\mathbf{p}] \\ (p_1, \dots, p_m) &\mapsto (H_1 p_1, \dots, H_m p_m). \end{aligned}$$

This allows us to reduce to the minor generic case:

$$\begin{aligned} \langle 2\text{- and } 3\text{-focals} |_{\bar{\mathbf{A}}} \rangle &= L_h(\langle 2\text{- and } 3\text{-focals} |_{h\bar{\mathbf{A}}} \rangle) && ([2, \text{Lemma 2.3, 3.6}]) \\ &= L_h(I_{h\bar{\mathbf{A}}, \mathbf{p}}^{m,1}) && (h\bar{\mathbf{A}} \text{ is minor generic}) \\ &= I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}. \end{aligned}$$

In the above calculation, we note that the results used from [2] are proven by Cauchy-Binet type arguments analogous to the proof of Theorem 2.8. The final equality follows directly from the definition of $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$. Since $\bar{\mathbf{A}}$ was an arbitrary camera arrangement with pairwise distinct centers, the proof is complete. \square

5. $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$: Constraints on cameras, world points, and image points for $n = 1$

We now move on to $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ for m cameras and one world point. Recall that this is the vanishing ideal of the image formation correspondence $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ (1.4), and the second bundle adjustment ideal in Figure 2.

For fixed i , the condition $A_i q \sim p_i$ is equivalent to the vanishing of all 2×2 minors of the matrix $(A_i q \ p_i)$. Thus, we may consider the ideal generated by all such minors, which we denote as:

$$M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = \sum_{i=1}^m \text{minors}(2, (A_i q \ p_i)). \quad (5.1)$$

Each summand, $\text{minors}(2, (A_i q \ p_i))$, appearing in (5.1) looks similar to a well-studied example in combinatorial commutative algebra, namely, the 2×2 minors of a 3×2 matrix of indeterminates, which form a quadratic universal Gröbner basis by [55, Example 1.4]. However, this analogy cannot be pursued too closely here, as the minors in (5.1) are *cubics* and the matrix entries are not indeterminates.

The ideal $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ is a natural candidate for $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$. However, in general $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} \subsetneq I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$.

Example 5.1. For $m = 2$ cameras, we can easily check in Macaulay2 that

$$I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{2,1} = M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{2,1} + I_{\mathbf{A}, \mathbf{p}}^{2,1} \supsetneq M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{2,1} = I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{2,1} \cap \mathfrak{m}_{\mathbf{q}}.$$

\square

In spite of Example 5.1, the ideals $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ and $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ are closely related. Our main result of this section, Theorem 5.1 below, establishes that $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}$.

Theorem 5.1. *The vanishing ideal $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ of the variety $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ may be described as follows:*

For any $1 \leq i \leq 4$,

$$I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : q[i], \quad (5.2)$$

and

$$I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}. \quad (5.3)$$

Thus, $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ is generated by the 2×2 minors generating $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ and the 2-, 3-, and 4-focals.

Once again, we use the recognition criterion of Proposition 2.3 to characterize $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$. The set-theoretic equalities

$$\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = V(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}) = V(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}) \quad (5.4)$$

follow along the same lines as in Proposition 3.4. To verify the remaining conditions in Proposition 2.3, we identify explicit Gröbner bases with squarefree initial ideals. At the core of our arguments is the following technical result, whose proof, unlike Theorem 3.2, is computer-assisted.

Proposition 5.2. *There exists a set $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$, which is a Gröbner basis for $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ with respect to all 12 Lex or GRevLex orders $<$ satisfying*

$$A_m[3, 4] < \dots < A_1[1, 1], \quad q[4] < \dots < q[1], \quad p_m[3] < \dots < p_1[1],$$

where $<$ refines the column-major order on each matrix A_i, q, p_i and the groups $\mathbf{A}, \mathbf{q}, \mathbf{p}$ may be permuted in any of the six possible ways.

Elements of the set $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ are listed explicitly in Appendix C and their degrees are tabulated in Figure 5.

Proof. To begin, we record several important properties of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$.

P1 For $m \geq 4$,

$$G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} = \bigcup_{\sigma \in \binom{[m]}{4}} G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\sigma,$$

where $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\sigma = G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \cap \mathbb{C}[A_{\sigma_1}, A_{\sigma_2}, A_{\sigma_3}, A_{\sigma_4}, q, p_{\sigma_1}, p_{\sigma_2}, p_{\sigma_3}, p_{\sigma_4}]$.

P2 $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ contains all elements of the form $q[4]$ times a 2, 3, or 4-focal.

P3 $\langle G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \rangle = M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$

P4 Aside from the $q[4]$ -bumped focals, no other elements of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ are divisible by any variable in $\mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}]$.

P5 The lead terms of all $g \in G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ are squarefree.

P1 and **P2** follow upon inspecting the polynomials listed in Appendix C, each of which depends on at most 4 cameras and their corresponding image points.

For **P3**, the inclusion $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} \subset \langle G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \rangle$ holds since $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ contains all the $3m$ cubic 2×2 minors generating $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$. The reverse inclusion and the assertions **P4** and **P5** may all be verified for $m = 1, \dots, 4$ using Macaulay2, and for $m > 4$ using **P1**. Here we are using the fact that $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ is the sum of all minor ideals $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{|\sigma|,1}$ as σ varies over all 4 element subsets of $[m]$.

Having established **P1-P5**, Macaulay2 verifies that $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is a Gröbner basis for all 12 term orders for each of the “base cases” $m = 1, \dots, 8$. For $m > 8$, we use Buchberger’s S-pair criterion: $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is a Gröbner basis for the monomial order $<$ if and only if for any $f, g \in G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$, the S -pair

$$S(f, g) = \text{lcm}(in_<(f), in_<(g))(f/in_<(f) - g/in_<(g))$$

has zero remainder after division by $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$. Consider any $f, g \in G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$. By **P1**, we have $f \in G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\sigma$ and $g \in G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\tau$ for some $\sigma, \tau \in \binom{[m]}{4}$. Thus

$$S(f, g) \in \langle G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\sigma \cup G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}^\tau \rangle.$$

This S-pair involves at most 8 cameras, so its remainder upon division by $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is zero by the base cases. \square

In contrast to the set of 2-, 3-, and 4-focals that form a Gröbner basis for $I_{\mathbf{A}, \mathbf{p}}^{m,1}$, the Gröbner basis $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is quite mysterious. We list three additional properties which may provide further clues to its structure.

P6 The elements of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ are linear in the variable group \mathbf{q} , and may be partitioned according to their support as follows:

Degree-3 elements have \mathbf{q} -support $\{q[1], q[2], q[3], q[4]\}$.

Degree-4, 5 elements have \mathbf{q} -support $\{q[2], q[3], q[4]\}$.

Degree-6, 7 elements which are not bumped 2-focals have \mathbf{q} -support $\{q[3], q[4]\}$.

Bumped 2-, 3-, and 4-focals have \mathbf{q} -support $\{q[4]\}$.

P7 Viewed as polynomials in \mathbf{q} and \mathbf{p} , the leading coefficient of any element of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ with respect to either of the induced Lex orders on $\mathbb{C}[\mathbf{A}][\mathbf{q}, \mathbf{p}]$ is always a $k \times k$ minor of $(A_1^\top \ \dots \ A_m^\top)$ for some $1 \leq k \leq 4$.

P8 With respect to the 12 monomial orders in Proposition 5.2, there are 4 distinct initial ideals $in_<(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1})$, realized by letting $<$ be one of the two GRevLex orders with $\mathbf{q} < \mathbf{A}, \mathbf{p}$ or one of the two Lex orders with $\mathbf{q} > \mathbf{A}, \mathbf{p}$.

By Proposition 5.2, it suffices to check these properties for $m = 4$ cameras. Property **P7** justifies the notion of ultra minor genericity in Definition 2.6, which we make use of in Section 6.

Definition 5.3. Let $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ be the set consisting of all 2-, 3-, and 4-focals together with the elements of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ which are not bumped focals.

degree	$G_{M_{A,q,p}^{m,1}}$	$G_{I_{A,q,p}^{m,1}}$
3	$3m$	$3m$
4	m	m
5	$9\binom{m}{2}$	$9\binom{m}{2}$
6	$6\binom{m}{2}$	$7\binom{m}{2}$
7	$\binom{m}{2} + 27\binom{m}{3}$	$54\binom{m}{3}$
8	$27\binom{m}{3}$	$81\binom{m}{4}$
9	$81\binom{m}{4}$	0

Figure 5. The number of generators of each degree in $G_{M_{A,q,p}^{m,1}}$ and $G_{I_{A,q,p}^{m,1}}$.

Alternatively, $G_{I_{A,q,p}^{m,1}}$ consists of the elements of $G_{M_{A,q,p}^{m,1}}$ after bumping down the elements of the form $q[4] \cdot k$ -focal where $k = 2, 3, 4$. See Figure 5 for the degrees of elements in $G_{I_{A,q,p}^{m,1}}$.

Our notation $G_{I_{A,q,p}^{m,1}}$ foreshadows the not-yet-proven fact that this set is a Gröbner basis of $I_{A,q,p}^{m,1}$. This will be established in the proof of of Theorem 5.1. First we write down some corollaries to Proposition 5.2.

Corollary 5.4. (1) $G_{I_{A,q,p}^{m,1}}$ is a Gröbner basis for the ideal

$$M_{A,q,p}^{m,1} + I_{A,p}^{m,1} = M_{A,q,p}^{m,1} : q[4] \quad (5.5)$$

with respect to any of the 12 monomial orders defined in Proposition 5.2.

(2) Moreover,

$$M_{A,q,p}^{m,1} + I_{A,p}^{m,1} = (M_{A,q,p}^{m,1} + I_{A,p}^{m,1}) : A_m[3,4] = (M_{A,q,p}^{m,1} + I_{A,p}^{m,1}) : p_m[3]. \quad (5.6)$$

Proof. Let $<$ be one of the two GRevLex orders appearing in property **P8** of $G_{M_{A,q,p}^{m,1}}$. Since $G_{M_{A,q,p}^{m,1}}$ is a Gröbner basis for $M_{A,q,p}^{m,1}$ with respect to $<$ by Proposition 5.2, combining the definition of $G_{I_{A,q,p}^{m,1}}$ and Lemma 2.4 says that $G_{I_{A,q,p}^{m,1}}$ is a Gröbner basis of $M_{A,q,p}^{m,1} : q[4]$ with respect to $<$. On the other hand, the ideal generated by $G_{I_{A,q,p}^{m,1}}$ is precisely $M_{A,q,p}^{m,1} + I_{A,p}^{m,1}$. Indeed, by the construction in Definition 5.3, $\langle G_{I_{A,q,p}^{m,1}} \rangle$ contains $M_{A,q,p}^{m,1} + I_{A,p}^{m,1}$, while every element of $G_{I_{A,q,p}^{m,1}}$ lies in $M_{A,q,p}^{m,1} + I_{A,p}^{m,1}$. Therefore, we get that $M_{A,q,p}^{m,1} + I_{A,p}^{m,1} = M_{A,q,p}^{m,1} : q[4]$.

