SIAM J. Applied Dynamical Systems
Vol. 23, No. 1, pp. 924–960

Solving Nonlinear Ordinary Differential Equations Using the Invariant Manifolds and Koopman Eigenfunctions*

Megan Morrison† and J. Nathan Kutz‡

Abstract. Nonlinear ODEs can rarely be solved analytically. Koopman operator theory provides a way to solve two-dimensional nonlinear systems, under suitable restrictions, by mapping nonlinear dynamics to a linear space using Koopman eigenfunctions. Unfortunately, finding such eigenfunctions is difficult. We introduce a method for constructing Koopman eigenfunctions from a two-dimensional nonlinear ODE's one-dimensional invariant manifolds. This method, when successful, allows us to find analytical solutions for autonomous, nonlinear systems. Previous data-driven methods have used Koopman theory to construct local Koopman eigenfunction approximations valid in different regions of phase space; our method finds analytic Koopman eigenfunctions that are exact and globally valid. We demonstrate our Koopman method of solving nonlinear systems on one-dimensional and two-dimensional ODEs. The nonlinear examples considered have simple expressions for their codimension-1 invariant manifolds which produce tractable analytical solutions. Thus our method allows for the construction of analytical solutions for previously unsolved ODEs. It also highlights the connection between invariant manifolds and eigenfunctions in nonlinear ODEs and presents avenues for extending this method to solve more nonlinear systems.

Key words. ordinary differential equations, dynamical systems, invariant manifolds, Koopman theory, Koopman eigenfunctions

MSC codes. 34A25, 34A05, 34A34

DOI. 10.1137/22M1516622

1. Introduction.

Aside from a few special cases, there are no general methods for solving nonlinear ODEs, thus necessitating numerical solution techniques [20, 44]. In contrast, autonomous linear ODEs are very well understood, with analytical solutions easily constructed from exponential solution forms, i.e., eigen-decompositions [5, 42, 41]. Most of the analysis, prediction, and control of nonlinear systems, in fact, relies on linearization around the fixed points of a given system [5] and numerical methods [42, 20, 44, 30]. Indeed, the approximation of nonlinear systems in terms of local linear systems is one of the few general methods available for characterizing nonlinear systems. However, linear analysis and numerical methods for nonlinear systems have limits in terms of their ability to predict and control dynamics. This

*Received by the editors August 18, 2022; accepted for publication (in revised form) by K. Lin November 2, 2023; published electronically March 21, 2024.
https://doi.org/10.1137/22M1516622
Funding: The work of the first author was supported by the National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship award 2103239. The work of the second author was supported by the Air Force Office of Scientific Research (AFOSR) grant FA9550-19-1-0386 and the National Science Foundation AI Institute in Dynamic Systems grant 2112085.
†Courant Institute of Mathematical Sciences, New York University, New York, NY 10012 USA (mjm1101@nyu.edu).
‡Department of Applied Mathematics, University of Washington, Seattle, WA 98195-3925 USA (kutz@uw.edu).

motivates our introduction of an approach for the construction of analytically tractable invariant manifolds which can be used for expressing solutions of nonlinear differential equations.

In 1931 Koopman found that nonlinear Hamiltonian systems could be mapped to an infinite-dimensional space of observables with linear dynamics [17, 18]. Thus, if one finds such a mapping, the dynamical system can be solved in the linear space and the solution can be mapped back to the original nonlinear space. In some instances, a finite number of Koopman eigenfunctions will form a closed, Koopman-invariant subspace, resulting in a finite linear model which can be easily solved, unlike infinite-dimensional linear models [6]. The existence and uniqueness of global Koopman eigenfunctions has been proven for stable fixed points and periodic orbits [22]. Koopman operator theory provides a way to find analytical solutions of previously unsolved nonlinear systems [28], yet finding the Koopman eigenfunctions necessary to construct solutions can be very difficult [6, 4]. When finding exact analytical solutions is impossible, Koopman operator theory still provides a way to analyze, predict, and control nonlinear systems better than what is possible using linear analysis [6, 16, 19]. Data-driven methods such as *dynamic mode decomposition* (DMD) [39, 38, 21, 2] and *extended dynamic mode decomposition* (EDMD) [46, 47, 21, 1] have provided a method for finding approximations to Koopman eigenfunctions that map nonlinear dynamics to linear dynamics. These mappings have been especially useful for applications in control [16, 19, 33]. Other approaches have divided the domain of nonlinear ODEs into separate "basins of attraction" where separate linearization transforms are constructed for different regions [23]. Koopman methods can also be useful for performing phase-amplitude reductions for nonlinear systems containing limit cycle oscillators [48, 49].

In this work, we develop a method for finding sets of Koopman eigenfunctions that map nonlinear dynamics to finite linear systems for certain planar nonlinear dynamical systems; these eigenfunctions allow us to construct solutions to nonlinear ODEs. Unlike the data-driven approaches which create approximate eigenfunctions by extending the linearization around fixed points, we construct exact, global Koopman eigenfunctions from codimension-1 invariant manifolds in the system. In the examples we present, the one-dimensional invariant manifolds we find have simple, closed-form analytic expressions which allow for closed-form Koopman eigenfunctions as well as closed-form analytical solutions. For most planar nonlinear systems, however, closed-form analytical expressions do not exist for their one-dimensional invariant manifolds; in these cases numerical methods become necessary to test for and construct eigenfunctions.

Exact analytical solutions are often preferable to numerical solutions as they do not generate errors which are inherent in numerical methods. Analytical solutions are also less computationally expensive than numerical solutions and are amenable to analysis techniques and control strategies that cannot otherwise be applied. Approximate analytical solutions have these same advantages over numerical solutions although they do contain errors.

Invariant manifolds are recognized as an important structure by which to analyze, reduce, and control nonlinear systems [12]. In fluid dynamics, Lagrangian coherent structures are robust structures that shape the patterns that appear in fluid flow [13, 40]. In unsteady vortical flows, transport is governed by invariant manifold structures in the flow [36]. In nonlinear systems containing heteroclinic networks, invariant manifolds are the dominant structures determining the system's behavior; analysis of the system is reduced to analyzing the heteroclinic orbits and the system's behavior within a neighborhood of these structures [12, 31]. Because they often dominate the dynamics, invariant manifolds can be used to dimension reduce dynamical systems [34, 15]. Model reduction is particularly useful in systems that have transient dynamics, such as chemical or biochemical systems [37]. Reduced order models of periodically excited nonlinear systems also utilize the system's invariant manifolds [11]. In systems with nonhyperbolic fixed points, center manifolds are used to analyze and reduce the system [12, 9]. In these systems, the nonhyperbolic equilibrium points can be stabilized by controlling the system's local center manifolds [43]. Following in this tradition of using invariant manifolds as a framework for understanding and describing dynamical systems, we extend the usability of invariant manifolds to eigenfunctions.

Our method for solving certain types of planar nonlinear systems is outlined in Figure 1(a). Figure 1(b) shows the method applied to the dynamical system $(\dot{x}, \dot{y}) = (x - xy, -x - y - y^2)$. We begin by finding closed-form analytical solutions for the one-dimensional invariant manifolds going through the system's fixed points and tangent to the eigenvector directions of the system linearized at the fixed points. The nonlinear system in this example has three invariant manifolds that fit this description (red lines). We then construct "M-functions" from the invariant manifold functions and use the M-functions to construct two globally valid Koopman eigenfunctions, $\varphi_1(x, y)$ and $\varphi_2(x, y)$, both of which have linear dynamics. Once we have a linearly independent set of eigenfunctions we solve for the state variables (x, y) in terms of the eigenfunctions, solve the linear dynamical system the state variables have been mapped to via the eigenfunctions, and then substitute the linear solutions into the state variable solutions to obtain analytical expressions for the state variables in terms of the initial conditions (x_0, y_0) and time t (Figure 1(d)). This example highlights how an algorithmic approach may be used to determine solutions to nonlinear systems of differential equations in select cases.

The manuscript is organized as follows. Section 2 provides an overview of Koopman theory, Koopman eigenfunctions, and the codimension-1 invariant manifolds of one-dimensional and two-dimensional nonlinear ODEs. Section 3 states necessary but not sufficient conditions for the construction of eigenfunctions from codimension-1 invariant manifolds. When eigenfunctions of the form we consider do exist we provide formulas for the eigenvalue-eigenfunction pairs. This section also considers the conditions necessary to obtain analytical solutions from sets of eigenfunctions once they are found. Section 4 outlines the Koopman eigenfunction approach to solving one-dimensional ODEs and provides several examples. Section 5 outlines the Koopman eigenfunction approach to solving two-dimensional ODEs and demonstrates the method with multiple examples. Section 6 discusses the method's limitations, possible extensions, and ramifications for data-driven discovery of eigenfunctions. Section 7 concludes the manuscript.

2. Background.

We consider how to construct solutions to nonlinear, autonomous, ODEs by using Koopman theory to map nonlinear dynamics to a space that has linear dynamics. We produce the nonlinear-to-linear mapping with Koopman eigenfunctions that we construct from the nonlinear ODE's invariant manifolds.

2.1. Koopman theory.

Consider the ODE

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

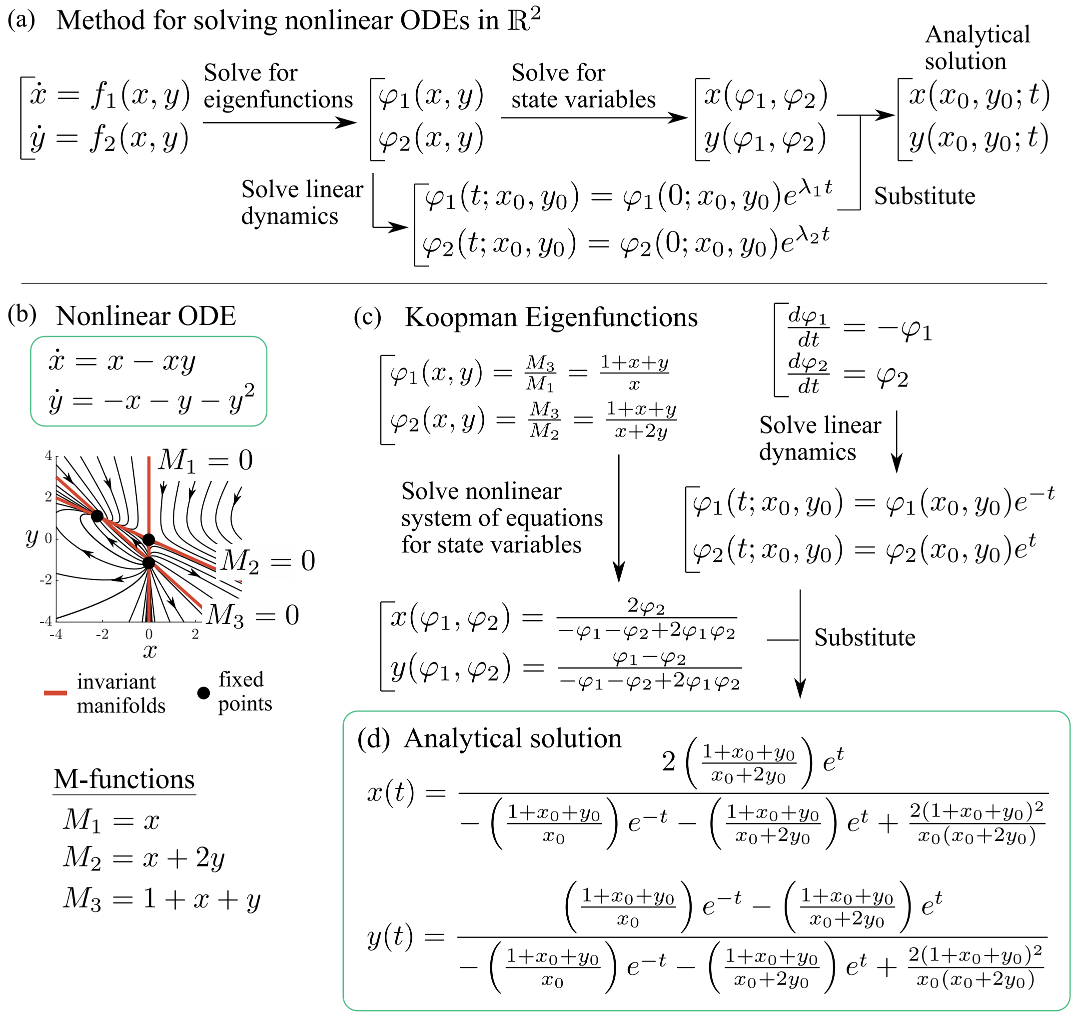

Figure 1. (a) *Outline of method for solving certain types of nonlinear ODEs in* $\mathbb{R}^2$. (b) *Example ODE with multiple fixed points and real invariant manifolds (red).* (c) *Global Koopman eigenfunctions are constructed from the M-functions. The state variables* (x, y) *can be solved for from the system of eigenfunctions. Solving the linear dynamics of the eigenfunctions and substituting the solutions into the state variable functions produces a solution for the original nonlinear ODE.* (d) *Analytical solution constructed from the global Koopman eigenfunctions.*

with the autonomous vector field $F:\mathbb{R}^n \to \mathbb{R}^n$ that operates on the state vector $\mathbf{x}$. The flow associated with (2.1) for each $t \in \mathbb{R}$ is the function $\mathbf{x}(t) := S_t(\mathbf{x}_0)$ for a trajectory starting at $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$. The Koopman operator describes the dynamics of "observables" or measurements of the state vector along its flow [17, 28, 8, 3]. The observable measurements $g : \mathbb{R}^n \to \mathbb{C}$ are elements of a space of observable functions $\mathcal{F}$. The Koopman operator $\mathcal{K}_t : \mathcal{F} \to \mathcal{F}$ is an infinite-dimensional linear operator that propogates observables g of the state vector $\mathbf{x}$ forward in time along trajectories of (2.1),

$$\mathcal{K}_t[g](\mathbf{x}) = g \circ S_t(\mathbf{x}). \tag{2.2}$$

The left-hand side of (2.2) states that the Koopman operator $\mathcal{K}_t$ pushes the observables g of the state vector $\mathbf{x}$ forward in time t. The resulting value is equivalent to the right-hand side of the equation, which states that the original variable $\mathbf{x}$ is pushed forward in time t according to flow S_t of (2.1) and then observed by g. Figure 2(a) illustrates the Koopman operator pushing observables g forward in time and how the result is equivalent to taking measurements of $\mathbf{x}$ along its flow.

Eigenfunctions $\varphi \in \mathcal{F}$ of the Koopman operator are special observables that have exponential time dependence [3]. An eigenvalue-eigenfunction pair (λ, φ) of $\mathcal{K}_t$ satisfies the equation

$$\mathcal{K}_t[\varphi](\mathbf{x}) = \varphi(\mathbf{x})e^{\lambda t}. \tag{2.3}$$

$\varphi(t;\mathbf{x})$ has exponential dynamics with eigenvalue (growth constant) λ and initial condition $\varphi(\mathbf{x})$ under the operator $\mathcal{K}_t$. Eigenfunctions have linear dynamics and satisfy the following equation:

$$\frac{d}{dt}\varphi(\mathbf{x}) = \lambda\varphi(\mathbf{x}). \tag{2.4}$$

Differentiating the left-hand side produces the linear first-order PDE

$$\nabla_{\mathbf{x}}\varphi(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{x}}\varphi(\mathbf{x}) \cdot F(\mathbf{x}) = \lambda\varphi(\mathbf{x}). \tag{2.5}$$

Solutions to (2.4) are eigenfunctions of $\mathcal{K}_t$ [3, 4]. Figure 2(b) illustrates that eigenfunctions under the Koopman operator have linear dynamics with the closed-form solution

$$\varphi(t;\mathbf{x}_0) = \varphi(\mathbf{x}_0)e^{\lambda t}. \tag{2.6}$$

Koopman eigenfunctions are extremely useful as they can be used to construct analytical solutions for nonlinear ODEs. There is no known method for finding analytical solutions for most nonlinear ODEs; finding eigenfunctions for such ODEs is a promising method for obtaining solutions. Unfortunately, despite the growing interest and usefulness of Koopman eigenfunctions, there are few methods available to discover explicit, closed-form expressions for Koopman eigenfunctions [4, 16, 3, 32].

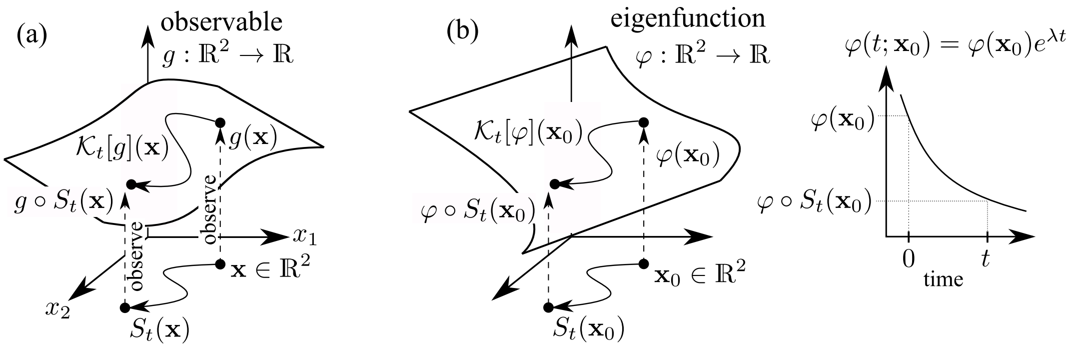

Figure 2. (a) The Koopman operator $\mathcal{K}_t$ *pushes observables* g *forward in time.* (b) *Eigenfunctions* φ *are special observables with linear dynamics,* $\dot{\varphi} = \lambda\varphi$.

