CONTACT HOMOLOGY AND HIGHER DIMENSIONAL CLOSING LEMMAS

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ABSTRACT. We develop methods for studying the smooth closing lemma for Reeb flows in any dimension using contact homology. As an application, we prove a conjecture of Irie, stating that the strong closing lemma holds for Reeb flows on ellipsoids. Our methods also apply to other Reeb flows, and we illustrate this for a class of examples introduced by Albers-Geiges-Zehmisch.

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1. INTRODUCTION

In [Pug67], Pugh proved a fundamental property of the periodic orbits of C^1 dynamical systems, called the C^1 *closing lemma*. Morally speaking, Pugh's closing lemma states that nearly periodic points become periodic after a slight C^1-perturbation of the dynamical system. Precisely, it is stated as follows.

Theorem 1. *[Pug67] Let V be a C^1 vector-field on a closed manifold X and let x be a non-wandering point of V. Then there is a C^1 vector-field V' that is C^1-close to V such that x is on a closed orbit of V'.*

Note that a point x is non-wandering if, for any open neighborhood U of x and every time S, there exists a $T \geqslant S$ such that $\phi^T(U) \cap U$ is non-empty, where ϕ is the flow of V. If ϕ preserves a volume form, then every point is non-wandering.

A key question in smooth dynamical systems (posed, for instance, by Smale [Sma98]) is whether or not Theorem 1 extends to the C^∞ setting. However, in [Gut87] Gutierrez proved that any compact manifold X containing an embedded punctured torus $\Sigma \subset X$ possesses a vector-field V and a non-wandering point p that does *not* become periodic under any small C^∞-perturbation of V. This follows the work of Herman [Her79] in the Hamiltonian setting, where he proved that all sufficiently small smooth Hamiltonian perturbations of a Diophantine rotation ϕ of the two-torus $\mathbb{T}^2$ have no periodic orbits. These negative results suggested that, for general smooth diffeomorphisms and flows, there is no analogue of the closing lemma.

Recently, dramatic progress has been made for area preserving diffeomorphisms of surfaces and Reeb flows of contact 3-manifolds . In [Iri15], Irie used spectral invariants coming from embedded contact homology (ECH) [Hut10] to prove the C^∞ closing lemma for Reeb flows on closed contact 3-manifolds. In fact, Irie's proof implied a *strong closing property*.

Definition 1.1 ([Iri22])**.** A manifold Y with contact form α satisfies the *strong closing property* if, for any non-zero smooth function

$$f : Y \to [0, \infty)$$

there is a $t \in [0, 1]$ such that $(1 + tf)\alpha$ has a closed Reeb orbit passing through the support of f.

Theorem 2. *[Iri15] Every closed 3-manifold with contact form (Y, α) has the strong closing property.*

Strong versions of the closing lemma were later proven for Hamiltonian surface maps [AI16] and more generally, area preserving surface maps [CGPZ21, EH21].

ECH is a fundamentally low-dimensional theory, and so the methods in [Iri15] are not directly applicable to studying the dynamics of higher-dimensional symplectomorphisms or Reeb flows. On the otherhand, ECH is part of a family of Floer theories collectively called symplectic field theory (or SFT) [EGH00], and other flavors of SFT (e.g. contact homology) generalize naturally to any dimension. In a recent work [Iri22], Irie described an abstract framework for proving strong closing properties using invariants satisfying formal properties in the spirit of SFT.

In this paper, we use contact homology to prove that the Reeb flow of any ellipsoid satisfies the strong closing property, as conjectured by Irie in [Iri22]. This is a first step towards applying the machinery of SFT to prove closing properties for more general classes of Reeb flows in higher dimensions.

1.1. **Spectral Gaps.** The strong closing property for Reeb flows is a consequence of an abstract criterion on contact homology. In order to explain this criterion, let us briefly review the structure of contact homology (for a detailed discussion, see §2).

The contact homology of a closed contact manifold (Y, ξ) with contact form α is a $\mathbb{Z}/2$-graded vector-space over $\mathbb{Q}$, denoted by

$$CH(Y, \xi) \qquad \text{with a filtration} \qquad CH^L(Y, \xi) \subset CH(Y, \xi) \qquad \text{determined by } \alpha.$$
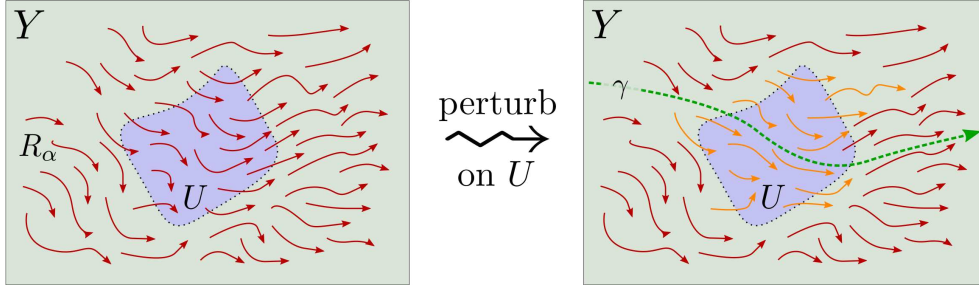
FIGURE 1. Morally speaking, the strong closing property states that any positive perturbation of a Reeb flow supported in U must produce a closed orbit γ through U.

If α is non-degenerate (i.e. if the linearized Poincaré return map of every closed Reeb orbit of α does not have 1 as eigenvalue) then $CH(Y, \xi)$ can be computed as the homology of a dg-algebra freely generated by good Reeb orbits. The differential counts genus 0 holomorphic curves in $\mathbb{R} \times Y$ with one puncture near $+\infty \times Y$ and any number of punctures near $-\infty \times Y$.

The contact homology algebra $CH(Y)$ comes with the additional structure of U-maps, which can be constructed as follows. An *abstract constraint P* of codimension $\operatorname{codim}(P)$ is a graded map

$$P : CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}[\operatorname{codim}(P)].$$

Here $(S^{2n-1}, \xi_{\text{std}})$ is the standard tight contact sphere. There is a filtered, graded map associated to any abstract constraint P, denoted by

$$U_P : CH(Y, \xi) \to CH(Y, \xi)[\operatorname{codim}(P)].$$

Intuitively, the *U-map* U_P counts holomorphic curves C in $\mathbb{R} \times Y$ passing through a point $p \in \Sigma$ in a small codimension 2 symplectic sub-manifold Σ of $\mathbb{R} \times Y$, where the number of branches of C through p and order of tangency of C at Σ is determined by P. Rigorously, U_P can be most easily constructed using the maps on contact homology induced by exact symplectic cobordisms (and this approach is related to the point constraint approach by Siegel [Sie19]).
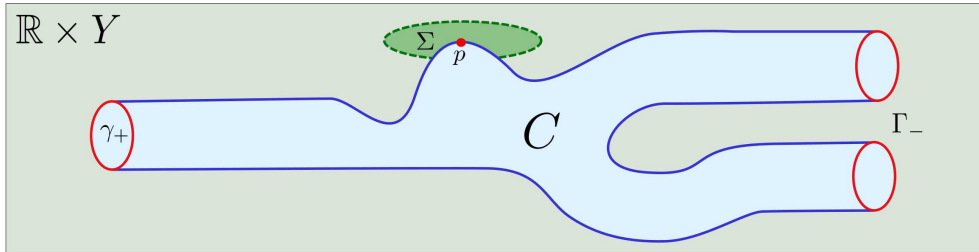
FIGURE 2. A cartoon of a U-map curve in contact homology. In general, the tangency of the curve C at the sub-manifold Σ can be arbitrarily complex.

Floer homologies typically have associated *spectral invariants* that track the minimal filtration at which a particular homology class appears. In symplectic geometry, these invariants have become pivotal tools in the study of quantitative and dynamical questions (cf. [GH18, Hut10, Sie19, CGHS21]). There are spectral invariants associated to contact homology, denoted by

$$\mathfrak{s}_\sigma(Y, \alpha) \qquad \text{for each} \qquad \sigma \in CH(Y, \xi).$$

The U-map U_P decreases this spectral invariant, in the sense that

$$\mathfrak{s}_{U_P\sigma}(Y, \alpha) \leqslant \mathfrak{s}_\sigma(Y, \alpha) \qquad \text{for any } \sigma \text{ and } P.$$

In particular, we can formulate an invariant that measures the minimal gap between the spectral invariants of a class σ and $U_P\sigma$.

Definition 1.2. The *spectral gap* of a contact homology class $\sigma \in CH(Y, \xi)$ is given by

$$\text{gap}_\sigma(Y, \alpha) := \inf_P \left\{ \frac{\mathfrak{s}_\sigma(Y, \alpha) - \mathfrak{s}_{U_P\sigma}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})} \right\} \in [0, \infty).$$

The normalizing constants $\mathfrak{c}_P(B^{2n})$ are certain symplectic capacities of the ball derived from the contact homology of its boundary. The *contact homology spectral gap* of a closed contact manifold Y with contact form α is given by

$$\text{gap}(Y, \alpha) := \inf_\sigma \left\{ \text{gap}_\sigma(Y, \alpha) \right\}.$$

The criterion for the strong closing property using the spectral gap can now be stated as follows.

Theorem 3. *Let (Y, ξ) be a closed contact manifold with contact form α, and suppose that*

$$gap(Y, \alpha) = 0.$$

Then (Y, α) satisfies the strong closing property.

This criterion is formulated in abstract terms in [Iri22]. We will provide our own concrete discussion of this condition, along with properties of the spectral gap, in §2.

1.2. **Spectral Gap Of Ellipsoids.** Given the spectral gap framework discussed above, we are naturally lead to the following question.

Question 4. Let (Y, ξ) be a closed contact manifold with non-trivial contact homology. Does the contact homology spectral gap vanish for any contact form α?

Even in simple cases, computing the U-map involves a difficult analysis of J-holomorphic curves, and so Question 4 is extremely difficult. As a first step, Irie conjectured an affirmative answer to Question 4 in the following family of examples of contact manifolds.

Example 1.3 (Ellipsoids). An *ellipsoid boundary* $(\partial E, \lambda|_{\partial E})$ is a contact manifold ∂E given as the boundary of an ellipsoid E in $\mathbb{C}^n$ of the form

$$E = \{z \in \mathbb{C}^n \ : \ \langle z, Az \rangle \leqslant 1\} \qquad \text{where} \qquad A \text{ is symmetric and positive definite.}$$

The contact form $\lambda|_{\partial E}$ is the restriction of the standard Liouville form λ on $\mathbb{C}^n$.

$$\lambda = \frac{1}{2} \cdot \sum_{j=1}^{n} x_j dy_j - y_j dx_j.$$

The Reeb vector-field R is given by $R(z) = 2JAz$ where $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is multiplication by i.
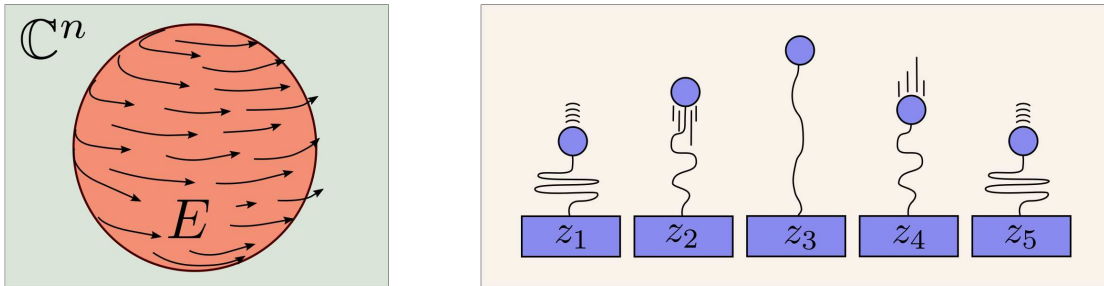

FIGURE 3. Two different visualizations of the Reeb flow of an ellipsoid $E \subset \mathbb{C}^n$. On the left, a flow on the boundary of the higher-dimensional domain E. On the right, as the dynamics of n independent harmonic oscillators (e.g. springs).

Remark 1.4 (Harmonic Oscillator). From the perspective of classical mechanics, the Reeb flow of an ellipsoid is the Hamiltonian dynamics of a harmonic oscillator on a fixed energy surface.

Indeed, up to a linear symplectomorphism, every ellipsoid is equivalent to one of the form

$$E(a) = E(a_1, \ldots, a_n) := \{(z_1, \ldots, z_n) \ : \ \pi \cdot \sum_i \frac{|x_i|^2 + |y_i|^2}{a_i} \leqslant 1\} \qquad \text{where} \qquad 0 < a_1 \leqslant \cdots \leqslant a_n.$$

The Reeb flow on the boundary of $E(a)$ is simply the Hamiltonian flow of the Hamiltonian

$$H : \mathbb{R}^{2n} \to \mathbb{R} \qquad \text{given by} \qquad H(x, y) = \pi \cdot \sum_i \frac{|x_i|^2 + |y_i|^2}{a_i}.$$

This is precisely the Hamiltonian for n independent harmonic oscillators with periods $a_1, \ldots, a_n$.

Ellipsoid boundaries are some of the most well-studied contact manifolds (cf. [CGHS21, Sie19, GH18]) and have provided a useful testing ground for many conjectures. Irie's conjecture from [Iri22] can be stated as follows.

Conjecture 5 ([Iri22, Conjecture 5.1]). *The boundary $(\partial E, \lambda|_{\partial E})$ of an ellipsoid $E \subset \mathbb{C}^n$ has the strong closing property.*

1.3. **Strong Closing Property For Ellipsoids.** The purpose of this paper is to prove Irie's conjecture via the following spectral gap result.

Theorem 6. *The boundary $(\partial E, \lambda|_{\partial E})$ of an ellipsoid $E \subset \mathbb{C}^n$ has vanishing spectral gap, and thus satisfies the strong closing property.*

Remark 1.5. After the posting of this paper, a different proof of the strong closing property for integrable systems was given by Xue [Xue22] using KAM theory. Our method of proof is different and applies to more general, non-integrable systems. See §1.4.

Our approach to Theorem 6 has two parts: the periodic (or integer) case and the general case.

1.3.1. *Periodic Case.* An *integer ellipsoid* E is an ellipsoid that is linearly symplectomorphic to a standard ellipsoid $E(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are all integers,

$$(a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

The Reeb flow of an ellipsoid with integer a_i is periodic, i.e., the flow φ_T is the identity for some T. The period is given by the least common multiple of $a_1, \ldots, a_n$,

$$T = \text{lcm}(a_1, \ldots, a_n).$$

The strong closing property is automatically satisfied for these flows, since every point goes through a periodic orbit of period bounded by T. We first prove that this property is reflected in contact homology as follows.

Theorem 7. *Let $E = E(a) \subset \mathbb{C}^n$ be an ellipsoid with $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then, there is a contact homology class $\sigma \in CH(\partial E)$ and a U-map $U = U_P$ such that*

$$\mathfrak{s}_{U\sigma}(\partial E, \lambda|_{\partial E}) = \mathfrak{s}_\sigma(\partial E, \lambda|_{\partial E}) = \text{lcm}(a_1, \ldots, a_n).$$

Theorem 7 is proven by a direct holomorphic curve calculation in contact homology. Let us briefly sketch the proof, as carried out in §4.

Proof Sketch. Note that ∂E does not have a non-degenerate contact form. Instead, the contact form is Morse-Bott, and the set of closed Reeb orbits of a given period T forms a sub-manifold

$$N_T \subset \partial E \qquad \text{with quotient} \qquad S_T := N_T/\mathbb{R} \qquad \text{by the Reeb flow.}$$

Any Morse-Bott contact form admits an arbitrarily small non-degenerate perturbation so that every closed Reeb orbit γ of period less than some fixed $L > 0$ corresponds to a pair

$$(T, p) \qquad \text{where } p \text{ is a critical point of a Morse function } f : S_T \to \mathbb{R}.$$

The period of γ is also approximately T. Note that S_T is an orbifold in general, and so morally we must work with orbifold Morse functions (in the appropriate sense). Moreover, gradient flow lines η between critical points on a fixed Morse-Bott family S_T lift to holomorphic cylinders u_η in the symplectization of ∂E between the correspoinding orbits.

When T is the period of the Reeb flow, i.e., the least common multiple of $a_1, \ldots, a_n$, the sub-manifold $N_T \subset \partial E(a)$ is simply the ellipsoid boundary $\partial E(a)$ itself and S_T is a closed orbifold of dimension $2n - 2$. If p_+ and p_- are the unique maximum and minimum of a Morse function f on S_T, then for any point $z \in S_T$, there is a unique gradient flow line

$$\eta : \mathbb{R} \to S_T \qquad \text{from } p_+ \text{ to } p_- \text{ passing through } z.$$

This flow line lifts to a cylinder u_η from the orbit γ_+ of (T, p_+) to the orbit γ_- of (T, p_-) passing through a point whose projection to S_T is z.

On the other hand, there is a U-map U_{P_0} that counts holomorphic curves satisfying a point constraint. Using intersection theory from [Sie11, MS19] and Wendl's automatic transversality [Wen10], we prove that the cylinder u_η is unique and transversely cut out. Therefore,

$$U_{P_0}(\gamma_+) \qquad \text{has a non-zero } \gamma_- \text{ coefficient.}$$

The orbits γ_+ and γ_- are both closed and non-exact in contact homology, and thus, the spectral invariant of $\sigma = [\gamma_+]$ and $U_{P_0}(\sigma)$ have the same action T, proving Theorem 7.   □

Remark 1.6. In principle, one could use Morse-Bott formulations of contact homology (cf. [Bou02]) to compute the U-map in Theorem 7 directly using gradient flow lines (and holomorphic cascades more generally). For the sake of completeness we provide here a direct analysis of the relevant moduli space and do not rely on the Morse-Bott formulation of contact homology.

1.3.2. *General Case.* The second step of our proof transfers the vanishing spectral gap of integer ellipsoids to general ellipsoids via the following approximation property.

Proposition 8. *Let (Y, ξ) be a closed contact manifold with a sequence of*

$$\text{contact forms } \alpha_i, \qquad \text{homology classes } \sigma_i \in CH(Y, \xi), \qquad \text{and} \qquad \text{real numbers } \epsilon_i > 0.$$

Suppose that, as $i \to \infty$, these sequences satisfy

$$\alpha_i \leqslant \alpha \leqslant (1 + \epsilon_i) \cdot \alpha_i \qquad \text{and} \qquad \epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i) \to 0.$$

Then, the contact homology spectral gaps satisfy

$$gap(Y, \alpha) \leqslant \liminf_{i \to \infty} gap_{\sigma_i}(Y, \alpha_i).$$

Proposition 8 can be proven in an entirely formal way from Definition 1.2. On the other hand, elementary results in Diophantine approximation can be used to prove the following result.

Proposition 9. *Let E be any ellipsoid. Then there exists a linear symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ and a sequence of rational ellipsoids E_i of period T_i such that*

$$(1 - \epsilon_i) \cdot E_i \subset \phi(E) \subset (1 + \epsilon_i) \cdot E_i \qquad \text{and} \qquad \epsilon_i \cdot T_i \to 0.$$

Rational ellipsoids are rescalings of integer ellipsoids and so by Theorem 7, there exist classes

$$\sigma_i \in CH(\partial E_i, \lambda|_{\partial E_i}) \qquad \text{with} \qquad \mathfrak{s}_{\sigma_i}(\partial E_i, \lambda|_{\partial E_i}) = T_i.$$

Theorem 6 is therefore an immediate consequence of Propositions 8 and 9.

1.4. **Strong Closing Property For A Non-Integrable Flow.** The methods of this paper are quite general, and can be applied to prove the strong closing lemma for flows that are very different from the ellipsoids. To illustrate this, in §6 we will prove the strong closing property for a family of non-periodic contact forms on a non-integrable contact 5-manifold Y. This family arises as a particular case of a construction by Albers-Geiges-Zehmisch [AGZ18].

Definition 1.7. We say that a Reeb flow on a contact manifold Y of dimension $2n - 1$ is *integrable* if it generates an S^1 action that extends to a Hamiltonian $\mathbb{T}^{2n-1}$ action on Y.