Now suppose $<$ is one of the two Lex orders appearing in **P8**. Using **P6**, the elements of $G_{M_{A,q,p}^{m,1}}$ are well-supported with respect to q , and hence we may apply Lemma 3.8 to conclude that $G_{I_{A,q,p}^{m,1}}$ is a Gröbner basis of $M_{A,q,p}^{m,1} : q[4]$ with respect to $<$.

Next we argue that $G_{I_{A,q,p}^{m,1}}$ is a Gröbner basis for $M_{A,q,p}^{m,1} + I_{A,p}^{m,1}$ with respect to any of the remaining 8 orders considered in Proposition 5.2. Letting $<'$ be such an order, **P8** lets us choose

$<$ from the 4 previously considered orders such that

$$\text{in}_{<'}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}) = \text{in}_{<}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}). \quad (5.7)$$

We may apply either Lemma 2.4 when $<$ is a GRevLex order or Lemma 3.8 when $<$ is a Lex order to the monomial ideal $\text{in}_{<}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1})$. In either case this gives

$$(\text{in}_{<}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1})) : q[4] = \langle \text{in}_{<}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}) \rangle, \quad (5.8)$$

where $\text{in}_{<}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}})$ is the set of leading terms of elements in $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$. Thus,

$$\begin{aligned} \text{in}_{<'} \langle G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \rangle &= \text{in}_{<'} (M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : q[4]) & \langle \langle G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \rangle \rangle &= M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : q[4] \\ &\subset (\text{in}_{<'}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1})) : q[4] & & \\ &= (\text{in}_{<}(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1})) : q[4] & & \text{(Equation (5.7))} \\ &= \langle \text{in}_{<}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}) \rangle & & \text{(Equation (5.8))} \\ &= \text{in}_{<}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}) & & (G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}} \text{ is a Gröbner basis with respect to } <.) \end{aligned}$$

The inclusion \subset follows from the definition of an ideal quotient. Since distinct initial ideals are incomparable with respect to inclusion, we conclude

$$\text{in}_{<'}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}) = \text{in}_{<}(G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}),$$

showing that $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is a Gröbner basis with respect to all 12 term orders.

Finally, we establish (5.6). Note that since no polynomial in $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ has $A_m[3, 4]$, $p_m[3]$ or $q[4]$ as a factor, by Lemma 2.4, $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is a GRevLex Gröbner basis of $(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}) : A_m[3, 4]$, $(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}) : q[4]$ and $(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}) : p_m[3]$. Hence, all three of these ideal quotients equal the ideal generated by $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ which is $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}$. \square

To complete the proof of Theorem 5.1, we need a basic lemma on the quotient by some variable of a polynomial ideal that is invariant under the action of a permutation group.

Lemma 5.5. *Let $P \subset S_k$ be a permutation group with its standard action on $\mathbb{C}[x_1, \dots, x_k]$. Suppose $I \subset \mathbb{C}[x_1, \dots, x_k]$ is an ideal which is P -invariant, meaning, $P \cdot I = I$. Then*

$$I = I : x_i \quad \Rightarrow \quad I = I : x_{\rho(i)} \quad \forall \rho \in P.$$

Proof. Suppose $f \in I : x_{\rho(i)}$. Equivalently,

$$(\rho \cdot x_i) f = x_{\rho(i)} f \in I.$$

We need to show $f \in I$ as well. Note that

$$\rho^{-1} \cdot (x_{\rho(i)} f) = x_i (\rho^{-1} \cdot f) \in P \cdot I = I.$$

This gives

$$\rho^{-1} \cdot f \in I : x_i = I.$$

Thus,

$$f = \rho \cdot (\rho^{-1} \cdot f) \in P \cdot I = I.$$

□

Corollary 5.6. *The following identities hold for any $i \in [m], j \in [3], k \in [4]$.*

$$\begin{aligned} M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : q[4] &= M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : q[k], \\ M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : A_m[3, 4] &= M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : A_i[j, k], \\ M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : p_m[3] &= M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} : p_i[j]. \end{aligned}$$

Proof. We deduce each identity from Lemma 5.5 by producing a suitable permutation $\rho \in P$, where the group $P \subset S_{4+15m}$ is generated by all permutations taking one of three forms:

- (1) Camera permutations, given by some $\rho \in S_m$ which sends A_i to $A_{\rho(i)}$ and p_i to $p_{\rho(i)}$.
- (2) World coordinate permutations, given by some $\rho \in S_4$ which sends $q[k]$ to $q[\rho(k)]$ and sends $A_i[:, k]$ to $A_i[:, \rho(k)]$.
- (3) Image coordinate permutations, given by $\rho \in S_3$ which sends the j th coordinate of $p_i[j]$ to $p_i[\rho(j)]$ and $A_i[j, :]$ to $A_i[\rho(j), :]$.

Note that $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}$ is invariant under the action of P . To deduce the first identity in the statement of the Corollary, apply a permutation of the form 2 above. For the second identity, apply 2, then 3, then 1. For the third identity, apply 2, then 1. □

Proof of Theorem 5.1. By Corollary 5.4, $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ is a Gröbner basis for $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}$ whose initial monomials are squarefree. Therefore, $M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}$ is a radical ideal by Proposition 2.2. By Corollary 5.4 and Corollary 5.6, this ideal is saturated with respect to all variables. This gives us both (5.2) and that

$$M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1}$$

is saturated with respect to the irrelevant ideal. Equation (5.4) gives the set-theoretic statement $\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} = V(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}) = V(M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1})$. Thus, (5.3), which states that

$$M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\mathbf{A}, \mathbf{p}}^{m,1} = I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1},$$

now follows from Proposition 2.3. □

6. $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$: Completing the square for $n = 1$

Just as our results on the ideal $I_{\mathbf{A}, \mathbf{p}}^{m,1}$ allow us to recover known results about the multiview ideal $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$, we may combine the results of the previous section with specialization arguments to study $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$. This ideal is an analogue of the multiview ideal that does not eliminate the world point variables \mathbf{q} . It is the final ideal in the square seen in Figure 2. The next two theorems characterize $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$ Gröbner-theoretically and ideal-theoretically.

Theorem 6.1. *Specializing the Gröbner basis $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ to an ultra minor generic camera arrangement $\bar{\mathbf{A}}$ yields a (non-reduced) Gröbner basis for $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$ with respect to the restrictions of Lex or GRevLex orders to $\mathbb{C}[\mathbf{q}, \mathbf{p}]$ such that $\mathbf{A} > \mathbf{q}, \mathbf{p}$.*

Proof. The strategy is much like the proof of Theorem 4.1. Property **P7** of $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ implies that the leading coefficients of the elements of $G_{I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ in $\mathbb{C}[\mathbf{A}]$ are nonzero after specialization to an ultra minor generic camera arrangement $\bar{\mathbf{A}}$. Since the Lex orders are product orders, the specialized Gröbner basis elements $G_{I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}}$ form a Gröbner basis for both Lex orders in the statement of the theorem. Moreover, the leading terms of $G_{I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}}$ with respect to either Lex order are the same for the corresponding GRevLex order. Comparing leading terms as in Corollary 5.4 proves that $G_{I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}}$ forms a Gröbner basis for the ideal it generates. Finally, to prove that this ideal is $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$, apply the recognition criterion of Proposition 2.3 in a manner analogous to the proof of Theorem 5.1. \square

Theorem 6.2. *If $\bar{\mathbf{A}}$ has pairwise distinct centers, then specializing the 2×2 minors of $(A_i q - p_i)$ and 2- and 3-focals yields generators for $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$. In particular,*

$$I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1} = M_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1} + I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}. \quad (6.1)$$

Proof. A group element $h = (H, H_1, \dots, H_m) \in \mathrm{PGL}_4 \times (\mathrm{PGL}_3)^m$ induces a linear automorphism of the \mathbb{Z}^{m+1} -graded ring $\mathbb{C}[\mathbf{q}, \mathbf{p}]$:

$$\begin{aligned} L_h : \mathbb{C}[\mathbf{q}, \mathbf{p}] &\rightarrow \mathbb{C}[\mathbf{q}, \mathbf{p}] \\ q &\mapsto Hq, \quad p_i \mapsto H_i p_i. \end{aligned}$$

We have already shown in Theorem 6.1 that (6.1) holds when $\bar{\mathbf{A}}$ is ultra minor generic. To extend this to $\bar{\mathbf{A}}$ with pairwise distinct centers, we apply Theorem 2.8 which reduces the proof to checking that the ideals $M_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1}$ and $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1}$ are preserved under coordinate change, ie.

$$M_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1} = L_h \left(M_{h\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,1} \right), \quad I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1} = L_h \left(I_{h\bar{\mathbf{A}}, \mathbf{p}}^{m,1} \right).$$

The second equality appeared already in the proof of Theorem 4.2. The first may be proven along similar lines: we calculate

$$\begin{aligned} (\bar{A}_i q - p_i) &= (H_i^{-1} (H_i A_i H^{-1}) Hq - H_i^{-1} (H_i p_i)) \\ &= H_i^{-1} \cdot ((h\bar{\mathbf{A}})_i \cdot L_h(q) - L_h(p_i)). \end{aligned}$$

A Cauchy-Binet argument analogous to that given in Appendix B shows that every 2×2 minor of the matrix on the left-hand side is a linear combination of 2×2 minors of the matrix on the right. This gives the inclusion $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,1} \subset L_h \left(I_{h\bar{\mathbf{A}}, \mathbf{p}}^{m,1} \right)$. The reverse inclusion is similar. \square

In closing, we note that the ideals appearing in this section and the last provide another route to understanding the ideals obtained by eliminating \mathbf{q} from them. For instance, we could use

the results of these sections to give a different proof of Theorem 4.2, or the ideal-theoretic part of Theorem 3.2.

7. $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}, I_{\mathbf{A}, \mathbf{p}}^{m,n}, I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}, I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$: The square for $n \geq 1$

So far, we have characterized the ideals in Figure 2 for the case of $n = 1$ world point and m cameras. In this section, we establish results for these ideals when $n \geq 1$.