We show that, when certain conditions are met, closed-form expressions for Koopman eigenfunctions can be constructed from invariant manifold generating functions. These are the first examples of combining multiple invariant manifolds to construct eigenfunctions that are rational expressions. These are also among the first examples of finding eigenfunctions for nonlinear ODEs that have multiple fixed points and that are not Hamiltonian.

Another way in which we deviate from previous methods is that upon finding eigenfunctions, we do not construct solutions for $\mathbf{x}(t)$ by using a linear combination of eigenfunctions. Rather, the closed-form solution for $\mathbf{x}(t)$ is constructed using a *nonlinear* combination of eigenfunctions which we find by inverting the eigenfunctions. This method allows for closed-form analytical solutions for $\mathbf{x}(t)$ that were previously not possible.

2.2. Invariant manifolds of ordinary differential equations. Our method for obtaining eigenfunctions for two-dimensional nonlinear ODEs requires finding one-dimensional invariant manifolds that go through the system's fixed points. An invariant manifold of a dynamical system is a topological manifold that is invariant under the actions of the dynamical system [10, 45, 44].

Definition 2.1 (invariant set). *A set of states $\Lambda \subseteq \mathbb{R}^n$ of (2.1) is called an invariant set of (2.1) if for all $\mathbf{x}_0 \in \Lambda$, and for all $t \in \mathbb{R}$, $\mathbf{x}(t) \in \Lambda$.*

We aim to define the invariant manifolds that go through the system's fixed points that are tangent to the eigenvector directions of the system linearized at the fixed points (Figure 3). Sometimes the invariant manifolds that go through a system's fixed points can be represented with closed-form analytical functions, $y = m_i(x)$ or $x = m_i(y)$. In section 5 we focus on nonlinear ODEs that have one-dimensional invariant manifolds that can be represented by closed-form analytical functions. This is a narrow focus; however, with this restriction we demonstrate how our method can produce exact analytical solutions without resorting to numerical methods if the invariant manifolds can be defined with closed-form solutions. In Appendix D we consider systems whose invariant manifolds can only be defined numerically and use regression to test for eigenfunctions.

A one-dimensional invariant manifold of a two-dimensional ODE, $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, can be defined by an implicit function $\Lambda = \{(x, y) : M(x, y) = 0\}$. Consider an invariant set defined

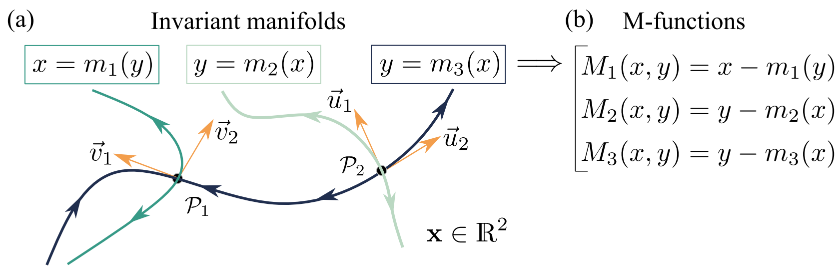

Figure 3. (a) *One-dimensional invariant manifolds of a two-dimensional nonlinear ODE defined by functions $x = m_1(y)$, $y = m_2(x)$, and $y = m_2(x)$. These functions go through the system's fixed points, $\mathcal{P}_1$ and $\mathcal{P}_2$, and are tangent to the eigenvectors of the system, $(\vec{v}_1, \vec{v}_2)$ and $(\vec{u}_1, \vec{u}_2)$, linearized at the fixed points.* (b) *M-functions constructed from the invariant manifolds have zero level sets along the invariant manifolds.*

by the curve $M(x,y)=0$. The flow $\mathbf{x}(t)=S_t(\mathbf{x}_0)$ will stay on the curve $M(x,y)=0$ for all time if the initial condition $\mathbf{x}_0$ is on the curve. While there are many functions $M(x,y)$ that have a zero level set along a given curve defining an invariant manifold $y=m(x)$, the simplest function that has this curve as its zero level set is $M(x,y)=y-m(x)$.

Definition 2.2 (invariant manifold generating function (M-function)). *A function $M:\mathbb{R}^2\to\mathbb{R}$ $M(x,y)=y-m(x)$ is called an M-function of the ODE $\dot{\mathbf{x}}=F(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^2$, if $y=m(x)$ defines a one-dimensional invariant manifold of the ODE.*

By definition, the zero level set of an M-function defines a one-dimensional invariant manifold of the nonlinear system. Note that a function of the form $x=m(y)$ can also define a one-dimensional invariant manifold; its corresponding M-function, in this case, would be $M(x,y)=x-m(y)$, which is also a valid form for the M-function and does not change the ensuing analysis. The invariant manifold generating functions (M-functions) of a two-dimensional ODE are the building blocks that we use for constructing eigenfunctions.

For two-dimensional nonlinear ODEs, we can use the fixed points $\mathcal{P}_i$ and the eigenvector directions to begin solving for the closed-form expressions of the invariant manifolds, $y=m_i(x)$ or $x=m_i(y)$, tangent to the eigenvectors [35, 14]. Figure 3(a) illustrates three invariant manifolds, ($x=m_1(y)$, $y=m_2(x)$, and $y=m_3(x)$), emanating from the fixed points ($\mathcal{P}_1$ and $\mathcal{P}_2$) of a system in $\mathbb{R}^2$. Figure 3(b) shows the M-functions we construct from the functions defining the invariant manifolds.

The M-functions of a two-dimensional ODE have corresponding N-functions that tell us how the M-functions must be combined in order to generate eigenfunctions for the system. We solve for these N-functions by differentiating $M(\mathbf{x})$:

$$N(\mathbf{x})=\frac{\frac{d}{dt}M(\mathbf{x})}{M(\mathbf{x})}. \tag{2.7}$$

$N(\mathbf{x})$ is the corresponding N-function of $M(\mathbf{x})$. The M-functions and their corresponding N-functions have the following interesting properties that aid us in constructing eigenfunctions.

Observation 1. If $\frac{d}{dt}M_1(\mathbf{x})=M_1(\mathbf{x})N_1(\mathbf{x})$ and $\frac{d}{dt}M_2(\mathbf{x})=M_2(\mathbf{x})N_2(\mathbf{x})$, then $\frac{d}{dt}(M_1(\mathbf{x})M_2(\mathbf{x}))=(N_1(\mathbf{x})+N_2(\mathbf{x}))M_1(\mathbf{x})M_2(\mathbf{x})$.

Observation 2. If $\frac{d}{dt}M_1(\mathbf{x})=M_1(\mathbf{x})N_1(\mathbf{x})$ and $\frac{d}{dt}M_2(\mathbf{x})=M_2(\mathbf{x})N_2(\mathbf{x})$, then $\frac{d}{dt}(\frac{M_1(\mathbf{x})}{M_2(\mathbf{x})})=(N_1(\mathbf{x})-N_2(\mathbf{x}))(\frac{M_1(\mathbf{x})}{M_2(\mathbf{x})})$.

Notice that if $N_1(\mathbf{x})+N_2(\mathbf{x})$ equals a constant, then $M_1(\mathbf{x})M_2(\mathbf{x})$ is an eigenfunction with corresponding eigenvalue $N_1(\mathbf{x})+N_2(\mathbf{x})$. If $N_1(\mathbf{x})-N_2(\mathbf{x})$ equals a constant, then $\frac{M_1(\mathbf{x})}{M_2(\mathbf{x})}$ is an eigenfunction with corresponding eigenvalue $N_1(\mathbf{x})-N_2(\mathbf{x})$. We will generalize this observation in the next section by using M-functions and N-functions to solve for eigenvalue-eigenfunction pairs of the Koopman operator $\mathcal{K}_t$. We construct eigenvalues from the N-functions and eigenfunctions from the M-functions.

3. Constructing eigenfunctions from invariant manifolds. Notice that if $N(\mathbf{x})=c\in\mathbb{R}$, a constant, then $M(\mathbf{x})$ is an eigenfunction of $\dot{\mathbf{x}}=F(\mathbf{x})$. If $N(\mathbf{x})$ is not a constant, then $M(\mathbf{x})$ cannot be an eigenfunction. Nonetheless, a combination of manifold generating functions, $M_1(\mathbf{x})$ and $M_2(\mathbf{x})$, may still create an eigenfunction so long as their corresponding N-functions can linearly combine to produce a constant. Theorem 3.1 allows us to construct eigenfunctions given there exists some linear combination of N-functions that results in a constant.

Theorem 3.1. *Let $M_1(\mathbf{x})$ and $M_2(\mathbf{x})$ be invariant manifold generating functions of $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, with corresponding N-functions $N_1(\mathbf{x})$ and $N_2(\mathbf{x})$. $\varphi(\mathbf{x}) = M_1^p(\mathbf{x})M_2^q(\mathbf{x})$, $p, q \in \mathbb{C}$, is an eigenfunction of $\dot{\mathbf{x}} = F(\mathbf{x})$ if and only if $pN_1(\mathbf{x}) + qN_2(\mathbf{x}) = \lambda$ for some constant $\lambda \in \mathbb{C}$.*

Proof. Let $\frac{d}{dt}M_1(\mathbf{x}) = M_1(\mathbf{x})N_1(\mathbf{x})$ and $\frac{d}{dt}M_2(\mathbf{x}) = M_2(\mathbf{x})N_2(\mathbf{x})$ for $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$. Let $\varphi(\mathbf{x}) = M_1^p(\mathbf{x})M_2^q(\mathbf{x})$ and $\lambda = pN_1(\mathbf{x}) + qN_2(\mathbf{x})$.

$$\begin{aligned}\frac{d}{dt}\varphi(\mathbf{x}) &= \frac{d}{dt}\left[M_1^p(\mathbf{x})M_2^q(\mathbf{x})\right] \\ &= M_1^p(\mathbf{x})\frac{d}{dt}M_2^q(\mathbf{x}) + M_2^q(\mathbf{x})\frac{d}{dt}M_1^p(\mathbf{x}) \\ &= M_1^p qM_2^{q-1}\frac{d}{dt}M_2 + M_2^q pM_1^{p-1}\frac{d}{dt}M_1 \\ &= M_1^p qM_2^{q-1}M_2N_2 + M_2^q pM_1^{p-1}M_1N_1 \\ &= M_1^p qM_2^q N_2 + M_2^q pM_1^p N_1 \\ &= (pN_1 + qN_2)M_1^p M_2^q \\ &= \lambda\varphi(\mathbf{x}).\end{aligned}$$

$\varphi(\mathbf{x})$ is an eigenfunction if and only if $\lambda \in \mathbb{C}$. Therefore, if $\lambda \in \mathbb{C}$, then $\varphi(\mathbf{x}) = M_1^p(\mathbf{x})M_2^q(\mathbf{x})$, $p, q \in \mathbb{C}$, is an eigenfunction of $\dot{\mathbf{x}} = F(\mathbf{x})$ with corresponding eigenvalue $\lambda = pN_1(\mathbf{x}) + qN_2(\mathbf{x})$. ∎

Theorem 3.1 says that if there exists a linear combination of $N_1(\mathbf{x})$ and $N_2(\mathbf{x})$ that results in a constant λ, then we can construct eigenfunctions from the corresponding M-functions. λ is the eigenvalue that corresponds to the resulting eigenfunction, producing the eigenvalue-eigenfunction pair $(\lambda, \varphi(\mathbf{x}))$. We note that if there exists a p and q such that $pN_1(\mathbf{x})+qN_2(\mathbf{x}) = \lambda$, where λ is a constant, then $c(pN_1(\mathbf{x}) + qN_2(\mathbf{x})) = c\lambda$, where $c\lambda$ is also a constant for any $c \in \mathbb{C}$. This tells us that from the single eigenvalue-eigenfunction pair $(\lambda, \varphi(\mathbf{x}))$, we can generate a family of eigenvalue-eigenfunction pairs $(c\lambda, \varphi^c(\mathbf{x}))$, where $c \in \mathbb{C}$. We observe from this eigenvalue-eigenfunction family that we may set any complex number to be the eigenvalue; however, the resulting corresponding eigenfunction may be quite complicated. Therefore, in the following examples we choose eigenvalues that will result in eigenfunctions that have simple expressions. Note that setting $p = 0$ and $q = 0$ results in the constant $\lambda = 0$, producing the trivial eigenvalue-eigenfunction pair $\lambda = 0$ and $\varphi(\mathbf{x}) = 1$. While the trivial solution is technically an eigenvalue-eigenfunction pair, it is not useful for solving the ODE.

Theorem 3.1 can be extended to include more invariant manifolds than simply two. Nonlinear systems may contain more than two invariant manifolds and a linear combination of N-functions from more than two of these invariant manifolds may be required to produce a constant.

Theorem 3.2. *Let $\{M_i(\mathbf{x})\}_i^k$ be a set of invariant manifold generating functions of $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, with corresponding N-functions $\{N_i(\mathbf{x})\}_i^k$. $\varphi(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$, $p_i \in \mathbb{C}$, is an eigenfunction of $\dot{\mathbf{x}} = F(\mathbf{x})$ if and only if $\sum_i^k p_iN_i(\mathbf{x}) = \lambda$ for some constant $\lambda \in \mathbb{C}$.*

Proof. Let $\frac{d}{dt}M_i(\mathbf{x}) = M_i(\mathbf{x})N_i(\mathbf{x})$, $i \in \{1,\ldots,k\}$, for $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$. Let $\varphi(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$ and $\lambda = \sum_i^k p_i N_i(\mathbf{x})$.

$$\begin{aligned}
\frac{d}{dt}\varphi(\mathbf{x}) &= \frac{d}{dt}\left(\prod_{i=1}^k M_i^{p_i}(\mathbf{x})\right)\\
&= \frac{d}{dt}M_1^{p_1}\left(\prod_{i=2}^k M_i^{p_i}\right) + M_1^{p_1}\frac{d}{dt}\left(\prod_{i=2}^k M_i^{p_i}\right)\\
&= p_1 M_1^{p_1} N_1\left(\prod_{i=2}^k M_i^{p_i}\right) + M_1^{p_1}\frac{d}{dt}\left(\prod_{i=2}^k M_i^{p_i}\right)\\
&= p_1 N_1\left(\prod_{i=1}^k M_i^{p_i}\right) + M_1^{p_1}\frac{d}{dt}\left(\prod_{i=2}^k M_i^{p_i}\right)\\
&= p_1 N_1\left(\prod_{i=1}^k M_i^{p_i}\right) + M_1^{p_1}\left(p_2 N_2\left(\prod_{i=2}^k M_i^{p_i}\right) + M_2^{p_2}\frac{d}{dt}\left(\prod_{i=3}^k M_i^{p_i}\right)\right)\\
&= p_1 N_1\left(\prod_{i=1}^k M_i^{p_i}\right) + p_2 N_2\left(\prod_{i=1}^k M_i^{p_i}\right) + M_1^{p_1}M_2^{p_2}\frac{d}{dt}\left(\prod_{i=3}^k M_i^{p_i}\right)\\
&= \left(\sum_{i=1}^k p_i N_i\right)\left(\prod_{i=1}^k M_i^{p_i}\right)\\
&= \left(\sum_{i=1}^k p_i N_i(\mathbf{x})\right)\varphi(\mathbf{x})\\
&= \lambda\varphi(\mathbf{x}).
\end{aligned}$$

$\varphi(\mathbf{x})$ is an eigenfunction if and only if $\lambda \in \mathbb{C}$. Therefore, if $\lambda \in \mathbb{C}$, then $\varphi(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$ is an eigenfunction of $\dot{\mathbf{x}} = F(\mathbf{x})$ with corresponding eigenvalue $\lambda = \sum_i^k p_i N_i(\mathbf{x})$. ■

We use the previous theorems to construct eigenfunctions for two-dimensional nonlinear ODEs by (1) finding closed-form expressions for the invariant manifolds, (2) solving for the corresponding N-functions, and (3) finding linear combinations of N-functions that reduce to a constant. Once we have solved for the N-function weights, p_i, we use these weights as exponents in the M-function product, producing an eigenfunction for the nonlinear ODE.