Consider the symplectic manifold $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the product symplectic form ω. Let A_1, A_2 denote the cohomology classes Poincare dual to $[\mathbb{C}P^1 \times \mathrm{pt}]$ and $[\mathrm{pt} \times \mathbb{C}P^1]$ respectively. Consider the prequantization bundle

$$S \to X \quad \text{with} \quad c_1(S) = A_1 + A_2$$

As a prequantization space, S admits a contact form α uniquely determined by $d\alpha = \pi^*\omega$. Moreover, S admits a Hamiltonian circle action generated by the vector-field X_H of the Hamiltonian

$$H : X \to \mathbb{R} \quad \text{given by} \quad H(x, y) = \frac{\pi}{2}\left(\frac{|x|^2}{(1 + |x|^2)} + \frac{|y|^2}{(1 + |y|^2)}\right) + 1$$

This action lifts to a circle action on S generated a Reeb vector-field R_2 (for a contact form α_2) that commutes with the Reeb vector-field R_1 of α. This yields a contact $\mathbb{T}^2$-action

$$\mathbb{T}^2 \curvearrowright (S, \xi)$$

Moreover, any vector $a = (a_1, a_2) \in \mathfrak{t}^2 \simeq \mathbb{R}^2$ with $a_i > 0$ corresponds to a contact form α_a and a vector-field R_a given by $a_1 \cdot R_1 + a_2 \cdot R_2$.

The space S described above is toric (since the S^1-action generated by H extends to a Hamiltonian $\mathbb{T}^2$-action). However, there is free $\mathbb{Z}_4$-action of S lifting the $\mathbb{Z}_4$-action on X generated by the following 4-periodic map.

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1 \quad \text{with} \quad f(x, y) = (-y, x)$$

The action on S commutes with the $\mathbb{T}^2$-action and we can show that

Lemma 1.8. *The quotient $Y = S/\mathbb{Z}_4$ does not admit a Hamiltonian $\mathbb{T}^3$-action extending the $\mathbb{T}^2$-action.*

The contact form α_a descends to Y, and is periodic if and only if a is (proportional to) a rational vector. By an analogous argument to §1.3, we prove that

Theorem 10. *The contact manifolds (Y, α_a) satisfy the strong closing property for each $a \in (0, \infty)^2$.*

In light of Lemma 1.8, we suspect that these Reeb flows cannot be approximated by integrable flows. If this is the case, methods from integrable systems (e.g. the KAM theory methods of [Xue22]) cannot be applied to these flows.

1.5. **Generalizing Irie's Conjecture.** We conclude this introduction by discussing several conjectures that are motivated by the results of this paper. These conjectures will be the subject of future work.

A contact form α on a contact manifold (Y, ξ) is *periodic* if the Reeb flow $\varphi : \mathbb{R} \times Y \to Y$ satisfies

$$\varphi_T = \mathrm{Id} \qquad \text{for some time} \qquad T > 0.$$

The closed Reeb orbits of a periodic contact form on a closed manifold Y form closed orbifolds

$$S_T := N_T/\mathbb{R} \qquad \text{where} \qquad N_T := \{y \in Y \ : \ \varphi_T(y) = y\}.$$

Every connected components S of S_T has an associated grading shift gr_S, given by the formula

$$\mathrm{gr}_S := \mathrm{RS}(S) - \frac{1}{2} \cdot \dim(S) \mod 2.$$

Here RS is the Robbin-Salamon index (see §2.1.3) of the linearized flow along any Reeb orbit in
S. To each component S, we also associate a filtered, graded vector-space

$$V(S) := H_{\bullet + \mathrm{gr}_S}(S; \mathbb{Q}).$$

We equip $V(S)$ with the homology grading shifted by gr_S and the trivial filtration where $V^L(S)$
is 0 if L is less than the period of the orbits in S and $V(S)$ otherwise.

Our first conjecture provides a simple formula for contact homology in the periodic setting.

Conjecture 11. *Let (Y, ξ) be a closed contact manifold with a periodic contact form α. Then*

$$CH(Y, \xi) \simeq \mathrm{Sym}\left(\bigoplus_S V(S)\right) \qquad \text{as filtered, graded vector-spaces.}$$

We expect that Conjecture 11 can be proven without Morse-Bott theory using Pardon's formula-
tion of contact homology [Par15] using an S^1-localization argument, analogous to the proof of
the Arnold conjecture given in [Par16].

Our next conjecture generalizes Theorem 7 to general periodic contact forms.

Conjecture 12. *Let (Y, ξ) be a closed contact manifold with a periodic contact form α of period T. Then,
there exists a class $\sigma \in CH(Y, \xi)$ and a U-map U_P such that*

$$\mathfrak{s}_{U_P\sigma}(Y, \alpha) = \mathfrak{s}_\sigma(Y, \alpha) = T.$$

We expect that Conjecture 12 admits a similar proof to Theorem 7. Namely, one should lift a
gradient flow line on the orbifold of closed orbits $S = S_T$ between the maximum and minimum
of an (orbifold) Morse function on S to a J-holomorphic cylinder counted by the U-map and then
verify uniqueness and transversality using intersection theory and automatic transversality.

Remark 1.9. A key technical difficulty is our use of a flag of Reeb invariant contact sub-manifolds
of ∂E in Theorem 7. A possible analogue of this flag can be constructed using the orbifold
Donaldson divisor construction [GMZ22] on the orbifold quotient $Y/\mathbb{R}$.

We conclude by noting that the spectral gap provides a criterion for periodicity; a similar
statement for ECH spectral invariants was proved by Cristofaro-Gardiner–Mazzuchelli [CGM20].
The proof of this criterion appears in §2. It follows from a simple modification of the formal
arguments used to prove Theorem 6.

Theorem 13 (Periodicity Criterion)**.** *Let (Y, ξ) be a closed contact manifold with contact form α where*

$$(1.1) \qquad\qquad \mathrm{gap}_\sigma(Y, \alpha) = 0 \qquad \text{for some} \qquad \sigma \in CH(Y, \xi).$$

Then the Reeb flow of α is periodic.

Note that Conjecture 12 states that (1.1) is also a necessary condition for periodicity. Therefore,
a slightly weaker version of Conjecture 12 can be reformulated as follows.

Conjecture 12′. *If (Y, ξ) is a closed contact manifold with contact form α, then the following are
equivalent.*

   (a) *The Reeb flow of the contact form α is periodic.*
   (b) *There is a class $\sigma \in CH(Y, \xi)$ such that $\mathrm{gap}_\sigma(Y, \alpha) = 0$.*

We say that a contact form α on (Y, ξ) is *near periodic* if there is a sequence of periodic contact
forms α_i of period T_i and $\epsilon_i > 0$ such that

$$\alpha_i \leqslant \alpha \leqslant (1 + \epsilon_i) \cdot \alpha_i \qquad \text{and} \qquad \epsilon_i \cdot T_i \to 0$$

The vanishing of the spectral gap of periodic contact forms can be transferred to near-periodic
contact contact forms via the approximation result in Proposition 8, as in the ellipsoid case. We
expect many new examples of non-periodic Reeb flows that satisfy Irie's strong closing property
to arise in this way.

Outline. The paper is organized as follows. In Section 2 we review necessary preliminaries from contact homology. Section 3 contains the exposition of abstract constraints in contact homology and the spectral gap. It also contains the proofs of Theorem 3, Proposition 8 and Theorem 13. Section 4 contains an analysis of the moduli space counted by the U_{P_1}-map. In Section 5 we use this analysis to prove Theorem 7. We also provide a proof of Proposition 9 and thus conclude the proof of the closing property for all ellipsoids, as stated in Theorem 6. Section 6 contains a proof of Theorem 10, the strong closing property for the non-integrable flows constructed in Section 1.4.

2. Contact Homology

In this section, we review the formalism of contact homology, which is a simple variant of symplectic field theory originally introduced by Eliashberg-Givental-Hofer [EGH00].

Remark 2.1. We will work with the transversality framework developed by Pardon [Par15], although all of the results discussed here should be independent of the specific transversality scheme.

2.1. **Reeb Orbits.** We start by discussing some preliminaries about Reeb dynamics. Throughout this section, we fix a contact manifold (Y, ξ) with a contact form α. We let R denote the Reeb vector-field of α and $\varphi : \mathbb{R} \times Y \to Y$ denote the Reeb flow.

2.1.1. *Reeb Orbits.* A closed or periodic *Reeb orbit* γ is a closed trajectory of the Reeb vector-field R, that is,

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y \qquad \text{satisfying} \qquad \frac{d\gamma}{dt} = R \circ \gamma.$$

Here T is called the the *period* or *action* of γ, and for any Reeb trajectory one has

$$T = \mathcal{A}(\gamma) \qquad \text{where} \qquad \mathcal{A}(\gamma) := \int_\gamma \alpha.$$

Two Reeb orbits γ and η are *equivalent* if they are related by translation in t, that is, if

$$\gamma(t + t_0) = \eta(t) \qquad \text{for some } t_0 \in \mathbb{R} \text{ and all } t \in \mathbb{R}.$$

Any closed trajectory γ factors into a covering map ϕ and a *simple* (i.e., injective) closed orbit η.

$$\mathbb{R}/T\mathbb{Z} \xrightarrow{\phi} \mathbb{R}/T'\mathbb{Z} \xrightarrow{\eta} Y.$$

The *covering multiplicity* κ_γ of γ is the degree of the covering map ϕ, namely,

$$(2.1) \qquad\qquad \kappa_\gamma := \deg(\phi).$$

We will consider tuples of Reeb orbits, possibly with repetition, of the form

$$\Gamma = (\gamma_1, \ldots, \gamma_N).$$

If Γ consists of m distinct orbits $\eta_1, \ldots, \eta_m$ occurring with multiplicity μ_i in the sequence and having covering multiplicity κ_i, respectively, then we let

$$\mathcal{A}(\Gamma) = \sum_i \mathcal{A}(\gamma_i), \qquad \mu_\Gamma = \mu_1 \mu_2 \cdots \mu_m, \qquad \text{and} \qquad \kappa_\Gamma = \kappa_1 \kappa_2 \cdots \kappa_m.$$

2.1.2. *Non-Degeneracy.* Let N_T denote the set of fixed points of the time T Reeb flow.

$$N_T := \{y \in Y \; : \; \varphi_T(y) = y\} \subset Y.$$

We say that N_T is *Morse-Bott* if it is a closed sub-manifold of Y with tangent bundle given by

$$TN_T = \ker(d\varphi_T - \text{Id})|_{N_T}.$$

A *Morse-Bott family* $S \subset N_T/S^1$ of Reeb orbits is a connected component of the quotient N_T/S^1 for some period T. Here $S^1 \simeq \mathbb{R}/T\mathbb{Z}$ acts on N_T by the Reeb flow. Note that any Morse-Bott family is automatically an effective orbifold. As a special case, a Reeb orbit γ is *non-degenerate* if

$$\ker(d\varphi_T - \text{Id})|_\gamma = T\gamma = \text{span}(R).$$

That is, if $S = \gamma/S^1$ is a 0-dimensional Morse-Bott family.

A contact form α is said to be *non-degenerate below action L* if every closed Reeb orbit γ with $\mathcal{A}(\gamma) \leqslant L$ is non-degenerate. The form α is called *non-degenerate* if every closed Reeb orbit is non-degenerate. Finally, α is *Morse-Bott* if every closed Reeb orbit is in a Morse-Bott family.

2.1.3. *Linearized Flow And Indices.* A *trivialization* $\tau : \gamma^*\xi \simeq \mathbb{C}^{n-1}$ of ξ along a Reeb orbit γ is a trivialization of $\gamma^*\xi$ as a symplectic vector-bundle, that is, τ is a 1-parameter family of symplectic diffeomorphism $\tau_{\gamma(t)} : \xi_{\gamma(t)} \to \mathbb{C}^{n-1}$. The *linearized flow* $\Phi_{\tau,\gamma}$ associated to γ and τ is the path of symplectic matrices

$$\Phi_{\tau,\gamma} : [0,T] \to \text{Sp}(2n-2) \quad \text{given by} \quad \Phi_{\tau,\gamma}(t) = \tau_{\gamma(t)} \circ d\varphi_t^R|_\xi \circ \tau_{\gamma(0)}^{-1}.$$

This path depends on the trivialization, but if σ and τ are isotopic trivializations, then the paths $\Phi_{\sigma,\gamma}$ and $\Phi_{\tau,\gamma}$ are isotopic via paths $\Phi^s : [0,T] \to \text{Sp}(2n-2)$ for $t \in [0,1]$ such that

$$\text{rank}(\ker(\Phi^s(T) - \text{Id})) \qquad \text{is constant in } s.$$

The path $\Phi_{\tau,\gamma}$ allows us to associate a Robbin-Salamon index (introduced by Robbin-Salamon in [RS93]) to any orbit with a trivialization.

Proposition 2.2. *[Gut14] For each $n \geqslant 1$, there exists an integer valued Robbin-Salamon index of paths of symplectic matrices*

$$\text{RS} : C^0([0,1], \text{Sp}(2n)) \to \mathbb{Z},$$

that is characterized by the following axioms.

   (a) (Homotopy) If $\Phi^s : [0,1] \to \text{Sp}(2n)$ for $s \in [0,1]$ is a family of paths such that $\Phi^s(0)$ and $\Phi^s(1)$ are independent of s, then

$$\text{RS}(\Phi^0) = \text{RS}(\Phi^1).$$

   (b) (Additive) RS is additive under concatenation and direct sum, that is,

$$\text{RS}(\Phi * \Psi) = \text{RS}(\Phi) + \text{RS}(\Psi) \qquad \text{and} \qquad \text{RS}(\Phi \oplus \Psi) = \text{RS}(\Phi) + \text{RS}(\Psi).$$

   (c) (Vanishing) If $\Phi : [0,1] \to \text{Sp}(2n)$ is such that $\text{rank}(\Phi(t) - \text{Id})$ is constant in t, then $\text{RS}(\Phi) = 0$.

The Robbin-Salamon index generalizes the Conley-Zehnder index, in the sense that $\text{RS}(\Phi) = \text{CZ}(\Phi)$ when $\ker(\Phi(1) - \text{Id})$ is 0-dimensional and $\Phi(0)$ is the identity.

For sufficiently nice paths, the Robbin-Salamon index can be explicitly computed using crossings of the Maslov cycle. To be precise, for a given path Φ and any $t \in [0,1]$, let Γ_t be the symmetric bilinear form on $\ker(\Phi(t) - \text{Id})$ given by

$$\Gamma_t(v,w) = \omega_0\left(\frac{d\Phi}{dt}(t)\Phi^{-1}(t)v, w\right).$$

Let $\text{sign}(\Gamma_t)$ be -1 raised to the power of the signature of Γ_t. A *crossing* is a time $t \in [0, T]$ such that $\det(\Phi(t) - \text{Id}) = 0$. A crossing t is *non-degenerate* if Γ_t is non-degenerate. When every crossing time of Φ is non-degenerate, the Robbin-Salamon index is given by

$$(2.2) \qquad \text{RS}(\Phi) = \frac{1}{2}\text{sign}(\Gamma_0) + \sum_{0 < t < T} \text{sign}(\Gamma_t) + \frac{1}{2}\text{sign}(\Gamma_T)$$

where the sum is over all crossings of Φ.

Let γ be an orbit lying in a Morse-Bott family of Reeb orbits S, and let τ be a trivialization of the contact structure ξ on a neighborhood of γ. The *Robbin-Salamon index* of a γ with respect to τ is given by

$$\text{RS}_\tau(\gamma) = \text{RS}(\Phi_{\tau,\gamma})$$

where $\Phi_{\tau,\gamma}$ is the linearized flow along γ, restricted to ξ and trivialized by τ. We will sometimes denote $\text{RS}_\tau(\gamma)$ by $\text{RS}_\tau(S)$ in order to distinguish the index of a degenerate orbit from the index of a non-degenerate perturbation of it. We remark that the pairity of the RS index is independent of the choice of trivialization[1]. If γ is non-degenerate, then we define the *Conley-Zehnder index* of γ with respect to τ as

$$\text{CZ}_\tau(\gamma) := \text{RS}(\Phi_{\tau,\gamma}).$$

Finally, the *SFT grading* $|\gamma|_\tau$ is given by

$$|\gamma|_\tau = (n - 3) + \text{CZ}_\tau(\gamma) \quad \mod 2$$

where $\dim(Y) = 2n - 1$.

2.2. Holomorphic Buildings.

We next establish basic notation for J-holomorphic curves and, more generally, J-holomorphic buildings.

2.2.1. Symplectic Cobordisms And Liouville Domains.

A *symplectic cobordism* $X : Y_+ \to Y_-$ between closed contact manifolds (Y_+, α_+) and (Y_-, α_-) is a compact symplectic manifold (X, ω) with boundary such that

$$\partial X = Y_+ \cup (-Y_-) \qquad \text{and} \qquad \omega|_{TY_\pm} = d\alpha_\pm.$$

The symplectic cobordism X is *exact* if $\omega = d\lambda$ where $\lambda|_{TY_\pm} = \alpha_\pm$. A *deformation* of exact cobordisms is simply a smooth 1-parameter family of exact cobordisms $(X, \lambda_t) : Y_+ \to Y_-$ parametrized by $t \in [0, 1]$. Two exact cobordisms are *deformation equivalent* if they are (up to isomorphism) connected by a deformation.

Given symplectic cobordisms $X : Y_1 \to Y_2$ and $X' : Y_2 \to Y_3$, there is a well-defined *composition* given by

$$X \circ X' := X \cup_{Y_1} X'.$$

This cobordism inherits a symplectic cobordism structure (induced by a standard collar neighborhood of the boundary) and is exact if X and X' are exact.

Any symplectic cobordism X can be completed to a non-compact, cylindrical manifold called the *completion* of X, denoted by

$$\hat{X} = (-\infty, 0] \times Y_- \cup_{Y_-} X \cup_{Y_+} [0, \infty) \times Y_+.$$

As a special case, if (Y, α) is a contact manifold with contact form, we have a trivial exact cobordism

$$[a, b]_r \times Y : (Y, e^b \alpha) \to (Y, e^a \alpha) \qquad \text{with Liouville form} \qquad \lambda = e^r \alpha.$$

The completion of the trivial cobordism is called the *symplectization* and is denoted by $\hat{Y}$.

---

[1] This follows from the loop property of the CZ index (see e.g. [Gut14]) and the concatenation property of the RS index

A *Liouville domain* (W, λ) is an exact symplectic cobordism from ∂W to the emptyset. If X is an exact symplectic manifold with Liouville form θ, then a symplectic embedding $\iota : W \to X$ from a Lioiville domain is called *exact* if

$$\iota^* \theta = \lambda$$

More generally, ι is *weakly exact* if

$$[\lambda|_{\partial W} - \iota^* \theta|_{\partial W}] = 0 \in H^1(\partial W; \mathbb{R}).$$

For any weakly exact embedding $\iota : W \to X$, the Liouville form θ of X is homotopic through Liouville forms in a neighborhood of W so that ι is exact. After such a deformation, $X \backslash W$ is a symplectic cobordism

$$X : Y_+ \to Y_- \cup \partial W$$

and we may write $X = (X \backslash W) \circ (W \cup [0, 1] \times Y_-)$ (up to deformation).

2.2.2. *Homology Classes.* Given a symplectic cobordism $X : Y_+ \to Y_-$ and sequences of Reeb orbits $\Gamma_\pm$ in $Y_\pm$, let

$$\Xi_\pm \subset Y_\pm$$

denote the 1-manifold in $Y_\pm$ given as the union of the underlying simple orbits of $\Gamma_\pm$. We may identify $\Xi_+ \cup \Xi_-$ as a sub-manifold of ∂X. We denote the following subset of the relative homology by

$$S(X; \Gamma_+, \Gamma_-) := \{ A \in H_2(X, \Xi_+ \cup \Xi_-) \ : \ \partial A = [\Gamma_+] - [\Gamma_-] \in H_1(\partial X) \}.$$

Given a homology class $A \in S(X; \Gamma_+, \Gamma_-)$ and a trivialization $\tau : \xi|_{\Gamma_+ \cup \Gamma_-} \simeq \mathbb{R}^{2n-2}$ of ξ over the collections of Reeb orbits Γ_+ and Γ_-, there is a well-defined *relative Chern number* (Definition 5.1 in [Wen15])

$$c_1(A, \tau) \in \mathbb{Z}.$$

Moreover, given a choice of genus g, there is a well-defined *Fredholm index* given by

$$\mathrm{ind}(A, g) := (n - 3) \cdot (2 - 2g - |\Gamma_+| - |\Gamma_-|) + c_1(A, \tau) + \sum_{\gamma_+ \in \Gamma_+} \mathrm{CZ}_\tau(\gamma_+) - \sum_{\gamma_- \in \Gamma_-} \mathrm{CZ}_\tau(\gamma_-).$$

Note that if $X : Y_0 \to Y_1$ and $X' : Y_1 \to Y_2$ are symplectic cobordisms, then there is a map of homology classes in X and X' to homology classes in the composition

$$S(X; \Gamma_0, \Gamma_1) \times S(X'; \Gamma_1, \Gamma_2) \to S(X \circ X'; \Gamma_0, \Gamma_2) \qquad \text{denoted by} \qquad (A, A') \mapsto A + A'.$$

This map is associative. Moreover, the Chern class and Fredholm index are both additive with respect to this operation.