For each $k = 1, \dots, n$, denote as follows the elimination ideals involving the k th world point:

$$\begin{aligned} I_{\mathbf{A}, \mathbf{q}_k, \mathbf{p}_k}^{m,1} &= I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} \cap \mathbb{C}[\mathbf{A}, \mathbf{q}_k, \mathbf{p}_{1k} \dots, \mathbf{p}_{mk}], \\ I_{\mathbf{A}, \mathbf{p}_k}^{m,1} &= I_{\mathbf{A}, \mathbf{p}}^{m,n} \cap \mathbb{C}[\mathbf{A}, \mathbf{p}_{1k} \dots, \mathbf{p}_{mk}], \\ I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1} &= I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n} \cap \mathbb{C}[\mathbf{q}_k, \mathbf{p}_{1k} \dots, \mathbf{p}_{mk}], \\ I_{\bar{\mathbf{A}}, \mathbf{p}_k}^{m,1} &= I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n} \cap \mathbb{C}[\mathbf{p}_{1k} \dots, \mathbf{p}_{mk}]. \end{aligned} \tag{7.1}$$

One naturally hopes that summing the appropriate extensions of the ideals in (7.1), over $k = 1, \dots, n$, would give us the ideals of Figure 2 for arbitrary n . If this were true, our results about explicit generators for these ideals in the $n = 1$ case would generalize effortlessly. For the triangulation ideals $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$ and $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$, this ideal scenario turns out to be true. The essential observation is that $\Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$ and $\Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$ are both direct products of the restricted varieties:

$$\begin{aligned} \Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n} &= V(I_{\bar{\mathbf{A}}, \mathbf{q}_1, \mathbf{p}_1}^{m,1}) \times \dots \times V(I_{\bar{\mathbf{A}}, \mathbf{q}_n, \mathbf{p}_n}^{m,1}), \\ \Gamma_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n} &= V(I_{\bar{\mathbf{A}}, \mathbf{p}_1}^{m,1}) \times \dots \times V(I_{\bar{\mathbf{A}}, \mathbf{p}_n}^{m,1}). \end{aligned}$$

Let $\mathbb{C}[\mathbf{q}, \mathbf{p}] I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}$ be the extension of $I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}$ in the ring $\mathbb{C}[\mathbf{q}, \mathbf{p}]$, i.e., the ideal generated by the generators of $I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}$ in $\mathbb{C}[\mathbf{q}, \mathbf{p}]$. The first result of this section determines explicit generators for the triangulation ideals $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$ and $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$ whenever $\bar{\mathbf{A}}$ has pairwise distinct centers.

Theorem 7.1 (Triangulation ideals when $n \geq 1$). *For $\bar{\mathbf{A}}$ with pairwise distinct centers and any $n \geq 1$, the triangulation ideals are obtained by summing the extensions of individual ideals in the following sense:*

$$I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n} = \sum_{k=1}^n \mathbb{C}[\mathbf{q}, \mathbf{p}] I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}, \tag{7.2}$$

$$I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n} = \sum_{k=1}^n \mathbb{C}[\mathbf{p}] I_{\bar{\mathbf{A}}, \mathbf{p}_k}^{m,1}. \tag{7.3}$$

Under appropriate genericity assumptions on $\bar{\mathbf{A}}$, the Gröbner bases studied in Section 4 and Section 6 may be combined to give Gröbner bases for any $n \geq 1$. A precise statement is the following result.

Theorem 7.2 (Gröbner bases for triangulation ideals when $n \geq 1$). *Suppose G_1, \dots, G_n are Gröbner bases for the restricted ideal $I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}$ (or $I_{\bar{\mathbf{A}}, \mathbf{p}_k}^{m,1}$) for $k = 1, \dots, n$, with respect to the term orders $<_1, \dots, <_n$ allowed in Theorem 6.1 (or Theorem 4.1). If $<$ is any term order on $\mathbb{C}[\mathbf{q}, \mathbf{p}]$ (or*

$\mathbb{C}[\mathbf{p}]$) refining $<_1, \dots, <_n$, then $G = G_1 \cup \dots \cup G_n$ is a Gröbner basis with respect to $<$ for the triangulation ideal $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$ (or $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$).

In particular, if $\bar{\mathbf{A}}$ is a minor generic arrangement, then the set of specialized 2-, 3-, and 4-focals form a universal Gröbner basis for $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$. Likewise, if $\bar{\mathbf{A}}$ is ultra minor generic, then the union of the Gröbner bases $G_{I_{\bar{\mathbf{A}}, \mathbf{q}_k, \mathbf{p}_k}^{m,1}}$ described in Theorem 6.1 is a Gröbner basis for $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$ with respect to a suitable refinement of the GRevLex or Lex orders used in Theorem 6.1.

In the case of the two bundle adjustment ideals in Figure 2 we can write down an ideal-theoretic result using our results for $n = 1$. These ideals may be obtained by summing the corresponding extended ideals for each individual world point, and then saturating by a polynomial defining an appropriate locus of non-generic cameras.

Theorem 7.3. *For any $n \geq 1$ we have*

$$I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} = \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}] I_{\mathbf{A}, \mathbf{q}_k, \mathbf{p}_k}^{m,1} \right) : s_{\text{ultra}}^{\infty}, \quad (7.4)$$

$$I_{\mathbf{A}, \mathbf{p}}^{m,n} = \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) : s^{\infty}, \quad (7.5)$$

where s is the product of all 4×4 minors of $(A_1^\top \ \dots \ A_m^\top)$, and s_{ultra} is the product of all minors (of any size) of $(A_1^\top \ \dots \ A_m^\top)$.

The following is an immediate corollary of Theorem 7.3.

Corollary 7.4. *Set-theoretically,*

$$\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} = \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}] I_{\mathbf{A}, \mathbf{q}_k, \mathbf{p}_k}^{m,1} \right) \setminus V(s_{\text{ultra}}) \right), \text{ and} \quad (7.6)$$

$$\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} = \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus V(s) \right). \quad (7.7)$$

However, we can also prove a slightly stronger set-theoretic statement.

Theorem 7.5. *Consider the subvariety $C_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} \subset \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ consisting of all tuples $(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}})$ where $\bar{\mathbf{A}}$ is such that the kernels of some pair of camera matrices intersect nontrivially. Let $C_{\mathbf{A}, \mathbf{p}}^{m,n}$ denote its projection into $\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n}$. Then*

$$\Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} = \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{q}, \mathbf{p}] I_{\mathbf{A}, \mathbf{q}_k, \mathbf{p}_k}^{m,1} \right) \setminus C_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} \right), \quad (7.8)$$

$$\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} = \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus C_{\mathbf{A}, \mathbf{p}}^{m,n} \right). \quad (7.9)$$

We note that Theorem 7.5 automatically implies Corollary 7.4, since every (ultra) minor generic camera arrangement has pairwise distinct centers. For instance, since $C_{\mathbf{A}, \mathbf{p}}^{m,n} \subsetneq V(s)$, we have the containments

$$\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} \subset \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus V(s) \right) \subset \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus C_{\mathbf{A}, \mathbf{p}}^{m,n} \right),$$

and Theorem 7.5 gives equality throughout. On the other hand, it remains unclear whether or not Theorem 7.5 can be “upgraded” to an ideal-theoretic result, since the technique used to prove the ideal-theoretic Theorem 7.3 requires working in rings where the various minors have inverses.

To summarize our results for $n \geq 1$ world point, the situation for the triangulation ideals $I_{\mathbf{A}, \mathbf{p}}^{m,n}$ and $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$ is as nice as possible. For the bundle adjustment ideals $I_{\mathbf{A}, \mathbf{p}}^{m,n}$ and $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n}$, Theorem 7.3 provides a characterization up to saturation by certain loci of non-generic cameras. However, we know little about explicit generators of the bundle adjustment ideals when $n > 1$. Below, we investigate the smallest interesting cases, $I_{\mathbf{A}, \mathbf{p}}^{2,2}$ and $I_{\mathbf{A}, \mathbf{p}}^{2,3}$, highlighting differences from the results for $n = 1$ established in Section 3.

Example 7.1. Consider the case of $(m, n) = (2, 2)$. The ideal $\mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_1}^{2,1} + \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_2}^{2,1}$ is minimally generated by the two 2-focals of degree 6 and is already saturated with respect to $\langle s \rangle$. Therefore, $I_{\mathbf{A}, \mathbf{p}}^{2,2} = \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_1}^{2,1} + \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_2}^{2,1}$ as stated in Theorem 7.3. The equality can also be verified computationally, by checking that the right-hand side is a prime ideal of Krull dimension $34 = 11m + 3n + m + mn$, where the right-hand side is the dimension of the affine cone $\widehat{\Gamma_{\mathbf{A}, \mathbf{p}}^{m,n}}$. The situation with Gröbner bases is more complicated. Using Macaulay2, we may verify that $I_{\mathbf{A}, \mathbf{p}}^{2,2}$ has a reduced Gröbner basis consisting of the 2-focals together with four additional elements of degrees 8 through 11 for a Lex order with $\mathbf{A} > \mathbf{p}$. This is not the union of any Lex Gröbner bases of the individual ideals, which would consist of just the two 2-focals. \square

Example 7.1 shows that Theorem 3.2 does not generalize for $n > 1$. Based on this example, it is tempting to conjecture that the k -focals might still generate $I_{\mathbf{A}, \mathbf{p}}^{m,n}$ for all m and n . However, such a conjecture is typically false for $n > 1$. In fact, for $m = 2$ cameras and $n = 3$ world points, the next example shows that *the k -focals do not even cut out $\Gamma_{\mathbf{A}, \mathbf{p}}^{2,3}$ set-theoretically!*

Example 7.2. Consider, in the case of $(m, n) = (2, 3)$, the ideal I generated by the three 2-focals associated to a partially-symbolic arrangement $\mathbf{B} = ((I \ 0), B_2)$ in the ring $\mathbb{C}[\mathbf{B}_2, \mathbf{p}_{11}, \dots, \mathbf{p}_{23}]$. The ideal I is radical, with the prime decomposition

$$I = I_{\mathbf{B}, \mathbf{p}}^{2,3} \cap \langle B_2[:, 4] \rangle.$$

Thus, the vanishing ideal $I_{\mathbf{B}, \mathbf{p}}^{2,3}$ may be obtained as an ideal quotient of the focal ideal:

$$I_{\mathbf{B}, \mathbf{p}}^{2,3} = I : \langle B_2[:, 4] \rangle.$$

The prime component $\langle B_2[:, 4] \rangle$ may be understood as defining the locus of camera pairs of the form \mathbf{B} , with coincident centers. The component $I_{\mathbf{B}, \mathbf{p}}^{2,3}$ is generated by the 2-focals and an additional polynomial $f(\mathbf{B}_2, \mathbf{p})$ of degree 9 given by

$$f(\mathbf{B}_2, \mathbf{p}) = \det(p_{21} \times (B_2[:, 1:3]p_{11}) \mid p_{22} \times (B_2[:, 1:3]p_{12}) \mid p_{23} \times (B_2[:, 1:3]p_{13})), \quad (7.10)$$

where \times denotes the usual cross product. This polynomial f *does not* lie in \mathbf{I} . However, if we specialize to $B_2 = \bar{B}_2$ generic, then the specialization of f is in the specialized focal ideal associated to (B_1, \bar{B}_2) by Theorem 7.1.