3.1. Obtaining independent eigenfunctions. To obtain solutions for two-dimensional ODEs, we must construct at least two independent eigenfunctions, $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ that have different level sets—that is, they cannot belong to the same "family" or equivalence class of eigenfunctions. Without two independent eigenfunctions we cannot create a unique mapping from the two eigenfunctions back to the original variables. An equivalence class of Koopman eigenvalue-eigenfunction pairs, $\overline{(\lambda, \varphi(\mathbf{x}))}$, have level sets that match between functions (although the level sets need not match to the same levels). Exponentiations of an eigenfunction belong to the same equivalence class [3, 8],

$$\{(p\lambda, \varphi^p(\mathbf{x})), p \in \mathbb{R}\} \subset \overline{(\lambda, \varphi(\mathbf{x}))}. \tag{3.1}$$

Multiples of an eigenfunction also belong to the same equivalence class,

$$\{(\lambda, \alpha\varphi(\mathbf{x})), \alpha \in \mathbb{R}\} \subset \overline{(\lambda, \varphi(\mathbf{x}))}.$$

If one of our eigenfunctions constructed from invariant manifolds is not an exponentiated multiple of the other, then the two eigenfunctions belong to different equivalence classes and we may use the pair of independent eigenfunctions to solve the ODE. However, we do not guarantee a unique analytical solution in all cases where a pair of independent eigenfunctions is found. We can use the following theorem to identify eigenfunctions that belong to different equivalence classes.

Theorem 3.3. *Let* $\lambda_1 = \sum_i^k p_i N_i(\mathbf{x})$ *and* $\lambda_2 = \sum_i^k q_i N_i(\mathbf{x})$ *be the eigenvalues corresponding to eigenfunctions* $\varphi_1(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$ *and* $\varphi_2(\mathbf{x}) = \prod_i^k M_i^{q_i}(\mathbf{x})$, *where the weights for the linear combination of* N*-functions are the vectors* $\mathbf{p} = [p_1 \quad p_2 \quad \cdots \quad p_k]^T$ *and* $\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_k]^T$. *Additionally, let* $\{\log(M_i(\mathbf{x}))\}_i^k$ *be a linearly independent set of functions. The eigenfunctions* $\varphi_1(\mathbf{x})$ *and* $\varphi_2(\mathbf{x})$ *belong to different equivalence classes if* $\mathbf{p}$ *and* $\mathbf{q}$ *are linearly independent vectors, that is,* $\mathbf{p} \neq \alpha\mathbf{q}$ *for any* $\alpha \in \mathbb{C}$.

Proof. If $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ are in the same equivalence class, then

$$\prod_i^k M_i^{p_i}(\mathbf{x}) = \left(\prod_i^k M_i^{q_i}(\mathbf{x})\right)^\alpha = \prod_i^k M_i^{\alpha q_i}(\mathbf{x}) \quad \text{for some} \quad \alpha \in \mathbb{R}.$$

Therefore, $1 = \prod_i^k M_i^{\alpha q_i - p_i}(\mathbf{x})$. Taking the log of both sides results in $0 = \sum_i^k (\alpha q_i - p_i) \log(M_i(\mathbf{x}))$. If the set of functions $\{\log(M_i(\mathbf{x}))\}_i^k$ is linearly independent, then the only solution for the weights is $\alpha q_i - p_i = 0$ for all $i \in \{1, 2, \ldots, k\}$. This implies that $p_i = \alpha q_i$ for all $i \in \{1, \ldots, k\}$. However, $\mathbf{p} \neq \alpha\mathbf{q}$ for any $\alpha \in \mathbb{C}$. Therefore, $\varphi_1(\mathbf{x}) \neq (\varphi_2(\mathbf{x}))^\alpha$ for any α; $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ are not in the same equivalence class. ■

Theorem 3.3 says that if the weighting vectors for the N-functions are linearly independent, then the eigenfunctions constructed from the M-functions are independent (in different equivalence classes). In order to use this test for independent eigenfunctions, we must first determine that the set of functions $\{\log(M_i(\mathbf{x}))\}_i^k$ is linearly independent, which can be determined by computing the generalized Wronskians of the list of functions $\phi = (\log(M_1(x,y)), \ldots, \log(M_k(x,y)))$ [50]. Reference [50] states that if a set of functions is linearly dependent, then all generalized Wronskians must vanish identically. Therefore, if any of the generalized Wronskians is not identically equal to zero, then the set of multivariable functions is linearly independent.

4. Koopman eigenfunctions for one-dimensional ODEs.

We first consider how to use a Koopman approach to solve nonlinear, autonomous, first-order ODEs. In the one-dimensional case, such equations are easily solvable via separation of variables. However, we will consider the alternative, Koopman approach to solving these differential equations in order to build an intuition for the method in the two-dimensional case, where separation of variables can no longer be used to construct a solution. Consider a nonlinear, autonomous, first-order, ODE:

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{C}. \tag{4.1}$$

This is a separable, first-order differential equation and so is solvable by separating the variables and then integrating,

$$\int \frac{dx}{f(x)} = \int dt = t + c. \tag{4.2}$$

Instead of solving this ODE directly, we can instead take the Koopman perspective and first map the nonlinear dynamics of x to a space with linear dynamics, find a solution in the linear space, and then map the solution back to x. We solve for the eigenfunction $\varphi : \mathbb{C} \to \mathbb{C}$ that maps the nonlinear dynamics to a space with linear dynamics,

$$\begin{aligned}
\frac{d}{dt}\varphi(x) = \frac{d}{dx}\varphi(x)\frac{dx}{dt} &= \varphi'(x)f(x) = \lambda\varphi(x) \\
\int \frac{\varphi'(x)}{\varphi(x)}dx &= \int \frac{\lambda}{f(x)}dx \\
\ln[\varphi(x)] + c_1 &= \int \frac{\lambda}{f(x)}dx \\
\varphi(x) &= c_2 e^{\int \frac{\lambda}{f(x)}dx}.
\end{aligned} \tag{4.3}$$

If $f(x)$ is a polynomial with simple roots, $f(x) = c\prod_{i=1}^{n}(x - x_i)$, where $x_i \in \mathbb{C}$, then we can solve further by integrating each of the resulting fractions separately,

$$\int \frac{\lambda}{f(x)}dx = \int \frac{\lambda}{c\prod_{i=1}^{n}(x - x_i)}dx = \int \sum_{i=1}^{n} \frac{p_i}{x - x_i}dx = \sum_{i=1}^{n} \log[(x - x_i)^{p_i}].$$

The numerators are determined via the method of partial fractions,

$$p_i = \frac{\lambda}{c\prod_{j=1:n, j\neq i}(x_i - x_j)}. \tag{4.4}$$

Therefore the solution to the eigenfunction is

$$\varphi(x) = c_2 e^{\int \frac{\lambda}{f(x)}dx} = c_2 \prod_{i=1}^{n}(x - x_i)^{p_i}, \tag{4.5}$$

where $\{x_i\}_i^n$ are the simple roots of $f(x)$ and the exponents $\{p_i\}_i^n$ are the constants generated by the method of partial fractions (equation (4.4)). Notice that $\varphi(x)$ is a composition of zeros and singularities and the signs of the p_i values determine which x_i are zeros versus singularities. The dynamics of φ are linear by construction,

$$\varphi(t, x_0) = \varphi(x_0)e^{\lambda t}. \tag{4.6}$$

Last, we solve for x as a function of $\varphi(t, x_0)$ by inverting $\varphi(x)$ (equation (4.5)). $\varphi(x)$ is often not invertible, meaning that multiple x map to a single φ value. Only by including knowledge of the initial condition x_0 can this ambiguity be resolved, allowing us to create a one-to-one mapping from $(\varphi, x_0) \mapsto x$.

4.1. One-dimensional ODE—Example 1. Let us first consider a nonlinear differential equation that has been used as an example in previous work on Koopman analysis [16, 4, 3],

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0. \tag{4.7}$$

The solution derived from separation of variables is

$$x(t) = \frac{x_0}{1 - x_0 t}. \tag{4.8}$$

Alternatively we can solve for $x(t)$ using the Koopman approach by first mapping x to a linear space. We choose our eigenvalue to be $\lambda = -1$ and solve for the corresponding eigenfunction,

$$\varphi(x) = e^{\int \frac{\lambda}{f(x)} dx} = e^{\int \frac{-1}{x^2} dx} = e^{\frac{1}{x}}, \tag{4.9}$$

$$\varphi(t; x_0) = \varphi(x_0) e^{\lambda t} = e^{\frac{1}{x_0}} e^{-t} = e^{\frac{1 - x_0 t}{x_0}}. \tag{4.10}$$

We have freedom in our choice of λ. Choosing $\lambda = -1$ results in a simple eigenfunction. According to property 3.1, any multiple of -1 could alternatively be selected as an eigenvalue which would result in an eigenfunction in the same equivalence class. Solving for x in terms of φ, using (4.9) gives us

$$x(t) = \frac{1}{\ln[\varphi(t; x_0)]} = \frac{1}{\ln\left[e^{\frac{1 - x_0 t}{x_0}}\right]} = \frac{x_0}{1 - x_0 t}. \tag{4.11}$$

Although the Koopman approach is less efficient than solving via separation of variables in the one-dimensional case, we will use a Koopman approach to solve two-dimensional systems which cannot be solved directly.

4.2. One-dimensional ODE—Example 2. Suppose we have a nonlinear differential equation of the form

$$\frac{dx}{dt} = -x^3 + x, \quad x(0) = x_0. \tag{4.12}$$

The solution can be found using separation of variables, resulting in

$$x(t) = \frac{\text{sign}(x_0) e^t}{\sqrt{-1 + e^{2t} + \frac{1}{x_0^2}}}. \tag{4.13}$$

Alternatively, we can also solve for $x(t)$ via the Koopman approach by first mapping x to a variable that has linear dynamics, solving the linear ODE, and then mapping the solution back to the nonlinear space. First let us factor the right-hand side of the differential equation,

$$\frac{dx}{dt} = -x(x+1)(x-1). \tag{4.14}$$

The fixed points of the nonlinear ODE are used to construct the solution. The system has two stable fixed points at $x = \pm 1$ and a source at $x = 0$ (Figure 4(a)). We can map both stable fixed points to the fixed point in the Koopman linear space by setting the eigenvalue to be $\lambda = -1$. The unstable fixed point is mapped to infinity (Figure 4(b)). We solve for the eigenfunction using the steps outlined above,

$$\begin{aligned}
\varphi(x) &= \exp \int \frac{\lambda}{f(x)} dx \\
&= \exp \int \frac{-1}{-x(x^2-1)} dx \\
&= \exp \left(\int \frac{-1}{x} dx + \int \frac{x}{x^2-1} dx \right) \\
&= \exp \left[-\ln(x) + \frac{1}{2} \ln(1-x^2) \right]
\end{aligned}$$

$$\varphi(x) = \frac{\sqrt{1-x^2}}{x}, \quad \lambda = -1. \tag{4.15}$$

The initial condition x_0 mapped to the eigenfunction space $\varphi(x)$ is

$$\varphi(x_0) = \frac{\sqrt{1-x_0^2}}{x_0}. \tag{4.16}$$

The dynamics of φ are linear (Figure 4(c)). Therefore the solution for $\varphi(t; x_0)$ is

$$\varphi(t; x_0) = \varphi(x_0) e^{\lambda t} = \frac{\sqrt{1-x_0^2}}{x_0} e^{-t}. \tag{4.17}$$

Using (4.15) we solve for x in terms of φ,

$$x(t) = \frac{\text{sign}(x_0)}{\sqrt{1+\varphi^2(t; x_0)}} = \frac{\text{sign}(x_0)}{\sqrt{1+\frac{1-x_0^2}{x_0^2} e^{-2t}}}. \tag{4.18}$$

The Koopman-derived solution is equivalent to the solution derived using separation of variables (4.13). In a previous Koopman approach to solving (4.12), separate linearization transforms were computed for each basin of attraction centered at each fixed point [23]. In contrast, we create a single nonlinear-to-linear mapping that is applicable to the entire domain of the nonlinear ODE.

The dynamics of nonlinear differential equations can be understood more fully by extending the dynamics into the complex plane. While all the fixed points of (4.12) are real, other differential equations have complex fixed points that impact the dynamics. Understanding the dynamics around fixed points is key to understanding the dynamics as a whole. We allow x to be a complex variable $x = a + ib$ and solve for the dynamics of the real and complex component of x,

$$\begin{aligned}
\frac{da}{dt} &= a - a^3 + 3ab^2, \\
\frac{db}{dt} &= b - 3a^2 b + b^3.
\end{aligned} \tag{4.19}$$

The dynamics in the complex plane is an extension of the dynamics along the real line. Figure 4(d) shows the dynamics in the complex plane; the linear dynamics of $\varphi(x)$ hold for

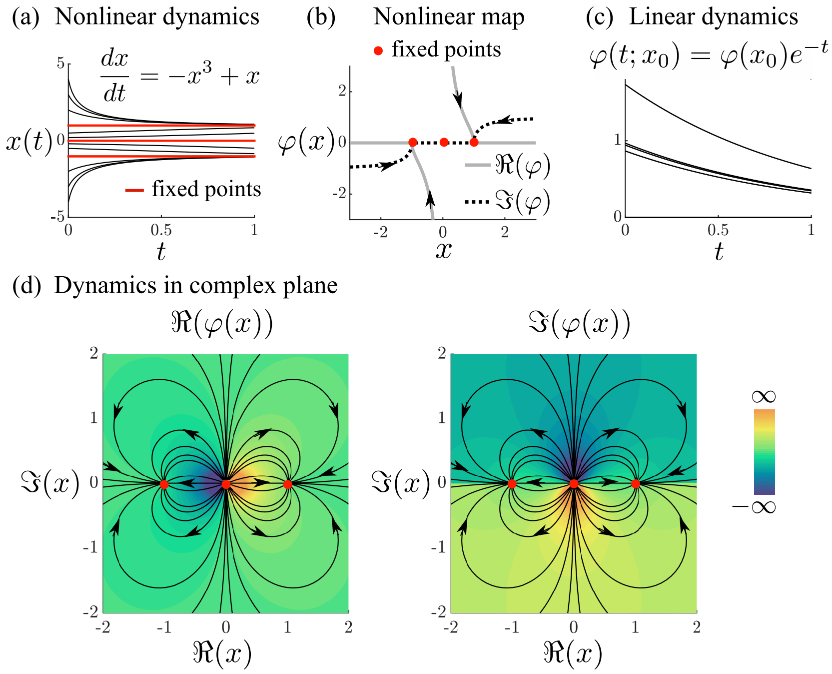

Figure 4. (a) *Equation* (4.12) *has nonlinear dynamics.* (b) *The nonlinear dynamics of* x *can be mapped to eigenfunction* $\varphi(x)$*.* (c) *The dynamics of* φ *is linear.* (d) *Dynamics of* $x = a + ib$ *viewed in the complex plane as well as mapping to complex-valued* φ*.*

complex values of x. By slicing the dynamics where the imaginary component is zero, $b = 0$, we recover the dynamics along the real line (Figure 4(b)). We see that $\lim_{x\to\pm 1} \varphi(x) = 0$ and $\lim_{x\to 0} \varphi(x) = \pm\infty$.

Appendix A contains two additional examples of one-dimensional ODEs solved via the Koopman approach.

5. Koopman eigenfunctions for two-dimensional ODEs.

We now turn our attention to solving two-dimensional nonlinear ODEs using Algorithm 5.1. Not all two-dimensional ODEs can be solved using this method. We must have multiple one-dimensional invariant manifolds, and we must be able to combine the M-functions in such a manner that the resulting functions are eigenfunctions. In order to solve a two-dimensional autonomous ODE using the following method, the ODE must satisfy the following requirements:

1. The ODE must have at least two distinct M-functions, $M_i(\mathbf{x})$ and $M_j(\mathbf{x})$, generated from distinct one-dimensional invariant manifolds, $m_i(x) \neq m_j(x)$.
2. There must exist at least two linear combinations of corresponding N-functions that result in nonzero constants.
3. At least two of the weight vectors, $\mathbf{p}_i$, corresponding to eigenfunctions φ_i, must be linearly independent; this requirement guarantees the eigenfunctions generated from the M-functions are from different equivalence classes.
4. The nonlinear system composed of the two independent eigenfunctions must be invertible, providing a unique solution to the state variables.

Algorithm 5.1 Koopman method for solving nonlinear ODEs in $\mathbb{R}^2$

1. Find independent eigenfunctions: $(x,y) \mapsto (\varphi_1(x,y), \varphi_2(x,y))$
2. Solve eigenfunction dynamics: $\varphi(t;x_0,y_0) = \varphi(x_0,y_0)e^{\lambda t}$
3. Compute initial condition: $\varphi_0 = \varphi(x_0,y_0)$
4. Solve for original variables: $(\varphi_1,\varphi_2) \mapsto (x(\varphi_1,\varphi_2), y(\varphi_1,\varphi_2))$
5. Substitute eigenfunction solutions into solutions for original variables:
   $x(t) = x(\varphi_1(t;x_0,y_0), \varphi_2(t;x_0,y_0)) = x(\varphi_1(x_0,y_0)e^{\lambda_1 t}, \varphi_2(x_0,y_0)e^{\lambda_2 t})$
   $y(t) = y(\varphi_1(t;x_0,y_0), \varphi_2(t;x_0,y_0)) = y(\varphi_1(x_0,y_0)e^{\lambda_1 t}, \varphi_2(x_0,y_0)e^{\lambda_2 t})$

This is a highly restrictive set of requirements; most planar nonlinear ODEs do not meet all of these requirements and so cannot be solved via this method. Once we have found a pair of independent eigenfunctions we can use Algorithm 5.1 to attempt to solve the nonlinear ODE. In all the examples we consider, closed-form expressions for the invariant manifolds exist, enabling us to derive closed-form expressions for the eigenfunctions and final solution. For most ODEs, however, closed-form expressions for one-dimensional invariant manifolds do not exist. When closed-form expressions do not exist the invariant manifolds may still be determined numerically and the eigenfunctions tested for numerically. We outline a method for numerically determining eigenfunctions constructed from M-functions in Appendix D.