2.2.3. *Complex Structures.* A *compatible almost complex structure* on a contact manifold (Y, ξ) is a bundle endomorphism $J : \xi \to \xi$ such that

$$J^2 = -\mathrm{Id} \qquad \text{and} \qquad d\alpha(-, J-)|_\xi \text{ is a metric for any contact form } \alpha.$$

Any such almost complex structure extends to an almost complex structure $\hat{J}$ on $\hat{Y}$ by

$$\hat{J}|_\xi = J \qquad \text{and} \qquad \hat{J}(\partial_r) = R \qquad \text{where} \qquad \hat{Y} = \mathbb{R}_r \times Y.$$

Likewise, for $J_\pm$ compatible almost complex structures on $Y_\pm$, a *compatible almost complex structure* $\hat{J}$ on the completion $\hat{X}$ of a symplectic cobordism $X : Y_+ \to Y_-$ is a bundle endomorphism $\hat{J} : T\hat{X} \to T\hat{X}$ such that

$$\hat{J}^2 = -\mathrm{Id}, \qquad \omega(-, \hat{J}-)|_X \text{ is a metric on } X, \qquad \hat{J}|_{(-\infty, 0] \times Y_-} = \hat{J}_-, \qquad \text{and } \hat{J}|_{[0, \infty) \times Y_+} = \hat{J}_+.$$

2.2.4. *Holomorphic Maps.* Fix a symplectic cobordism $X : Y_+ \to Y_-$ and a compatible almost complex structure $\hat{J}$ on $\hat{X}$. Consider a closed Riemann surface

$$(\bar{\Sigma}, j) \qquad \text{with a finite set of punctures} \qquad P \subset \bar{\Sigma}.$$

A *J-holomorphic map* from the punctured surface $\Sigma := \bar{\Sigma} \backslash P$ to the symplectization $(\hat{X}, \hat{J})$ is a map

$$u : \Sigma \to \hat{X} \qquad \text{satisfying} \qquad J \circ du = du \circ j.$$

The *λ-energy* $\mathcal{E}_\lambda(u)$ and *area* $\mathcal{A}(u)$ of a $\hat{J}$-holomorphic map are defined as follows.

$$\mathcal{E}_\lambda(u) = \sup_{\phi_-} \int_{u^{-1}((-\infty,0)\times Y)} u^*(\phi_- ds \wedge \alpha_-)) + \sup_{\phi_+} \int_{u^{-1}((0,\infty)\times Y)} u^*(\phi_+ ds \wedge \alpha_+)),$$

$$\mathcal{A}(u) = \int_{u^{-1}((-\infty,0)\times Y)} u^*(d\alpha_-) + \int_{u^{-1}(X)} u^*\omega + \int_{u^{-1}((0,\infty)\times Y)} u^*(d\alpha_+).$$

Here the supremums are over compactly supported functions $\phi_- : (-\infty, 0] \to [0, \infty)$ and $\phi_+ : [0, \infty) \to [0, \infty)$ with integral 1. The *energy* $\mathcal{E}(u)$ is simply the sum

$$\mathcal{E}(u) = \mathcal{E}_\lambda(u) + \mathcal{A}(u).$$

A holomorphic map is said to have *finite energy* if $\mathcal{E}(u)$ is well-defined. Any finite energy, proper holomorphic map u is asymptotic to sequences of closed Reeb orbits $\Gamma_\pm$ as $r \to \pm\infty$, $u \to \Gamma_\pm$ as $\pm\infty$. To be precise, for each puncture $p \in P$, there is a neighborhood U of p and a holomorphic chart $\phi : [0, \infty)_s \times (S^1)_t \simeq U\backslash p \subset \Sigma$ such that

$$u \circ \phi([0, \infty) \times S^1) \subset [0, \infty) \times Y_+ \qquad \text{or} \qquad u \circ \phi([0, \infty) \times S^1)) \subset (-\infty, 0] \times Y_-,$$

and

$$\pi_\mathbb{R} \circ u \circ \phi(s, -) \xrightarrow[s\to\pm\infty]{} \pm\infty, \qquad \pi_Y \circ u \circ \phi(s, -) \xrightarrow[s\to\pm\infty]{C^0} \gamma(\pm T-) \qquad \text{as a map} \quad S^1 \to Y_\pm.$$

Here γ is a closed Reeb orbit of $Y_\pm$, $\pi_\mathbb{R} : \hat{X} \to \mathbb{R}$ is the projection to the $\mathbb{R}$-factor, and $\pi_Y : \hat{X} \to Y_\pm$ is the projection to the $Y_\pm$-factor where both projections are defined on the cylindrical ends of $\hat{X}$. Hence, Stokes theorem implies that

$$\mathcal{A}(u) = \mathcal{A}(\Gamma_+) - \mathcal{A}(\Gamma_-).$$

Since the area of holomorphic maps is always non-negative, this implies that $\mathcal{A}(\Gamma_+) \geqslant \mathcal{A}(\Gamma_-)$. Finally, any holomorphic curve u from Γ_+ to Γ_- represents a class

$$[u] \in S(X; \Gamma_+, \Gamma_-)$$

acquired as the fundamental class of the composition $\pi \circ u$ where $\pi : \hat{X} \to X$ is the continuous map sending X to itself and the ends $(-\infty, 0) \times Y_-$ and $(0, \infty) \times Y_+$ to Y_- and Y_+, respectively.

2.2.5. *Asymptotic Markers and Matchings.* Consider a cobordism $X : Y_+ \to Y_-$ and equip each simple Reeb orbit η in Y_+ and Y_- with a basepoint $b_\eta \in \eta \subset Y_\pm$.

Given a finite energy holomorphic map $u : \Sigma \to \hat{X}$ asymptotic to a Reeb orbit γ at a puncture p, let S_p denote the unit circle bundle at the puncture p

$$S_p := (T_p\bar{\Sigma}\backslash 0)/\mathbb{R}_+.$$

The complex structure on $\bar{\Sigma}$ induces an S^1-action on S_p and there is a natural map of the form

$$\pi_{u,p} : S_p \to \eta, \qquad \text{where } \eta \text{ is the simple orbit of } \gamma.$$

An *asymptotic marker* m_γ at a puncture $p \in P$ is a choice of element

$$m_\gamma \in S_p, \qquad \text{such that} \qquad \pi_{u,p}(m_\gamma) = b_\eta.$$

Given two holomorphic curves u and v asymptotic to γ at punctures p and q, respectively, a *matching isomorphism* μ between their ends is a map

$$\mu : S_p \to S_q, \qquad \text{such that} \qquad \pi_{v,q} \circ \mu(\pi_{u,p}^{-1}(b_\eta)) = b_\eta.$$

A holomorphic map u is said to be *equipped with asymptotic markers* if each of its punctures is. Note that a biholomorphism $\phi : \Sigma \to \Sigma'$ induces a map on the set of asymptotic markers and matching isomorphisms along the punctures. We denote these maps by ϕ_*.

2.2.6. *Holomorphic Curves.* Given an integer g, a homology class $A \in S(X; \Gamma_+, \Gamma_-)$, and tuples of Reeb orbits $\Gamma_\pm$ in $Y_\pm$, we have an associated *moduli space* of finite energy J-holomorphic curves

$$\mathcal{M}_{g,A}(X, J; \Gamma_+, \Gamma_-) \qquad \text{or} \qquad \mathcal{M}_{g,A}(\hat{X}, \hat{J}; \Gamma_+, \Gamma_-).$$

The points in this moduli space are J-holomorphic maps $u : \Sigma \to \hat{X}$ equipped with asymptotic markers that satisfy

$$g(\bar{\Sigma}) = g, \qquad [u] = A \quad \text{and} \quad u \to \Gamma_\pm \text{ as } r \to \pm\infty,$$

modulo the relation that u is equivalent to u' if there is a biholomorphism $\phi : \Sigma' \simeq \Sigma$ such that

$$u' = u \circ \phi$$

and the induced ϕ^*'s respect the asymptotic markers. We refer to such an equivalence class u as a *J-holomorphic curve* (with asymptotic markers).

In the case where $\hat{X} = \hat{Y}$ is the symplectization of a contact manifold Y and $\hat{J}$ is the symplectization of a compatible almost complex structure on ξ, the moduli space admits a natural $\mathbb{R}$-action given by $\mathbb{R}$-translation in $\mathbb{R} \times Y$. Then we adopt the notation

$$\mathcal{M}_{g,A}(Y, J; \Gamma_+, \Gamma_-) := \mathcal{M}_{g,A}(\mathbb{R} \times Y, \hat{J}; \Gamma_+, \Gamma_-)/\mathbb{R}.$$

Example 2.3. A *trivial cylinder* in a symplectization $(\hat{Y}, \hat{J})$ is any map $u : \mathbb{R} \times S^1 \to \hat{Y}$ of the form $u(s,t) = (s, \gamma(T \cdot t))$ where γ is a closed Reeb orbit of period T.

2.2.7. *Holomorphic Buildings.* A *J-holomorphic building* $\bar{u}$ in a contact manifold Y from Γ_+ to Γ_- is a finite sequence of orbit tuples

$$\Gamma_1, \ldots, \Gamma_m \qquad \text{with} \qquad \Gamma_+ = \Gamma_1 \quad \text{and} \quad \Gamma_- = \Gamma_m,$$

and a sequence of finite energy J-holomorphic maps in $\hat{Y}$, called the *levels* of $\bar{u}$, denoted by

$$u_i : \Sigma_i \to \hat{Y} \qquad \text{with} \qquad u_i \to \Gamma_{i+1} \text{ at } -\infty \quad \text{and} \quad u_i \to \Gamma_i \text{ at } +\infty,$$

where each level u_i is non-trivial, i.e., not a union of trivial cylinders. Moreover, the levels asymptotic to Γ_i are equipped with asymptotic markers for all $i \in \{1, \ldots, m\}$, and the (pairs of) punctures of the levels asymptotic to orbits in Γ_i for $i = 2, \ldots, m-1$ are equipped with matching isomorphisms.

Two buildings $\bar{u}$ and $\bar{v}$ are equivalent if, up to $\mathbb{R}$-translations, there is a biholomorphism of domains on each level respecting the holomorphic map, asymptotic markers and matching isomorphisms.

Any building $\bar{u}$ has a well-defined genus $g = g(\bar{u})$ and homology class $[\bar{u}] = A$ determined by gluing the curves along the matching punctures. In particular, the homology class is given by

$$A = \sum_i A_i \in S(Y; \Gamma_+, \Gamma_-).$$

The moduli space of equivalence classes of J-holomorphic buildings in Y from Γ_+ to Γ_- of genus g and homology class A is denoted by

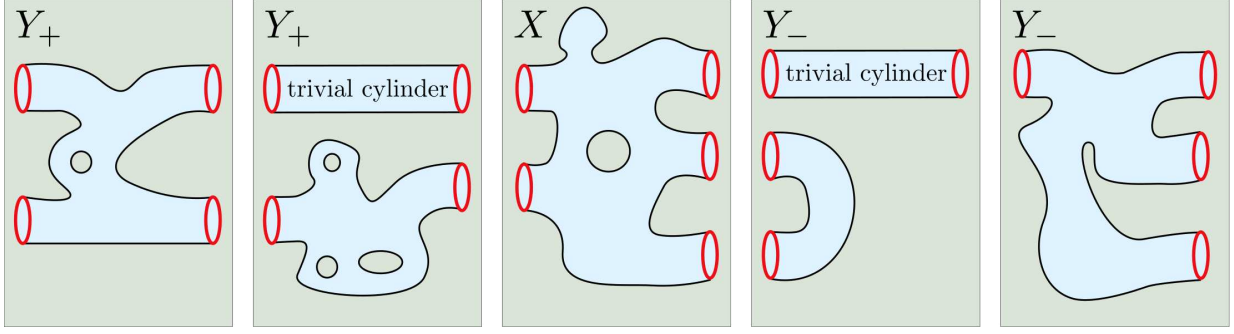$$\overline{\mathcal{M}}_{g,A}(Y, J; \Gamma_+, \Gamma_-).$$

FIGURE 4. A simplified picture of a possible holomorphic building. Note that all of the holomorphic buildings and curves of interest in this paper will be genus 0.

More generally, a J-holomorphic building $\bar{u}$ in a symplectic cobordism $X : Y_+ \to Y_-$ from Γ_+ to Γ_- is a sequence of Reeb orbit tuples

$$\Gamma_1^+, \ldots, \Gamma_a^+, \Gamma_1^-, \ldots, \Gamma_b^- \qquad \text{with} \qquad \Gamma_+ = \Gamma_1^+ \text{ and } \Gamma_- = \Gamma_b^-$$

and a sequence of finite energy J-holomorphic maps of the form

$$u_i^\pm : \Sigma_i^\pm \to \hat{Y}_\pm, \qquad \text{with} \qquad u_i^\pm \to \Gamma_i^\pm \text{ at } +\infty \quad \text{and} \quad u_i^\pm \to \Gamma_{i+1}^\pm \text{ at } -\infty,$$
$$u_X : \Sigma_X \to \hat{X}, \qquad \text{with} \qquad u_X \to \Gamma_a^+ \text{ at } +\infty \quad \text{and} \quad u_X \to \Gamma_1^- \text{ at } -\infty,$$

equipped with the same asymptotic markers and matching isomorphisms as in the symplectization case. Equivalence of a pair of buildings is defined as in the symplectization case, but we only quotient by the $\mathbb{R}$-direction in the symplectization levels. The moduli space of J-holomorphic buildings in X from Γ_+ to Γ_- of genus g and homology class A is denoted by

$$\overline{\mathcal{M}}_{g,A}(X, J; \Gamma_+, \Gamma_-).$$

The moduli spaces $\overline{\mathcal{M}}_{g,A}(Y, J; \Gamma_+, \Gamma_-)$ and $\overline{\mathcal{M}}_{g,A}(X, J; \Gamma_+, \Gamma_-)$ admit a Gromov topology described by [BEH$^+$03, §9.1]. Moreover, both spaces are compact [BEH$^+$03, §10.1].

2.2.8. *Marked Moduli Spaces.* J-holomorphic curves and buildings can be decorated to include marked points. More precisely, we can formulate a moduli space

$$\mathcal{M}_{g,A,m}(X, J; \Gamma_+, \Gamma_-).$$

The points in this moduli space are genus g J-holomorphic maps $u : \Sigma \to \hat{X}$ in homology class A (with asymptotic markers) and an ordered tuple of m *marked points* $s_1, \ldots, s_m$ in Σ, modulo reparametrizations that respect the marked points. There is an *evaluation map*

$$\text{ev} : \mathcal{M}_{g,A,m}(X, J; \Gamma_+, \Gamma_-) \to \hat{X}^m$$

that takes an equivalence class $[u, s_1, \ldots, s_m]$ to the point $(u(s_1), \ldots, u(s_m)) \in \hat{X}^m$. One may also form a moduli space of buildings

$$\overline{\mathcal{M}}_{g,A,m}(X, J; \Gamma_+, \Gamma_-)$$

of J-holomorphic buildings where each level is a J-holomorphic curve with marked points and the total number of marked points over all the levels is m. This moduli space is compact with respect to the topology in [BEH$^+$03].

2.2.9. *Parametric Moduli Spaces.* Let P be a compact manifold with boundary and let J_P be a P-parameter family of compactible complex structures on X, consisting of a compatible complex structure J_p for each $p \in P$.

There is a *parametric moduli space* of J_p-holomorphic curves ranging over all p, namely,

$$\mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) := \bigcup_{p \in P} \{p\} \times \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)$$

That is, a point in this moduli space is a pair consisting of a point $p \in P$ and a J_p-holomorphic curve (with marked points). Likewise, there is a compactified moduli space of buildings

$$\overline{\mathcal{M}}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) := \bigcup_{p \in P} \{p\} \times \overline{\mathcal{M}}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-).$$

These parametric moduli spaces inherit evaluation maps ev and an additional continuous projection map

$$\pi : \overline{\mathcal{M}}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) \to P, \qquad (p, u) \mapsto p$$

given by projection to the P-factor. We will consider parametric moduli spaces of buildings with marked points in §4.4.

2.2.10. *Generic Tranversality.* Let P be a compact manifold with boundary ∂P and let $J_P := \{J_p\}_{p \in P}$ be a P-family of compatible almost complex structures on X. We now briefly review some generic transversality results that are standard in the literature on SFT.

Recall that a point $(p, [u, s_1, \ldots, s_m])$ in the moduli space

$$\mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)$$

is called *parametrically regular* or *parametrically transverse* if the parametric linearized operator $D_{u,p}$, incorporating both variations in the map u and variations in the parameter space P, is surjective.

Proposition 2.4. *(cf. [Wen15, Thm. 7.1 and Rmk. 7.4]) The set of parametrically regular points*

$$\mathcal{M}_{\mathrm{reg}} \subset \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)$$

is an open set and a smooth orbifold of dimension

$$\dim(\mathcal{M}_{\mathrm{reg}}) = \mathrm{vdim}(\mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)) := \mathrm{ind}(A, g) + \dim(P) + 2m.$$

The local isotropy group at an orbifold point $(s, [u, s_1, \ldots, s_m]) \in \mathcal{M}_{\mathrm{reg}}$ is given by

$$\mathrm{Aut}(p, u, s_1, \ldots, _m) := \{\phi : \Sigma \to \Sigma \ : \ j \circ d\phi = d\phi \circ j, u \circ \phi = u, u(s_i) = s_i\}.$$

Finally, the evaluation map ev *and projection map* π *are both smooth on* $\mathcal{M}_{\mathrm{reg}}$.

As a special case, an unparametrized J-holomorphic curve $[u, s_1, \ldots, s_m])$ is simply called *regular* if it is parametrically regular with respect to the 0-parameter family J.

Given a compact, closed submanifold $Z \subset \hat{X}^m$, a parametrically regular $(p, [u_1, \ldots, u_m])$ is *parametrically Z-regular* if the evaluation map

$$\mathrm{ev} : \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) \to \hat{X}^m$$

from the parametric moduli space is transverse to Z at u.

Proposition 2.5. *There is a comeager set $\mathcal{J}_{\mathrm{reg}}(X, P)$ of P-families of compatible almost complex structures J_P such that the space of somewhere injective curves*

$$\mathcal{M}^{\mathrm{i}}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) := \{u \in \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma) \ : \ u \text{ is somewhere injective}\}$$

consists of (parametrically) Z-regular curves.

The parametric regularity part of this result (without accounting for the evaluation map) is proven in [Wen15, Thm. 7.1-7.2, Rmk. 7.4]. The transversality of the evaluation map is proven in [Wen15, §4.6] for the case of closed curves in a closed symplectic manifold X. The approach used in [Wen15, §4.6] is a standard one, using the Sard-Smale theorem, and can be adapted to the symplectization case with minimal modifications. Mainly, we need to work in the appropriate analytic set up, for example, by working with weighted Sobolev spaces instead of Sobolev spaces.

2.2.11. *Buildings in Contact Homology.* We primarily consider holomorphic buildings arising in contact homology. These are genus 0 buildings with a single positive end in an exact cobordism (X, λ) or in (the symplectization of) a contact manifold Y. These are curves in the moduli spaces

(2.3) $$\overline{\mathcal{M}}_{0,A}(X, J; \gamma, \Gamma) \qquad \text{and} \qquad \overline{\mathcal{M}}_{0,A}(Y, J; \gamma, \Gamma).$$

Lemma 2.6. *Let $\bar{u}$ be a J-holomorphic building in one of the moduli spaces (2.3). Then, each level u_i is a disjoint union of curves*

$$v : \Sigma \to \hat{Y}_{\pm} \qquad or \qquad v : \Sigma \to \hat{X}$$

where Σ is connected and of genus 0, and v has exactly one positive puncture.

Proof. Let $\bar{u} = (u_1, \dots, u_m)$ be a holomorphic building in (the symplectization of) Y. The case of buildings in a cobordism is similar. Let $\bar{u}_k$ denote the building

$$\bar{u}_k = (u_1, \dots, u_k) \qquad \text{for} \qquad 1 \leqslant k \leqslant m.$$

We prove by induction on k that each building $\bar{u}_k$ is connected and that each component curve v satisfies the conclusion of the lemma.

For the base case, note that every component v of every level u_i is genus 0, since otherwise the entire building would have positive genus. Moreover, since $\hat{X}$ and $\hat{Y}$ are both exact, every non-constant finite energy holomorphic curve v must have at least one positive puncture. Thus, the top level $\bar{u}_1 = u_1$ is a connected genus 0 curve with one positive puncture.