In the above, fixing the first camera allows us to decompose \mathbf{I} more easily. However, we note this example also allows us to prove the strict containment

$$I_{\mathbf{A}, \mathbf{p}}^{2,3} \supsetneq I_{\mathbf{A}, \mathbf{p}_1}^{2,1} + I_{\mathbf{A}, \mathbf{p}_2}^{2,1} + I_{\mathbf{A}, \mathbf{p}_3}^{2,1} \quad (7.11)$$

when both cameras A_1 and A_2 are indeterminate. To see this, define the following 4×4 symbolic matrix in block form:

$$H(A_1) = \begin{pmatrix} \text{adj } A_1[:, 1:3] & -\text{adj } A_1[:, 1:3] A_1[:, 4] \\ 0 & \det A_1[:, 1:3] \end{pmatrix}.$$

This is an explicit formula for a world coordinate change $H \in \text{PGL}_4$ fixing the first camera to be a multiple of $(I \ 0)$; if we write

$$B_1 = A_1 H(A_1) \sim (I \ 0), \quad B_2 = A_2 H(A_1),$$

then $g(\mathbf{A}, \mathbf{p}) := f(B_2, \mathbf{p})$ lies in the left, but not the right, ideal in (7.11). \square

In the vision literature, the polynomial f in (7.10) defines a scalar triple product constraint that was considered in several previous works, eg. [1, 30]. We emphasize that, after *specializing* \mathbf{A} to an arrangement $\bar{\mathbf{A}}$ with pairwise distinct centers, the specialized polynomial $g(\bar{\mathbf{A}}, \mathbf{p})$ will lie in the triangulation ideal $I_{\bar{\mathbf{A}}, \mathbf{p}}^{2,3}$ (see eg. [1, Appendix 2].) This *does not* imply the corresponding statement for the bundle adjustment ideal $I_{\mathbf{A}, \mathbf{p}}^{2,3}$. Thus, we observe a similar phenomenon as in Example 4.1, in which the 4-focals were required to cut out $\Gamma_{\mathbf{A}, \mathbf{p}}^{4,1}$; equations that are necessary to define a bundle adjustment ideal may generically specialize to redundant constraints in the associated triangulation ideal.

Returning to the triangulation ideals $I_{\bar{\mathbf{A}}, \mathbf{p}}^{m,n}$ and $I_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n}$, we now prove Theorem 7.1 and Theorem 7.2, making crucial use of the results for $n = 1$ world point given in Section 4 and Section 6.

Proof of Theorem 7.1 and Theorem 7.2. Equations (7.2) and (7.3) follow by applying Theorem 4.2 and Theorem 6.2, respectively, and the fact that the vanishing ideal of the direct product of closed subvarieties in projective space is the sum of their individual vanishing ideals. This proves Theorem 7.1.

For Theorem 7.2, we may show that G is a Gröbner basis using Buchberger's S-pair criterion. Two $g, g' \in G$ are either in the same G_i or they have disjoint variable support. In the first case, $S(g, g') \rightarrow_{G_i} 0$ implies $S(g, g') \rightarrow_G 0$; in the latter case, the lead terms are relatively prime, so Buchberger's first criterion [15, Proposition 4, pp. 106] implies $S(g, g') \rightarrow_G 0$. The explicit descriptions of G for suitably generic $\bar{\mathbf{A}}$ follow from Theorem 4.1 and Theorem 6.1, respectively. \square

Now it only remains to prove our results about the varieties $\Gamma_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$, $\Gamma_{\mathbf{A},\mathbf{p}}^{m,n}$ and their vanishing ideals $I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$ and $I_{\mathbf{A},\mathbf{p}}^{m,n}$. As in the case of the triangulation ideals treated in the previous proof, the $n = 1$ results of Section 3 and Section 5 are used in an essential way. As a warm-up to the ideal-theoretic Theorem 7.3, we first prove the simpler, more geometric result of Theorem 7.5.

Proof of Theorem 7.5. We prove the equality

$$\Gamma_{\mathbf{A},\mathbf{p}}^{m,n} = \text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus C_{\mathbf{A}, \mathbf{p}}^{m,n} \right). \quad (7.12)$$

The proof of the second equality in the theorem statement is identical. First, we observe that

$$\Gamma_{\mathbf{A},\mathbf{p}}^{m,n} = \text{cl} \left(\Gamma_{\mathbf{A},\mathbf{p}}^{m,n} \setminus C_{\mathbf{A},\mathbf{p}}^{m,n} \right). \quad (7.13)$$

This follows since $\Gamma_{\mathbf{A},\mathbf{p}}^{m,n}$ is an irreducible variety, and $\Gamma_{\mathbf{A},\mathbf{p}}^{m,n} \cap C_{\mathbf{A},\mathbf{p}}^{m,n}$ is a proper subvariety of $\Gamma_{\mathbf{A},\mathbf{p}}^{m,n}$. The equality

$$\text{cl} \left(V \left(\sum_{k=1}^n \mathbb{C}[\mathbf{A}, \mathbf{p}] I_{\mathbf{A}, \mathbf{p}_k}^{m,1} \right) \setminus C_{\mathbf{A}, \mathbf{p}}^{m,n} \right) = \text{cl} \left(\Gamma_{\mathbf{A},\mathbf{p}}^{m,n} \setminus C_{\mathbf{A},\mathbf{p}}^{m,n} \right), \quad (7.14)$$

follows by the same arguments used in the proof of Proposition 3.4. Combining (7.13) and (7.14) gives (7.12). \square

Finally, we prove the ideal-theoretic Theorem 7.3. In the proof below, we argue the result for $I_{\mathbf{A},\mathbf{p}}^{m,n}$ in (7.5). A nearly identical argument, relying on Property **P7** of the Gröbner basis $G_{I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,1}}$ in Section 5, allows us to show (7.4). The general idea is to mimic the proof of Theorem 7.2 by extending scalars $\mathbb{C}[\mathbf{A}] \hookrightarrow \mathbb{C}[\mathbf{A}]_s$ so that G is a Gröbner basis for the ideal it generates in the polynomial ring $\mathbb{C}[\mathbf{A}]_s[\mathbf{p}]$. We emphasize that G need not be a Gröbner basis in the polynomial ring $\mathbb{C}[\mathbf{A}, \mathbf{p}]$, due to Example 7.1.

Proof of Theorem 7.3. Let G_1, \dots, G_n be the sets of 2-, 3-, and 4-focals which respectively generate the ideals $I_{\mathbf{A}, \mathbf{p}_1}^{m,1}, \dots, I_{\mathbf{A}, \mathbf{p}_n}^{m,1}$. Set $G = G_1 \cup \dots \cup G_n$. We extend scalars by localizing $\mathbb{C}[\mathbf{A}]$ at the powers of s :

$$\mathbb{C}[\mathbf{A}]_s = \{f/s^k \text{ s.t. } f \in \mathbb{C}[\mathbf{A}], k \geq 0\}.$$

Let ι be the natural inclusion

$$\iota : \mathbb{C}[\mathbf{A}][\mathbf{p}] \hookrightarrow \mathbb{C}[\mathbf{A}]_s[\mathbf{p}]$$

Note that saturating by $\langle s \rangle$ is the same as extending and contracting through the map ι ,

$$\langle G \rangle : s^\infty = \iota^{-1} (\mathbb{C}[\mathbf{A}]_s \langle G \rangle).$$

We claim that

$$I_{\mathbf{A},\mathbf{p}}^{m,n} = \langle G \rangle : s^\infty. \quad (7.15)$$

To prove this, we first argue that $\iota(G)$ forms a Gröbner basis for the extended ideal $\mathbb{C}[\mathbf{A}]_s \langle G \rangle$ with respect to any fixed monomial order $<$. This requires some care. Arguments based on Buchberger's algorithm, although valid over an arbitrary field, do not extend to arbitrary rings

of scalars. However, we may first extend the coefficient ring to the fraction field $\mathbb{C}(A)$, extend $<$ to a monomial order $<'$ on $\mathbb{C}(A)[\mathbf{p}]$, and extend $\iota(G)$ to the set $\iota(G)' \subset \mathbb{C}(A)[\mathbf{p}]$. The same argument as in Theorem 7.1 shows that $\iota(G)'$ is a Gröbner basis with respect to $<'$. Moreover, the initial ideal $in_{<'}(\iota(G)')$ is generated by the leading terms of $\iota(G)$ with respect to $<$, since the leading coefficients are units in either of the coefficient rings $\mathbb{C}[A]_s$ or $\mathbb{C}(A)$. Thus, $in_{<'}(f)$ for any $f \in \mathbb{C}[\mathbf{A}]_s \langle G \rangle \subset \mathbb{C}(\mathbf{A}) \langle G \rangle$ is divisible by $in_{<'}(g) \in \mathbb{C}(A)[\mathbf{p}]$ for some $g \in \iota(G)$, showing that $\iota(G)$ is a Gröbner basis for the extended ideal $\mathbb{C}[\mathbf{A}]_s \langle G \rangle$.

Finally, we verify that the contracted ideal $\iota^{-1}(\mathbb{C}[\mathbf{A}]_s \langle G \rangle) = \langle G \rangle : s^\infty$ satisfies the conditions for being the vanishing ideal of $\Gamma_{\mathbf{A}, \mathbf{p}}^{m, n}$ given in the recognition criterion of Proposition 2.3:

- (1) The argument for the set-theoretic Condition 1 is the same as in the proof of Theorem 7.5.
- (2) Since $s \in \mathfrak{m}_A$, it follows that $\langle G \rangle : s^\infty$ is saturated with respect to \mathfrak{m}_A . Moreover, Lemma 2.4 can be applied to show that the extended ideal $\mathbb{C}[\mathbf{A}]_s \langle G \rangle$ is saturated with respect to the extension of the irrelevant ideal $\mathbb{C}[\mathbf{A}]_s \mathfrak{m}_\mathbf{p}$. This implies the contracted ideal $\langle G \rangle : s^\infty$ is saturated with respect to $\mathfrak{m}_\mathbf{p}$, since

$$\begin{aligned}
 (\langle G \rangle : s^\infty) : \mathfrak{m}_\mathbf{p}^\infty &= (\langle G \rangle : \mathfrak{m}_\mathbf{p}^\infty) : s^\infty \\
 &= \iota^{-1}(\mathbb{C}[\mathbf{A}]_s (\langle G \rangle : \mathfrak{m}_\mathbf{p}^\infty)) \\
 &\subset \iota^{-1}((\mathbb{C}[\mathbf{A}]_s \langle G \rangle) : (\mathbb{C}[\mathbf{A}]_s \mathfrak{m}_\mathbf{p}^\infty)) && ([6, \text{Exercise 1.18}]) \\
 &= \iota^{-1}((\mathbb{C}[\mathbf{A}]_s \langle G \rangle)) && (\text{Lemma 2.4}) \\
 &= \langle G \rangle : s^\infty.
 \end{aligned}$$

- (3) The extended ideal $\mathbb{C}[\mathbf{A}]_s \langle G \rangle$ is radical by Proposition 2.2, since it has a squarefree initial ideal. Thus the contracted ideal $\iota^{-1}(\mathbb{C}[\mathbf{A}]_s \langle G \rangle)$ is also radical.

□

8. Open Problems and New Directions

In this section we list some open questions about the atlas (Figure 1) that we hope will guide future research, starting with specific questions related to those studied in this paper.