5.1. Nonlinear example 1—Linear invariant manifolds. We extend the method we used to solve the linear ODE to nonlinear ODEs. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= y^2 - x - 1 \end{aligned} \tag{5.1}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. The fixed points of this system are at $(x,y) = (0,\pm 1), (-1,0)$. Three one-dimensional invariant manifolds go through the fixed points and are tangent to the eigenvectors of the system linearized at the fixed points. These one-dimensional invariant manifolds can be defined by the functions $x = 0$, $y = x+1$, and $y = -x-1$. The resulting M-functions $M_1(x,y) = x$, $M_2(x,y) = y - x - 1$, and $M_3(x,y) = y + x + 1$ define these three invariant manifolds with their zero level sets $\Lambda_1 = \{(x,y) : M_1(x,y) = 0\}$, $\Lambda_2 = \{(x,y) : M_2(x,y) = 0\}$, and $\Lambda_3 = \{(x,y) : M_3(x,y) = 0\}$ (Figure 5(a)). We can confirm that these are indeed invariant manifolds of (5.1) by checking that they satisfy $\frac{d}{dt}M_i(\mathbf{x}) = 0$ when $M_i(\mathbf{x}) = 0$. We use (2.7) to solve for the N-functions.

$$N_1(x,y) = \frac{\frac{d}{dt}M_1(x,y)}{M_1(x,y)} = \frac{\frac{dx}{dt}}{x} = \frac{xy}{x} = y,$$

$$N_2(x,y) = \frac{\frac{d}{dt}M_2(x,y)}{M_2(x,y)} = \frac{\frac{d(y-x-1)}{dt}}{y-x-1} = \frac{y^2 - x - 1 - xy}{y-x-1} = \frac{(y-x-1)(y+1)}{y-x-1} = y+1,$$

$$N_3(x,y) = \frac{\frac{d}{dt}M_3(x,y)}{M_3(x,y)} = \frac{\frac{d(y+x+1)}{dt}}{y+x+1} = \frac{y^2 - x - 1 + xy}{y+x+1} = \frac{(y+x+1)(y-1)}{y+x+1} = y-1.$$

The N-functions corresponding to the M-functions are $N_1(x,y) = y$, $N_2(x,y) = y+1$, and $N_3(x,y) = y - 1$. None of the N-functions are constants, and therefore none of the

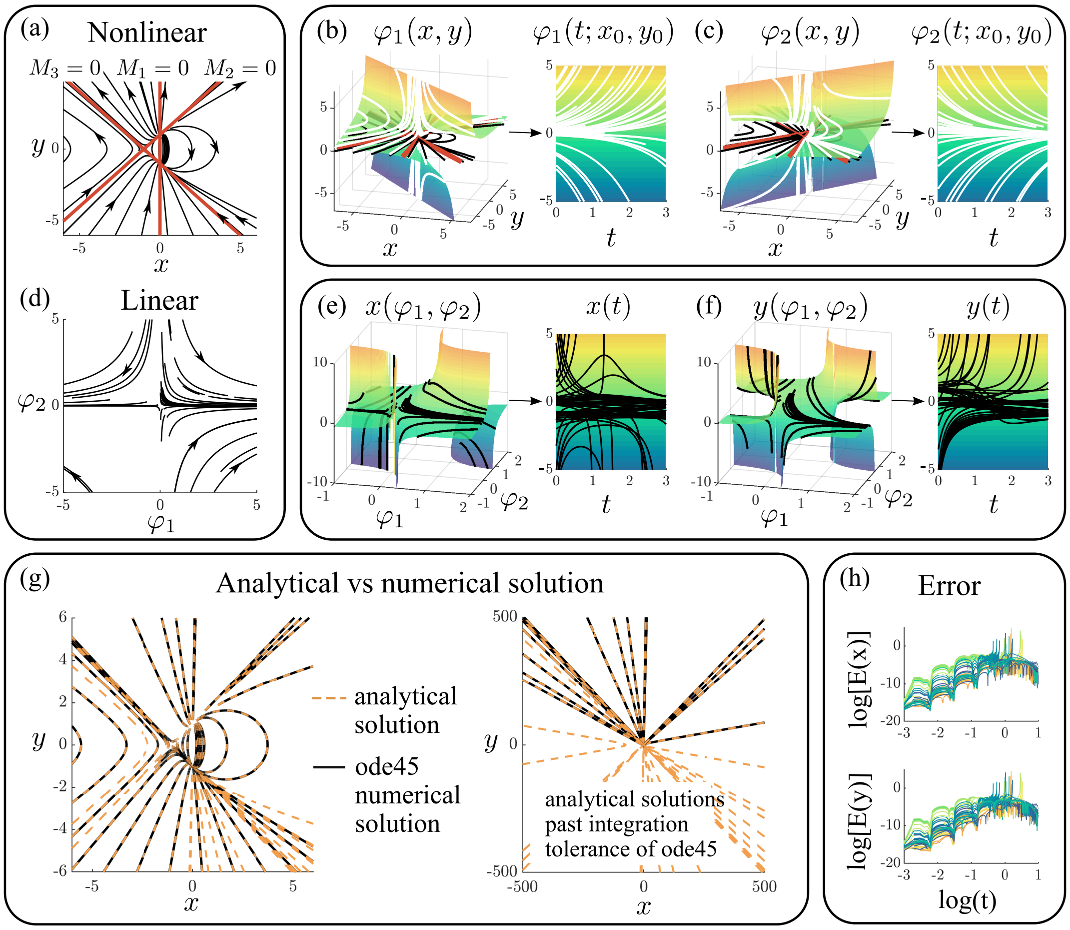

Figure 5. *Dynamics in original space and eigenfunction space.* (a) *Phase plane for* (5.1). (b)–(c) *Flow lines projected onto* φ_1 *and* φ_2. (d) *Phase plane of* (φ_1, φ_2). (e)–(f) *Flow lines of* (φ_1, φ_2) *mapped back to original variables* (x, y). (g) *Analytical solution compared to numerical solution produced using ode*45. (h) *Error in trajectories over time.*

M-functions are eigenfunctions. However, multiple linear combinations of the N-functions result in constants,

$$\begin{aligned}
\lambda_1 &= N_1(x,y) - N_3(x,y) = y - (y-1) &&= 1,\\
\lambda_2 &= N_1(x,y) - N_2(x,y) = y - (y+1) &&= -1,\\
\lambda_3 &= N_2(x,y) - N_3(x,y) = y + 1 - (y-1) &&= 2.
\end{aligned}$$

Therefore, according to Theorem 3.1, we can construct eigenfunctions from the M-function quotients

$$\varphi_1(x,y) = \frac{M_1}{M_3} = \frac{x}{1+x+y}, \quad \lambda_1 = 1, \tag{5.2}$$

$$\varphi_2(x,y) = \frac{M_1}{M_2} = \frac{x}{1+x-y}, \quad \lambda_2 = -1, \tag{5.3}$$

$$\varphi_3(x,y) = \frac{M_2}{M_3} = \frac{y-x-1}{y+x+1}, \quad \lambda_3 = 2. \tag{5.4}$$

The N-function weighting vectors for φ_1, φ_2, and φ_3 are $\mathbf{p}_1 = [1\ \ 0\ -1]^T$, $\mathbf{p}_2 = [1\ -1\ \ 0]^T$, and $\mathbf{p}_3 = [0\ \ 1\ -1]^T$ since $\lambda_1 = 1N_1 + 0N_2 - 1N_3$, $\lambda_2 = 1N_1 - 1N_2 + 0N_3$, and $\lambda_3 = 0N_1 + 1N_2 - 1N_3$. Theorem 3.3 states that if the set of functions $\{\log(M_1(x,y)), \log(M_2(x,y)), \log(M_3(x,y))\}$ is linearly independent and a set of $\mathbf{p}$ vectors $\{\mathbf{p}_i, \mathbf{p}_j\}$ is linearly independent, then their corresponding eigenfunctions φ_i and φ_j are in different equivalence classes (independent). In Appendix B we show that the set of functions $\{\log(M_1(x,y)), \log(M_2(x,y)), \log(M_3(x,y))\}$ is linearly independent. All pairs of $\mathbf{p}$ vectors are linearly independent sets. Therefore, according to Theorem 3.3, all pairs of eigenfunctions φ_1, φ_2, and φ_3 are in different equivalence classes and so any pair of these eigenfunctions can be used to solve for $\mathbf{x}(t)$. We confirm that these are eigenvalue-eigenfunction pairs of (5.1) by checking that they satisfy (2.5):

$$\begin{aligned} \nabla_{\mathbf{x}}\varphi_1(\mathbf{x}) \cdot F(\mathbf{x}) &= \lambda_1 \varphi_1(\mathbf{x}), \\ \begin{bmatrix} \frac{1+y}{(1+x+y)^2} & \frac{-x}{(1+x+y)^2} \end{bmatrix} \begin{bmatrix} xy \\ y^2 - x - 1 \end{bmatrix} &= \frac{x}{1+x+y}, \\ \frac{x}{1+x+y} &= \frac{x}{1+x+y}, \\ &\implies \varphi_1 \text{ is an eigenfunction.} \end{aligned}$$

$$\begin{aligned} \nabla_{\mathbf{x}}\varphi_2(\mathbf{x}) \cdot F(\mathbf{x}) &= \lambda_2 \varphi_2(\mathbf{x}), \\ \begin{bmatrix} \frac{1-y}{(1+x-y)^2} & \frac{x}{(1+x-y)^2} \end{bmatrix} \begin{bmatrix} xy \\ y^2 - x - 1 \end{bmatrix} &= \frac{(-1)x}{1+x-y}, \\ \frac{-x}{1+x-y} &= \frac{-x}{1+x-y}, \\ &\implies \varphi_2 \text{ is an eigenfunction.} \end{aligned}$$

The dynamics on these mappings are linear, giving us solutions in the eigenfunction space,

$$\varphi_1(t; x_0, y_0) = \varphi_1(x_0, y_0) e^{\lambda_1 t} = \varphi_1(x_0, y_0) e^{t}, \tag{5.5}$$

$$\varphi_2(t; x_0, y_0) = \varphi_2(x_0, y_0) e^{\lambda_2 t} = \varphi_2(x_0, y_0) e^{-t}. \tag{5.6}$$

The initial conditions in the eigenfunction space can be found by mapping the initial conditions in the (x, y) space, (x_0, y_0), to the eigenfunction space,

$$\varphi_1(x_0, y_0) = \frac{x_0}{1 + x_0 + y_0}, \tag{5.7}$$

$$\varphi_2(x_0, y_0) = \frac{x_0}{1 + x_0 - y_0}. \tag{5.8}$$

We now need to map the solution back to the original (x, y) space. We use the system (5.2) and (5.3) to solve for (x, y). Notice that if these eigenfunctions are not independent we will not be able to map back to the (x, y) space. This give us

$$x(\varphi_1, \varphi_2) = \frac{2\varphi_1\varphi_2}{\varphi_1 + \varphi_2 - 2\varphi_1\varphi_2}, \tag{5.9}$$

$$y(\varphi_1, \varphi_2) = \frac{-\varphi_1 + \varphi_2}{\varphi_1 + \varphi_2 - 2\varphi_1\varphi_2}. \tag{5.10}$$

Substituting the analytical solutions for $\varphi_1(t;x_0,y_0)$ and $\varphi_2(t;x_0,y_0)$ into the expressions for x and y gives us analytical solutions for x and y,

$$x(t) = \frac{2\left(\frac{x_0}{1+x_0+y_0}\right)\left(\frac{x_0}{1+x_0-y_0}\right)}{\left(\frac{x_0}{1+x_0+y_0}\right)e^t + \left(\frac{x_0}{1+x_0-y_0}\right)e^{-t} - 2\left(\frac{x_0}{1+x_0+y_0}\right)\left(\frac{x_0}{1+x_0-y_0}\right)}, \tag{5.11}$$

$$y(t) = \frac{-\left(\frac{x_0}{1+x_0+y_0}\right)e^t + \left(\frac{x_0}{1+x_0-y_0}\right)e^{-t}}{\left(\frac{x_0}{1+x_0+y_0}\right)e^t + \left(\frac{x_0}{1+x_0-y_0}\right)e^{-t} - 2\left(\frac{x_0}{1+x_0+y_0}\right)\left(\frac{x_0}{1+x_0-y_0}\right)}. \tag{5.12}$$

Notice that, similarly to the linear case, the solution is constructed from combinations of exponentials originating from the linear eigenfunction solutions. Instead of linear combinations of exponentials, however, the solution is formed from a nonlinear combination of these exponentials. Figure 5(b)–(c) shows the flow lines in the (x,y) space projected onto the eigenfunctions, resulting in linear dynamics. The system (φ_1,φ_2) has simple linear dynamics (Figure 5(d)) which can be projected back to the original space (Figure 5(e)–(f)). The analytical solution for $\mathbf{x}(t)$ initially matches the numerical solution from ode45 (Figure 5(g)); however, the error in the numerical solution accumulates over time (Figure 5(h)). In addition to the accumulation of errors, ode45 cannot produce a numerical solution past the threshold of its integration tolerance. The analytical solution shows that the trajectories go to infinity and back in finite time, which is a phenomena that is unobservable using ode45. This is due to the cessation in integration when the integration tolerance is met (Figure 5(g)).

5.2. Nonlinear example 2. Consider another nonlinear ODE that, as in the previous example, has three real linear invariant manifolds,

$$\begin{aligned} \dot{x} &= x - xy, \\ \dot{y} &= -x - y - y^2 \end{aligned} \tag{5.13}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. Three one-dimensional invariant manifolds go through the fixed points and are tangent to the eigenvectors of the system linearized at the fixed points. These one-dimensional invariant manifolds can be defined by the functions $x = 0$, $y = -\frac{1}{2}x$, and $y = -x - 1$. The M-functions produced from these invariant manifolds are $M_1 = x$, $M_2 = x + 2y$, and $M_3 = 1 + x + y$. The invariant manifolds are generated by the zero level sets of the M-functions $\Lambda_1 = \{(x,y) : M_1(x,y) = 0\}$, $\Lambda_2 = \{(x,y) : M_2(x,y) = 0\}$, and $\Lambda_3 = \{(x,y) : M_3(x,y) = 0\}$ (Figure 6(a)). We solve for the corresponding N-functions using (2.7).

$$N_1(x,y) = \frac{\frac{d}{dt}M_1(x,y)}{M_1(x,y)} = \frac{\frac{dx}{dt}}{x} = \frac{x - xy}{x} = \frac{x(1-y)}{x} = 1 - y,$$

$$N_2(x,y) = \frac{\frac{d}{dt}M_2(x,y)}{M_2(x,y)} = \frac{\frac{d(x+2y)}{dt}}{x+2y} = \frac{-x - xy - 2y - y^2}{x+2y} = \frac{(x+2y)(-1-y)}{x+2y} = -1 - y,$$

$$N_3(x,y) = \frac{\frac{d}{dt}M_3(x,y)}{M_3(x,y)} = \frac{\frac{d(1+x+y)}{dt}}{1+x+y} = \frac{-xy - y - y^2}{1+x+y} = \frac{(1+x+y)(-y)}{1+x+y} = -y.$$

The N-functions corresponding to the M-functions are $N_1(x,y)=1-y$, $N_2(x,y)=-1-y$, and $N_3(x,y)=-y$. Multiple differences of the N-functions result in constants,

$$\begin{aligned}
\lambda_1 &= N_3(x,y)-N_1(x,y) = -y-(1-y) &&= -1,\\
\lambda_2 &= N_3(x,y)-N_2(x,y) = -y-(-1-y) &&= 1,\\
\lambda_3 &= N_1(x,y)-N_3(x,y) = 1-y-(-y) &&= 1,\\
\lambda_4 &= N_1(x,y)-N_2(x,y) = 1-y-(-1-y) &&= 2.
\end{aligned}$$

Therefore, according to Theorem 3.1, (5.13) has eigenfunctions

$$\varphi_1(x,y)=\frac{M_3}{M_1}=\frac{1+x+y}{x},\quad \lambda_1=-1, \tag{5.14}$$

$$\varphi_2(x,y)=\frac{M_3}{M_2}=\frac{1+x+y}{x+2y},\quad \lambda_2=1, \tag{5.15}$$

$$\varphi_3(x,y)=\frac{M_1}{M_3}=\frac{x}{1+x+y},\quad \lambda_3=1, \tag{5.16}$$

$$\varphi_4(x,y)=\frac{2M_1}{M_2}=\frac{2x}{x+2y},\quad \lambda_4=2. \tag{5.17}$$