For the induction case, assume that $\bar{u}_k$ satisfies the induction hypothesis. By the above reasoning, each component v of the level u_{k+1} is genus 0 with at least one positive puncture. This positive puncture must connect to a negative puncture of $\bar{u}_k$ so that $\bar{u}_{k+1}$ is connected. If a component v has more than one positive puncture, then attaching v to the connected building $\bar{u}_k$ contributes genus to $\bar{u}_{k+1}$. Therefore, v has exactly one puncture. □

Remark 2.7. In [Par15], a slightly different compactification of $\mathcal{M} = \mathcal{M}_{g,A}(X, J; \Gamma_+, \Gamma_-)$ is used (and similarly for $\mathcal{M}_{g,A}(Y, J; \Gamma_+, \Gamma_-)$). In Pardon's compactification, if $u_i : \Sigma_i \to \hat{Y}_{\pm}$ is a symplectization level of a building $\bar{u}$ and Σ_i breaks into disconnected components $\Sigma_{i,1}, \dots, \Sigma_{i,k}$, then each component $u_i|_{\Sigma_{i,j}}$ is separately regarded as a holomorphic curve modulo translation and any trivial cylinder components are eliminated.

The differences between the BEHWZ compactification of [BEH$^+$03] and the Pardon compactification [Par15] will not be important for this paper. In particular, we will treat them as equivalent in Constructions 2.8 and 2.9 below.

However, we do note that any BEHWZ building corresponds to a unique Pardon building by only remembering the constituent maps of the building on each connected component of the levels (and eliminating the trivial components of every level). Conversely, any Pardon building can be lifted to a BEHWZ building by adding trivial levels and grouping the connected components of the building appropriately. In particular

$$\mathcal{M} = \overline{\mathcal{M}} \text{ in the BEHWZ topology} \qquad \Longleftrightarrow \qquad \mathcal{M} = \overline{\mathcal{M}} \text{ in the Pardon topology.}$$

2.3. **Basic Formalism.** We can now discuss the basic construction of the contact dg-algebra of a contact manifold and the cobordism map of an exact cobordism. This construction was introduced in [EGH00]. Here we discuss the specific foundational setup of Pardon [Par15].

Construction 2.8. The *contact dga* of a closed contact manifold Y with a non-degenerate contact form α, denoted by

$$(A(Y, \alpha), \partial_{J,\theta}) \qquad \text{or more simply} \qquad A(Y)$$

is the filtered dg-algebra formulated as follows. Associate a generator x_γ to each good Reeb orbit γ (see [Par15, Definition 2.49] for a definition). Each generator x_γ is given a standard SFT grading and action filtration

$$(2.4) \qquad |x_\gamma| = -\,\mathrm{CZ}(\gamma) - n + 3 \mod 2 \quad \text{and} \quad \mathcal{A}(x_\gamma) = \int_\gamma \alpha, \qquad \text{respectively.}$$

The algebra $A(Y, \alpha)$ is the graded-symmetric algebra freely generated by these generators,

$$(2.5) \qquad A(Y, \alpha) := \mathrm{Sym}_\bullet \big[x_\gamma \ : \ \gamma \text{ is a good orbit} \big].$$

The SFT-grading $|\cdot|$ and the action filtration $\mathcal{A}$ on $x_\Gamma = x_{\gamma_1} \dots x_{\gamma_k}$ are given by

$$|x_\Gamma| := \sum_i |\gamma_i| \qquad \text{and} \qquad \mathcal{A}(x_\Gamma) := \sum_i \mathcal{A}(\gamma_i).$$

We let $A^L(Y, \alpha) \subset A(Y, \alpha)$ denote the graded subspace given by

$$A^L(Y, \alpha) := \mathbb{Q}\langle x_\Gamma \ : \ \mathcal{A}(x_\Gamma) \leqslant L \rangle$$

The differential on $A(Y, \alpha)$ is the unique derivation such that, for any good orbit γ, we have

$$\partial_{J,\theta}(x_\gamma) := \sum_{A,\Gamma} \frac{\#_\theta \overline{\mathcal{M}}_{0,A}(Y, J; \gamma, \Gamma)}{\mu_\Gamma \cdot \kappa_\Gamma} \cdot x_\Gamma.$$

Here, $\#_\theta \overline{\mathcal{M}}_{0,A}(Y, J; \gamma, \Gamma)$ is a (virtual) point count of index 1 holomorphic buildings in the symplectization of Y with one positive puncture at γ and negative punctures at Γ (see [Par15]). This count depends on a choice of the VFC data θ. The sum is over all ordered lists Γ of good orbits and all homology classes $A \in S(Y; \gamma, \Gamma)$ such that $\mathrm{ind}(A, 0) = 1$.

Construction 2.9. The *cobordism map* of an exact cobordism $X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)$, denoted

$$\Phi_{X,\lambda,J,\theta} : A(Y_+, \alpha_+) \to A(Y_-, \alpha_-), \qquad \text{or more simply} \quad \Phi_X,$$

is the unique filtered dg-algebra map such that, for any good closed orbit γ of Y_+, we have

$$\Phi_{X,\lambda,J,\theta}(x_\gamma) := \sum_{A,\Gamma} \frac{\#_\theta \overline{\mathcal{M}}_{0,A}(X, J; \gamma, \Gamma)}{\mu_\Gamma \cdot \kappa_\Gamma} \cdot x_\Gamma.$$

Here $\#\overline{\mathcal{M}}_{0,A}(X, J; \gamma, \Gamma)$ is a (virtual) point count of index 0 holomorphic buildings in the completion of X with 1 positive puncture at γ and negative punctures at Γ (see [Par15]), and the sum is over all ordered lists Γ of good orbits in Y_- and all homology classes $A \in S(X; \gamma, \Gamma)$ such that $\mathrm{ind}(A, 0) = 0$.

Remark 2.10. In both Constructions 2.8 and 2.9, the virtual point count is equal to an actual (oriented) point count when the relevant moduli space $\mathcal{M}$ is compact (i.e. $\overline{\mathcal{M}} = \mathcal{M}$) and each holomorphic curve is transversely cut out (i.e. the relevant linearized $\overline{\partial}$ operator is surjective). See [Par15, Thm. 1.1(iv)].

The main results of Pardon's construction [Par15] can now be summarized as follows.

Theorem 2.11. *[Par15] Let (Y, α) and $X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)$ be as in Construction 2.8-2.9. Then the following hold.*

(a) *The map $\partial_{J,\theta} : A(Y) \to A(Y)$ is a filtered differential. That is,*

$$\partial_{J,\theta}^2 = 0 \qquad and \qquad \mathcal{A}(\partial_{J,\theta}(x)) \leqslant \mathcal{A}(x).$$

(b) *The map $\Phi_{X,\lambda,J,\theta} : A(Y_+) \to A(Y_-)$ is a filtered chain map. That is,*

$$\Phi_{X,\lambda,J,\theta} \circ \partial_{J_+,\theta_+} = \partial_{J_-,\theta_-} \circ \Phi_{X,\lambda,J,\theta} \qquad and \qquad \mathcal{A}(\Phi_{X,\lambda,J,\theta}(x)) \leqslant \mathcal{A}(x).$$

Furthermore, $\Phi_{X,\lambda,J,\theta}$ is independent of J and θ up to filtered chain homotopy.

(c) *The composition of cobordism maps is filtered homotopic to the cobordism map of the composition,*

$$\Phi_{X \circ X'} \simeq \Phi_X \circ \Phi_{X'}.$$

(d) *If (X, λ) is the trivial cobordism $X = [a, b] \times Y, \lambda = e^t \alpha$, J is a translation invariant almost complex structure induced by a compatible complex structure on ξ, and θ is any VFC data, then*

$$\Phi_{X,\lambda,J,\theta} : H(A(Y), \partial_{J,\theta}) \to H(A(Y), \partial_{J,\theta})$$

is the identity map on the level of unfiltered graded dg-algebras.

By Theorem 2.11, we can now define contact homology as an invariant of contact manifolds.

Definition 2.12. The *contact homology $CH(Y)$* of a closed contact manifold (Y, ξ) is given by

$$CH(Y) := H(A(Y), \partial_{J,\theta}), \qquad \text{for any choice of } \alpha, J, \theta.$$

The map $\Phi_X : CH(Y_+) \to CH(Y_-)$ induced by an exact cobordism $X : Y_+ \to Y_-$ is similarly defined with respect to any choice of J, θ. Any choice of contact form α induces a filtration of $CH(Y)$ by sub-spaces

$$CH^L(Y) \quad \text{or} \quad CH^L(Y, \alpha) \qquad \text{for any} \qquad L \in [0, \infty)$$

Here $CH^L(Y, \alpha)$ is the image of the map

$$H(A^L(Y)) \to H(A(Y)) = CH(Y)$$

if the contact form α is non-degenerate. In general, the $CH^L(Y)$ is defined as the colimit

$$CH^L(Y, \alpha) = \text{colim}_\beta CH^L(Y, \beta)$$

where the colimit is taken over all non-degenerate contact forms $\beta = f\alpha$ with $f > 1$ pointwise. The cobordism maps Φ_X are filtered with respect to this filtration.

Remark 2.13. More generally, given a contact form α on Y that is non-degenerate below action T and any $L < T$, we can still define the graded algebra

$$A^L(Y) := \mathbb{Q}\langle x_\Gamma \; : \; \mathcal{A}(\Gamma) \leqslant L \rangle.$$

We may equip this algebra with a differential $\partial_{J,\theta}$ given a choice of compatible complex structure and VFC data. Likewise, if $X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)$ are non-degenerate below action T, then we have a cobordism dg-algebra map

$$\Phi_X : A^L(Y_+) \to A^L(Y_-)$$

for any $L \leqslant T$, well-defined up to filtered chain homotopy. In particular, $CH^L(Y, \alpha)$ is the image of the map

$$H(A^L(Y, \alpha)) \to CH(Y)$$

Moreover, for any contact forms α_+ and α_- that are non-degenerate up to action T, the following diagram commutes.

(2.6)
$$
\begin{array}{ccc}
H(A^L(Y_+)) & \xrightarrow{\Phi_X} & H(A^L(Y_-)) \\
\Phi \downarrow & & \downarrow \Phi \\
CH^L(Y_+) & \xrightarrow[\Phi_X]{} & CH^L(Y_-)
\end{array}
\quad .
$$

2.4. **The Tight Sphere.** We now calculate the contact homology algebra of the standard sphere, which is a key example in later constructions.

Consider $\mathbb{C}^n$ equipped with the standard Liouville form and associated Liouville vector-field

$$
\lambda_{\text{std}} = \frac{1}{2} \sum_i x_i dy_i - y_i dx_i, \text{ and} \qquad Z_{\text{std}} = \frac{1}{2} \sum_i x_i \partial_{x_i} + y_i \partial_{y_i}.
$$

Definition 2.14. A *star-shaped domain* $W \subset \mathbb{C}^n$ is an embedded Liouville sub-domain of $\mathbb{C}^n$. Equivalently, W is a codimension zero submanifold with smooth boundary that is transverse to the Liouville vector field Z, $Z \pitchfork \partial W$.

Every pair of star-shaped domains W and W' are equivalent through a canonical deformation, i.e., by deforming the boundary along the radial direction. In particular, the contact boundaries are all contactomorphic to the standard tight sphere.

Definition 2.15. The *standard tight sphere* $(S^{2n-1}, \xi_{\text{std}})$ is the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ equipped with the contact structure $\xi_{\text{std}} = \ker(\lambda_{\text{std}}|_{S^{2n-1}})$.

If $W \subset W'$ is an inclusion of star-shaped domains, then the exact symplectic cobordism

$$
X : (\partial W', \lambda|_{\partial W'}) \to (\partial W, \lambda|_{\partial W}) \qquad \text{given by} \qquad X := W' \backslash W
$$

is isomorphic (as an exact cobordism) to a cylindrical cobordism $X : (S^{2n-1}, \alpha) \to (S^{2n-1}, \beta)$ for a pair of contact forms α and β on (S^{2n-1}, ξ).

Example 2.16. Choose a sequence of rationally independent, positive real numbers

$$
0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n.
$$

The *standard ellipsoid* $E = E(a_1, \ldots, a_n) \subset \mathbb{C}^n$ is the star-shaped domain given by

$$
E(a_1, \ldots, a_n) = \{z \in \mathbb{C}^n \; : \; \sum_i \frac{\pi}{a_i} \cdot |z_i|^2 \leqslant 1\}.
$$

The Reeb dynamics on ∂E is very explicit and easy to determine (cf. [GH18, §2.1]). Specifically, there are exactly n non-degenerate, simple, closed Reeb orbits given by

$$
\gamma_i = \partial E \cap \mathbb{C}_i \qquad \text{for} \qquad i = 1, \ldots, n.
$$

Here $\mathbb{C}_i \subset \mathbb{C}^n$ is the ith complex axis in $\mathbb{C}^n$. Every Reeb orbit γ is an iterate of one of these orbits. The action and Conley-Zehnder index of an orbit $\gamma = \gamma_i^j$ is given by

$$
\mathcal{A}(\gamma) = j \cdot a_i \qquad \text{and} \qquad CZ(\gamma) = n - 1 + 2 \cdot \#\{\text{orbits } \eta \text{ of } \partial E \; : \; \mathcal{A}(\eta) \leqslant \mathcal{A}(\gamma)\}.
$$

Note that the Conley-Zehnder index is well-defined without reference to a trivialization, since $c_1(S^3, \xi_{\text{std}}) = 0$ and $\pi_1(S^3) = 0$.

Lemma 2.17. *The contact homology algebra $CH(S^{2n-1}, \xi_{\text{std}})$ is isomorphic to a graded-symmetric algebra freely generated by generators x_k of grading $-2n + 2 - 2(k-1)$ for each $k \geqslant 1$, namely,*

$$
CH(S^{2n-1}, \xi_{\text{std}}) \simeq \text{Sym}_\bullet [x_k \; ; \; k \geqslant 1].
$$

Proof. Consider S^{2n-1} equipped with the contact form induced as the boundary of an irrational ellipsoid $E(a_1, \ldots, a_n)$. The (cohomological) SFT grading of a closed Reeb orbit γ is given by

$$|\gamma| = -2n + 2 - 2 \cdot \#\{\text{orbits } \eta \text{ of } \partial E \; : \; \mathcal{A}(\eta) \leqslant \mathcal{A}(\gamma)\}.$$

In particular, the SFT grading is even for all generators of $A(\partial E)$ and the differential is trivial. □

Remark 2.18. Note that the isomorphism in Lemma 2.17 is not claimed to be canonical.

In §5, we will require the following property of cobordism maps given by inclusion.

Lemma 2.19. *Let $E(a) \subset E(b)$ be two ellipsoids and consider the map $\Phi : A(\partial E(a)) \to A(\partial E(b))$ corresponding to the cobordism $E(b) \backslash E(a)$. Then, Φ is an isomorphism on the chain level and its inverse is word-length non-decreasing.*

Proof. The exact cobordism $E(b) \backslash E(a)$ is isomorphic to a cylindrical cobordism, and so the map

$$\Phi : A(\partial E(a)) \to A(\partial E(b))$$

is a quasi-isomorphism inducing the natural isomorphism on homology. Since the differentials of $A(\partial E(a))$ and $A(\partial E(b))$ are trivial, Φ is in fact an isomorphism of dg-algebras. Moreover, by Theorem 2.11(c)-(d), the inverse Φ^{-1} is the cobordism map induced by $E(b) \backslash E(c \cdot a)$ for any $c > 0$ sufficiently small.

It remains to show that cobordism maps between ellipsoids are word-length non-decreasing. Since cobordism maps are algebra maps, this is equivalent to having a non-constant generator being mapped to the constants. In the case of the ellipsoids this is impossible, since the constants have grading zero, the non-constant generators lie in positive degrees and the cylindrical cobordism maps preserve the $\mathbb{Z}$-grading. □

3. SPECTRAL GAPS

In this section, we discuss contact homology spectral gaps and related structures, including abstract constraints, constrained cobordism maps and spectral invariants.

3.1. **Abstract Constraints.** An abstract constraint provides a purely homological tool for tracking the ways in which a holomorphic curve can be tangent to (or asymptotic to) a set of points in a symplectic cobordism. Rigorously, we have the following definition.

Definition 3.1. An *abstract constraint P in contact homology* with m points, dimension n and codimension $\text{codim}(P) = k$ is a degree k map

$$P : \bigotimes_{i=1}^{m} CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}[k],$$

or equivalently, a cohomology class $P \in CH(\cup_1^m S^{2n-1})^\vee$ of grading $\text{codim}(P) = k$.

Example 3.2 (Empty Constraint). The *empty constraint $P_\varnothing$* is the codimension 0 map given by

$$P_\varnothing(1) = 1 \qquad \text{and} \qquad P_\varnothing(x) = 0 \quad \text{if} \quad |x| > 0.$$

Alternatively, $P_\varnothing$ is the unique $\mathbb{Z}$-graded algebra map $CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}$.

Example 3.3 (Tangency Constraints). It follows from Lemma 2.17 that

$$\ker(P_\varnothing)/\ker(P_\varnothing)^2 \simeq \bigoplus_{k=0}^{\infty} \mathbb{Q}[2n - 2 + 2k].$$

In particular, there is (up to multiplication by a non-zero constant) a unique surjective map

$$\Pi_k : \ker(P_\varnothing)/\ker(P_\varnothing)^2 \to \mathbb{Q}[2n - 2 + 2k] \qquad \text{for each } k \geqslant 0.$$

Therefore, we have an abstract constraint (well-defined up to multiplication by a constant)

$$P_k : CH(S^{2n-1}, \xi_{std}) = \mathbb{Q} \oplus \ker(P_\varnothing) \to \ker(P_\varnothing) \to \ker(P_\varnothing)/\ker(P_\varnothing)^2 \xrightarrow{\Pi_k} \mathbb{Q}[2n-2+2k].$$

Remark 3.4. As observed by Siegel [Sie19, §5.5], the constraint P_k coincides with the map acquired by counting genus 0 holomorphic curves in a star-shaped domain, with one positive puncture, passing through a point p and tangent to a local divisor D through p to order k.

In §5, we will use a version of this fact in the proof of our main results. We also give an alternate argument for a weak version of this correspondence (for certain curves counted in constrained cobordism maps and counts of cylinders passing through a point) in §4.4.

Example 3.5 (Dual Constraints)**.** Let E be an irrational ellipsoid and let Ξ be an orbit tuple on ∂E. There is a dual abstract constraint

$$P_\Xi : CH(\partial E) \simeq CH(S^{2n-1}, \xi_{std}) \to \mathbb{Q}[|\Xi|]$$

determined by Ξ. This is defined in the usual way, with

$$P_\Xi(x_\Gamma) = 1 \text{ if } \Gamma = \Xi \qquad \text{and} \qquad P_\Xi(x_\Gamma) = 0 \qquad \text{otherwise}$$

As a special case, the abstract constraint P_k in Example 3.3 coincides with P_γ where γ is the unique closed orbit of E with $|\gamma| = 2n - 2 + 2k$.

3.2. **Constrained Cobordism Maps.** By using abstract constraints, we can formulate a generalization of the cobordism maps in contact homology, which morally counts curves satisfying a number of tangency constraints.

Definition 3.6. Let $X : Y_+ \to Y_-$ be a connected exact cobordism and let P be an abstract constraint. The *P-constrained cobordism map*

$$\Phi_{X,P} : A(Y_+) \to A(Y_-)[\mathrm{codim}(P)]$$

is the filtered chain map of degree $\mathrm{codim}(P)$ constructed by the following procedure. Choose a star-shaped domain W_i for $i = 1, \ldots, m$ and an embedding

$$\iota : W = W_1 \cup \cdots \cup W_m \to \mathrm{int}(X).$$

Since W_i are star-shaped domains, ι is automatically weakly exact (see §2.2.1). Thus we may assume (after deformation) that ι is exact and $X \backslash W$ is an exact cobordism. Also, choose a cochain $\pi_{\partial W, P}$ representing P, i.e., a chain map

$$\pi_{\partial W, P} : A(\partial W) \simeq A(\cup_1^m S^{2n-1}) \to \mathbb{Q}[\mathrm{codim}(P)]$$

which is in the cohomology class P. Then, we define $\Phi_{X,P}$ to be the composition

$$A(Y_+) \xrightarrow{\Phi_{X \backslash W}} A(Y_- \cup \partial W) = A(Y_-) \otimes A(\partial W) \xrightarrow{\mathrm{Id} \otimes \pi_{\partial W, P}} A(Y_-) \otimes \mathbb{Q}[\mathrm{codim}(P)].$$

Definition 3.7. Let P be an abstract tangency constraint and Y be a connected, closed contact manifold. The *U-map*

$$U_P : A(Y) \to A(Y)[\mathrm{codim}(P)].$$

is the P-constrained cobordism map $\Phi_{X,P}$ where $X = [0,1] \times Y$ is the trivial cobordism.