8.1. Algebra.

- (1) In view of Theorem 4.1, it is natural to ask whether Theorem 3.2 can be strengthened. Is the set of 2-, 3-, and 4-focals a universal Gröbner basis of $I_{\mathbf{A}, \mathbf{p}}^{m, 1}$?
- (2) The Gröbner basis of $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, 1}$ (Appendix C) is complicated. Is it possible to obtain a better understanding of its structure? What is a universal Gröbner basis of $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, 1}$?
- (3) What are generators for $I_{\mathbf{A}, \mathbf{p}}^{m, n}$ and $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ for all m and n ?
- (4) What are set-theoretic and ideal-theoretic descriptions of the resectioning varieties?
- (5) All of our results about the specialized ideals $I_{\mathbf{A}, \mathbf{p}}^{m, n}$ and $I_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m, n}$ make genericity assumptions implying that the camera arrangement $\bar{\mathbf{A}}$ has pairwise distinct centers. What can be said when these assumptions are relaxed?
- (6) What are the Euclidean distance degrees of the various varieties in Figure 1? In particular what is the Euclidean distance degree of the bundle adjustment and resectioning varieties?

Note that, generally speaking the Euclidean distance degree is studied by considering nearness in *all* the coordinates of points on the variety. For us only nearness in \mathbf{p} matters. In general, it would be interesting to compare the nodes in the atlas via their ED degrees.

8.2. Geometry.

- (1) What can be said about the geometry of the multiprojective varieties in the atlas? For instance, what are their multidegrees and their singular loci?
- (2) The atlas (Figure 1) is drawn in a vertically symmetric manner. This is reminiscent of a classical principle in multiview geometry known as *Carlsson-Weinshall duality* [11, 60]. Various formulations of this principle express a duality between world points and cameras. Here we ask what ideal-theoretic consequences of this duality are. Can we deduce significant results about the ideals $I_{\mathbf{q},\mathbf{p}}^{m,n}$, $I_{\bar{\mathbf{q}},\mathbf{p}}^{m,n}$, etc, from what we already know about the ideals in Figure 2?
- (3) How can the results related to $I_{\bar{\mathbf{A}},\mathbf{p}}^{m,1}$ and its Hilbert scheme in [5, 34] be extended to $I_{\bar{\mathbf{A}},\mathbf{p}}^{m,n}$ and $I_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}}^{m,n}$, beyond Theorem 7.2? Are there more Hilbert schemes lurking in the atlas? More precisely, when does the flat locus of the associated family give an open immersion into the Hilbert scheme as in [34]? Does this always happen?

We note that the ideals $I_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}}^{m,1}$ appearing in Theorem 6.1 and Theorem 6.2 have the same \mathbb{Z}^{m+1} -graded Hilbert function and Hilbert series for any camera arrangement $\bar{\mathbf{A}}$ with pairwise distinct centers. This is immediate if $\bar{\mathbf{A}}$ is ultra minor generic, since the initial ideals for a given term order are all the same. For $\bar{\mathbf{A}}$ with pairwise distinct centers, this follows since the map L_h defined in the proof preserves the grading. For $m = 2$ cameras, the first few terms of this Hilbert series are

$$1 + 4T_{\mathbf{q}} + 3T_{\mathbf{p}_1} + 3T_{\mathbf{p}_2} + 10T_{\mathbf{q}}^2 + 9T_{\mathbf{q}}T_{\mathbf{p}_1} + \\ 9T_{\mathbf{q}}T_{\mathbf{p}_2} + 6T_{\mathbf{p}_1}^2 + 8T_{\mathbf{p}_1}T_{\mathbf{p}_2} + 6T_{\mathbf{p}_2}^2 + \dots$$

This mirrors the corresponding result for the multiview variety given in [5, Theorem 3.7]. This could be a starting point for studying the Hilbert scheme parametrizing these ideals, parallel to the study of the ideals $I_{\bar{\mathbf{A}},\mathbf{p}}^{m,1}$ in [5].

- (4) Degenerate camera configurations may yield very different ideals for the $I_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}}^{m,n}$ than the specializations of $I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$. Example 4.3 shows that the multiview variety of a tuple of concentric cameras has dimension 2; it is really parametrizing a tuple of planar homographies. This tells us that nice behavior for $I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$ almost ensures that we cannot capture specializations properly in degenerate cases, or, equivalently, that $I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$ does not represent an obvious moduli problem. Is there a moduli problem that we can describe that *does* naturally give rise to $I_{\mathbf{A},\mathbf{q},\mathbf{p}}^{m,n}$ or some modification of it? What happens if we attempt to flatten après Raynaud and Gruson [46]? Can the result be computed or interpreted? More generally, what is the “correct” moduli problem for capturing degenerations?

8.3. Generalizations.

(1) When constructing the atlas studied in this paper, we assumed the simplest possible camera model, the *projective camera*, where the only constraint is that the camera matrix be rank 3. There are a number of specializations of this camera model which are of significant practical interest, each of which will lead to their own atlas. We mention a few:

- (a) Euclidean cameras, $A_i = [R_i \ t_i]$, where $R_i \in SO(3)$ and $t_i \in \mathbb{R}^3$.
- (b) Calibrated cameras with varying intrinsics, $A_i = K_i [R_i \ t_i]$, where $R_i \in SO(3)$ and $t_i \in \mathbb{R}^3$. Here K_i is the so called calibration/intrinsics matrix, which is an upper triangular matrix with positive diagonals.
- (c) Calibrated cameras with common intrinsics, $A_i = K [R_i \ t_i]$, where $R_i \in SO(3)$ and $t_i \in \mathbb{R}^3$. Here all cameras share the same calibration/intrinsics matrix K . This is the case for example when we are using a fixed focus video camera.

The camera models described above are *linear*, and they do not account for the non-linear distortion caused by lenses. Accounting for them gives rise to polynomial and rational cameras and their corresponding atlases.

(2) Just like atlases can be studied as we vary the kind of camera, we can also vary the scene *objects* being imaged. The first would be to replace world points (\mathbf{q}) and image points (\mathbf{p}) with lines [10]. This is a well studied topic in 3D computer vision [7, 57].

We can go further and consider cameras imaging a world where the objects are quadrics and their *images* are conics in the image plane [16, 28]. Unlike the point and line case, where the world objects map directly to the image space objects, in this case we take a quadric surface to the boundary of its shadow in the plane.

More precisely, given a camera $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, a general quadric Q in \mathbb{P}^3 has the property that the camera defines a finite morphism $Q \rightarrow \mathbb{P}^2$ whose set of critical values (the branch curve) is a conic curve $C \subset \mathbb{P}^2$. This defines a rational map of linear spaces $\mathbb{P}^9 \dashrightarrow \mathbb{P}^5$ from the space of quadrics in \mathbb{P}^3 to the space of conics in \mathbb{P}^2 . Similarly, a tuple of cameras defines a rational map in this way from $\mathbb{P}^9 \rightarrow \text{Hilb}_{(\mathbb{P}^2)^m}$. When $m = 1$, the relevant component of the Hilbert scheme of \mathbb{P}^2 is the linear space of conics.

(3) The atlas can also be generalized by replacing the matrix-vector multiplication Aq with arbitrary matrix-matrix multiplication, i.e.

$$C_{ij} \sim A_i B_j \tag{8.1}$$

where $A_i \in \mathbb{P}^{\alpha \times \beta}$, $B_j \in \mathbb{P}^{\beta \times \gamma}$ and $C_{ij} \in \mathbb{P}^{\alpha \times \gamma}$. This seems like a very general object and worthy of study on its own.

References

1. Sameer Agarwal, Hon-leung Lee, Bernd Sturmfels, and Rekha R. Thomas, *On the existence of epipolar matrices*, Int. J. Comput. Vis. **121** (2017), no. 3, 403–415. 7
2. Sameer Agarwal, Andrew Pryhuber, and Rekha R. Thomas, *Ideals of the Multiview Variety*, IEEE Transactions on Pattern Analysis & Machine Intelligence **43** (2021), no. 04, 1279–1292. 1, 1.2, 2.5, 3, 4, 4.2, 4, 4, 4, 4, 1

3. Chris Aholt, Sameer Agarwal, and Rekha Thomas, *A QCQP approach to triangulation*, European Conference on Computer Vision, Springer-Verlag, 2012, p. 654–667. 1.1
4. Chris Aholt and Luke Oeding, *The ideal of the trifocal variety*, Mathematics of Computation **83** (2014), no. 289, 2553–2574. 1.1
5. Chris Aholt, Bernd Sturmfels, and Rekha Thomas, *A Hilbert scheme in computer vision*, Canad. J. Math. **65** (2013), no. 5, 961–988. 1, 1.1, 1.2, 2.5, 3, 4, 4.1, 4, 4.2, 3
6. Michael F. Atiyah and Ian G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. 2
7. Adrien Bartoli and Peter Sturm, *Structure-from-motion using lines: Representation, triangulation, and bundle adjustment*, Computer Vision and Image Understanding **100** (2005), no. 3, 416–441. 2
8. David Bernstein and Andrei Zelevinsky, *Combinatorics of maximal minors*, J. Algebraic Combin. **2** (1993), no. 2, 111–121. 4
9. Adam Boocher, *Free resolutions and sparse determinantal ideals*, Math. Res. Lett. **19** (2012), no. 4, 805–821. 3
10. Paul Breiding, Felix Rydell, Elima Shehu, and Angélica Torres, *Line multiview varieties*, 2022. 2
11. Stefan Carlsson and Daphna Weinshall, *Dual computation of projective shape and camera positions from multiple images*, International Journal of Computer Vision **27** (1998), no. 3, 227–241. 2
12. Dustin Cartwright and Bernd Sturmfels, *The Hilbert scheme of the diagonal in a product of projective spaces*, International Mathematics Research Notices **2010** (2010), no. 9, 1741–1771. 4
13. Aldo Conca, Emanuela De Negri, and Elisa Gorla, *Universal Gröbner bases for maximal minors*, International Mathematics Research Notices **2015** (2015), no. 11, 3245–3262. 4
14. David A. Cox, *The moving curve ideal and the Rees algebra*, Theoretical Computer Science **392** (2008), no. 1-3, 23–36. 2.1
15. David A. Cox, John Little, and Donal O’Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 4 ed., Undergraduate Texts in Mathematics, Springer, Cham, 2015. 2.1, 2.1, 4, 7, B.3
16. Song De Ma, *Conics-based stereo, motion estimation, and pose determination*, International Journal of Computer Vision **10** (1993), no. 1, 7–25. 2
17. Michel Demazure, *Sur deux problèmes de reconstruction*, Tech. Report RR-0882, INRIA, July 1988. 1.1
18. Jan Draisma, Emil Horobeț, Giorgio Ottaviani, Bernd Sturmfels, and Rekha R Thomas, *The Euclidean distance degree of an algebraic variety*, Foundations of Computational Mathematics **16** (2016), no. 1, 99–149. 1.1
19. Timothy Duff, Kathlén Kohn, Anton Leykin, and Tomas Pajdla, *PL₁P-Point-Line minimal problems under partial visibility in three views*, European Conference on Computer Vision, Springer, 2020, pp. 175–192. 2
20. Olivier D. Faugeras and Bernard Mourrain, *On the Geometry and Algebra of the Point and Line Correspondences Between N Images*, IEEE International Conference on Computer Vision, 1995, pp. 951–956. 1, 4, 4