The weighting vectors for these eigenfunctions are $\mathbf{p}_1=[-1\ 0\ 1]^T$, $\mathbf{p}_2=[0\ -1\ \ 1]^T$, $\mathbf{p}_3=[1\ \ 0\ -1]^T$, and $\mathbf{p}_4=[1\ -1\ \ 0]^T$ since $\lambda_1=-1N_1+0N_2+1N_3$, $\lambda_2=0N_1-1N_2+1N_3$, $\lambda_3=1N_1+0N_2-1N_3$, and $\lambda_4=1N_1-1N_2+0N_3$. The set of functions $\{\log(M_1(x,y),\log(M_2(x,y),\log(M_3(x,y)\}$ is linearly independent because at least one of the generalized Wronskians is not identically equal to zero [50]. Not all pairs of vectors are linearly independent. For example, $\mathbf{p}_1$ and $\mathbf{p}_3$ are linearly dependent, meaning that the eigenfunction pair (φ_1,φ_3) cannot be used to solve for $\mathbf{x}(t)$. $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent; we therefore select the corresponding pair of eigenfunctions (φ_1,φ_2) to solve for $\mathbf{x}(t)$. Using $\varphi_1(x,y)$ and $\varphi_2(x,y)$ to solve for x and y gives us

$$x(\varphi_1,\varphi_2)=\frac{2\varphi_2}{-\varphi_1-\varphi_2+2\varphi_1\varphi_2}, \tag{5.18}$$

$$y(\varphi_1,\varphi_2)=\frac{\varphi_1-\varphi_2}{-\varphi_1-\varphi_2+2\varphi_1\varphi_2}. \tag{5.19}$$

Substituting the solutions $\varphi_1(t;x_0,y_0)=\varphi_1(x_0,y_0)e^{\lambda_1 t}$ and $\varphi_2(t;x_0,y_0)=\varphi_2(x_0,y_0)e^{\lambda_2 t}$ for φ_1 and φ_2 in the formulas for x and y gives us

$$x(t)=\frac{2\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t}{-\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t}-\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t+2\left(\frac{1+x_0+y_0}{x_0}\right)\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)}, \tag{5.20}$$

$$y(t)=\frac{\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t}-\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t}{-\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t}-\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t+2\left(\frac{1+x_0+y_0}{x_0}\right)\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)}. \tag{5.21}$$

Simplifying we get

$$x(t) = \frac{2\left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t}{-\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t} - \left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t + \frac{2(1+x_0+y_0)^2}{x_0(x_0+2y_0)}}, \tag{5.22}$$

$$y(t) = \frac{\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t} - \left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t}{-\left(\frac{1+x_0+y_0}{x_0}\right)e^{-t} - \left(\frac{1+x_0+y_0}{x_0+2y_0}\right)e^t + \frac{2(1+x_0+y_0)^2}{x_0(x_0+2y_0)}}. \tag{5.23}$$

Figure 6(b)–(c) shows the flow of (5.13) in the (x, y) space projected onto the eigenfunctions which have linear dynamics. The flow in the (φ_1, φ_2) space can be solved and projected back onto the original variables (Figure 6(d)–(f)). The numerical solutions to $\mathbf{x}(t)$ match the analytical solutions initially, but error does accumulate; when the integration tolerance of ode45 is met the numerical solution ends (Figure 6(g)–(h)).

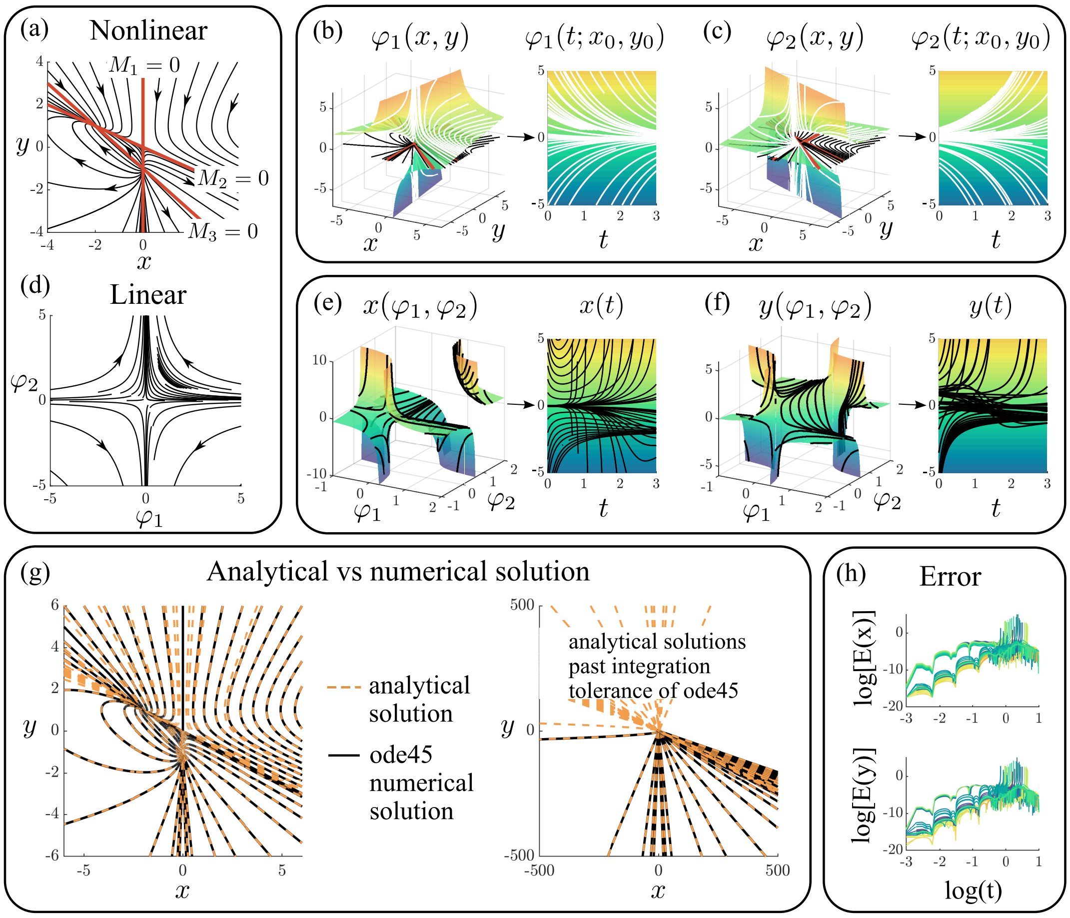

Figure 6. *Dynamics in original space and eigenfunction space.* (a) *Phase plane for* (5.13). (b)–(c) *Flow lines projected onto* φ_1 *and* φ_2. (d) *Phase plane of* (φ_1, φ_2). (e)–(f) *Flow lines of* (φ_1, φ_2) *mapped back to original variables* (x, y). (g) *Analytical solution compared to numerical solution produced using ode45.* (h) *Error in trajectories over time.*

5.3. Nonlinear example 3—Quadratic invariant manifolds. The examples we have considered thus far all have linear invariant manifolds. We now consider a system with quadratic manifolds intersecting the system's fixed points,

$$\begin{aligned}\dot{x} &= x - xy,\\ \dot{y} &= -y + x^2 - 2y^2\end{aligned} \tag{5.24}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. Three one-dimensional invariant manifolds go through the fixed points and are tangent to the eigenvectors of the system linearized at the fixed points. These one-dimensional invariant manifolds can be defined by the functions $x = 0$, $y = -\frac{1}{3}x^2$, and $y = \frac{1}{2}x^2 - \frac{1}{2}$. The invariant manifold generating functions for (5.24) are $M_1 = x$, $M_2 = x^2 - 3y$, and $M_3 = 1 - x^2 + 2y$. We use (2.7) to solve for the N-functions:

$$\begin{aligned}
N_1(x,y) &= \frac{\frac{d}{dt}M_1(x,y)}{M_1(x,y)} = \frac{\frac{dx}{dt}}{x} = \frac{x - xy}{x} = \frac{x(1-y)}{x} = 1 - y,\\
N_2(x,y) &= \frac{\frac{d}{dt}M_2(x,y)}{M_2(x,y)} = \frac{\frac{d(x^2-3y)}{dt}}{x^2 - 3y} = \frac{(x^2-3y)(-1-2y)}{x^2-3y} = -1 - 2y,\\
N_3(x,y) &= \frac{\frac{d}{dt}M_3(x,y)}{M_3(x,y)} = \frac{\frac{d(1-x^2+2y)}{dt}}{1 - x^2 + 2y} = \frac{(1-x^2+2y)(-2y)}{1-x^2+2y} = -2y.
\end{aligned}$$

The N-functions corresponding to the M-functions are $N_1(x,y) = 1 - y$, $N_2(x,y) = -1 - 2y$, and $N_3(x,y) = -2y$. The following differences in the N-functions result in constants:

$$\begin{aligned}
\lambda_1 &= N_3(x,y) - N_2(x,y) = -2y - (-1 - 2y) = 1,\\
\lambda_2 &= N_3(x,y) - 2N_1(x,y) = -2y - (1-y) - (1-y) = -2.
\end{aligned}$$

According to Theorem 3.1, we can construct the following eigenfunctions from the M-functions:

$$\begin{aligned}
\varphi_1(x,y) &= \frac{-3M_3}{M_2} = \frac{-3(1-x^2+2y)}{x^2-3y}, \quad && \lambda_1 = 1,\\
\varphi_2(x,y) &= \frac{M_3}{M_1^2} = \frac{1-x^2+2y}{x^2}, && \lambda_2 = -2.
\end{aligned}$$

The set of functions $\{\log(M_1(x,y), \log(M_2(x,y), \log(M_3(x,y)\}$ is linearly independent because at least one of the generalized Wronskians is not identically equal to zero [50]. The weight vectors for φ_1 and φ_2, $\mathbf{p}_1 = [0 \;\; -1 \;\; 1]^T$ and $\mathbf{p}_2 = [2 \;\; 0 \;\; 1]^T$, are linearly independent, and therefore (φ_1, φ_2) can be used to solve for $\mathbf{x}(t)$. Solving for x and y gives us two solutions this time,

$$x = \frac{\pm\sqrt{3\varphi_1}}{\sqrt{\varphi_1 - 6\varphi_2 + 3\varphi_1\varphi_2}}, \tag{5.25}$$

$$y = \frac{\varphi_1 + 3\varphi_2}{\varphi_1 - 6\varphi_2 + 3\varphi_1\varphi_2}. \tag{5.26}$$

The correct sign for x is resolved by taking the sign of the initial condition, x_0, resulting in the solution

$$x(t) = \frac{\operatorname{sign}(x_0)\sqrt{\frac{9(-1+x_0^2-2y_0)}{x_0^2-3y_0}e^t}}{\sqrt{\frac{3(-1+x_0^2-2y_0)}{x_0^2-3y_0}e^t - \frac{6(1-x_0^2+2y_0)}{x_0^2}e^{-2t} + \frac{9(-1+x_0^2-2y_0)(1-x_0^2+2y_0)}{(x_0^2-3y_0)x_0^2}e^{-t}}}, \tag{5.27}$$

$$y(t) = \frac{\frac{3(-1+x_0^2-2y_0)}{x_0^2-3y_0}e^t + \frac{3(1-x_0^2+2y_0)}{x_0^2}e^{-2t}}{\frac{3(-1+x_0^2-2y_0)}{x_0^2-3y_0}e^t - \frac{6(1-x_0^2+2y_0)}{x_0^2}e^{-2t} + \frac{9(-1+x_0^2-2y_0)(1-x_0^2+2y_0)}{(x_0^2-3y_0)x_0^2}e^{-t}}. \tag{5.28}$$

Figure 7(a) shows the flow lines of (5.24) and the system's quadratic invariant manifolds. The nonlinear trajectories are projected onto the eigenfunction space, resulting in linear dynamics (Figure 7(b)–(c)). The linear system can be solved and the trajectories projected back to the original space (Figure 7(d)–(f)). The numerical solution matches the analytical solution initially with growing error over time (Figure 7(g)–(h)).

5.4. Eigenfunctions from invariant manifolds in previous work. Eigenfunctions constructed from a system's invariant manifolds can be found in previous work [6, 4]. In [6] the nonlinear ODE

$$\begin{aligned} \dot{x} &= -0.05x, \\ \dot{y} &= -(y - x^2) \end{aligned} \tag{5.29}$$

has eigenfunctions $\varphi_1 = M_1 = x$ and $\varphi_2 = M_2 = y - \frac{10}{9}x^2$ with corresponding eigenvalues $\lambda_1 = N_1 = -0.05$ and $\lambda_2 = N_2 = -1$. Both of these eigenfunctions are invariant manifold generating functions with zero level sets that go through the fixed point at the origin. Eigenfunctions could be constructed from individual M-functions because the corresponding N-functions are constants, making them eigenvalues.

In [4] the nonlinear ODE

$$\begin{aligned} \frac{dx}{dt} &= -2y(x^2 - y - 2xy^2 + y^4) + (x + 4x^2y - y^2 - 8xy^3 + 4y^5), \\ \frac{dy}{dt} &= 2(x - y^2)^2 - (x^2 - y - 2xy^2 + y^4) \end{aligned} \tag{5.30}$$

was found to have eigenfunctions $\varphi_1 = M_1 = x - y^2$ and $\varphi_2 = M_2 = -x^2 + y + 2xy^2 - y^4$ with corresponding eigenvalues $\lambda_1 = N_1 = 1$ and $\lambda_2 = N_2 = 1$. Once again, these eigenfunctions are M-functions whose zero level sets describe invariant manifolds that go through the system's fixed points. The corresponding N-functions are constants, once again allowing the M-functions to be eigenfunctions.

6. Discussion. We present a novel method for solving planar nonlinear ODEs when certain restrictive criteria are met. The crux of our method relies on finding explicit, globally valid expressions for eigenfunctions by composing invariant manifold generating functions in the manner outlined in Theorems 3.1 and 3.2. We demonstrate our method by finding analytical solutions for two-dimensional nonlinear ODEs that were previously analytically intractable.

This method only applies to a restrictive subset of planar ODEs. First, the ODE must have one-dimensional invariant manifolds that can be described by functions of the form

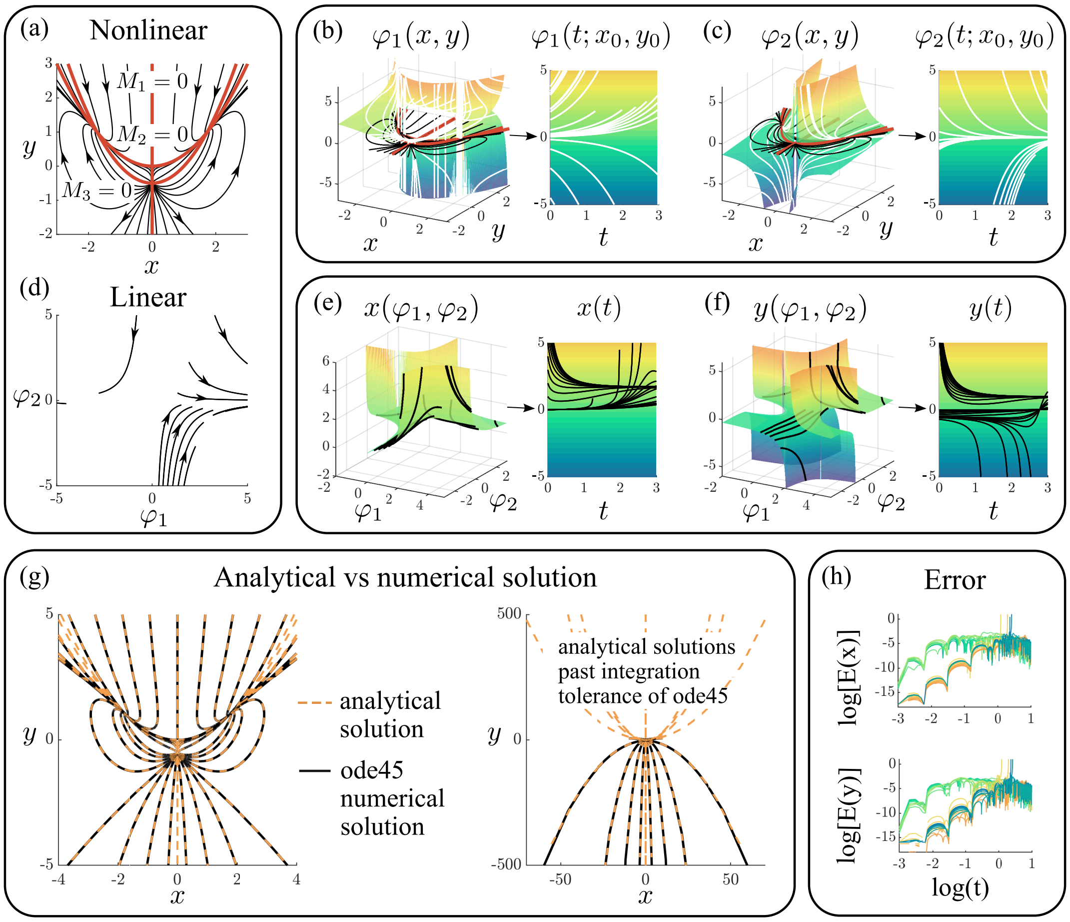

Figure 7. *Dynamics in original space and eigenfunction space.* (a) *Phase plane for* (5.24). (b)–(c) *Flow lines projected onto φ_1 and φ_2.* (d) *Phase plane of (φ_1, φ_2).* (e)–(f) *Flow lines of (φ_1, φ_2) mapped back to original variables (x, y).* (g) *Analytical solution compared to numerical solution produced using ode*45. (h) *Error in trajectories over time.*

$y = m_i(x)$ or $x = m_i(y)$ that go through the system's fixed points. If closed-form expressions for these functions cannot be obtained, then the eigenfunctions must be generated numerically (Appendix D). Second, according to Theorem 3.2, the variables in the N-functions must be cancelable, producing a nonzero constant. This is not guaranteed to be the case, even with a closed-form expression for the invariant manifolds. Third, at least two independent eigenfunctions must be attainable to map from the eigenfunction space back to the original state variables. Fourth, the system of eigenfunctions must be invertible, resulting in a unique solution for the state variables $(x(t), y(t))$. These conditions are restrictive, and few nonlinear ODEs meet this criteria. However, for those that do, we have shown that a method for obtaining an analytical solution exists.