Remark 3.8. The constrained cobordism maps $\Phi_{X,P}$ are *not* dg-algebra maps in general. However, some constrained cobordism maps are compatible with the algebra structure in other ways (see Lemma 3.12).

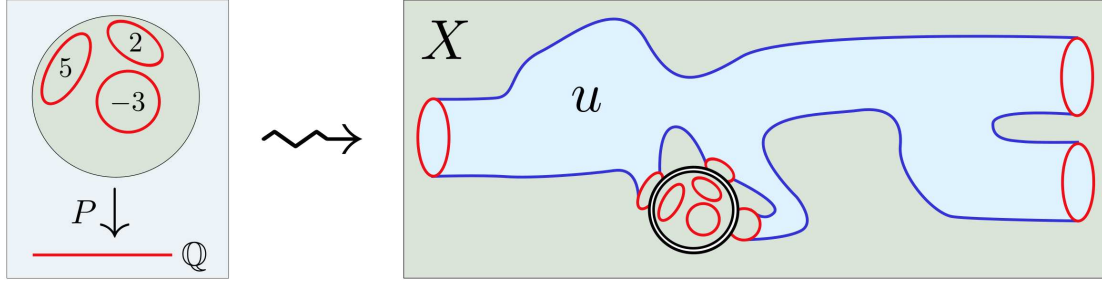**Lemma 3.9.** *The cobordism maps $\Phi_{X,P}$ are well-defined up to filtered chain homotopy.*

FIGURE 5. A cartoon of (a) an abstract constraint P, visualized as a weighting of the orbits on the sphere and (b) the curves counted in constrained cobordism maps, asymptotic to the orbits with non-zero weights under P.

Proof. Choose a disjoint union $W = W_1 \cup \cdots \cup W_m$ of star-shaped domains U_i, an embedding $W \subset X$, Floer data (J, θ) on $X \backslash W$ and a cochain representative $\pi_{\partial W, P}$ of P. We adopt the notation

$$\Phi_{X,P,W,J,\theta} := (\mathrm{Id}_{Y_-} \otimes \pi_{\partial W, P}) \circ \Phi_{X \backslash W, J, \theta}.$$

Since chain homotopy is a closed relation under composition, $\Phi_{X,P,W,J,\theta}$ is independent of the choice of $\pi_{\partial W, P}$ up to filtered chain homotopy. Note that we are viewing $\pi_{\partial W, P}$ as a filtered map by equipping $\mathbb{Q}[\mathrm{codim}(P)]$ with the trivial filtration.

To show independence of W and (J, θ) up to filtered homotopy, we start by considering two special cases and then move on to address the general case.

Case 1. Let $\iota : [0, 1] \times W \to X$ be a family of symplectic embeddings. This family ι induces a family of exact symplectic cobordisms

$$X \backslash \iota_t(W) : Y_+ \to Y_- \cup \partial W$$

By Theorem 2.11(b), the induced cobordism maps of $X \backslash \iota_0(W)$ and $X \backslash \iota_1(W)$ are homotopic (for any choices of Floer data). Since filtered homotopy is a closed relation under composition with filtered chain maps, we see that

$$\Phi_{X,P,\iota_0(W),J,\theta} = (\mathrm{Id}_{Y_-} \otimes \pi_{\partial W, P}) \circ \Phi_{X \backslash \iota_0(W), J, \theta} \simeq (\mathrm{Id}_{Y_-} \otimes \pi_{\partial W, P}) \circ \Phi_{X \backslash \iota_1(W), J', \theta'} = \Phi_{X,P,\iota_1(W),J',\theta'}$$

Case 2. Let $V = V_1 \cup \cdots \cup V_m$ be a collection of star-shaped domains with inclusions $V_i \subset W_i$ that are strictly exact, i.e., that intertwine the Liouville forms. Let $W \backslash V : \partial W \to \partial V$ be the difference cobordism. Consider the cobordism map

$$\Phi_{W \backslash V, I, \varphi} : A(\partial W) \to A(\partial V)$$

for some choice of Floer data (I, φ) on $W \backslash V$. We may choose the cochains $\pi_{\partial W, P}$ and $\pi_{\partial V, P}$ so that

$$\pi_{\partial W, P} \simeq \pi_{\partial V, P} \circ \Phi_{W, I, \varphi}.$$

On the other hand, by Theorem 2.11(c), we have

$$\Phi_{X \backslash V, J, \theta} \simeq \Phi_{(Y_- \times [0,1]) \cup W \backslash V} \circ \Phi_{X \backslash W, J', \theta'} = (\mathrm{Id}_{Y_-} \otimes \Phi_{W \backslash V, I, \varphi}) \circ \Phi_{X \backslash W, J', \theta'}$$

for appropriate choices of Floer data (J, θ) on $X \backslash V$ and (J', θ') on $X \backslash W$. Therefore,

$$\begin{aligned}
\Phi_{X,P,V,J',\theta'} &= (\mathrm{Id}_{Y_-} \otimes \pi_{\partial V, P}) \circ \Phi_{X \backslash V, J', \theta'} \\
&\simeq (\mathrm{Id}_{Y_-} \otimes \pi_{\partial V, P}) \circ (\mathrm{Id}_{Y_-} \otimes \Phi_{W \backslash V, I, \varphi}) \circ \Phi_{X \backslash W, J', \theta'} \\
&\simeq (\mathrm{Id}_{Y_-} \otimes \pi_{\partial W, P}) \circ \Phi_{X \backslash W, J, \theta} = \Phi_{X,P,W,J,\theta}.
\end{aligned}$$

General Case. Let $W = W_1 \cup \cdots \cup W_m$ and $V = V_1 \cup \cdots \cup V_m$ be any two choices of m disjoint star-shaped domains in X. We may choose star-shaped domains E_i (e.g. sufficiently small ellipsods) that include into $E_i \subset W_i$ and $E_i \subset V_i$, and such that the embeddings

$$E_i \to W_i \to X \qquad \text{and} \qquad E_i \to V_i \to X$$

are homotopic through a homotopy of embeddings $\iota : [0,1] \times E_i \to X$. Thus, by Cases 1 and 2, $\Phi_{X,P,W,J,\theta}$ and $\Phi_{X,P,V,J',\theta'}$ are filtered chain homotopic (for any Floer data). $\square$

As mentioned above, an abstract constraint P is an assignment of numerical weights to the Reeb orbits on the boundary of a star-shaped domain W. Constrained cobordism maps are acquired by deleting W from a cobordism X and counting curves with ends on ∂W with non-zero weights.

Let us make this intuition precise in a specific case. Fix a connected exact cobordism $X : Y_+ \to Y_-$ between contact manifolds with non-degenerate contact forms. Let $E \subset X$ be an embedded irrational ellipsoid and let Ξ be an orbit tuple in ∂E. Consider the exact cobordism

$$X \backslash E : Y_+ \to Y_- \cup \partial E$$

Finally, fix an orbit γ of Y_+ and an orbit tuple Γ of Y_-. The following lemma is immediate from Construction 2.9, Remark 2.10 and Definition 3.6.

Lemma 3.10. *Let P_Ξ be the dual constraint to Ξ (see Example 3.5). Let J be a compatible almost complex structure on J such that*

$$\mathcal{M}_{0,A}(X \backslash E; \gamma, \Gamma \cup \Xi)$$

is regular and compact in the BEHWZ topology for each homology class $A \in S(X \backslash E; \gamma, \Gamma \cup \Xi)$. Then the x_Γ-coefficient of $\Phi_{X,P_\Xi}(x_\gamma)$ is given by

$$\langle x_\Gamma, \Phi_{X,P_\Xi}(x_\gamma) \rangle := \frac{1}{\mu_\Gamma \cdot \kappa_\Gamma} \cdot \sum_A \#\mathcal{M}_{0,A}(X \backslash E, J; \gamma, \Gamma \cup \Xi)$$

Here the sum is over all classes with $\mathrm{ind}(0, A) = 0$ and $\#$ denotes an oriented point count.

Constrained cobordism maps satisfy a number of useful (and expected) axioms presented in the following lemma.

Lemma 3.11. *The constrained cobordism maps $\Phi_{X,P}$ satisfy the following properties.*

A. *(Functoriality) If $X : Y \to Y'$ and $X' : Y' \to Y''$ are two exact cobordisms, and P and Q are two tangency constraints, then*

$$\Phi_{X \circ X', P \otimes Q} \simeq \Phi_{X,P} \circ \Phi_{X',Q} \qquad \text{and} \qquad U_Q \circ \Phi_{X,P} = \Phi_{X,P \otimes Q} = \Phi_{X,P} \circ U_Q.$$

B. *(Additivity) If $X : Y \to Y'$ is an exact cobordism, and P and Q are two tangency constraints of the same dimension, then*

$$\Phi_{X,P+Q} \simeq \Phi_{X,P} + \Phi_{X,Q}.$$

C. *(Empty Constraint) Let $P_\varnothing$ be the empty constraint. Then*

$$\Phi_{X,P_\varnothing} = \Phi_X \qquad \text{and} \qquad U_{P_\varnothing} = \mathrm{Id}.$$

Proof. It suffices to prove these properties for $\Phi_{X,P}$, as the U-maps are a special case.

Axiom A. Choose collections of disjoint star-shaped domains $W = W_1 \cup \cdots \cup W_m \subset X$ and $V = V_1 \cup \cdots \cup V_n \subset X'$. Up to deformation of exact cobordisms, we may write

$$(X \circ X') \backslash (W \cup V) = \big((X \backslash W) \cup ([0,1] \times \partial V)\big) \circ (X' \backslash V).$$

Therefore, by Theorem 2.11(c) we have (up to filtered chain homotopy) the equivalence

$$\Phi_{(X \circ X') \backslash (W \cup V)} \simeq \Phi_{(X \backslash W) \cup (\partial V \times [0,1])} \circ \Phi_{X' \backslash V} \simeq (\Phi_{X \backslash W} \otimes \mathrm{Id}_{\partial V}) \circ \Phi_{X' \backslash V}.$$

We may choose the cochains representing P, Q and $P \otimes Q$ to satisfy

$$(3.1) \qquad \qquad \pi_{\partial(W \cup V), P \otimes Q} = \pi_{\partial W, P} \otimes \pi_{\partial V, Q}.$$

This implies the desired composition property, by the following calculation.

$$\begin{aligned}
\Phi_{X \circ X', P \cup Q} &= (\mathrm{Id}_{Y''} \otimes \pi_{\partial(W \cup V), P \cup Q}) \circ \Phi_{(X \circ X') \setminus (W \cup V)} \\
&\simeq \big((\mathrm{Id}_{Y''} \otimes \pi_{\partial W, P}) \otimes \pi_{\partial V, Q}\big) \circ (\Phi_{X \setminus W} \otimes \mathrm{Id}_{\partial V}) \circ \Phi_{X' \setminus V} \\
&= (\mathrm{Id}_{Y''} \otimes \pi_{\partial W, P}) \circ \Phi_{X \setminus W} \circ (\mathrm{Id}_{Y'} \otimes \pi_{\partial V.Q}) \circ \Phi_{X' \setminus V} = \Phi_{X, P} \circ \Phi_{X', Q}.
\end{aligned}$$

Axiom B. Choose a collection of disjoint star-shaped domains $W = W_1 \cup \cdots \cup W_m \subset X$. We may choose the cochain representatives of P, Q, and $P + Q$ so that

$$\pi_{\partial W, P+Q} = \pi_{\partial W, P} + \pi_{\partial W, Q}.$$

Thus, we can calculate that

$$\begin{aligned}
\Phi_{X, P+Q} &= (\mathrm{Id}_{Y'} \otimes (\pi_{\partial W, P} + \pi_{\partial W, Q})) \circ \Phi_{X \setminus W} \\
&= (\mathrm{Id}_{Y'} \otimes \pi_{\partial W, P}) \circ \Phi_{X \setminus W} + (\mathrm{Id}_{Y'} \otimes \pi_{\partial W, Q}) \circ \Phi_{X \setminus W} = \Phi_{X, P} + \Phi_{X, Q}.
\end{aligned}$$

Axiom C. Any star-shaped domain W determines a unital, $\mathbb{Z}$-graded dg-algebra map

$$\Phi_W : CH(\partial W) \to CH(\varnothing) = \mathbb{Q}.$$

The cohomology class of this map is unique, and equal to $P_\varnothing$. Therefore,

$$\Phi_{X, P_\varnothing} = (\mathrm{Id} \otimes \Phi_W) \circ \Phi_{X \setminus W} = \Phi_X. \qquad \square$$

The constrained cobordism maps Φ_{X, P_k} with respect to the tangency constraints P_k defined in Example 3.3 satisfy, in addition, a chain level Leibniz rule.

Lemma 3.12. *The constrained cobordism map Φ_{X, P_k} of the tangency constraints P_k satsifies*

$$(3.2) \qquad \Phi_{X, P_k}(xy) = \Phi_{X, P_k}(x) \cdot \Phi_X(y) + \Phi_X(x) \cdot \Phi_{X, P_k}(y).$$

As a special case, the U-maps U_{P_k} satisfy the Leibniz rule.

$$(3.3) \qquad U_{P_k}(xy) = U_{P_k}(x) \cdot y + x \cdot U_{P_k}(y).$$

Proof. Let $W \subset X$ be an embedded irrational ellipsoid. Let y_j denote the generators of $CH(\partial W)$. We first note that the map $\Phi_{X \setminus W}$ in Definition 3.6 satisfies

$$\Phi_{X \setminus W}(z) = \Phi_X(z) \otimes 1 + \sum_{k=0}^{\infty} \Phi_{X, P_k}(z) \otimes y_{k+1} + r$$

where r is a sum of factors $x \otimes y$ where y is a monomial of word length $\geqslant 2$. This follows from Definition 3.6, the fact that P_k is the dual constraint to the $(k + 1)$-th generator y_{k+1} (see Example 3.5) and the fact that $P_\varnothing$ is dual to $1 \in CH(\partial W)$. Since $\Phi_{X \setminus W}$ is an algebra map, we thus have

$$(3.4) \qquad \Phi_{X \setminus W}(zz') = \Phi_X(zz') \otimes 1 + \sum_{k=1}^{\infty} (\Phi_{X, P_k}(z) \Phi_X(z') + \Phi_X(z) \Phi_{X, P_k}(z')) \otimes y_{k+1} + r'$$

where r' is a remainder term of the same form as r. On the otherhand, the map $\Pi = \mathrm{Id} \otimes \pi_{\partial W, P_k}$ from Definition 3.6 is given by

$$(3.5) \qquad \Pi(x \otimes y_{k_1} \ldots y_{k_m}) = \begin{cases} x & \text{if } m = 1, k_1 = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

By Definition 3.6, we have $\Phi_{X, P_k} = \Pi \circ \Phi_{X, W}$. Thus it follows from (3.4) and (3.5) that

$$\Phi_{X, P_k}(zz') = \Pi\big((\Phi_{X, P_k}(z) \Phi_X(z') + \Phi_X(z) \Phi_{X, P_k}(z')) \otimes y_{k+1}\big) = \Phi_{X, P_k}(z) \Phi_X(z') + \Phi_X(z) \Phi_{X, P_k}(z')$$

which is the desired Leibniz rule. $\qquad \square$

3.3. **Spectral Invariants.** We now recall the definitions and properties of spectral invariants and capacities in the setting of contact homology.

Definition 3.13. The *contact homology spectral invariant* $\mathfrak{s}_\sigma(Y, \alpha)$ of a closed contact manifold Y with contact form α and a class $\sigma \in CH(Y)$ is given by

(3.6) $$\mathfrak{s}_\sigma(Y, \alpha) := \mathcal{A}(\sigma) = \min\{L \; : \; \sigma \in CH^L(Y) \subset CH(Y)\} \in (0, \infty).$$

Definition 3.14. The *contact homology capacity* $\mathfrak{c}_P(W)$ of a Liouville domain (W, λ) and an abstract constraint P is given by

$$\mathfrak{c}_P(W) := \inf\{\mathfrak{s}_\sigma(\partial W, \lambda|_{\partial W}) \; : \; \Phi_{W,P}(\sigma) \neq 0\}.$$

Note that here we are viewing W as an exact cobordism from ∂W to $\varnothing$.

Remark 3.15. If (Y, α) is a closed contact manifold where α is non-degenerate, (3.6) is equivalent to the minimum action of a cycle in the dg-algebra $A(Y)$ representing σ,

$$\mathfrak{s}_\sigma(Y, \alpha) = \min\{\mathcal{A}(x) \; : \; x \in A(Y) \text{ satisfying } \partial x = 0 \text{ and } [x] = \sigma\}.$$

Theorem 3.16. *The contact homology spectral invariants $\mathfrak{s}_\sigma$ satisfy the following properties.*

    A. *(Conformality) If (Y, α) is a contact manifold with contact form α and $a > 0$ is a constant, then*

$$\mathfrak{s}_\sigma(Y, a \cdot \alpha) = a \cdot \mathfrak{s}_\sigma(Y, \alpha).$$

    B. *(Cobordism Map) If $X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)$ is an exact symplectic cobordism, P is an abstract constraint and $W \subset X$ is a (weakly) exactly embedded Liouville domain, then*

$$\mathfrak{s}_{\Phi_{X,P}(\sigma)}(Y_-, \alpha_-) + \mathfrak{c}_P(W) \leqslant \mathfrak{s}_\sigma(Y_+, \alpha_+).$$

    C. *(U-Map) If (Y, α) is a contact manifold with contact form α and P is an abstract constraint, then*

$$\mathfrak{s}_{U_P(\sigma)}(Y, \alpha) \leqslant \mathfrak{s}_\sigma(Y, \alpha).$$

    D. *(Monotonicity) Let $f : Y \to [0, \infty)$ be a smooth non-negative function on Y. Then,*

$$\mathfrak{s}_\sigma(Y, \alpha) \leqslant \mathfrak{s}_\sigma(Y, e^f \cdot \alpha).$$

    Moreover, $\mathfrak{s}_\sigma(Y, -)$ is continuous in the C^0-topology on contact forms.

    E. *(Reeb Orbits) For each class $\sigma \in CH(Y)$, there is a Reeb orbit tuple Γ such that*

$$\mathfrak{s}_\sigma(Y, \alpha) = \mathcal{A}(\Gamma).$$

    Furthermore, if α is non-degenerate and σ has grading $|\sigma|$, then

$$|\Gamma| = |\sigma|.$$

Proof. We demonstrate each of these axioms individually.

    Axiom A. This follows immediately from the definition and the fact that the contact homology groups of $CH(Y)$ with respect to α and $a \cdot \alpha$ are canonically identified with action filtrations differing by the scaling factor a.

    Axiom B. By the functoriality property of constrained cobordism maps stated in Lemma 3.11, $\Phi_{X,P}$ can be written as the composition

$$CH(Y_+) \xrightarrow{\Phi_{X\backslash W}} CH(Y_-) \otimes CH(\partial W) \xrightarrow{\text{Id} \otimes \Phi_{W,P}} CH(Y_-) \otimes CH(\varnothing)[\text{codim}(P)] = CH(Y_-)[\text{codim}(P)].$$

Choose basis x_i of $CH(Y_-)$ of pure action filtration, that is, x_i is a linear combination of generators which have the same action. Then, for some set of elements $y_i \in CH(\partial W)$, all but finitely many of which vanish), we may write

$$\Phi_{X\backslash W}(\sigma) = \sum_i x_i \otimes y_i \qquad \text{and} \qquad \Phi_{X,P}(\sigma) = \sum_i \Phi_{W,P}(y_i) \cdot x_i.$$

Since x_i is a basis of pure filtration, we know that

$$(3.7) \qquad \mathcal{A}(\Phi_{X\setminus W}(\sigma)) = \mathcal{A}\Big(\sum_i x_i \otimes y_i\Big) = \max\{\mathcal{A}(x_i \otimes y_i) \ : \ y_i \neq 0\}.$$

$$(3.8) \qquad \mathcal{A}(\Phi_{X,P}(\sigma)) = \mathcal{A}\left(\sum_i \Phi_{W,P}(y_i) \cdot x_i\right) = \max\{\mathcal{A}(x_i) \ : \ \Phi_{W,P}(y_i) \neq 0\}.$$

Let m be the index such that $\mathcal{A}(x_m) = \mathcal{A}(\Phi_{X,P}(\sigma))$ and $\Phi_{W,P}(y_m) \neq 0$. By Definitions 3.13 and 3.14, we have

$$(3.9) \qquad \mathfrak{s}_{\Phi_{X,P}(\sigma)}(Y_-, \alpha_-) = \mathcal{A}(\Phi_{X,P}(\sigma)) = \mathcal{A}(x_m) \qquad \text{and} \qquad c_P(W) \leqslant \mathcal{A}(y_m).$$

On the other hand, by (3.7) and the monotonicity of the action filtration under cobordism maps, we know that

$$(3.10) \qquad \mathcal{A}(x_m) + \mathcal{A}(y_m) = \mathcal{A}(x_m \otimes y_m) \leqslant \mathcal{A}(\Phi_{X\setminus W}(\sigma)) \leqslant \mathcal{A}(\sigma) = \mathfrak{s}_\sigma(Y_+, \alpha_+).$$

The constrained monotonicity property follows immediately from (3.9) and (3.10).