21. M. Giusti and M. Merle, *Singularités isolées et sections planes de variétés déterminantielles. II. Sections de variétés déterminantielles par les plans de coordonnées*, Lecture Notes in Math., vol. 961, pp. 103–118, Springer, Berlin, 1982. 3
22. Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <https://math.uiuc.edu/Macaulay2/>. 4.1
23. Richard Hartley and Andrew Zisserman, *Multiple View Geometry in Computer Vision*, Cambridge University Press, 2004. 1.1, 1.1, 1.1, 1.1, 1.1, 3, A
24. Richard I Hartley, *An investigation of the essential matrix*, Tech. report, General Electric CRD, 1995. 1.1
25. ———, *Lines and points in three views and the trifocal tensor*, International Journal of Computer Vision **22** (1997), no. 2, 125–140. 1.1
26. Richard I Hartley and Peter Sturm, *Triangulation*, Computer Vision and Image Understanding **68** (1997), no. 2, 146–157. 1.1, 1.1
27. Anders Heyden and Kalle Åström, *Algebraic properties of multilinear constraints*, Math. Methods Appl. Sci. **20** (1997), no. 13, 1135–1162. 1, 1.1, 4, 4
28. Fredrik Kahl and Anders Heyden, *Affine structure and motion from points, lines and conics*, International Journal of Computer Vision **33** (1999), no. 3, 163–180. 2
29. Joe Kileel, *Minimal problems for the calibrated trifocal variety*, SIAM Journal on Applied Algebra and Geometry **1** (2017), no. 1, 575–598. 2
30. Laurent Kneip, Roland Siegwart, and Marc Pollefeys, *Finding the exact rotation between two images independently of the translation*, European Conference on Computer Vision, vol. 7577, Springer, 2012, pp. 696–709. 7
31. Zuzana Kukelova, Martin Bujnak, and Tomas Pajdla, *Polynomial eigenvalue solutions to minimal problems in computer vision*, IEEE Transactions on Pattern Analysis and Machine Intelligence **34** (2011), no. 7, 1381–1393. 1.1
32. Viktor Larsson, Kalle Astrom, and Magnus Oskarsson, *Efficient solvers for minimal problems by syzygy-based reduction*, IEEE Conference on Computer Vision and Pattern Recognition, 2017, pp. 820–829. 1.1
33. Vincent Lepetit, Francesc Moreno-Noguer, and Pascal Fua, *EPnP: An accurate $O(n)$ solution to the PnP problem*, International Journal of Computer Vision **81** (2009), no. 2, 155–166. 1.1
34. Max Lieblich and Lucas Van Meter, *Two Hilbert schemes in computer vision*, SIAM J. Appl. Algebra Geom. **4** (2020), no. 2, 297–321. 1, 3
35. Peter Lindstrom, *Triangulation made easy*, 2010 IEEE Conference on Computer Vision and Pattern Recognition, 2010, pp. 1554–1561. 1.1
36. H Christopher Longuet-Higgins, *A computer algorithm for reconstructing a scene from two projections*, Nature **293** (1981), no. 5828, 133–135. 1.1
37. Quan-Tuan Luong and Olivier D Faugeras, *The fundamental matrix: Theory, algorithms, and stability analysis*, International Journal of Computer Vision **17** (1996), no. 1, 43–75. 1.1
38. Yi Ma, Stefano Soatto, Jana Košeká, and Shankar Sastry, *An Invitation to 3-D Vision: from Images to Geometric Models*, vol. 26, Springer, 2004. 1, 1.1, 1.2, 4
39. Laurentiu G Maxim, Jose I Rodriguez, and Botong Wang, *Euclidean distance degree of the multiview variety*, SIAM Journal on Applied Algebra and Geometry **4** (2020), no. 1, 28–48. 1.1

40. Mateusz Michałek and Bernd Sturmfels, *Invitation to Nonlinear Algebra*, vol. 211, American Mathematical Society, 2021. 1
41. Behrooz Nashatkon, Richard Hartley, and Jochen Trumpf, *A generalized projective reconstruction theorem and depth constraints for projective factorization*, International Journal of Computer Vision **115** (2015), no. 2, 87–114. 1.1
42. David Nistér, *An efficient solution to the five-point relative pose problem*, IEEE Transactions on Pattern Analysis and Machine Intelligence **26** (2004), no. 6, 756–770. 1.1
43. Luke Oeding, *The quadrifocal variety*, Linear Algebra and Its Applications **512** (2017), 306–330. 1.1
44. Patrik Persson and Kalle Åström, *Global trifocal adjustment*, Scandinavian Conference on Image Analysis, Springer, 2019, pp. 287–298. 1.1
45. Long Quan, *Invariants of six points and projective reconstruction from three uncalibrated images*, IEEE Transactions on Pattern Analysis and Machine Intelligence **17** (1995), no. 1, 34–46. 1.1
46. Michel Raynaud and Laurent Gruson, *Criteres de platitude et de projectivité*, Inventiones Mathematicae **13** (1971), no. 1, 1–89. 4
47. Antonio L Rodríguez, Pedro E López-de Teruel, and Alberto Ruiz, *Reduced epipolar cost for accelerated incremental SfM*, IEEE Conference on Computer Vision and Pattern Recognition, 2011, pp. 3097–3104. 1.1
48. Ewelina Rupnik and Marc Deseilligny, *Towards structureless bundle adjustment with two-and three-view structure approximation*, ISPRS Annals of the Photogrammetry, Remote Sensing and Spatial Information Sciences, vol. 2, 2020, pp. 71–78. 1.1
49. Frederik Schaffalitzky, Andrew Zisserman, Richard I. Hartley, and Philip H. S. Torr, *A six point solution for structure and motion*, European Conference on Computer Vision (David Vernon, ed.), vol. 1842, Springer, 2000, pp. 632–648. 1.1
50. J. Schneider, C. Stachniss, and W. Förstner, *On the quality and efficiency of approximate solutions to bundle adjustment with epipolar and trifocal constraints*, ISPRS Annals of the Photogrammetry, Remote Sensing and Spatial Information Sciences **IV-2/W3** (2017), 81–88. 1.1
51. Amnon Shashua and Lior Wolf, *On the structure and properties of the quadrifocal tensor*, European Conference on Computer Vision, Springer, 2000, pp. 710–724. 1.1, 1.2
52. Richard Steffen, Jan-Michael Frahm, and Wolfgang Förstner, *Relative bundle adjustment based on trifocal constraints*, European Conference on Computer Vision, Springer, 2010, pp. 282–295. 1.1
53. Henrik Stewénius, *Gröbner basis methods for minimal problems in computer vision*, PhD thesis 2005. 1.1
54. Henrik Stewénius, Frederik Schaffalitzky, and David Nistér, *How hard is 3-view triangulation really?*, IEEE International Conference on Computer Vision, vol. 1, 2005, pp. 686–693. 1.1
55. Bernd Sturmfels, *Gröbner bases and Convex Polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. 2.4, 3, 5
56. Bernd Sturmfels and Andrei Zelevinsky, *Maximal minors and their leading terms*, Adv. Math. **98** (1993), no. 1, 65–112. 4
57. Camillo J Taylor and David J Kriegman, *Structure and motion from line segments in multiple images*, IEEE Transactions on Pattern Analysis and Machine Intelligence **17** (1995), no. 11, 1021–1032. 2

58. Sebastian Thrun, Wolfram Burgard, and Dieter Fox, *Probabilistic robotics*, MIT Press, 2005. 1.1
59. Matthew Trager, Martial Hebert, and Jean Ponce, *The joint image handbook*, IEEE International Conference on Computer Vision, 2015, pp. 909–917. 1, 1.1, 1.1, 3, 3, 4, 4, 4
60. Matthew Trager, Martial Hebert, and Jean Ponce, *Coordinate-free carlsson-weinshall duality and relative multi-view geometry*, IEEE Conference on Computer Vision and Pattern Recognition, 2019, pp. 225–233. 2
61. Bill Triggs, *Matching Constraints and the Joint Image*, IEEE International Conference on Computer Vision, 1995, pp. 338–343. 1, 1.1, 3, 4
62. Bill Triggs, *Factorization methods for projective structure and motion*, IEEE Conference on Computer Vision and Pattern Recognition, 1996, pp. 845–851. 1.1
63. Bill Triggs, Philip F McLauchlan, Richard I Hartley, and Andrew W Fitzgibbon, *Bundle Adjustment—a Modern Synthesis*, International Workshop on Vision Algorithms, Springer, 1999, pp. 298–372. 1.1, 1.1
64. Cynthia Vinzant, *Real radical initial ideals*, J. Algebra **352** (2012), 392–407. 2.1

Appendix A. Dimension counts

Proposition A.1. *Below, $\bar{\mathbf{A}} \in (\mathbb{P}^{11})^m$, $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$, $\bar{\mathbf{p}} \in \Gamma_{\mathbf{p}}^{m,n}$ are generic whenever they appear.*

$$\dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} = 3n + 11m \quad (\text{A.1})$$

$$\dim \Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n} = \min(2mn + 11m, 3n + 11m) \quad (\text{A.2})$$

$$\dim \Gamma_{\mathbf{q}, \bar{\mathbf{p}}}^{m,n} = \min(3n + 2mn, 3n + 11m) \quad (\text{A.3})$$

$$\dim \Gamma_{\mathbf{p}}^{m,n} = \min(2mn, 11m + \max(3n - 15, 0)) \quad (\text{A.4})$$

$$\dim \Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n} = \dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{A}}^{m,n} = 3n \quad (\text{A.5})$$

$$\dim \Gamma_{\mathbf{A}, \mathbf{q}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{p}}^{m,n} \quad (\text{A.6})$$

$$\dim \Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{m,n} = \dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{q}}^{m,n} = 11m \quad (\text{A.7})$$

$$\dim \Gamma_{\bar{\mathbf{A}}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n} - \dim \Gamma_{\mathbf{A}}^{m,n} = \min(2mn, 3n) \quad (\text{A.8})$$

$$\dim \Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{p}}^{m,n} \quad (\text{A.9})$$

$$\dim \Gamma_{\mathbf{q}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{p}}^{m,n} \quad (\text{A.10})$$

$$\dim \Gamma_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n} = \dim \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} - \dim \Gamma_{\mathbf{q}}^{m,n} \quad (\text{A.11})$$

Remark A.2. For m and n sufficiently large, the formulas above involving \min and \max expressions can be simplified as follows:

$$\dim \Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} = \dim \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} = 3n + 11m \quad (\text{A.12})$$

$$\dim \Gamma_{\mathbf{p}}^{m,n} = 3n + 11m - 15 \quad (\text{A.13})$$

$$\dim \Gamma_{\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}}^{m,n} = \dim \Gamma_{\bar{\mathbf{A}}, \bar{\mathbf{p}}}^{m,n} = 3n \quad (\text{A.14})$$

$$\dim \Gamma_{\mathbf{A}, \mathbf{q}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n} = \dim \Gamma_{\mathbf{q}, \bar{\mathbf{p}}}^{m,n} = 15 \quad (\text{A.15})$$

$$\dim \Gamma_{\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}}^{m,n} = \dim \Gamma_{\bar{\mathbf{q}}, \mathbf{p}}^{m,n} = 11m \quad (\text{A.16})$$

Proof.