6.1. Extensions to a larger class of two-dimensional ODEs. There are several ways in which it may be possible to generalize the method to a larger class of ODEs. One avenue for extending the method to more systems of ODEs is to allow the M-functions to have a less restrictive form while maintaining a zero level set along the one-dimensional invariant manifold of interest. We restrict our invariant manifold generating functions to have the form $M(x,y) = y - m(x)$, where the zero level set of $M(x,y)$ is the one-dimensional invariant manifold $y = m(x)$; however, a broader class of M-functions of the form $M(x,y) = (y - m(x))g(x,y)$ also have a zero level set along the curve $y = m(x)$, for example, $M(x,y) = (y - m(x))(1 + x^2)$. M-functions of this more general form may produce eigenfunctions when M-functions of the restrictive form fail to do so. In addition to this, many planar systems of ODEs have one-dimensional invariant manifolds that can only be described by implicit functions $m(x,y) = 0$. The M-functions could be generalized to take this form as well, $M(x,y) = m(x,y)$. Another restriction we set is how the M-functions can be combined to form eigenfunctions. Relaxing this restriction may also expand the class of eigenfunctions that can be represented.

6.2. Complex-valued eigenfunctions. In the one-dimensional case, mapping an ODE in $\mathbb{R}$ to a space with linear dynamics sometimes requires the eigenfunction to be complex-valued, $\varphi : \mathbb{R} \to \mathbb{C}$ (section 4.2, Appendix A). This is unequivocally the case when the ODE contains complex fixed points, (A.1, A.6), but also can occur when the ODE contains only real fixed points, (4.12). Just as in the one-dimensional case, we expect that for two-dimensional nonlinear ODEs, complex-valued eigenfunctions will sometimes be necessary to obtain linear dynamics, $\varphi : \mathbb{R}^2 \to \mathbb{C}$. Eigenfunctions may be able to be constructed from complex-valued invariant manifolds and manifolds eminating from complex-valued fixed points in two-dimensional ODEs. We already have an example of this in the linear case. Linear systems in $\mathbb{R}^2$ that are spiral sinks or sources do not have real invariant manifolds that are tangent to the fixed point eigenvectors, and yet these systems can be solved using complex-valued eigenvalues and eigenfunctions, $\varphi : \mathbb{R}^2 \to \mathbb{C}$, constructed from complex-valued invariant manifolds. We aim to investigate how to find and construct complex-valued invariant manifolds as well as how they may be composed to produce eigenfunctions.

6.3. Ramifications for data-driven approximations of eigenfunctions. When analytical solutions are not possible, our method highlights an alternative approach for approximating eigenfunctions and the ensuing solutions to ODEs. Approximate analytical solutions can be preferable to numerical integration as they are less computationally expensive, can be analyzed, and expand the number of control techniques that can be used for the ODE [20, 7]. M-functions may be an excellent set of functions for approximating eigenfunctions in certain cases. The basis chosen for a given approximation can have a significant impact on the quality of the approximation and its accuracy, stability, and simplicity. Using M-functions to focus the search for eigenfunctions may result in simpler functional forms and more accurate predictions.

Polynomials or other smooth continuous functions are often used as basis functions for the data-driven discovery of eigenfunctions [46, 21, 16, 6, 25, 24]. Choosing smooth and continuous basis functions for data-driven discovery assumes the resulting eigenfunctions, linearly composed from the basis functions, are smooth and continuous. Many of the eigenfunctions found in this work are not continuous; it is impossible to generate continuous eigenfunctions that are globally valid for some ODEs. In these instances, forcing the approximate eigenfunctions to be continuous by using smooth continuous basis functions would result in a poor approximation, particularly around discontinuities.

Deriving globally valid eigenfunctions or exact expressions may not be necessary depending on the desired use for the eigenfunctions. Nonlinear dynamics can be linearized for certain regions around fixed points with continuous approximate eigenfunctions [23, 26, 32, 25]; these methods, in essence, create approximate local eigenfunctions for different regions of the domain that extend farther and are more accurate than the simple approximate eigenfunctions obtained through linearization using the Jacobian [42, 6, 16]. DMD methods are often used to approximate eigenfunctions and produce different subspaces of observables for the region around each fixed point in the dynamical system [21, 32, 16, 4]. Our method, in contrast, produces a single set of observables for the entire space. If approximate solutions and control are desired only for a region around a particular fixed point, then using smooth functions to approximate eigenfunctions for a particular region may be suitable. Otherwise, data-driven discovery of eigenfunctions that allow for the discovery of rational eigenfunctions and eigenfunctions that have discontinuities may be useful in finding globally valid eigenfunctions and more exact and simpler mappings between spaces with nonlinear and linear dynamics. In Appendix D we outline and implement a numerical method for finding approximate eigenfunctions that are compositions of M-functions. This method can be applied to approximate eigenfunctions when closed-form expressions for the one-dimensional invariant manifolds do not exist. We aim to investigate the usefulness of this alternative means of constructing approximate eigenfunctions.

6.4. Comparison to the method of characteristics solution for eigenfunctions.

The eigenfunctions of a nonlinear ODE, (2.1), are solutions to (2.5), a linear PDE which is solvable via the method of characteristics [4, 3, 27]. The method of characteristics propagates an initial condition curve, Γ, forward in time creating an integral surface that is a solution to the PDE [27]. In general, the method of characteristics does not produce simple, closed-form solutions. The Γ chosen to be the initial condition curve must traverse the flow. As there is an infinite number of curves that can be chosen for the initial condition, there is an infinite number of resulting eigenfunctions [4]. Most of the transverse flows chosen to be initial conditions will result in eigenfunctions that do not have simple expressions. The eigenfunctions with simple analytical expressions found in this manuscript correspond to a method of characteristics solution for a specially chosen initial conditions curve Γ. We have not solved for these curves, and it remains an open question how to select Γ such that the resulting eigenfunction will have a simple closed-form expression. This line of questioning is not addressed in this paper and is an avenue for further investigation.

7. Conclusion.

We solve two-dimensional nonlinear ODEs that have closed-form expressions for some of their one-dimensional invariant manifolds by finding Koopman eigenfunctions, from which we can construct solutions. Koopman eigenfunctions are generally difficult to find; we outline how Koopman eigenfunctions can be constructed from a system's invariant manifolds when certain conditions are met. We provide formulas for sets of eigenvalue-eigenfunction pairs which can then be used to solve the nonlinear system. We demonstrate the Koopman eigenfunction method of solving nonlinear ODEs on one-dimensional and two-dimensional ODEs. This method solves nonlinear ODEs in $\mathbb{R}^2$ that thus far had no known analytical solution. It is possible this method may be extended to solve a larger class of nonlinear ODEs and we offer several avenues for improvement; our method may also prove useful for the data-driven discovery of eigenfunctions.

Building exact or approximate eigenfunctions from invariant manifolds may be a useful framework for further solving nonlinear systems and approximating solutions. Invariant manifolds are well established as important underlying structures in dynamical systems that can be used for analysis, dimension reduction, and control; the method and ideas we explore reveal another way in which invariant manifolds are a useful framework for understanding and solving nonlinear systems.

Appendix A. Koopman eigenfunctions for 1-dimensional ODEs—More examples.

A.1. One-dimensional ODE—Example 3. Suppose we take (4.12) and add two more fixed points to the system but on the imaginary axis, still resulting in a real-valued ODE,

$$\begin{aligned}\frac{dx}{dt} &= -x^5 + x, \quad x(0) = x_0,\\ \frac{dx}{dt} &= (-x^3 + x)(x^2 + 1),\\ \frac{dx}{dt} &= -x(x+1)(x-1)(x+i)(x-i).\end{aligned} \tag{A.1}$$

We use the eigenvalue $\lambda = -1$ to map all stable fixed points, $x = \pm 1, \pm i$, to the stable fixed point in the eigenfunction space $\varphi = 0$ (Figure 8(a)–(b)). We map the unstable fixed point, $x = 0$, to unstable fixed point $\varphi = \infty$,

$$\varphi(x) = x^{-1}(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}(x-i)^{\frac{1}{4}}(x+i)^{\frac{1}{4}} = \frac{(x^4-1)^{\frac{1}{4}}}{x}. \tag{A.2}$$

The initial condition x_0 mapped to $\varphi(x)$ is $\varphi_0 = \frac{(x_0^4-1)^{\frac{1}{4}}}{x_0}$. The solution for $\varphi(t; x_0)$ is

$$\varphi(t; x_0) = \varphi(x_0)e^{\lambda t} = \frac{(x_0^4-1)^{\frac{1}{4}}}{x_0}e^{-t}. \tag{A.3}$$

Using (A.2) we solve for x in terms of φ,

$$x(t) = \frac{1^{\frac{1}{4}}}{(1-\varphi^4)^{\frac{1}{4}}} = \frac{1^{\frac{1}{4}}}{\left(1 - \left(\frac{(x_0^4-1)^{\frac{1}{4}}}{x_0}e^{-t}\right)^4\right)^{\frac{1}{4}}}. \tag{A.4}$$

Figure 8(b)–(c) shows that the nonlinear mapping of the eigenfunction $\varphi(x)$ generates linear dynamics for $\varphi(t; x_0)$. The dynamics when $x \in \mathbb{C}$, where $x = a + ib$, can be written as

$$\begin{aligned}\frac{da}{dt} &= a - a^5 + 10a^3b^2 - 5ab^4,\\ \frac{db}{dt} &= b - 5a^4b + 10a^2b^3 - b^5.\end{aligned} \tag{A.5}$$

We observe that (A.1) must map to a complex-valued eigenfunction in order to obtain linear dynamics. If $\dot{x} = f(x)$ has complex fixed points, then $\varphi(x)$ will be a complex-valued function even for real inputs $x \in \mathbb{R}$. Figure 8(d)–(e) shows $x = \Re(x) + i\Im(x)$ mapped to the real and imaginary components of the eigenfunction $\varphi(x) = \Re(\varphi) + i\Im(\varphi)$.

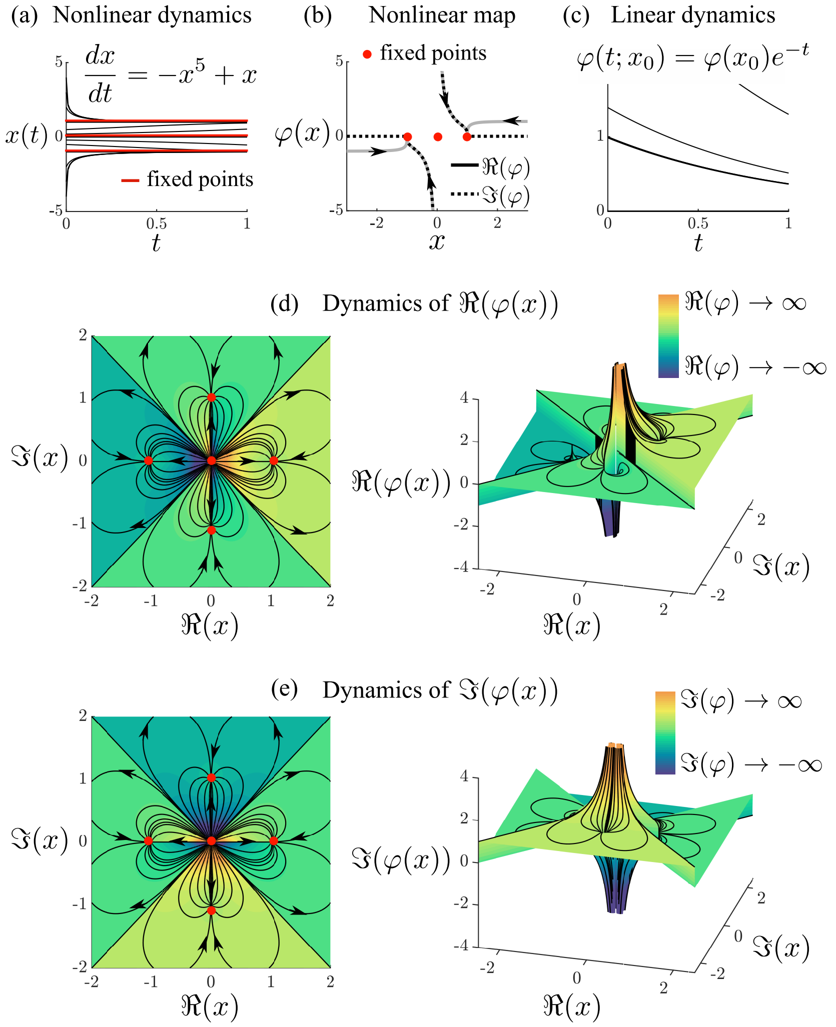

Figure 8. (a) *Equation* (A.1) *has nonlinear dynamics.* (b) x *mapped to complex-valued eigenfunction* $\varphi(x)$. (c) *The dynamics of* φ *is linear.* (d)–(e) *Dynamics of* $x = a + ib$ *viewed in the complex plaine as well as mapped to complex-valued* φ.

A.2. One-dimensional ODE—Example 4. Consider the following nonlinear differential equation and its factored form:

$$\text{(A.6)} \qquad \begin{aligned} \frac{dx}{dt} &= x^3 + 2x^2 + 2x, \quad x(0) = x_0, \\ \frac{dx}{dt} &= x(x-(-1-i))(x-(-1+i)). \end{aligned}$$

Set $\lambda = 1$, and then using (4.5) we get

$$\text{(A.7)} \qquad \varphi(x) = x^{\frac{1}{2}}[x-(-1-i)]^{\frac{1}{-2+2i}}[x-(-1+i)]^{\frac{1}{-2-2i}} = \frac{e^{-\frac{1}{2}\arctan(1+x)}\sqrt{x}}{(2+2x+x^2)^{\frac{1}{4}}}.$$

The dynamics of φ are $\varphi(t;x_0)=\varphi(x_0)e^t$ with initial condition

$$\varphi(x_0)=\frac{e^{-\frac{1}{2}\arctan(1+x_0)}\sqrt{x_0}}{(2+2x_0+x_0^2)^{\frac{1}{4}}}. \tag{A.8}$$

The mapping from x to φ, (A.7), does not have a nice inverse; we cannot find an explicit solution for x, only an implicit solution. The solution for $x(t)$ is the implicit solution to

$$\frac{e^{-\frac{1}{2}\arctan(1+x_0)}\sqrt{x_0}}{(2+2x_0+x_0^2)^{\frac{1}{4}}}e^t=\frac{e^{-\frac{1}{2}\arctan(1+x(t))}\sqrt{x(t)}}{(2+2x(t)+x(t)^2)^{\frac{1}{4}}}. \tag{A.9}$$

Implicit solutions can be solved using a numerical method.

Appendix B. Testing the linear independence of multivariable functions with generalized Wronskians. Reference [50] states that if a set of functions $\{g_i(x,y)\}_i^k$ is linearly dependent, then all generalized Wronskians of the list of functions $(g_1(x,y),\dots g_k(x,y))$ must be identically equal to zero. We test the set of $\log(M)$-functions from Example 5.1 for linear independence.

$$\begin{aligned}(g_1(x,y),g_2(x,y),g_3(x,y))&=(\log(M_1(x,y)),\log(M_2(x,y)),\log(M_3(x,y)))\\&=(\log(x),\log(y-x-1),\log(y+x+1)).\end{aligned}$$

We compute one of the generalized Wronskians and find that it is not identically equivalent to zero.

$$W=\begin{vmatrix}g_1(x,y) & g_2(x,y) & g_3(x,y)\\ \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_3(x,y)}{\partial x}\\ \frac{\partial g_1(x,y)}{\partial y} & \frac{\partial g_2(x,y)}{\partial y} & \frac{\partial g_3(x,y)}{\partial y}\end{vmatrix}=\begin{vmatrix}\log(x) & \log(y-x-1) & \log(y+x+1)\\ \frac{1}{x} & \frac{-1}{y-x-1} & \frac{1}{y+x+1}\\ 0 & \frac{1}{y-x-1} & \frac{1}{y+x+1}\end{vmatrix} \tag{B.1}$$

$$=\frac{-2\log(x)}{(y+x+1)(y-x-1)}-\frac{\log(y-x-1)}{x(y+x+1)}+\frac{\log(y+x+1)}{x(y-x-1)}\not\equiv 0. \tag{B.2}$$

Therefore, $\{\log(x),\log(y-x-1),\log(y+x+1)\}$ is a linearly independent set of functions.