Axiom C. Fix $\epsilon > 0$ and consider the cobordism $X : (Y, e^\epsilon \cdot \alpha) \to (Y, \alpha)$ given by $X = [0, \epsilon]_r \times Y$ equipped with the standard Liouville form $\lambda = e^r \alpha$. The U-map U_P is the constrained map

$$\Phi_{X,P} : CH(Y) \to CH(Y).$$

By the usual monotonicity and scaling axioms, we therefore know that

$$\mathfrak{s}_{U_P(\sigma)}(Y, \alpha) \leqslant \lim_{\epsilon \to 0} \mathfrak{s}_\sigma(Y, e^\epsilon \cdot \alpha) = \mathfrak{s}_\sigma(Y, \alpha).$$

Axiom D. To prove monotonicity, assume that $f : Y \to [0, \infty)$ and choose $\epsilon > 0$. Consider the cobordism $X : (Y, e^f \cdot \alpha) \to (Y, e^{-\epsilon} \cdot \alpha)$ given by

$$X := \{(r, y) \ : \ 0 \leqslant r \leqslant f(y)\} \subset \mathbb{R}_r \times Y \quad \text{with Liouville form} \quad \lambda = e^r \lambda$$

The cobordism map and scaling axioms imply that $\epsilon^{-1} \cdot \mathfrak{s}_\sigma(Y, \alpha) \leqslant \mathfrak{s}_\sigma(Y, e^f \cdot \alpha)$. Thus, we take $\epsilon \to 0$ to acquire the monotonicity inequality.

To deduce continuity, let $\alpha_i = f_i \cdot \alpha$ be a sequence of contact forms that C^0 converges to α. Then, $f_i \xrightarrow{C^0} 1$ and so there exists a sequence of constants $C_i > 0$ such that

$$C_i \to 1 \text{ as } i \to \infty \qquad \text{and} \qquad C_i > f_i > C_i^{-1}.$$

By the cobordism map and scaling axioms, we see that

$$C_i^{-1} \cdot \mathfrak{s}_\sigma(Y, \alpha) = \mathfrak{s}_\sigma(Y, C_i^{-1} \cdot \alpha) \leqslant \mathfrak{s}_\sigma(Y, f_i \cdot \alpha) \leqslant \mathfrak{s}_\sigma(Y, C_i \cdot \alpha) = C_i \cdot \mathfrak{s}_\sigma(Y, \alpha).$$

By taking the limit as $i \to \infty$, we see that $\mathfrak{s}_\sigma(Y, \alpha_i) \to \mathfrak{s}_\sigma(Y, \alpha)$.

Axiom E. Assume that (Y, α) is non-degenerate and let $\sigma \in CH(Y)$. Then there exists a cycle x representing σ such that

$$x = \sum_\Gamma c_\Gamma \cdot x_\Gamma \qquad \text{with} \qquad \mathcal{A}(x) = \max\{x_\Gamma \ : \ c_\Gamma \neq 0\} = \mathfrak{s}_\sigma(Y, \alpha).$$

If σ has pure homological grading $|\sigma|$, then we can assume that $c_\Gamma = 0$ for $|\Gamma| \neq |\sigma|$. Let Γ be the maximal action orbit tuple with $c_\Gamma \neq 0$, then

$$\mathfrak{s}_\sigma(Y, \alpha) = \mathcal{A}(\Gamma) \qquad \text{and} \qquad |\Gamma| = |\sigma|$$

as desired.

If α is degenerate, we can take a sequence of non-degenerate contact forms α_i that C^∞ converges to α. Then, the corresponding orbit tuples Γ_i have bounded total action and thus converge to an orbit tuple Γ of α as $i \to \infty$. This convergence can be seen via arguments similar to those in

the Proof of Theorem 13 (3.4.1). Since $\mathfrak{s}_\sigma$ is continuous in the C^∞-topology on contact forms, this implies that

$$\mathfrak{s}_\sigma(Y,\alpha) = \lim_{i\to\infty} \mathfrak{s}_\sigma(Y,\alpha_i) = \lim_{i\to\infty} \mathcal{A}(\Gamma_i) = \mathcal{A}(\Gamma). \qquad \square$$

Theorem 3.17. *The contact homology capacities $\mathfrak{c}_P$ satisfy the following properties.*

   A. *(Conformality) If (W,λ) is a Liouville domain and $a > 0$ is a constant, then*

$$\mathfrak{c}_P(W, a\cdot\lambda) = a\cdot\mathfrak{c}_P(W,\lambda).$$

   B. *(Monotonicity) If $(W,\lambda) \to (V,\mu)$ is a (weakly) exact embedding of Liouville domains, then*

$$\mathfrak{c}_P(W,\lambda) \leqslant \mathfrak{c}_P(V,\mu).$$

   C. *(Tensor Product) If P and Q are two abstract constraints, then*

$$\mathfrak{c}_P(W,\lambda) \leqslant \mathfrak{c}_{P\otimes Q}(W,\lambda).$$

   D. *(Reeb Orbits) If P is an abstract constraint, then there is a tuple of Reeb orbits Γ of ∂W such that*

$$\mathfrak{c}_P(W,\lambda) = \mathcal{A}(\Gamma).$$

   Furthermore, if $\lambda|_{\partial W}$ is non-degenerate, then

$$|\Gamma| = \operatorname{codim}(P) \mod 2m \qquad where \qquad m := \min|c_1(W)\cdot A|.$$

Proof. Axioms A, B, and D are proven by approaches that are essentially identical to the analogous properties (respectively A, D and E) in Theorem 3.16. For Axiom C, we note that

$$\Phi_{W,P\otimes Q} = \Phi_{W,P} \circ U_Q.$$

Therefore, if $\Phi_{W,P\otimes Q}(\sigma) \neq 0$, we have $\Phi_{W,P}(U_Q(\sigma)) \neq 0$. We thus acquire the inequality

$$c_P(W,\lambda) \leqslant \min\{\mathcal{A}(\tau) \,:\, \Phi_{W,P}(\tau) \neq 0\} \leqslant \min\{\mathcal{A}(U_Q(\sigma)) \,:\, \Phi_{W,P\otimes Q}(\sigma) \neq 0\}$$

$$\leqslant \min\{\mathcal{A}(\sigma) \,:\, \Phi_{W,P\otimes Q}(\sigma) \neq 0\} = c_{P\otimes Q}(W,\lambda). \qquad \square$$

3.4. **Spectral Gap.** We are now ready to introduce the contact homology spectral gap.

Definition 3.18. Let Y be a closed contact manifold with contact form α and let $\sigma \in CH(Y)$ be a contact homology class. The *spectral gap* of (Y,α) in class σ is defined to be

$$(3.11) \qquad \operatorname{gap}_\sigma(Y,\alpha) := \inf\left\{ \frac{\mathfrak{s}_\sigma(Y,\alpha) - \mathfrak{s}_{U_P(\sigma)}(Y,\alpha)}{\mathfrak{c}_P(B^{2n})} \,:\, P \text{ is an abstract constraint}\right\}.$$

The *(total) spectral gap* of the contact manifold (Y,α) is given by

$$(3.12) \qquad \operatorname{gap}(Y,\alpha) := \inf_{\sigma\in CH(Y)} \operatorname{gap}_\sigma(Y,\alpha).$$

Theorem 3.19 (Theorem 3). *Let (Y,α) be a closed contact manifold with contact form such that*

$$gap(Y,\alpha) = 0.$$

Then, (Y,α) satisfies the strong closing property, namely, for every non-zero $f : Y \to [0,\infty)$ there exists $t \in [0,1]$ such that $(1 + tf)\alpha$ has a closed Reeb orbit passing through the support of f. Moreover, if $gap_\sigma(Y,\alpha) = 0$ for some $\sigma \in CH(Y)$, then the period of this orbit is bounded by $\mathfrak{s}_\sigma(Y,\alpha)$.

Proof. Let $f : Y \to [0,\infty)$ and fix $L \in [0,\infty]$. Assume, if possible, that for all $t \in [0,1]$, the contact form $(1 + tf)\alpha$ does not have a periodic Reeb orbit of action up to L through the support of f. In this case, the action spectrum of $(1 + tf)\alpha$ up to L remains the same as t varies. Note that the action spectrum of α is a measure zero set. As observed in [Iri15, Lemma 2.2], this is a consequence of the fact that the critical values of the contact action functional are contained in the critical values of a smooth function on a finite dimensional manifold, which can be constructed

by adapting the proof of [Sch00, Lemma 3.8] from the Hamiltonian setting. So, the continuity (in t) of the spectral invariants, stated in Theorem 3.16, guarantees that

$$\mathfrak{s}_\sigma(Y, (1 + tf)\alpha) = \mathfrak{s}_\sigma(Y, \alpha), \tag{3.13}$$

for all $t \in [0, 1]$ and $\sigma \in CH(Y)$ such that $\mathfrak{s}_\sigma(Y, \alpha) \leqslant T$. We will show that the cobordism property gives a positive lower bound for the spectral gap of such contact homology classes.

Fix $\varepsilon > 0$ small and let X be the cobordism from $(Y, e^\varepsilon(1 + f)\alpha)$ to (Y, α) given by

$$X := [0, \varepsilon] \times Y \, \cup \, \{(\varepsilon + \log(1 + t \cdot f(y)), y) : t \in [0, 1]\} \subset \hat{Y}.$$

There exists a number $r = r(f, Y, \alpha)$ such that the ball $B^{2n}(r)$ of radius r embeds into X. By the cobordism property of spectral invariants stated in Theorem 3.16, for any homology class $\sigma \in CH(Y)$ and an abstract constraint P it holds that

$$\mathfrak{s}_\sigma(Y, e^\varepsilon(1 + f)\alpha) - \mathfrak{s}_{\Phi_{X,P}(\sigma)}(Y, \alpha) \geqslant \mathfrak{c}_P(B^{2n}(r)).$$

By the conformality property of the capacities stated in Theorem 3.17, we have $\mathfrak{c}_P(B^{2n}(r)) = r^2 \cdot \mathfrak{c}_P(B^{2n})$. Rearranging the above inequality we obtain

$$r^2 \leqslant \frac{\mathfrak{s}_\sigma(Y, e^\varepsilon(1 + f)\alpha) - \mathfrak{s}_{\Phi_{X,P}(\sigma)}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})}. \tag{3.14}$$

We will show that the latter lower bound contradicts the vanishing of the spectral gap. Let $X' := [0, \varepsilon/2] \times Y$ be a trivial cobordism and decompose X as $X = X' \cup (X \backslash X')$. By the functoriality property of the constrained cobordism map we have

$$\Phi_{X,P} \cong \Phi_{X',P} \circ \Phi_{X \backslash X'} = U_P \circ \Phi_{X \backslash X'}.$$

Combining this with inequality (3.14) and equation (3.13), and using again the properties of spectral invariants, we conclude that for every homology class $\sigma \in CH(Y)$ such that $\mathfrak{s}_\sigma(Y, \alpha) \leqslant T$,

$$r^2 \leqslant \frac{\mathfrak{s}_\sigma(Y, e^\varepsilon(1 + f)\alpha) - \mathfrak{s}_{U_P \circ \Phi_{X \backslash X'}\sigma}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})} = \frac{e^\varepsilon \mathfrak{s}_\sigma(Y, (1 + f)\alpha) - \mathfrak{s}_{U_P\sigma}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})}$$

$$\overset{(3.13)}{=} \frac{e^\varepsilon \mathfrak{s}_\sigma(Y, \alpha) - \mathfrak{s}_{U_P\sigma}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})} \leqslant \frac{\mathfrak{s}_\sigma(Y, \alpha) - \mathfrak{s}_{U_P\sigma}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})} + \frac{r^2}{2},$$

where the last inequality holds when we take $\varepsilon \leqslant \log\left(1 + \frac{r^2}{2\mathfrak{s}_\sigma(Y,\alpha)}\right)$. Applying this estimate for every abstract constraint P, we get a positive lower bound for the spectral gap,

$$\frac{r^2}{2} \leqslant \inf_P \frac{\mathfrak{s}_\sigma(Y, \alpha) - \mathfrak{s}_{U_P}(Y, \alpha)}{\mathfrak{c}_P(B^{2n})} = \mathrm{gap}_\sigma(Y, \alpha). \tag{3.15}$$

To prove the first assertion of the theorem, take $T = \infty$. Then the above lower bound for $\mathrm{gap}_\sigma(Y, \alpha)$ for all σ implies that $\mathrm{gap}(Y, \alpha)$ is positive, in contradiction with the hypothesis. To prove the second part of the theorem, suppose $\mathrm{gap}_\sigma(Y, \alpha) = 0$ for some $\sigma \in CH(Y)$ and take $T = \mathfrak{s}_\sigma(Y, \alpha)$. Then (3.15) yields a contradiction. □

Remark 3.20 (Semi-continuity of the spectral gap)**.** The spectral gap is upper semi-continuous, that is, for every sequence of contact forms α_i converging to α and for every class $\sigma \in CH(Y)$ we have

$$\limsup_{i \to \infty} \mathrm{gap}_\sigma(Y, \alpha_i) \leqslant \mathrm{gap}_\sigma(Y, \alpha),$$

$$\limsup_{i \to \infty} \mathrm{gap}(Y, \alpha_i) \leqslant \mathrm{gap}(Y, \alpha).$$

This is due to the fact that the spectral gaps are defined as an infimum over continuous functions.

In fact, for a fixed class σ, the σ-spectral gap is continuous. More generally, under some conditions on the rate of convergence of α_i to α, the spectral gap of α can be bounded from above by spectral gaps of α_i. In order to give a more precise statement of this fact we adapt the following notation.

Notation 3.21. Let α and α' be two contact forms on Y. We write $\alpha \leqslant \alpha'$ if there exists an exact cobordism from (Y, α') to (Y, α).

Proposition 3.22. *Let (Y, α) be a contact manifold and suppose there exist sequences $\{\alpha_i\}$ of contact forms, $\{\sigma_i\}$ of contact homology classes, and $\{\epsilon_i\}$ of positive numbers, such that:*

(i) $\alpha_i \leqslant \alpha \leqslant (1 + \epsilon_i)\alpha_i$, and
(ii) $\epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i) \xrightarrow[i \to \infty]{} 0.$

Then, $gap(Y, \alpha) \leqslant \liminf_{i \to \infty} gap_{\sigma_i}(Y, \alpha_i)$. In particular, if $gap_{\sigma_i}(Y, \alpha_i) \to 0$, then $gap(Y, \alpha) = 0$.

Proof. Fix $\delta > 0$. For each i, let P_i be an abstract constraint such that

$$(3.16) \qquad \frac{\mathfrak{s}_{\sigma_i}(Y, \alpha_i) - \mathfrak{s}_{U_{P_i}(\sigma_i)}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})} < gap_{\sigma_i}(Y, \alpha_i) + \delta.$$

By our assumption that $\alpha_i \leqslant \alpha \leqslant (1 + \epsilon_i)\alpha_i$ we have the following commutative diagram of filtered homologies

$$(3.17)$$

$$
\begin{array}{ccc}
CH(Y, (1 + \epsilon_i)\alpha_i) & \xrightarrow{\;U_{P_i}^{(1+\epsilon_i)\alpha_i}\;} & CH(Y, (1 + \epsilon_i)\alpha_i) \\[2mm]
\Phi_1 \downarrow & & \downarrow \Phi_1 \\[2mm]
CH(Y, \alpha) & \xrightarrow{\;U_{P_i}^{\alpha}\;} & CH(Y, \alpha) \\[2mm]
\Phi_2 \downarrow & & \downarrow \Phi_2 \\[2mm]
CH(Y, \alpha_i) & \xrightarrow{\;U_{P_i}^{\alpha_i}\;} & CH(Y, \alpha_i).
\end{array}
$$

Here, in order to distinguish between the U-maps on contact homologies filtered by the contact forms α and α_i, we adapt the notations $U_{P_i}^{\alpha}$ and $U_{P_1}^{\alpha_i}$ respectively. By definition, for $\sigma_i \in CH(Y, (1 + \epsilon_i)\alpha_i)$ we have

$$
gap(Y, \alpha) \leqslant \frac{\mathfrak{s}_{\Phi_1(\sigma_i)}(Y, \alpha) - \mathfrak{s}_{U_{P_i}^{\alpha}\Phi_1(\sigma_i)}(Y, \alpha)}{\mathfrak{c}_{P_i}(B^{2n})} = \frac{\mathfrak{s}_{\Phi_1(\sigma_i)}(Y, \alpha) - \mathfrak{s}_{\Phi_1 U_{P_i}^{(1+\epsilon_i)\alpha_i}(\sigma_i)}(Y, \alpha)}{\mathfrak{c}_{P_i}(B^{2n})}
$$

$$
\leqslant \frac{\mathfrak{s}_{\sigma_i}(Y, (1 + \epsilon_i)\alpha_i) - \mathfrak{s}_{\Phi_2\Phi_1 U_{P_i}^{(1+\epsilon_i)\alpha_i}\sigma_i}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})},
$$

where the last inequality follows from the cobordism property of spectral invariants, stated in Theorem 3.16. By Theorem 2.11, the composition $\Phi_2\Phi_1 : CH(Y, (1+\epsilon_i)\alpha_i) \to CH(Y, \alpha_i)$ of maps between the contact homologies is the indentity map with filtration rescaled by $1/(1 + \epsilon_i)$. Using

the conformality property of spectral invariants (Theorem 3.16) we obtain

$$
\begin{aligned}
\mathrm{gap}(Y, \alpha) &\leqslant \frac{\mathfrak{s}_{\sigma_i}(Y, (1 + \epsilon_i)\alpha_i) - \mathfrak{s}_{U_{P_i}^{\alpha_i}\sigma_i}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})} = \frac{(1 + \epsilon_i)\mathfrak{s}_{\sigma_i}(Y, \alpha_i) - \mathfrak{s}_{U_{P_i}^{\alpha_i}\sigma_i}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})} \\
&= \left(\frac{\mathfrak{s}_{\sigma_i}(Y, \alpha_i) - \mathfrak{s}_{U_{P_i}^{\alpha_i}\sigma_i}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})} \right) + \frac{\epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i)}{\mathfrak{c}_{P_i}(B^{2n})} \\
&\leqslant \mathrm{gap}_{\sigma_i}(Y, \alpha_i) + \delta + c\epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i),
\end{aligned}
$$

where the last inequality follows from our choice of the abstract constraint P_i, and the fact that the capacities $\mathfrak{c}_{P_i}(B^{2n})$ have a positive lower bound which is uniform in i. Since our δ can be taken to be arbitrarily small, we conclude that if the product $\epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i)$ converges to zero, then $\mathrm{gap}(Y, \alpha) \leqslant \liminf_{i \to \infty} \mathrm{gap}_{\sigma_i}(Y, \alpha_i)$. □

3.4.1. *Detecting periodicity via spectral gaps.* Theorem 13 from the introduction states that if there exists a homology class $\sigma \in CH(Y, \xi)$ such that $\mathrm{gap}_\sigma(Y, \alpha) = 0$, then the Reeb flow of α is periodic. The following proof is similar to an argument from ECH that was pointed out to us by Oliver Edtmair.

Proof of Theorem 13. A classical theorem of Wadsley [Wad75] implies that if all Reeb orbits are closed then the flow is periodic. Therefore, it is sufficient to show that there is a periodic orbit passing through every point of Y.