(A.1): This follows at once from the birational equivalence

$$\begin{aligned} \pi_{\mathbf{p}} : \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} &\rightarrow (\mathbb{P}^{11})^m \times (\mathbb{P}^3)^n \\ (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) &\mapsto (\bar{\mathbf{A}}, \bar{\mathbf{q}}). \end{aligned}$$

(A.2): Consider the projection

$$\begin{aligned} \pi_{\mathbf{q}} : \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} &\rightarrow (\mathbb{P}^{11})^m \times (\mathbb{P}^2)^{mn} \\ (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) &\mapsto (\bar{\mathbf{A}}, \bar{\mathbf{p}}). \end{aligned}$$

A fiber $\pi_{\mathbf{q}}^{-1}(\bar{\mathbf{A}}, \bar{\mathbf{p}})$ can be identified with the projective linear space of all q satisfying $A_i q_j \sim p_{ij}$. Equivalently, each of the $2mn$ matrices $(A_i q_j \ p_{ij})$ is of rank one. For generic $(\bar{\mathbf{A}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{A}, \bar{\mathbf{p}}}^{m,n}$, the 2×2 minors impose 2 linear conditions on $\bar{\mathbf{q}}$, so that $\pi_{\mathbf{q}}^{-1}(\bar{\mathbf{A}}, \bar{\mathbf{p}})$ is a projective linear space

of dimension $\max(3n - 2mn, 0)$. Hence, using (A.1) and the fiber-dimension theorem,

$$\begin{aligned}\dim \Gamma_{\mathbf{A}, \mathbf{p}}^{m,n} &= \dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} - \max(3n - 2mn, 0) \\ &= \min(2mn + 11m, 3n + 11m).\end{aligned}$$

(A.3): Similar to (A.2), consider

$$\begin{aligned}\pi_{\mathbf{A}} : \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} &\rightarrow (\mathbb{P}^{11})^m \times (\mathbb{P}^2)^{mn} \\ (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) &\mapsto (\bar{\mathbf{q}}, \bar{\mathbf{p}}).\end{aligned}$$

For generic $(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n}$, the fibers $\pi_{\mathbf{A}}^{-1}(\bar{\mathbf{q}}, \bar{\mathbf{p}})$ are projective linear spaces of dimension $\max(11m - 2mn, 0)$ which are defined by the \mathbf{A} -linear 2×2 minors of $(A_i q_j \ p_{ij})$. Hence

$$\begin{aligned}\dim \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} &= \dim \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} - \max(11m - 2mn, 0) \\ &= \min(3n + 2mn, 3n + 11m).\end{aligned}$$

(A.4): For generic $q_1, \dots, q_5 \in \mathbb{P}^3$, define $\sigma = \{\sigma_1, \dots, \sigma_4\} \subset [5]$,

$$[\sigma_1 \ \sigma_2 \ \sigma_3 \ \sigma_4]^{-1} := \det(q_{\sigma_1} \ q_{\sigma_2} \ q_{\sigma_3} \ q_{\sigma_4})^{-1},$$

and consider the *projective change of basis matrix*

$$C_{q_1, \dots, q_5} = \text{diag}([5234]^{-1}, [1534]^{-1}, [1254]^{-1}, [1235]^{-1}) \cdot (q_1 \ q_2 \ q_3 \ q_4)^{-1} \quad (\text{A.17})$$

where \bullet^{-1} denotes matrix inversion. When defined, the projective transformation defined by C_{q_1, \dots, q_5} maps $q_1 \dots q_5$ onto the *standard projective basis*:

$$\begin{aligned}C_{q_1, \dots, q_5} \cdot q_1 &\sim e_1, \\ C_{q_1, \dots, q_5} \cdot q_2 &\sim e_2, \\ C_{q_1, \dots, q_5} \cdot q_3 &\sim e_3, \\ C_{q_1, \dots, q_5} \cdot q_4 &\sim e_4, \\ C_{q_1, \dots, q_5} \cdot q_5 &\sim e_1 + e_2 + e_3 + e_4.\end{aligned}$$

Consider the projection

$$\begin{aligned}\pi_{\mathbf{A}, \mathbf{q}} : \Gamma_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,n} &\rightarrow \Gamma_{\mathbf{p}}^{m,n} \\ (\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) &\mapsto \bar{\mathbf{p}}.\end{aligned}$$

We observe a version of *projective ambiguity* [23, p 265], stating that the fibers of $\pi_{\mathbf{A}, \mathbf{q}}$ are invariant under the action of PGL_4 described in Section 2.2. Suppose first that $n < 6$. We need to show $\dim \Gamma_{\mathbf{p}}^{m,n} = 2mn$. Let $\bar{\mathbf{p}} \in \Gamma_{\mathbf{p}}^{m,n}$ be generic and suppose $(\bar{\mathbf{A}}, \bar{\mathbf{q}}, \bar{\mathbf{p}}) \in \pi_{\mathbf{A}, \mathbf{q}}^{-1}(\bar{\mathbf{p}})$. Then for generic $\bar{\mathbf{q}}' \in (\mathbb{P}^3)^n$, we may find $(\bar{\mathbf{A}}', \bar{\mathbf{q}}', \bar{\mathbf{p}}) \in \pi_{\mathbf{A}, \mathbf{q}}^{-1}(\bar{\mathbf{p}})$ by projective change of basis $H = C_{\bar{\mathbf{q}}'}^{-1} C_{\bar{\mathbf{q}}}$, where $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{q}}'$ extend $\bar{\mathbf{q}}$ and $\bar{\mathbf{q}}'$ to projective bases when $n < 5$. In other words, the generic fiber of $\Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} \rightarrow \Gamma_{\mathbf{p}}^{m,n}$ has dimension $3n$. Applying (A.1),

$$\dim \Gamma_{\mathbf{p}}^{m,n} = \dim \Gamma_{\mathbf{q}, \mathbf{p}}^{m,n} - 3n = (3n + 2mn) - 3n = 2mn.$$

Now suppose $n \geq 6$. For $m = 1$ camera, (A.4) asserts that $\dim \Gamma_p^{m,n} = 2n$, which follows since there are no constraints on image points. Otherwise, observe that the quantity

$$2mn - (11m + 3n - 15) = (2m - 3)n - 11m + 15 = (2n - 11)m + 15 - 3n$$

is increasing in n for fixed $m \geq 2$ and increasing in m for fixed $n \geq 6$. Moreover, this quantity equals zero precisely in the *minimal cases* $(m, n) = (2, 7), (3, 6)$. Thus, (A.4) asserts that $\dim \Gamma_p^{m,n} = 11m + 3n - 15$ whenever either $m \geq 2$ and $n \geq 7$ or $m \geq 3$ and $n \geq 6$. This leaves one exceptional case for $n \geq 6$, which is $(m, n) = (2, 6)$; here, to show that $\dim \Gamma_p^{m,n} = 2mn = 24$, it suffices to verify that the Jacobian of $\pi_{A,q}$ evaluated at some point in local coordinates has rank 24. The same Jacobian check gives us $\dim \Gamma_p^{m,n} = 11m + 3n - 15$ for the two minimal cases; equivalently, $\dim \pi_{A,q}^{-1}(\bar{p}) = 15$ for generic $\bar{p} \in \Gamma_p^{m,n}$. Finally, if either $m \geq 2$ and $n \geq 8$ or $m \geq 3$ and $n \geq 7$, note that the fiber $\pi_{A,q}^{-1}(\bar{p})$ for generic $\bar{p} \in \Gamma_p^{m,n}$ is nonempty, and thus has dimension at least 15 by projective ambiguity. Since $\pi_{A,q}^{-1}(\bar{p})$ projects onto a fiber for one of the minimal cases, we also have $\dim \pi_{A,q}^{-1}(\bar{p}) \leq 15$. Thus

$$\dim \Gamma_p^{m,n} = \dim \Gamma_{A,q,p}^{m,n} - \dim \dim \pi_{A,q}^{-1}(\bar{p}) = 11m + 3n - 15.$$

(A.5)–(A.11) In all cases, $\Gamma_{X,Y}^{m,n}$ is the generic fiber of $\Gamma_{X,Y}^{m,n} \rightarrow \Gamma_X^{m,n}$, so these formulas follow from the fiber dimension theorem and (A.1)–(A.4). \square

Appendix B. Miscellaneous Proofs

B.1. Generic Cameras.

Proposition B.1. *If a camera arrangement $\bar{A} = (\bar{A}_1, \dots, \bar{A}_m)$ is ultra minor generic then it is minor generic, and if \bar{A} is minor generic, then it has pairwise distinct centers.*

Proof. If \bar{A} is ultra minor generic, then all $k \times k$ minors of $(\bar{A}_1^\top \mid \dots \mid \bar{A}_m^\top)$ are nonzero for any $k \in [4]$. In particular, all 4×4 minors are nonzero and \bar{A} is minor generic. If \bar{A} is minor generic, then for any $1 \leq i < j \leq m$, the 4×6 matrix $(\bar{A}_i^\top \mid \bar{A}_j^\top)$ has rank 4. This implies that \bar{A}_i and \bar{A}_j have distinct centers. \square

Theorem B.2. (1) *A camera arrangement \bar{A} has pairwise distinct centers if and only if it is equivalent to a minor generic camera arrangement under the group action (2.1).*
 (2) *A camera arrangement \bar{A} is minor generic if and only if it is equivalent to an ultra minor generic arrangement under the group action (2.2).*
 (3) *A camera arrangement \bar{A} has pairwise distinct centers if and only if it is equivalent to an ultra minor generic camera arrangement under the group action (2.3).*

Proof. (1) This statement was proved in [2, Lemma 3.6].

(2) We already saw in Proposition B.1 that ultra minor genericity implies minor genericity. For the other direction, fix a minor generic arrangement $(\bar{A}_1, \dots, \bar{A}_m)$. Let $\sigma \in \binom{[4]}{k}$ and $\tau \in \binom{[3m]}{k}$ be subsets indexing the rows and columns of some $k \times k$ minor of

$(\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top)$. Using the Cauchy-Binet theorem,

$$\begin{aligned} \det \left(\left((\bar{A}_1 H)^\top \mid \cdots \mid (\bar{A}_m H)^\top \right) [\sigma, \tau] \right) = \\ \sum_{v \in \binom{[4]}{k}} \det \left(H^\top [\sigma, v] \right) \cdot \det \left((\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top) [v, \tau] \right). \end{aligned} \quad (\text{B.1})$$

The minors $\det((\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top) [v, \tau])$ which occur in this sum range over all $k \times k$ minors of the $4 \times k$ matrix $(\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top) [[4], \tau]$. This $4 \times k$ matrix has full rank k since if it did not, we could add $4 - k$ additional columns from $(\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top)$ to get a rank-deficient 4×4 matrix, contradicting our assumption that $(\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top)$ is minor generic. Thus, $\det(\bar{A}_1^\top \mid \cdots \mid \bar{A}_m^\top) [v, \tau] \neq 0$ for some $v \in \binom{[4]}{k}$. Hence the expressions in (B.1) are not zero, and setting (B.1) to 0 we obtain a hypersurface in GL_4 . Any choice of H lying outside the union of the finitely many hypersurfaces, obtained by varying over all k, σ, τ yields an arrangement $(A_1 H, \dots, A_m H)$ satisfying the conclusion.