Appendix C. Koopman eigenfunctions for two-dimensional ODEs—More examples.

C.1. Linear systems. Linear systems have well-known analytical solutions and are used ubiquitously in the applied sciences for prediction and control [10, 42, 41]. Although solutions to linear systems are well-known, we will revisit the method here in a way that highlights the Koopman eigenfunctions and invariant manifolds and their connection to the ensuing solution. Previous work has considered linear systems from a Koopman perspective [8, 3, 29]; we consider linear systems here once again as an introduction for solving nonlinear systems.

The solution to a linear ODE $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ is $\mathbf{x}(t)=\exp(\mathbf{A}t)\mathbf{x}_0$. The solution is typically constructed by finding the eigenvalues and eigenvectors and then linearly composing them [42, 10]. We will instead use the Koopman approach, Algorithm 5.1, to find a solution. Consider the two-dimensional linear ODE

$$\begin{aligned}\dot{x}&=x-y,\\ \dot{y}&=-2x\end{aligned} \tag{C.1}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. This system can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{C.2}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{C.3}$$

The eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = 2$, $\lambda_2 = -1$, $v_1 = [-1 \;\; 1]^T$, and $v_2 = [1 \;\; 2]^T$. We use the eigenvectors to construct two invariant manifold generating functions, $M_1 = y + x$ and $M_2 = y - 2x$, that are zero along the eigenvector directions; $\Lambda_1 = \{(x,y) : M_1(x,y) = 0\}$ and $\Lambda_2 = \{(x,y) : M_2(x,y) = 0\}$ are the two one-dimensional invariant manifolds of (C.1) that pass through the fixed point at the origin (Figure 9(a)). We use (2.7) to solve for the N-functions,

$$N_1(x,y) = \frac{\frac{d}{dt}M_1(x,y)}{M_1(x,y)} = \frac{\frac{d(y+x)}{dt}}{y+x} = \frac{-2x + x - y}{y+x} = -\frac{y+x}{y+x} = -1,$$
$$N_2(x,y) = \frac{\frac{d}{dt}M_2(x,y)}{M_2(x,y)} = \frac{\frac{d(y-2x)}{dt}}{y-2x} = \frac{-2x - 2(x-y)}{y-2x} = 2\frac{y-2x}{y-2x} = 2.$$

Because N_1 and N_2 are both constants, $M_1(x,y)$ and $M_2(x,y)$ are both eigenfunctions of the linear system (C.1) with eigenvalues $\lambda_1 = N_1(x,y) = -1$ and $\lambda_2 = N_2(x,y) = 2$,

$$\varphi_1(x,y) = y + x, \qquad \lambda_1 = -1, \tag{C.4}$$

$$\varphi_2(x,y) = y - 2x, \quad \lambda_2 = 2. \tag{C.5}$$

Figure 9(b)–(c) shows eigenfunctions φ_1 and φ_2 with the dynamics in the (x,y) space projected onto the eigenfunction surfaces. The dynamics in the eigenfunction space are linear. The weight vector of the N-functions for φ_1 is $\mathbf{p}_1 = [1 \;\; 0]^T$ since $\lambda_1 = 1N_1 + 0N_2$, while the weight vector for φ_2 is $\mathbf{p}_2 = [0 \;\; 1]^T$ since $\lambda_2 = 0N_1 + 1N_2$. $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent; therefore, according to Theorem 3.3, φ_1 and φ_2 belong to different equivalence classes and can be used in conjunction to solve for $\mathbf{x}(t)$. We can confirm that these are in fact eigenvalue-eigenfunction pairs of (C.1) by checking that they satisfy (2.5).

$$\begin{aligned}
\nabla_{\mathbf{x}}\varphi_1(\mathbf{x}) \cdot F(\mathbf{x}) &= \lambda_1\varphi_1(\mathbf{x}), \\
\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x - y \\ -2x \end{bmatrix} &= -1(y+x), \\
-y - x &= -y - x \\
&\implies \varphi_1 \text{ is an eigenfunction.}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\mathbf{x}}\varphi_2(\mathbf{x}) \cdot F(\mathbf{x}) &= \lambda_2\varphi_2(\mathbf{x}), \\
\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} x - y \\ -2x \end{bmatrix} &= 2(y - 2x), \\
2y - 4x &= 2y - 4x \\
&\implies \varphi_2 \text{ is an eigenfunction.}
\end{aligned}$$

According to (2.6), the eigenfunction solutions are

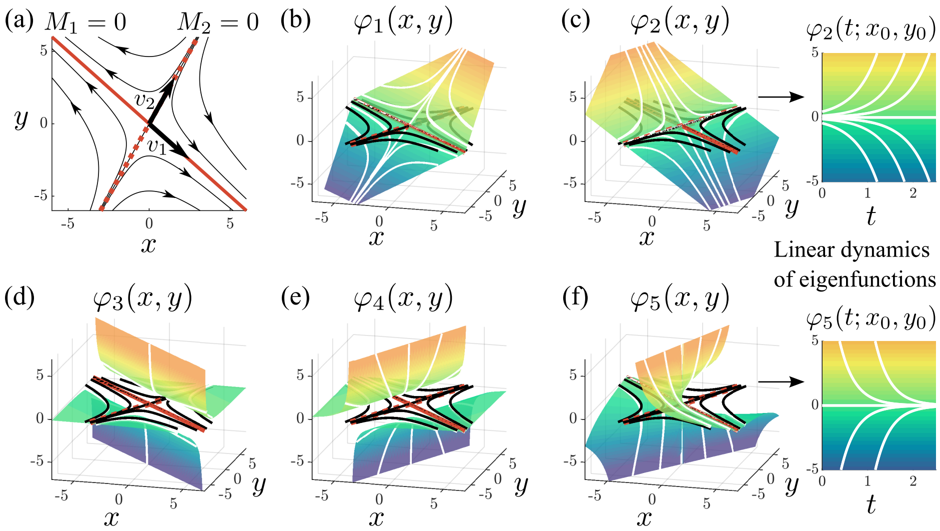

Figure 9. *Linear dynamics and eigenfunctions.* (a) *Phase plane of* (C.1). (b) $\varphi_1(x,y)$, (C.4), *formed from the first eigenvector.* (c) $\varphi_2(x,y)$, (C.5), *formed from the second eigenvector.* (d)–(f) *Additional eigenfunctions. The eigenfunctions have linear dynamics.*

$$\varphi_1(t;x_0,y_0) = \varphi_1(x_0,y_0)e^{-t} = \varphi_1(x_0,y_0)e^{-t}, \tag{C.6}$$
$$\varphi_2(t;x_0,y_0) = \varphi_2(x_0,y_0)e^{2t} = \varphi_2(x_0,y_0)e^{2t}. \tag{C.7}$$

Equations (C.4) and (C.5) give us initial conditions in terms of x and y, $\varphi_1(x_0,y_0) = y_0 - x_0$, and $\varphi_2(x_0,y_0) = y_0 - 2x_0$. Now that we have analytical solutions to the dynamics in the eigenfunction space we can map the solutions back to the original (x,y) space. We use the system of equations (C.4) and (C.5) to solve for x and y as functions of φ_1 and φ_2,

$$x(t) = \varphi_1(t;x_0,y_0) - \varphi_2(t;x_0,y_0), \tag{C.8}$$
$$y(t) = 2\varphi_1(t;x_0,y_0) - \varphi_2(t;x_0,y_0). \tag{C.9}$$

We have analytical solutions for φ_1 (C.6) and φ_2 (C.7) and may substitute these solutions into the equations for x and y. With this substitution we get the analytical solution for $(x(t), y(t))$ in terms of time t and the initial condition (x_0, y_0),

$$x(t) = (y_0 - x_0)e^{2t} - (y_0 - 2x_0)e^{-t}, \tag{C.10}$$
$$y(t) = 2(y_0 - x_0)e^{2t} - (y_0 - 2x_0)e^{-t}. \tag{C.11}$$

The key to obtaining a solution was finding eigenfunctions, which have linear dynamics. Because observables with linear dynamics have known solutions, this process allows us to find an analytical solution in the $(\varphi_1(t;x_0,y_0), \varphi_2(t;x_0,y_0))$ space and then map the solution back to the $(x(t), y(t))$ space.

Although we are finished solving this problem, we will consider what other eigenfunctions we might have used, as φ_1 and φ_2 were not our only options. Bollt [3] demonstrates that additional eigenfunctions can be generated from primary eigenfunctions [3]; the inverse and product of eigenfunctions are also eigenfunctions. From these properties we generate additional eigenfunctions for (C.1) out of eigenfunctions φ_1 and φ_2,

$$\varphi_3 = \frac{1}{\varphi_1}, \quad \lambda_3 = -\lambda_1, \tag{C.12}$$

$$\varphi_4 = \frac{1}{\varphi_2}, \quad \lambda_4 = -\lambda_2, \tag{C.13}$$

$$\varphi_5 = \frac{\varphi_1}{\varphi_2}, \quad \lambda_5 = \lambda_1 + \frac{1}{\lambda_2}. \tag{C.14}$$

Figure 9 shows the phase plane of (C.1), the dynamics mapped onto the eigenfunctions, and the resulting linear dynamics that occur on these special observables. φ_1 and φ_2 are both a linear mapping from the (x, y) plane with a null space along v_1 and v_2, respectively. Conversely, φ_3 and φ_4 are nonlinear with respect to x and y. Both of these functions have a discontinuity along the eigenvector directions; the eigenfunctions go to infinity as they approach the invariant manifolds. φ_5 has a null space along v_1 and is undefined along v_2 as it is composed of φ_1 in the numerator and φ_2 in the denominator. Although φ_3, φ_4, and φ_5 are all *nonlinear* with respect to x and y, they are *linear* with respect to time. This means that they all display exponential growth or exponential decay as the eigenfunction observables are measured along the flow of the original variables. We extend this process to nonlinear ODEs and use the invariant manifolds of nonlinear systems to generate eigenfunctions.

C.2. Nonlinear example 4—Quasi-two-dimensional system. This next example connects the one-dimensional and two-dimensional cases and clarifies the connection between the system's invariant manifolds and the eigenfunctions,

$$\begin{aligned} \dot{x} &= x^2 - x, \\ \dot{y} &= xy - 2y \end{aligned} \tag{C.15}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. The surfaces that generate one-dimensional invariant manifolds that include the fixed points along their zero level-set are $M_1 = x$, $M_2 = x - 1$, and $M_3 = y$; these one-dimensional invariant manifolds are $\Lambda_1 = \{(x, y) : M_1(x, y) = 0\}$, $\Lambda_2 = \{(x, y) : M_2(x, y) = 0\}$, and $\Lambda_3 = \{(x, y) : M_3(x, y) = 0\}$. We use (2.7) to solve for the N-functions:

$$\begin{aligned} N_1(x, y) &= \frac{\frac{d}{dt} M_1(x, y)}{M_1(x, y)} = \frac{\frac{dx}{dt}}{x} = \frac{x^2 - x}{x} = \frac{x(x-1)}{x} = x - 1, \\ N_2(x, y) &= \frac{\frac{d}{dt} M_2(x, y)}{M_2(x, y)} = \frac{\frac{d(x-1)}{dt}}{x - 1} = \frac{x^2 - x}{x - 1} = \frac{x(x-1)}{x - 1} = x, \\ N_3(x, y) &= \frac{\frac{d}{dt} M_3(x, y)}{M_3(x, y)} = \frac{\frac{dy}{dt}}{y} = \frac{xy - 2y}{y} = \frac{y(x-2)}{y} = x - 2. \end{aligned}$$

The N-functions corresponding to the M-functions are $N_1(x, y) = x - 1$, $N_2(x, y) = x$, and $N_3(x, y) = x - 2$. Multiple differences of the N-functions result in constants,

$$\begin{aligned}\lambda_1 &= N_2(x,y) - N_1(x,y) = x - (x-1) &&= 1,\\ \lambda_2 &= N_1(x,y) - N_3(x,y) = x - 1 - (x-2) &&= 1,\\ \lambda_3 &= N_2(x,y) - N_3(x,y) = x - (x-2) &&= 2.\end{aligned}$$

According to Theorem 3.1, we can construct eigenfunctions from the M-function quotients,

$$\varphi_1(x,y) = -\frac{M_2}{M_1} = \frac{1-x}{x}, \quad \lambda_1 = 1, \tag{C.16}$$

$$\varphi_2(x,y) = \frac{M_1}{M_3} = \frac{x}{y}, \quad \lambda_2 = 1. \tag{C.17}$$

The corresponding weight vectors are $\mathbf{p}_1 = [-1\ \ 1\ \ 0]$ and $\mathbf{p}_2 = [1\ \ 0\ -1]$. $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent, meaning we may use φ_1 and φ_2 to solve for $\mathbf{x}(t)$ according to Theorem 3.3. Solving for x and y as functions of φ_1 and φ_2 and substituting the solutions for $\varphi_1(t; x_0, y_0)$ and $\varphi_2(t; x_0, y_0)$ gives us

$$x(t) = \frac{1}{1+\varphi_1(t;x_0,y_0)} = \frac{1}{1+\frac{1-x_0}{x_0}e^t}, \tag{C.18}$$

$$y(t) = \frac{1}{\varphi_2(t;x_0,y_0)(1+\varphi_1(t;x_0,y_0))} = \frac{y_0}{x_0e^t + (1-x_0)e^{2t}}. \tag{C.19}$$

Notice that the dynamics of x and the solution for x do not depend on y, making it a one-dimensional ODE. Its eigenfunction, (C.16), has a zero at $x = 1$ and a singularity at $x = 0$. Now when x is viewed as the first variable in a two-dimensional ODE, the first eigenfunction is now zero not only at a single point, $x = 1$, but along the entire curve $x = 1$ in the (x,y) plane. Likewise its singularity now no longer exists only at a single point, $x = 0$, but along the entire curve $x = 0$ in the (x,y) plane. This example demonstrates how zero and singular level-sets of eigenfunctions in $\mathbb{R}^2$ are natural extensions of zero and singular values of eigenfunctions in $\mathbb{R}$.

C.3. Nonlinear example 5. Consider the system with three linear invariant manifolds,

$$\begin{aligned}\dot{x} &= x - y - x^2,\\ \dot{y} &= -x - y - xy.\end{aligned} \tag{C.20}$$

The invariant manifolds that are one-dimensional, go through the system's fixed points, and are tangent to the fixed point eigenvector directions occur along $y = (1+\sqrt{2})x$, $y = (1-\sqrt{2})x$, and $y = x - 2$. The invariant manifold generating functions are $M_1 = y - (1+\sqrt{2})x$, $M_2 = y - (1-\sqrt{2})x$, and $M_3 = y - x + 2$; the invariant manifolds are $\Lambda_1 = \{(x,y) : M_1(x,y) = 0\}$, $\Lambda_2 = \{(x,y) : M_1(x,y) = 0\}$, and $\Lambda_3 = \{(x,y) : M_1(x,y) = 0\}$. We use (2.7) to solve for the corresponding N-functions,

$$\begin{aligned}N_1(x,y) &= \frac{\frac{d}{dt}M_1(x,y)}{M_1(x,y)} = \frac{\frac{d(y-(1+\sqrt{2})x)}{dt}}{y-(1+\sqrt{2})x} = \frac{(y-(1+\sqrt{2})x)(\sqrt{2}-x)}{y-(1+\sqrt{2})x} = \sqrt{2} - x,\\ N_2(x,y) &= \frac{\frac{d}{dt}M_2(x,y)}{M_2(x,y)} = \frac{\frac{d(y-(1-\sqrt{2})x)}{dt}}{y-(1-\sqrt{2})x} = \frac{(y-(1-\sqrt{2})x)(-\sqrt{2}-x)}{y-(1-\sqrt{2})x} = -\sqrt{2} - x,\\ N_3(x,y) &= \frac{\frac{d}{dt}M_3(x,y)}{M_3(x,y)} = \frac{\frac{d(y-x+2)}{dt}}{y-x+2} = \frac{(y-x+2)(-x)}{y-x+2} = -x.\end{aligned}$$

The N-functions are $N_1(x,y) = \sqrt{2} - x$, $N_2(x,y) = -\sqrt{2} - x$, and $N_3(x,y) = -x$. The following linear combinations of N-functions result in constants: $\lambda_1 = N_3(x,y) - N_2(x,y) = \sqrt{2}$, $\lambda_2 = N_3(x,y) - N_1(x,y) = -\sqrt{2}$, and $\lambda_3 = N_1(x,y) - N_2(x,y) = 2\sqrt{2}$. We construct the eigenfunctions

$$\varphi_1(x,y) = \frac{M_3}{2M_2} = \frac{y - x + 2}{2(y - (1-\sqrt{2})x)}, \quad \lambda_1 = \sqrt{2}, \tag{C.21}$$

$$\varphi_2(x,y) = \frac{M_3}{2M_1} = \frac{y - x + 2}{2(y - (1+\sqrt{2})x)}, \quad \lambda_2 = -\sqrt{2}. \tag{C.22}$$