Fix a point $y \in Y$, let $\{V_n\}_{n=1}^\infty$ be open sets in Y such that $\cap_n V_n = \{y\}$. Let $f_n : Y \to [0, \infty)$ be a smooth non-zero function supported in V_n, such that $\|f_n\|_{C^3} \leqslant \frac{1}{n}$. Recall our assumption that $\mathrm{gap}_\sigma(Y, \alpha) = 0$ and denote $T := \mathfrak{s}_\sigma(Y, \alpha)$. By Theorem 3.19, there exists $t_n \in [0, 1]$ such that the contact form $\alpha_n := (1 + t_n f_n)\alpha$ has a periodic orbit γ_n of period $T_n \leqslant T$ passing through the support of f_n. By extracting a subsequence we may assume that T_n converge to some $T_\star \leqslant T$. We think of γ_n as a map $\mathbb{R} \to Y$, and denote $\underline{\gamma}_n(-) := \gamma_n(T_n \cdot -) : S^1 \cong \mathbb{R}/\mathbb{Z} \to Y$. Denote by R_{α_n} and R_α the Reeb vector fields of α_n and α, respectively. The sequence $\underline{\gamma}_n$ is equicontinuous since the derivatives $\dot{\underline{\gamma}}_n = T_n \cdot R_{\alpha_n}$ are uniformly bounded. Therefore, we can apply Arzelà-Ascoli theorem and conclude that there is a subsequence $\{\underline{\gamma}_{n_k}\}_k$ that converges to a limit $\underline{\gamma} : S^1 \to Y$. Clearly, $\underline{\gamma}$ passes through y since $\underline{\gamma}_n$ passes through V_n. Let us show that $\underline{\gamma}$ is differentiable and that its derivative is $T_\star \cdot R_\alpha$. Fix $t \in S^1$ and consider a chart in Y around $\underline{\gamma}(t)$. For large enough n, $\underline{\gamma}_n(t)$ lies in this chart and satisfies

$$
\underline{\gamma}_n(t + \varepsilon) - \underline{\gamma}_n(t) = \int_0^\varepsilon T_n \cdot R_{\alpha_n} \circ \underline{\gamma}_n(t + \tau)d\tau,
$$

for small enough ε. Taking the limit when $n \to \infty$ over this equation, we obtain

$$
\underline{\gamma}(t + \varepsilon) - \underline{\gamma}(t) = \int_0^\varepsilon T_\star \cdot R_\alpha \circ \underline{\gamma}(t + \tau)d\tau
$$

for every $\varepsilon > 0$ small enough. Therefore, $\underline{\gamma}$ is indeed differentiable, its derivative is $T_\star \cdot R_\alpha$, and its reparamterization $\gamma(t) := \underline{\gamma}(t/T_\star)$ is a periodic orbit of R_α that passes through y, as required. □

4. ANALYSIS OF THE CONSTRAINED MODULI SPACES

In this section, we analyze the compactified moduli spaces of holomorphic cylinders between select orbits of a Morse-Bott contact form. We prove that, under some conditions, these moduli spaces are cut-out transversely and consist of a single point. Before formally stating the main result for this section, let us fix the setting and present the relevant definitions and notations.

Setup 4.1. Throughout this section we fix the following structural assumptions.

(i) A contact manifold (Y, ξ) of dimension $2n - 1$, for $n \geqslant 3$.

(ii) A contact form α on Y such that the flow φ_t of the corresponding Reeb vector field R is periodic.

(iii) An almost complex structure J on ξ and a Riemannian metric $g_J(-, -) = \alpha \otimes \alpha + d\alpha(-, J-)$ on Y.

(iv) A Morse-Bott function $f : Y \to \mathbb{R}$ that is φ_t-invariant and whose critical manifolds are 1-dimensional. Note that the φ_t-invariance of f implies that the critical manifolds are disjoint unions of periodic Reeb orbits. For such f, we let:
   - X_f be the vector field defined by $\alpha(X_f) \equiv 0$ and $d\alpha(-, X_f) = df(-)$; and
   - $\nabla_J f$ be the vector field defined by $g_J(-, \nabla_J f) = df(-)$. Note that $J\nabla_J f = -X_f$.

(v) Given f as above and $\varepsilon > 0$ we define perturbations of α, R and J as follows.
   - The perturbation of α is $\alpha_f := e^{\varepsilon f} \alpha$.
   - The Reeb vector field of α_f is $R_f = e^{-\varepsilon f}(R - \varepsilon X_f)$.
   - Let J_f be the $\mathbb{R}$-invariant almost complex structure on $\hat{Y} = \mathbb{R}_r \times Y$ satisfying:
     * $J_f(\partial_r) = R_f$,
     * J_f preserves the bundle ξ, and
     * the restriction of J_f to ξ is equal to J.

(vi) Let γ_+ and γ_- be circles of global maxima and minima of f, respectively. Considering $\gamma_\pm$ as (not necessarily simple) Reeb orbits of α, assume that their periods are the same and coincide[2] with the minimal period of the flow φ_t.

(vii) Let $z \in Y$ be a point in the intersection of the unstable manifold of γ_+ and the stable manifold of γ_-.

(viii) Consider a sequence of nested manifolds $Y_3 \subset \cdots \subset Y_{2n-1} = Y$, such that for each j:
   (a) $\dim(Y_{2j-1}) = 2j - 1$ and $H_2(Y_{2j-1}) = 0$;
   (b) Y_{2j-1} is invariant under the Reeb flow φ_t;
   (c) $(Y_{2j-1}, \xi \cap TY_{2j-1})$ is a contact manifold and $\alpha|_{TY_{2j-1}}$ is a contact form for it. Moreover, $\xi \cap TY_{2j-1}$ is J-invariant;
   (d) $\nabla_J f$ is tangent to Y_{2j-1} and $f|_{Y_{2j-1}}$ is Morse-Bott. In particular, Y_{2j-1} is invariant under the gradient flow of f;
   (e) Along any critical circle γ of f, other than γ_+ which lies in Y_{2n-3}, the restriction of the Hessian of f to the symplectic orthogonal of $\xi \cap TY_{2j-1}$ in $\xi \cap TY_{2j+1}$ is positive definite[3];
   (f) $z \in Y_3 \subset \cdots \subset Y_{2n-1}$.
   We call such a sequence $\{Y_{2j-1}\}_{j=2}^n$ a *contact flag*. This is illustrated in Figure 4.

Example 4.2. The example of interest of this setting is when $Y = \partial E$ is the boundary of an ellipsoid with rationally dependent entries. The contact flag is given by intersecting ∂E with complex linear subspaces: $Y_{2j-1} = \partial E \cap (\mathbb{C}^{j-1} \times \{0\}^{n-j} \times \mathbb{C})$. In Section 5.1 we study this example in detail and explain how it fits into Setup 4.1.

Remark 4.3. Assumption (viii).(a) that $H_2(Y_{2j-1}) = 0$ has two purposes:
   - To ensure that the relative CZ index is well defined over $\mathbb{Z}$,
   - To guarantee that the homological intersection of any 2-torus in Y_{2j+1} with the hypersurface Y_{2j-1} is zero (see Lemma 4.16 and Figure 8).

---

[2]For the arguments in Sections 4.1-4.3, it is enough to assume that the periods of $\gamma_\pm$ are the same and *divide* the minimal period of the flow φ_t. The assumption that the periods $\gamma_\pm$ coincide with the minimal period of the flow is used only in Section 4.4 for regularity purposes. See Remark 5.8

[3]In particular, this means that γ_+ is the only maximum of f in Y_{2n-3}. This assumption will be used for computing certain Conley-Zehnder indices of an f-perturbation of the Reeb flow.
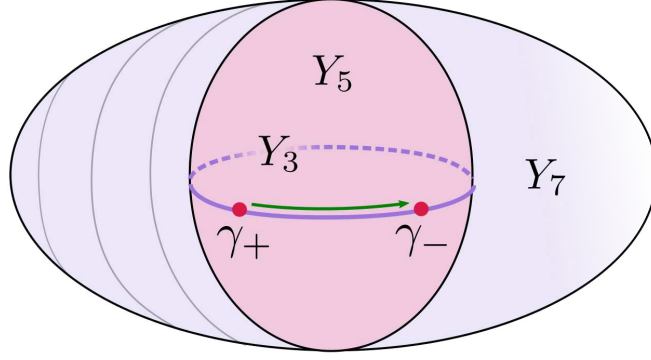
FIGURE 6. An illustration of a contact flag, as defined in Setup 4.1. The lavender sphere Y_7, the pink disk Y_5 and the purple circle Y_3 represent consecutive nested sub-manifolds in the flag. The orbits γ_+ and γ_- lie in Y_3, and are connected by a Morse flow line in green.

As a consequence, assumption (viii).(a) can be replaced with any weaker conditions that ensure the two above properties. This is done in Section 6.

The main purpose of this section is to prove the following claim.

Proposition 4.4. *Consider Setup 4.1. The compactified moduli space[4]*

$$\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$$

of J_f-holomorphic cylinders in $\hat{Y}$ between γ_- and γ_+ containing $(0, z)$ consists of a single point which is transversely cut-out.

Remark 4.5. The single point in the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is a certain lift of the gradient flow line of f from γ_- to γ_+ that passes through z. This lift is described in section 4.1 below.

Remark 4.6.
- For any periodic flow, the fixed point set at any time T is a submanifold satisfying the Morse–Bott condition. This is due to the fact that N any periodic diffeomorphism ψ is an isometry for the metric $\hat{g} := \frac{1}{N} \sum_{j=1}^{N} \psi^* g$, where g is any Riemannian metric.
- The contact form $\alpha_f := e^{\varepsilon f} \alpha$ is non-degenerate in a finite action window when ε is small enough. That is, fixing $T_\star > 0$, there exists $\varepsilon > 0$ such that every periodic orbit of α_f with action less than $T_\star$ is non-degenerate. These periodic orbits lie over the critical circles of f, and thus correspond to pairs (S, p) of a Morse–Bott family S of orbits of α and a circle p of f-critical points. The Conley-Zehnder index of such an orbit can be calculated from the Robbin-Salamon index of the family S, its dimension, and a Morse-type index of p, as shown in the following lemma.

Lemma 4.7. *Fix a trivialization τ of the contact structure on a neighborhood of the perturbed orbit $\gamma = (S, p)$ corresponding to a pair of a Morse–Bott family S of orbits of α and a circle p of f-critical points. The Conley-Zehnder index of $\gamma = (S, p)$ with respect to τ is given by*

$$(4.1) \qquad \mathrm{CZ}_\tau(S, p) = \mathrm{RS}_\tau(S) + \frac{1}{2} \dim(S) - \mathrm{ind}_{Morse}^{S}(f; p),$$

where $\mathrm{ind}_{Morse}^{S}(f; p)$ is the number of negative eigenvalues of the restriction of the Hessian of f to the tangent space of the image of S in Y.

---

[4]See section 2.2.11 for the definition of this moduli space.

Proof. To see this, write the linearized Reeb flow of α_f as a composition

$$d\varphi_t^f = (d\varphi_t^f \circ d\varphi_t^{-1}) \circ d\varphi_t.$$

As explained in [Gut14], the composition is homotopic to the concatenation of paths. By the additivity and homotopy invariance of the Conley-Zehnder index (Proposition 2.2), the index of the orbit (S, p) with respect to α_f is equal to the sum of the Robbin-Salomon index of the family S with respect to α and the CZ-index of the path $d\phi_t := (d\varphi_t^{-1} \circ d\varphi_t^f) \circ d\varphi_T$, where T is the period of S. Let us compute the latter. First, notice that for every $t \leqslant T$, the image $\phi_t(\gamma(0))$ lies in a small neighborhood of the point $\varphi_T(\gamma(0))$, since the flows φ_t and φ_t^f differ by a small time reparametrization on critical circles of f. Identifying a neighborhood of $\varphi_T \gamma(0)$ with its Darboux chart, the path $d\phi_t$ solves the ODE $\dot{d\phi_t} = J_0 \operatorname{Hess}(\varepsilon f)_{\gamma(0)}|_\xi d\phi_t$. When ε is sufficiently small, the path $\Phi(t)$ crosses the Maslov cycle only at the origin and the crossing form is $\Gamma(\gamma(0)) := -\operatorname{Hess}(\varepsilon f)_{\gamma(0)}|_{\ker(d\phi_0 - \operatorname{Id})}$. The kernel of $d\phi_0 - \operatorname{Id} = d\varphi_T - \operatorname{Id}$ coincides with the tangent space to the image of S in Y by our assumption that S is a Morse–Bott family. Since the Hessian of f is degenerate only in the Reeb direction, the signature of this crossing form is by definition the number of positive eigenvalues minus the number of negative eigenvalues, and hence coincides $\dim(S) - 2\operatorname{ind}_{Morse}^S(f; p)$. This shows that (4.1) holds. $\square$

The next lemma shows that the parity of the CH-grading coincides with the parity of the Morse indices of the perturbing function f. We will use it to conclude that the differential in CH vanishes.

Lemma 4.8. *The contact homology grading of a pair (S, p) of a Morse–Bott family S of orbits of α and a circle p of f-critical points satisfies:*

$$(4.2) \qquad\qquad |(S, p)| := n - 3 + CZ(S, p) = \operatorname{ind}_{Morse}^S(f; p) \mod 2.$$

Proof. We start by showing that the dimension of the family S is even. Indeed, let T be the period of the family S and denote by $N_T \subset Y$ the fixed-point set of φ_T, or equivalently, the submanifold composed of the orbits in S. Its tangent space is

$$TN_T = \langle R \rangle \oplus \ker(d\varphi_T - \operatorname{Id}).$$

Therefore, $\dim(S) = \dim(N_T) - 1 = \dim(\ker(d\phi_T - \operatorname{Id}))$, which is even, since it is the 1-eigenspace of a linear symplectic map.

Set $d := \frac{1}{2}\dim(S) \in \mathbb{N}$. We will show that

$$(4.3) \qquad\qquad RS(S) = n - d - 1 \mod 2.$$

Together with the formula for $CZ(S, p)$ given in Lemma 4.7, this yields the required result. Our first step towards (4.3) is to notice that the parity of the Robbin-Salamon index of a path of matrices depends only on the ends and not on the path it self. This can be seen, for example, from the definition of the Robbin-Salamon index given at (2.2). Given a path with non-degenerate crossings, the index is a sum of the signatures of the crossing forms. These forms are non-degenerate, and are defined over even dimensional spaces, $\ker(d\varphi_t - \operatorname{Id})$. Therefore, the signatures are all even. Since the signatures of the ends of the path are are multiplied by $1/2$, they determine the parity of the index.

So now we focus on $d\varphi_T$ and identify it with a symplectic matrix. We can write it as a $\operatorname{Id}_{\xi \cap TN_T} \oplus \Psi$. By the additivity of the RS index, The Morse–Bott condition implies that Ψ does not have 1 as an eigenvalue. Moreover, there exists N such that $\Psi^N = \operatorname{Id}$, since the Reeb flow on Y is periodic. Let us show that, up to conjugation, the matrix Ψ decomposes to a direct sum of elements of $U(1)$. Fix a compatible metric g, then $\hat{g} := \frac{1}{N}\sum_{k=1}^N (\Psi^k)^* g$ is a Ψ-invariant compatible metric. We conclude that up to a change of basis, Ψ is orthogonal, and hence unitary.

In particular, Ψ is diagonalizable, and after another change of basis can be written as a direct sum of two-dimensional rotations

$$\Psi = \oplus_{j=1}^{n-1-d} e^{i\theta_j}$$

, with $\theta_j \neq 0 \mod 2\pi$ due to the Morse–Bott condition. The RS index of the path $\Psi(t) = \oplus_{j=1}^{n-1-d} e^{i\theta_j t}$ is $\sum_{j=1}^{n-1-d} 1 = n - 1 - d$. Therefore,

$$RS(S) = RS(\Psi(t)) \mod 2$$
$$= n - 1 - d \mod 2.$$

This shows that (4.3) holds and concludes the proof. $\qquad\square$

The rest of this section is dedicated to proving the above proposition. In subsection 4.1 we describe an element of $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$. In subsection 4.2 we prove that there are no other elements in this moduli space. In subsection 4.3 we show that this moduli space is cut-out transversely.

4.1. **Lifting flow lines to holomorphic cylinders.** In this section we lift a gradient flow line of the Morse-Bott function f in Y to a J_f-holomorphic curve in the symplectization $\hat{Y}$. Let $\eta : \mathbb{R} \to Y$ be a gradient flow line of f, that is, $\dot{\eta}(s) = \nabla_J f(\eta(s))$, such that $\lim_{s \to \pm\infty} \eta(s)$ is contained in $\gamma_{\pm}$, respectively. Further, assume that $\eta(0) = z$. Such a gradient flow line exists by our choice of the point z stated in Setup 4.1 and is clearly unique. Denoting by $T > 0$ the common period of $\gamma_{\pm}$ with respect to the contact form α, we can define a J_f holomorphic map $u_\eta : \mathbb{R} \times S^1 \to \hat{Y}$ by

(4.4) $$u_\eta(s, t) := (a(s), \varphi_{T \cdot t}(\eta(\varepsilon T s))) ,$$

where $a(s)$ is defined by the ODE

$$\dot{a}(s) = T \cdot e^{\varepsilon f(\eta(\varepsilon T s))}, \qquad a(0) = 0.$$

Note that u_η is indeed a J_f-holomorphic curve that limits to $\gamma_{\pm}$ at the ends, and contains $(0, z)$ in its image, since $u_\eta(0, 0) = (0, \eta(0))$. Therefore, u_η lies in the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$.

4.2. **Uniqueness.** In this section we show that the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ has no elements other than the lift u_η constructed above.

Proposition 4.9. *Consider Setup 4.1. The only point in the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is the curve u_η constructed in Section 4.1.*

The proof of Proposition 4.9 uses the intersection theories in [Sie11, MS19]. We will show inductively that any element of $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is contained in the symplectizations of all of the submanifolds $Y_3 \subset \cdots \subset Y_{2n-1} = Y$. Then it will follow from a result in [Sie11] that this element is in fact contained in the image of u_η. The structure of this section is as follows. First, we show that the buildings in $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ consist only of cylinders, then we give an overview of the holomorphic intersection theory of Moreno-Siefring, and finally we explain how to use it to show that $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ consists of only one element.

4.2.1. *Ruling out non-cylindrical buildings.* Our first step towards proving that u_η is the unique element in the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is to show that all buildings in $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ consist only of cylinders.

Lemma 4.10. *For $\varepsilon > 0$ sufficiently small, each building in $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ consists solely of J_f-holomorphic cylinders.*

Proof. Fix a building $\overline{u} \in \overline{\mathcal{M}}(\gamma_+, \gamma_-; J_f)$ with $k \geqslant 1$ levels. Let $\Gamma^1, \Gamma^2, \ldots, \Gamma^{k+1}$ be the associated limits and notice that $\Gamma^1 = (\gamma_-)$ and $\Gamma^{k+1} = (\gamma_+)$.

We start by finding a sequence of orbits $\{\gamma^j \in \Gamma^j\}_{j=1}^{k+1}$ of non-decreasing actions:

$$\int_{\gamma^1} \alpha_f \leqslant \int_{\gamma^2} \alpha_f \leqslant \ldots \leqslant \int_{\gamma^{k+1}} \alpha_f.$$

This sequence is constructed inductively as follows. Pick $\gamma^1 = \gamma_-$. Given γ^j, there exists a unique connected J_f-holomorphic curve u^j in $\overline{u}$ such that γ^j is one of its negative ends. Recall that every connected component of a level in $\overline{u}$ has a single positive end, as explained in section 2.2.11. We take γ^{j+1} to be the unique positive end of u^j. See Figure 7 for an illustration.
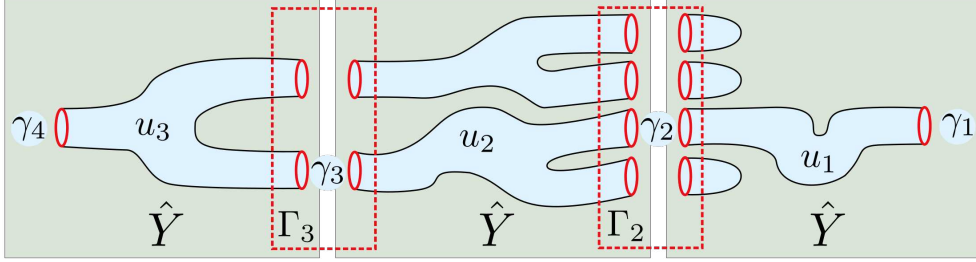

FIGURE 7. An illustration of a holomorphic building $\overline{u}$, and the sequence of orbits $\{\gamma^j\}$.