(3) This statement follows from the first two. □

B.2. Proof of Proposition 2.2, part 1.

Proof. Let $I = \langle g_1, \dots, g_s \rangle$. By definition, g_1, \dots, g_s forming a Gröbner basis with respect to $<$ means that

$$\text{in}_<(I) = \langle \text{in}_<(g_1), \dots, \text{in}_<(g_s) \rangle.$$

First, we verify that this monomial ideal in $R[x_1, \dots, x_k]$ is radical. To see this, let $\text{in}_<(g_s) = x_{i_1} \cdots x_{i_l}$ and note that

$$\text{in}_<(I) = \bigcap_{j=1}^l \langle \text{in}_<(g_1), \dots, \text{in}_<(g_{s-1}), x_{i_j} \rangle.$$

Iterating this argument, we obtain $\text{in}_<(I)$ as an intersection of prime ideals generated by subsets of the variables.

Now, to show that $I = \langle g_1, \dots, g_s \rangle$ is radical, suppose that $f \in \sqrt{I}$, so that $f^n \in I$ for some positive integer n . We need to argue that $f \in I$. We have

$$\text{in}_<(f)^n = \text{in}_<(f^n) \in \text{in}_<(I) \quad \Rightarrow \quad \text{in}_<(f) \in \text{in}_<(I),$$

where the first equality of leading terms uses the fact that R is a domain and the implication holds since $\text{in}_<(I)$ is radical. Thus there exists $f_0 \in I \subseteq \sqrt{I}$ such that $\text{in}_<(f_0) = \text{in}_<(f)$. Now $f - f_0 \in \sqrt{I}$ is an element whose leading term is strictly smaller than $\text{in}_<(f)$. Replacing f with $f - f_0$ and iterating the argument, we obtain $f_0, \dots, f_l \in I$ such that $f = f_0 + \cdots + f_l \in I$. □

B.3. Proof of the recognition criterion: Proposition 2.3.

Proof. A point $x \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ may be represented in homogeneous coordinates by a point \hat{x} in the affine space $\mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_k+1}$. Consider the *affine cone*

$$\hat{X} = \text{cl}\{\hat{x} \in \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_k+1} \text{ s.t. } x \in X\}.$$

This is an *affine* variety whose vanishing ideal is precisely the vanishing ideal of X .

Suppose that Conditions 1–3 are satisfied; we must show that $\langle f_1, \dots, f_s \rangle$ is the vanishing ideal of X , or equivalently that of \hat{X} . Condition 3 and the Nullstellensatz [15, Ch. 4, §2] imply that $\langle f_1, \dots, f_s \rangle$ is the vanishing ideal of the affine variety

$$V_a(f_1, \dots, f_s) = \{\hat{x} \in \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_k+1} \text{ s.t. } f_1(\hat{x}) = \cdots = f_s(\hat{x})\}.$$

Moreover, Conditions 2 and 3 together with standard properties of ideal quotients and saturation [15, Ch. 4, §4] imply $\langle f_1, \dots, f_s \rangle$ is the vanishing ideal of

$$\text{cl}(V_a(f_1, \dots, f_s) \setminus V_a(\mathfrak{m}_{x_1} \cap \cdots \cap \mathfrak{m}_{x_k})).$$

Since affine varieties are uniquely determined by their vanishing ideals, it is now enough to observe the following equality, which holds whenever Condition 1 is satisfied:

$$\hat{X} = \text{cl}(V_a(f_1, \dots, f_s) \setminus V_a(\mathfrak{m}_{x_1} \cap \cdots \cap \mathfrak{m}_{x_k})).$$

Conversely, we verify Conditions 1–3 when $\langle f_1, \dots, f_s \rangle$ is the vanishing ideal of X :

- (1) $X \subset V(f_1, \dots, f_s)$ since each f_i vanishes on all points of X . On the other hand, we have $X = V(g_1, \dots, g_s)$ for certain homogeneous polynomials g_1, \dots, g_s , all of which must be contained in $\langle f_1, \dots, f_s \rangle$. If f_1, \dots, f_s vanish at a point, so must g_1, \dots, g_s , and thus $V(f_1, \dots, f_s) \subset X$.
- (2) Let $f \in \langle f_1, \dots, f_s \rangle : (\mathfrak{m}_{x_1} \cap \cdots \cap \mathfrak{m}_{x_k})^\infty$. To show $f \in \langle f_1, \dots, f_s \rangle$, it is enough to show that each of the homogeneous components of f vanish on X , so suppose further that f is homogeneous. If X is empty, then f vanishing on X holds vacuously. Otherwise, for any point in X there exists some monomial of the form

$$m(x) = x_{1,i_1} \cdots x_{k,i_k} \in \mathfrak{m}_{x_1} \cap \cdots \cap \mathfrak{m}_{x_k}$$

which does not vanish at that point. Since $m(x)^n f$ is in the vanishing ideal for some $n \geq 1$, we see that that f must vanish at this point.

- (3) If $f^n(x) = 0$ for some $n \geq 1$ and all $x \in X$, then $f(x) = 0$ for all $x \in X$.

□

Appendix C. Generators of $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$

The Gröbner basis $G_{M_{\mathbf{A}, \mathbf{q}, \mathbf{p}}^{m,1}}$ of Proposition 5.2 contains elements in degree 3, 4, 5, 6, 7, 8, 9. Here, for completeness, we give explicit formulas for all of them.

Degree 3: $3m$ elements—for $1 \leq i \leq m$, $1 \leq i_1 < i_2 \leq 3$, $\det \begin{pmatrix} (A_i q & p_i) [\{i_1, i_2\}, :] \end{pmatrix}$.

Degree 4: m elements—for $1 \leq i \leq m$, $\det (A_i[:, 1] \ A_i q \ p_i)$.

Degree 5: $9 \binom{m}{2}$ elements—for $1 \leq i < j \leq m$, $1 \leq i_1 < i_2 \leq 3$, $1 \leq j_1 < j_2 \leq 3$,

$$\det \left((A_i[:, 1] \ p_i) [\{i_1, i_2\}, :] \right) \cdot \det \left((A_j q \ p_j) [\{j_1, j_2\}, :] \right) - \\ \det \left((A_j[:, 1] \ p_j) [\{j_1, j_2\}, :] \right) \cdot \det \left((A_i q \ p_i) [\{i_1, i_2\}, :] \right).$$

Degree 6: $6 \binom{m}{2}$ elements—for $1 \leq k_1 < k_2 \leq 3$, $1 \leq i, j \leq m$, $i \neq j$,

$$\sum_{l=1}^3 (-1) \cdot \det \left((A_j q \ p_j) [\{1, 2, 3\} \setminus l, :] \right) \cdot \left(p_i[k_1] \det \left(\begin{matrix} A_i[k_2, 1:2] \\ A_j[l, 1:2] \end{matrix} \right) - p_i[k_2] \det \left(\begin{matrix} A_i[k_1, 1:2] \\ A_j[l, 1:2] \end{matrix} \right) \right) \\ + \det \left((A_j[:, 1] \ A_j[:, 2] \ p_j) \cdot \det \left((A_i q \ p_i) [\{k_1, k_2\}, :] \right) \right).$$

Degree 7: $\binom{m}{2}$ elements of the form q_4 times a 2-focal, plus an additional $27 \binom{m}{3}$ elements—for $1 \leq i < j < k \leq m$, $1 \leq i_1 < i_2 \leq 3$, $1 \leq j_1 < j_2 \leq 3$, $1 \leq k_1 < k_2 \leq 3$,

$$\det \left((A_i q \ p_i) [\{i_1, i_2\}, :] \right) \cdot \left(p_j[j_1] p_k[k_1] \cdot \det \left(\begin{matrix} A_j[j_2, 1:2] \\ A_k[k_2, 1:2] \end{matrix} \right) - p_j[j_1] p_k[k_2] \cdot \det \left(\begin{matrix} A_j[j_2, 1:2] \\ A_k[k_1, 1:2] \end{matrix} \right) - \right. \\ \left. p_j[j_2] p_k[k_1] \cdot \det \left(\begin{matrix} A_j[j_1, 1:2] \\ A_k[k_2, 1:2] \end{matrix} \right) + p_j[j_2] p_k[k_2] \cdot \det \left(\begin{matrix} A_j[j_1, 1:2] \\ A_k[k_1, 1:2] \end{matrix} \right) \right) - \\ \det \left((A_j q \ p_j) [\{j_1, j_2\}, :] \right) \cdot \left(p_i[i_1] p_k[k_1] \cdot \det \left(\begin{matrix} A_i[i_2, 1:2] \\ A_k[k_2, 1:2] \end{matrix} \right) - p_i[i_1] p_k[k_2] \cdot \det \left(\begin{matrix} A_i[i_2, 1:2] \\ A_k[k_1, 1:2] \end{matrix} \right) - \right. \\ \left. p_i[i_2] p_k[k_1] \cdot \det \left(\begin{matrix} A_i[i_1, 1:2] \\ A_k[k_2, 1:2] \end{matrix} \right) + p_i[i_2] p_k[k_2] \cdot \det \left(\begin{matrix} A_i[i_1, 1:2] \\ A_k[k_1, 1:2] \end{matrix} \right) \right) + \\ \det \left((A_k q \ p_k) [\{k_1, k_2\}, :] \right) \cdot \left(p_i[i_1] p_j[j_1] \cdot \det \left(\begin{matrix} A_i[i_2, 1:2] \\ A_j[j_2, 1:2] \end{matrix} \right) - p_i[i_1] p_j[j_2] \cdot \det \left(\begin{matrix} A_i[i_2, 1:2] \\ A_j[j_1, 1:2] \end{matrix} \right) - \right. \\ \left. p_i[i_2] p_j[j_1] \cdot \det \left(\begin{matrix} A_i[i_1, 1:2] \\ A_j[j_2, 1:2] \end{matrix} \right) + p_i[i_2] p_j[j_2] \cdot \det \left(\begin{matrix} A_i[i_1, 1:2] \\ A_j[j_1, 1:2] \end{matrix} \right) \right)$$

Degrees 8 & 9: q_4 times $27 \binom{m}{3}$ 3-focals and q_4 times $81 \binom{m}{4}$ 4-focals, respectively.

Google Inc., Seattle
Email address: sameeragarwal@google.com

Department of Mathematics, University of Washington, Seattle
Email address: timduff@uw.edu

Department of Mathematics, University of Washington, Seattle
Email address: lieblich@uw.edu

Department of Mathematics, University of Washington, Seattle
Email address: rrthomas@uw.edu