The weight vectors $\mathbf{p}_1 = [0 \ \ -1 \ \ 1]^T$ and $\mathbf{p}_2 = [-1 \ \ 0 \ \ 1]^T$ are linearly independent, meaning that we can solve for $\mathbf{x}(t)$ using (φ_1, φ_2). Solving the nonlinear system of equations (φ_1, φ_2) for (x, y) gives us

$$x = \frac{-\sqrt{2}(\varphi_1 - \varphi_2)}{-\varphi_1 - \varphi_2 + 4\varphi_1\varphi_2}, \tag{C.23}$$

$$y = \frac{(2-\sqrt{2})\varphi_1 + (2+\sqrt{2})\varphi_2}{-\varphi_1 - \varphi_2 + 4\varphi_1\varphi_2}. \tag{C.24}$$

Substituting the analytical solutions for φ_1 and φ_2 and the initial condition (x_0, y_0) gives us

$$x(t) = \frac{-\sqrt{2}\left(\frac{y_0 - x_0 + 2}{2(y_0 - (1-\sqrt{2})x_0)} e^{\sqrt{2}t} - \frac{y_0 - x_0 + 2}{2(y_0 - (1+\sqrt{2})x_0)} e^{-\sqrt{2}t}\right)}{-\frac{y_0 - x_0 + 2}{2(y_0 - (1-\sqrt{2})x_0)} e^{\sqrt{2}t} - \frac{y_0 - x_0 + 2}{2(y_0 - (1+\sqrt{2})x_0)} e^{-\sqrt{2}t} + \frac{(y_0 - x_0 + 2)^2}{(y_0 - (1-\sqrt{2})x_0)(y_0 - (1+\sqrt{2})x_0)}},$$

$$y(t) = \frac{(2-\sqrt{2})\frac{y_0 - x_0 + 2}{2(y_0 - (1-\sqrt{2})x_0)} e^{\sqrt{2}t} + (2+\sqrt{2})\frac{y_0 - x_0 + 2}{2(y_0 - (1+\sqrt{2})x_0)} e^{-\sqrt{2}t}}{-\frac{y_0 - x_0 + 2}{2(y_0 - (1-\sqrt{2})x_0)} e^{\sqrt{2}t} - \frac{y_0 - x_0 + 2}{2(y_0 - (1+\sqrt{2})x_0)} e^{-\sqrt{2}t} + \frac{(y_0 - x_0 + 2)^2}{(y_0 - (1-\sqrt{2})x_0)(y_0 - (1+\sqrt{2})x_0)}}.$$

Appendix D. Koopman eigenfunctions for two-dimensional ODEs—Numerical methods. Many two-dimensional nonlinear ODEs have invariant manifolds that cannot be easily defined. We consider how we may test for eigenfunctions that are composed of M-functions by numerically determining the functions $y = m_i(x)$ that define the invariant manifolds. We then use these functions to numerically construct the M-functions and N-functions and use regression to solve for a linear combination of N-functions that equals one. If the regression error is minimal, then by Theorem 3.2 we can use the resulting constants to construct an eigenfunction.

We use the following steps to test for eigenfunctions:

1. Find fixed points and eigenvectors.
2. Select starting values from linearization along eigenvector directions.
3. Numerically integrate to determine invariant manifolds.
4. Interpolate points along invariant manifolds to numerically define invariant manifold functions $y = m_i(x)$.
5. Numerically define M-functions $M_i(\mathbf{x})$ from the interpolated functions $y = m_i(x)$.

6. Numerically differentiate the M-functions $M_i(\mathbf{x})$ to compute $N_i(\mathbf{x})$.
7. Use regression to solve for p_i in the linear equation $\mathbf{1} = \sum_{i=1}^{k} p_i N_i(x,y)$, using random samples x, y drawn from some distribution (e.g., $x, y \sim \mathcal{N}(0,3)$).
8. If the regression error is minimal, then the p_i coefficients determine the eigenfunction $\varphi(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$; otherwise there is no eigenfunction of this form.

Testing our procedure on various nonlinear systems, we find that often the regression results in a large error, indicating that there does not exist an eigenfunction of the form $\varphi(\mathbf{x}) = \prod_i^k M_i^{p_i}(\mathbf{x})$. Other times our procedure yields a near zero error, indicating an approximate, not exact, eigenfunction. Figure 10 shows eigenfunctions identified for two nonlinear ODEs; one eigenfunction identified is exact, the other approximate. The left ODE (Figure 10(a)) was previously analyzed in section 5.3 and was found to have exact eigenfunctions constructed from M-functions. The right ODE has invariant manifolds that can only be numerically defined. Figure 10(b) shows the regression results; the regression produces zero error on the left and minimal error on the right. The resulting constants are used to construct eigenfunctions (Figure 10(c)).

Exact eigenfunctions usually cannot be constructed from invariant manifolds using the method we outline. However, approximate eigenfunctions may sometimes be found and may be useful in constructing approximate analytical solutions. One direction of further research would be to perform error analysis on approximate analytical solutions derived from numerically determined approximate eigenfunctions found using this method.

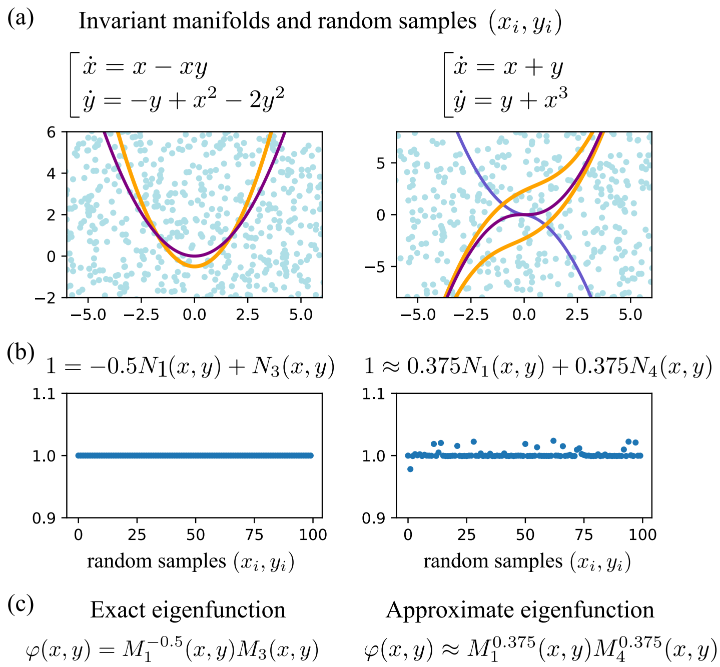

Figure 10. (a) *Numerically determined invariant manifolds for two nonlinear ODEs and sample points used for the regression.* (b) *Linear combination of N-functions results in exactly* 1 *with no error on the left and some error on the right.* (c) *The regression with no error produces an exact eigenfunction; the regression with some error produces an approximate eigenfunction.*

Our process for testing for eigenfunctions is implemented in the scripts contained in the following Github repository: **invariant_manifolds_eigenfunctions**.

Acknowledgment. We would like to thank Marko Budišić for his valuable comments on our manuscript.

REFERENCES

[1] H. Arbabi and I. Mezić, *Ergodic theory, dynamic mode decomposition, and computation of spectral properties of the Koopman operator*, SIAM J. Appl. Dyn. Syst., 16 (2017), pp. 2096–2126, https://doi.org/10.1137/17M1125236.

[2] T. Askham and J. N. Kutz, *Variable projection methods for an optimized dynamic mode decomposition*, SIAM J. Appl. Dyn. Syst., 17 (2018), pp. 380–416.

[3] E. M. Bollt, *Geometric considerations of a good dictionary for Koopman analysis of dynamical systems: Cardinality, "primary eigenfunction," and efficient representation*, Commun. Nonlinear Sci. Numer. Simul., 100 (2021), 105833, https://doi.org/10.1016/j.cnsns.2021.105833.

[4] E. M. Bollt, Q. Li, F. Dietrich, and I. Kevrekidis, *On matching, and even rectifying, dynamical systems through Koopman operator eigenfunctions*, SIAM J. Appl. Dyn. Syst., 17 (2018), pp. 1925–1960, https://doi.org/10.1137/17M116207X.

[5] W. E. Boyce, R. C. DiPrima, and D. B. Meade, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, New York, 2021.

[6] S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz, *Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control*, PLOS ONE, 11 (2016), pp. 1–19, https://doi.org/10.1371/journal.pone.0150171.

[7] S. L. Brunton and J. N. Kutz, *Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control*, Cambridge University Press, Cambridge, 2019.

[8] M. Budišić, R. Mohr, and I. Mezić, *Applied Koopmanism*, Chaos, 22 (2012), 047510, https://doi.org/10.1063/1.4772195.

[9] J. Carr, *Applications of Centre Manifold Theory*, Appl. Math. Sci. 35, Springer, New York, 1981, https://doi.org/10.1007/978-1-4612-5929-9.

[10] C. Chicone, J. E. Marsden, and L. Sirovich, *Ordinary Differential Equations with Applications*, Springer, New York, 2006.

[11] A. P. Gabale and S. C. Sinha, *Model reduction of nonlinear systems with external periodic excitations via construction of invariant manifolds*, J. Sound Vib., 330 (2011), pp. 2596–2607, https://doi.org/10.1016/j.jsv.2010.12.013.

[12] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Appl. Math. Sci. 42, Springer, New York, 1983.

[13] G. Haller, *Lagrangian coherent structures*, Annu. Rev. Fluid Mech., 47 (2015), pp. 137–162.

[14] A. J. Homburg, H. M. Osinga, and G. Vegter, *On the computation of invariant manifolds of fixed points*, Z. Angew. Math. Phys., 46 (1995), pp. 171–187, https://doi.org/10.1007/BF00944751.

[15] S. Jain and G. Haller, *How to compute invariant manifolds and their reduced dynamics in high-dimensional finite element models*, Nonlinear Dynam., 107 (2022), pp. 1417–1450, https://doi.org/10.1007/s11071-021-06957-4.

[16] E. Kaiser, J. N. Kutz, and S. L. Brunton, *Data-driven discovery of Koopman eigenfunctions for control*, Mach. Learn. Sci. Technol., 2 (2021), 035023, https://doi.org/10.1088/2632-2153/abf0f5.

[17] B. O. Koopman, *Hamiltonian systems and transformation in Hilbert space*, Proc. Natl. Acad. Sci. USA, 17 (1931), pp. 315–318.

[18] B. O. Koopman and J. von Neumann, *Dynamical systems of continuous spectra*, Proc. Natl. Acad. Sci. USA, 18 (1932), pp. 255–263.

[19] M. Korda and I. Mezić, *Optimal construction of Koopman eigenfunctions for prediction and control*, IEEE Trans. Automat. Control, 65 (2020), pp. 5114–5129, https://doi.org/10.1109/TAC.2020.2978039.

[20] J. N. Kutz, *Data-Driven Modeling and Scientific Computation: Methods for Complex Systems and Big Data*, Oxford University Press, Oxford, 2013.

[21] J. N. Kutz, S. L. Brunton, B. W. Brunton, and J. L. Proctor, *Dynamic Mode Decomposition*, SIAM, Philadelphia, 2016, https://doi.org/10.1137/1.9781611974508.
[22] M. D. Kvalheim and S. Revzen, *Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits*, Phys. D, 425 (2021), 132959, https://doi.org/10.1016/j.physd.2021.132959.
[23] Y. Lan and I. Mezić, *Linearization in the large of nonlinear systems and Koopman operator spectrum*, Phys. D, 242 (2013), pp. 42–53, https://doi.org/10.1016/j.physd.2012.08.017.
[24] Q. Li, F. Dietrich, E. M. Bollt, and I. G. Kevrekidis, *Extended dynamic mode decomposition with dictionary learning: A data-driven adaptive spectral decomposition of the Koopman operator*, Chaos, 27 (2017), 103111, https://doi.org/10.1063/1.4993854.
[25] B. Lusch, J. N. Kutz, and S. L. Brunton, *Deep learning for universal linear embeddings of nonlinear dynamics*, Nat. Commun., 9 (2018), 4950, https://doi.org/10.1038/s41467-018-07210-0.
[26] A. Mauroy and I. Mezić, *Global stability analysis using the eigenfunctions of the Koopman operator*, IEEE Trans. Automat. Control, 61 (2016), pp. 3356–3369, https://doi.org/10.1109/TAC.2016.2518918.
[27] R. C. McOwen, *Partial Differential Equations: Methods and Applications*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2003.
[28] I. Mezić, *Spectral properties of dynamical systems, model reduction and decompositions*, Nonlinear Dynam., 41 (2005), pp. 309–325, https://doi.org/10.1007/s11071-005-2824-x.
[29] I. Mezić, *Spectrum of the Koopman operator, spectral expansions in functional spaces, and state-space geometry*, J. Nonlinear Sci., 30 (2020), pp. 2091–2145, https://doi.org/10.1007/s00332-019-09598-5.
[30] M. Morrison, C. Fieseler, and J. N. Kutz, *Nonlinear control in the nematode C. elegans*, Front. Comput. Neurosci., 14 (2021).
[31] M. Morrison and L.-S. Young, *Chaotic heteroclinic networks as models of switching behavior in biological systems*, Chaos, 32 (2022), 123102, https://doi.org/10.1063/5.0122184.
[32] J. Page and R. R. Kerswell, *Koopman mode expansions between simple invariant solutions*, J. Fluid Mech., 879 (2019), pp. 1–27, https://doi.org/10.1017/jfm.2019.686.
[33] J. L. Proctor, S. L. Brunton, and J. N. Kutz, *Generalizing Koopman theory to allow for inputs and control*, SIAM J. Appl. Dyn. Syst., 17 (2018), pp. 909–930, https://doi.org/10.1137/16M1062296.
[34] G. Rega and H. Troger, *Dimension reduction of dynamical systems: Methods, models applications*, Nonlinear Dynam., 41 (2005), pp. 1–15, https://doi.org/10.1007/s11071-005-2790-3.
[35] A. J. Roberts, *The utility of an invariant manifold description of the evolution of a dynamical system*, SIAM J. Math. Anal., 20 (1989), pp. 1447–1458, https://doi.org/10.1137/0520094.
[36] V. Rom-Kedar, A. Leonard, and S. Wiggins, *An analytical study of transport, mixing and chaos in an unsteady vortical flow*, J. Fluid Mech., 214 (1990), pp. 347–394, https://doi.org/10.1017/S0022112090000167.
[37] M. R. Roussel and S. J. Fraser, *Invariant manifold methods for metabolic model reduction*, Chaos, 11 (2001), pp. 196–206, https://doi.org/10.1063/1.1349891.
[38] C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson, *Spectral analysis of nonlinear flows*, J. Fluid Mech., 641 (2009), pp. 115–127.
[39] P. J. Schmid, *Dynamic mode decomposition of numerical and experimental data*, J. Fluid Mech., 656 (2010), pp. 5–28.
[40] S. C. Shadden, *Lagrangian coherent structures*, in Transport and Mixing in Laminar Flows, John Wiley & Sons, New York, 2011, pp. 59–89, https://doi.org/10.1002/9783527639748.ch3.
[41] E. D. Sontag, J. E. Marsden, L. Sirovich, M. Golubitsky, and W. Jager, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Springer, New York, 2013.
[42] S. H. Strogatz and M. Dichter, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, Taylor & Francis, Boulder, CO, 2016.
[43] F. E. Udwadia, *Control of nonhyperbolic dynamical systems through center manifold control*, in Nonlinear Approaches in Engineering Application, L. Dai and R. N. Jazar, eds., Springer, Cham, 2022, pp. 69–86, https://doi.org/10.1007/978-3-030-82719-9_3.
[44] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Classics Appl. Math. 42, SIAM, Philadelphia, 2002.
[45] S. Wiggins, *Introduction to Dynamical Systems*, Springer, New York, 2003.

[46] M. Williams, I. Kevrekidis, and C. Rowley, *A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition*, J. Nonlinear Sci., 25 (2015), pp. 1307–1346, https://doi.org/10.1007/s00332-015-9258-5.

[47] M. O. Williams, C. W. Rowley, and I. G. Kevrekidis, *A kernel-based method for data-driven Koopman spectral analysis*, J. Comput. Dyn., 2 (2015), pp. 247–265, https://www.aimsciences.org/article/doi/10.3934/jcd.2015005.

[48] D. Wilson, *Isostable reduction of oscillators with piecewise smooth dynamics and complex Floquet multipliers*, Phys. Rev. E (3), 99 (2019), 022210, https://doi.org/10.1103/PhysRevE.99.022210.

[49] D. Wilson, *Phase-amplitude reduction far beyond the weakly perturbed paradigm*, Phys. Rev. E (3), 101 (2020), 022220, https://doi.org/10.1103/PhysRevE.101.022220.

[50] K. Wolsson, *Linear dependence of a function set of m variables with vanishing generalized Wronskians*, Linear Algebra Appl., 117 (1989), pp. 73–80, https://doi.org/10.1016/0024-3795(89)90548-X.