We will now show that the connected components u^j must be all cylinders. This will conclude the proof, since the positive end of u^{k+1} is γ_+. Let $\gamma_1^j := \gamma^j, \gamma_2^j, \ldots, \gamma_{N_j}^j$ be the negative ends of u^j. We need to show that $N_j = 1$ for all j. For any given punctured J_f-holomorphic curve in $\hat{Y}$, the total integral of α_f over the positive limits is greater than or equal to the total integral of α_f over the negative limits. Applying this to u^j, we find that

(4.5)
$$\int_{\gamma^{j+1}} \alpha_f \geqslant \sum_{i=1}^{N_j} \int_{\gamma_i^j} \alpha_f \geqslant \int_{\gamma^j} \alpha_f + (N_j - 1) \cdot T_{min},$$

where T_{min} is the minimal period of a periodic orbit of α_f. In particular, the α_f-integrals are non-decreasing along the sequence $\{\gamma^j\}$. The α_f integrals of the first and last orbits in this sequence are

$$\int_{\gamma_\pm} \alpha_f = e^{\varepsilon f(\gamma_\pm)} T$$

where $T > 0$ is the common period of $\gamma_\pm$ under the Reeb flow of α. Combining this with (4.5) we find that for all j,

$$e^{\varepsilon f(\gamma_+)} T = \int_{\gamma_+} \alpha_f \geqslant \int_{\gamma^{j+1}} \alpha_f$$

$$\overset{(4.5)}{\geqslant} \int_{\gamma^j} \alpha_f + (N_j - 1) \cdot T_{min}$$

$$\geqslant \int_{\gamma_-} \alpha_f + (N_j - 1) \cdot T_{min}$$

$$= e^{\varepsilon f(\gamma_-)} T + (N_j - 1) \cdot T_{min}.$$

Rearranging the above, we obtain

(4.6)
$$(N_j - 1) \cdot T_{min} \leqslant (e^{\varepsilon f(\gamma_+)} - e^{\varepsilon f(\gamma_-)}) T.$$

The minimal action T_{min} of a Reeb orbit of $\alpha_f = e^{\varepsilon f} \alpha$ is uniformly bounded (in ε) below by a constant depending only on α, as long as ε is taken to be smaller than some constant depending

only on f. This implies that the inequality (4.6) can only hold if $N_j - 1 = 0$ for all $j \in \{1, \ldots, k+1\}$. We thus conclude that u^j are all cylinders, and therefore $\overline{u}$ is composed solely of cylinders as well. □

4.2.2. *An overview of holomorphic intersection theory.* Our main tool in proving the uniqueness result stated in Proposition 4.9 is the holomorphic intersection theory developed in [Sie11, MS19]. We give here an overview of this theory, following [MS19] and adapted to our case.

Let (Y, ξ, α) be a closed contact manifold with a contact form and denote the Reeb vector field by R. Let J be an almost complex structure on $\hat{Y}$ and consider a codimension-2 submanifold Z of $\hat{Y}$ such that:

- there exist closed codim-2 submanifolds $Z_\pm \subset Y$ such that Z is *asymptotically cylindrical* over $Z_\pm$ (see Definition 4.11 below),
- the sub-bundles $\xi_{Z_\pm} := \xi \cap TZ_\pm$ are J-invariant,
- $(Z_\pm, \xi_{Z_\pm})$ are contact manifolds and $\alpha|_{TZ_\pm}$ are contact forms on them, and
- Z is invariant under the flow of R.

The contact structure ξ splits along $Z_\pm$ into a pair of symplectic vector bundles

$$(\xi|_{Z_\pm}, d\alpha) \simeq (\xi_{Z_\pm}, \omega_Z) \oplus (\xi_N, \omega_N)$$

where $\xi_{Z_\pm} = TZ_\pm \cap \xi$ as mentioned above, ξ_N is the symplectic complement of ξ_Z in ξ, and ω_Z, ω_N are the restrictions of $d\alpha$ to ξ_Z, ξ_N respectively.

Definition 4.11. The hypersurface $Z \subset \hat{Y}$ is *asymptotically cylindrical* over $Z_\pm \subset Y$ if there exists s-families of sections of ξ_N, i.e. $\zeta_+ : [R', \infty) \to \Gamma(\xi_N|_{Z_+})$ and $\zeta_- : (-\infty, -R'] \to \Gamma(\xi_N|_{Z_-})$ such that

$$Z \cap ([R', \infty) \times Y) = \bigcup_{(r,p) \in [R,\infty) \times Z_+} \widetilde{\exp}_{(r,p)}(\zeta_+(r)(p)),$$

$$Z \cap ((-\infty, R']) \times Y) = \bigcup_{(r,p) \in (-\infty, R'] \times Z_-} \widetilde{\exp}_{(r,p)}(\zeta_-(r)(p)).$$

w

Remark 4.12. We will consider two simple cases of asymptotically cylindrical hypersurfaces. The first is a symplectization of a contact submanifold, namely $Z := \hat{Z}_+$. The second case is when Y is 3-dimensional and Z is a pseudoholomorphic cylinder. It follows from [MS19, Theorem 2.2] that any pseudoholomorphic cylinder is asymptotically cylindrical over its ends (see also Definition 4.14 below).

Given a J-holomorphic cylinder u in $\hat{Y}$ with non-degenerate ends $\gamma^u_\pm$, the works [MS19, Sie] define a "holomorphic intersection number" $u * Z$ of u with the manifold Z and prove a "positivity of intersection" property (see Theorem 4.15). The holomorphic intersection number is defined as a sum of a "relative intersection number" and a contribution from the Conley-Zehnder index of the ends in the normal contact direction ξ_N.

Definition 4.13 (Holomorphic intersection number, [MS19, Sie]). Let Y, Z, u and $\gamma^u_\pm$ be as above. Fix a trivialization of ξ_N along the orbits of R that lie in Z (if there are any) and collectively denote it by τ.

- (Relative intersection number). Let u^τ be a deformation of u as described in Definition 4.14 below. Roughly speaking, this deformation uses a τ-constant section of ξ_N to push any ends of u that lie in Z off of it. In particular, if the ends of u do not lie in Z then $u^\tau = u$. Define the *relative intersection number* of u with Z to be

(4.7) $$\iota^\tau(u, Z) := u^\tau \cdot Z \in \mathbb{Z},$$

namely, the standard transverse intersection number of the deformation u^τ with the hypersurface Z.

- (Normal Conley–Zehnder index). Let γ be a non-degenerate Reeb orbit which lies in $Z_\pm$. Its *normal Conley–Zehnder index* $CZ_\tau^N(\gamma)$ is the Conley–Zehnder index of the path of 2×2 symplectic matrices defined by applying the trivialization τ to the projection of the linearized Reeb flow to ξ_N. For an orbit γ that does not lie in Z, we define $CZ_\tau^N(\gamma) := 0$.

- The *holomorphic intersection number* of u and Z is defined to be

$$(4.8) \qquad u * Z := \iota^\tau(u, Z) + \lfloor CZ_\tau^N(\gamma_+^u)/2 \rfloor - \lceil CZ_\tau^N(\gamma_-^u)/2 \rceil.$$

The deformation u^τ required for the definition of the relative intersection number (4.7) is constructed as follows.

Definition 4.14 (Deformation u^τ of u). We write down the deformation at the positive end, the deformation at the negative end is analogous with the appropriate changes in sign. If γ_+^u is not contained in Z_+, then we do not deform u near the positive end. Suppose γ_+^u is contained in Z_+ and extend the trivialization τ to a trivialization $\tau : U \times \mathbb{R}^2 \to \xi_N|_U$ on some open neighborhood $U \subset Z_+$ of γ_+^u. Then, [MS19, Theorem 2.2] states that the map u may be written as

$$(s, t) \mapsto \widetilde{\exp}_{u_T(s,t)}(u_N(s, t)),$$

where:

- $(s, t) \in [R, +\infty) \times S^1$ for $R \gg 1$ large enough,
- $u_T(s, t)$ lies in $\mathbb{R} \times U \subset \mathbb{R} \times Z_+$,
- u_N is a smooth section of $u_T^* \pi^* \xi_N$, where π is the projection $\mathbb{R} \times Z_+ \to Z_+$.

We then perturb the map u by replacing the above parametrization of u near $+\infty$ by the map

$$(4.9) \qquad (s, t) \mapsto \widetilde{\exp}_{u_T(s,t)}(u_N(s, t) + \beta(|s|)\tau(u_T(s, t), \epsilon))$$

where $\beta : [0, \infty) \to [0, 1]$ is a smooth cut-off function equal to 0 for $s < R + 1$ and equal to 1 for $s > R + 2$, and $0 \neq \epsilon \in \mathbb{R}^2$.

The following theorem is a restriction of Theorem 2.5 from [MS19] adapted to our notations.

Theorem 4.15 ([MS19, Theorem 2.5]). *Let Y, Z, u be as above, and assume further that the image of u is not contained in Z. Then, $u * Z \geqslant 0$ and it is equal to 0 if and only if the image of u does not intersect Z.*

4.2.3. *Proof of uniqueness.* In this section we use the intersection theory reviewed above to prove Proposition 4.9. We continue with Setup 4.1. Our first step is to show that the relative intersection number ι^τ of any J_f-holomorphic cylinder u in Y_{2i+1} with a Reeb invariant codimension-2 hypersurface Z is zero. We will later apply this lemma to $Z = \hat{Y}_{2i-1}$ when $i > 1$ and to $Z = im(u_\eta)$ when $i = 1$.

Lemma 4.16. *Let u be a J_f-holomorphic cylinder in $\hat{Y}_{2i+1}$, and let $Z \subset \hat{Y}_{2i+1}$ be a codimension-2 asymptotically cylindrical hypersurface that is invariant under φ_t. Then, $\iota^\tau(u, Z) = 0$.*

Proof. Let u^τ be the deformation of u as in Definition 4.14, and denote by $\gamma_\pm^{u^\tau}$ the ends after the deformation. Let $\ell : \mathbb{R} \to Y_{2i+1}$ be a curve that connects $\gamma_\pm^{u^\tau}$, namely $\lim_{s \to \pm\infty} \ell(s) \in \gamma_\pm^{u^\tau}$. Since $Y_{2i+1} \backslash Z$ is connected, we can choose the path ℓ to not intersect Z. Consider the cylinder

$$\sigma : \mathbb{R} \times S^1 \to \hat{Y}_{2i+1}, \qquad \sigma(s, t) := (s, \varphi_t \ell(s)) \in \mathbb{R} \times Y_{2i+1} = \hat{Y}_{2i+1}.$$

Since ℓ does not intersect the φ_t-invariant submanifold Z, its orbit under this action does not intersect it as well. Therefore, the cylinder σ is disjoint from Z (see Figure 8). Moreover,

our assumption that $H_2(Y_{2i+1}) = 0$ guarantees that the union of u^τ and $\overline{\sigma}$ is null-homologous. Consider the compactification $\overline{Y}$ of $\hat{Y}$ into a manifold with boundary, diffeomorphic to $[0,1] \times Y$. By the homology invariance of the standard algebraic intersection number (e.g., [Bre13, Part VI, Section 11]), the intersection number of $u\#\overline{\sigma}$ with the compactification of Z vanishes. Thus,

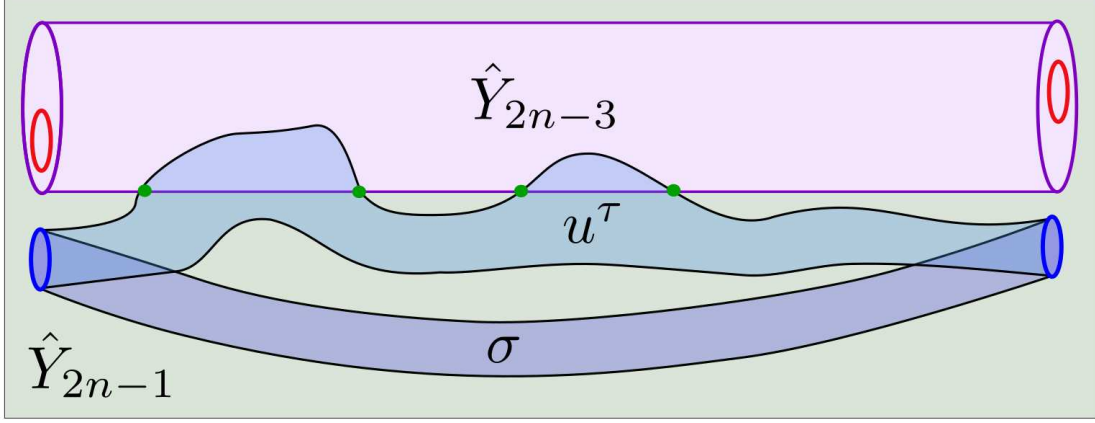$$\iota^\tau(u, Z) = u^\tau \cdot Z = \sigma \cdot Z = 0. \qquad \square$$

FIGURE 8. An illustration of the deformation u^τ of u, whose ends are disjoint from the hypersurface, and the cylinder σ that has the same ends and does not intersect $\hat{Y}_{2n-3}$.

The next lemma concerns the normal Conley-Zehnder indices of the periodic orbits of the f-perturbed Reeb flow φ_t^f. This will be useful for computing the holomorphic intersection number of a J_f-holomorphic cylinder with the submanifolds in the contact flag.

Lemma 4.17. *Take $Y = Y_{2i+1}$ and $Z = \hat{Y}_{2i-1}$ for $i \in \{2, \ldots, n-1\}$. Let γ be a periodic orbit of the perturbed contact form α_f that lies in Y_{2i-1}. There exits a trivialization τ of ξ_N such that the normal CZ index of γ is*

$$\mathrm{CZ}_\tau^N(\gamma) = \begin{cases} +1 & \text{if } \gamma = \gamma_+, \\ -1 & \text{if } \gamma \neq \gamma_+. \end{cases}$$

Proof. Let τ be any trivialization of ξ_N along γ that is invariant under the periodic Reeb flow φ_t, namely, $\tau : \mathbb{R}^2 \times \gamma \to \xi_N$ satisfying

$$(4.10) \qquad\qquad d\varphi_t \circ \tau(\cdot, x) = \tau(\cdot, \varphi_t(x))$$

for all $x \in \gamma$. In this trivialization the linearized flow of R_f is given by

$$\Phi(t) := \tau(\cdot, \gamma(t))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0))$$
$$= \tau(\cdot, \varphi_t^f \gamma(0))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)).$$

Since γ is a periodic orbit of R_f, it is a critical circle of f. By the definition of R_f stated in (v), it is proportional to R wherever X_f vanishes. Therefore,

$$\Phi(t) = \tau(\cdot, \varphi_{e^{\varepsilon f} \cdot t} \gamma(0))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0))$$
$$\overset{(4.10)}{=} \tau(\cdot, \gamma(0))^{-1} \circ (d\varphi_{e^{\varepsilon f} \cdot t})^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)).$$

Denoting $\phi_t := (\varphi_{e^{\varepsilon f} \cdot t})^{-1} \circ \varphi_t^f$, the linearization $d\phi_t$ is conjugate to $\Phi(t)$ by $\tau(\cdot, \gamma(0))$. Identifying a neighborhood of $\gamma(0)$ with its Darboux chart, the path $d\phi_t$ of matrices solves the ODE $d\dot{\phi}_t = -J_0 \cdot \mathrm{Hess}(\varepsilon f) d\phi_t$. When ε is sufficiently small, the path $\Phi(t)$ crosses the Maslov cycle only at the origin and the crossing form is $\Gamma(\gamma(0)) := -\mathrm{Hess}(\varepsilon f)_{\gamma(0)}|_{\xi_N}$. By assumption (viii).(e) from

Setup 4.1, the Hessian of f is positive definite on ξ_N unless $\gamma = \gamma_+$, in which case it is negative definite. Therefore, the normal Conley-Zehnder index of γ is $\frac{1}{2} \cdot (+2) = +1$ if $\gamma = \gamma_+$ and $\frac{1}{2} \cdot (-2) = -1$ otherwise. $\qquad \square$

We are now ready to prove that, under the assumptions of Setup 4.1, the only point in the compactified moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is u_η.

Proof of Proposition 4.9. Let $\overline{u} = (u_1, \dots, u_k) \in \overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ be a building of J_f-holomorphic curves. Lemma 4.10 above states that the components u_j of $\overline{u}$ are all cylinders. Therefore, we may apply Lemma 4.16 and conclude that $\iota^\tau(u_j, \hat{Y}_{2n-3}) = 0$ for all j. Denoting the negative and positive ends of u_j by γ_{j-1} and γ_j respectively, we notice that $\gamma_{j-1} \neq \gamma_+$ for all j. This is due to the fact that γ_+ is a global maximum for f and that the α_f-action is decreasing along J_f-holomorphic curves. Therefore, Lemma 4.17 asserts that, for a properly chosen trivialization τ, $\mathrm{CZ}_\tau^N(\gamma_{j-1}) = -1$ and $\mathrm{CZ}_\tau^N(\gamma_j) \leqslant 1$ for all j. It follows that the holomorphic intersection numbers of each piece with the hypersurface $Z = \hat{Y}_{2n-3}$ is non-positive. Indeed,

$$u_j * \hat{Y}_{2n-3} := \iota^\tau(u_j, \hat{Y}_{2n-3}) + \lfloor \mathrm{CZ}_\tau^N(\gamma_j)/2 \rfloor - \lceil \mathrm{CZ}_\tau^N(\gamma_{j-1})/2 \rceil$$
$$= 0 + \lfloor \mathrm{CZ}_\tau^N(\gamma_j)/2 \rfloor - \lceil -1/2 \rceil = \lfloor \mathrm{CZ}_\tau^N(\gamma_j)/2 \rfloor \leqslant 0.$$

Theorem 4.15 (which is a special case of [MS19, Theorem 2.5]) states that, in this case, each u_j is either disjoint from $\hat{Y}_{2n-3}$ or contained in it. By definition of $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$, the image of $\overline{u}$ contains the point $(0, z) \in \hat{Y}_3 \subset \cdots \subset \hat{Y}_{2n-3}$. Denoting by u_{j_0} the component of $\overline{u}$ whose image contains $(0, z)$, we find that u_{j_0} is not disjoint from $\hat{Y}_{2n-3}$, and thus is contained in it.
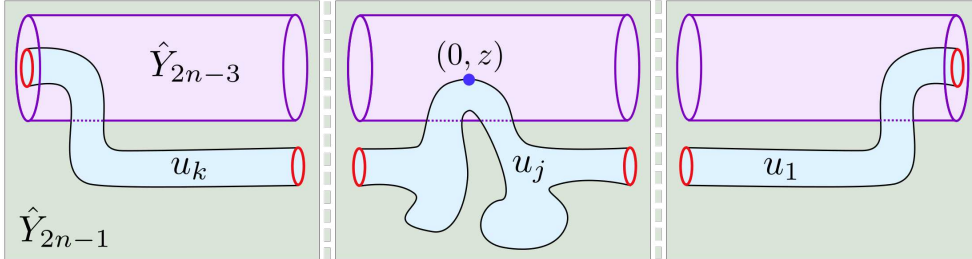

FIGURE 9. The proof of Proposition 4.9 uses positivity of intersection to rule out the illustrated scenario, in which the piece of the building that intersects the hypersurface is not contained in it.

We now consider the curve u_{j_0} in $\hat{Y}_{2n-3}$. Arguing as above, we see that its holomorphic intersection number with the hypersurface $\hat{Y}_{2n-5}$ is again non-positive. Since its image intersects $\hat{Y}_3 \subset \cdots \hat{Y}_{2n-5}$ we conclude that $u_{j_0} \subset \hat{Y}_{2n-5}$. Continuing by induction we conclude that the image of u_{j_0} is contained in $\hat{Y}_3$.

To finish the proof, notice that the above arguments imply that the holomorphic intersection number of u_{j_0} and u_η in $\hat{Y}_3$ is non-positive. Since these two J_f-holomorphic curves intersect at $(0, z)$, it follows from Theorem 4.15 that $u_{j_0} = u_\eta$. In particular, their ends agree: $\gamma_j = \gamma_+$ and $\gamma_{j-1} = \gamma_-$. Recalling that γ_+ and γ_- are global maxima and minima for f, it follows from action considerations that the only ends that could coincide with them are γ_k and γ_0 respectively, we conclude that the building $\overline{u}$ consists of a single cylinder, $u_{j_0} = u_\eta$, which concludes the proof. $\qquad \square$

4.3. **Transversality.** The aim of this section is to show that the J_f-holomorphic curve u_η, described in Section 4.1, is cut-out transversely.

4.3.1. *Local transversality of the moduli space without a point constraint.* We start by showing that the moduli space without the point constraint is locally transversely cut-out at u_η. The notion of a moduli space being locally transversely cut-out at a curve appears in [Par15, Section 2.11]; we give a brief overview here for convenience.

Definition 4.18 ([Par15, Section 2.11]). Let u be a J_f-holomorphic curve in $\hat{Y}$.

- For $k \geqslant 0$ and $\delta < 1$, define the associated *weighted Sobolev space*

(4.11) $$W^{k,2,\delta}(u^*T\hat{Y}) = \left\{ f \mid \mu \cdot f \in W^{k,2}(u^*T\hat{Y}) \right\}, \qquad \|f\|_{k,2,\delta} := \|\mu \cdot f\|_{k,2},$$

where $\mu : \mathbb{R} \times S^1 \to \mathbb{R}_{>0}$ equals 1 away from the ends and equals $e^{\delta|s|}$ near the ends.
- The linearized operator[5]

(4.12) $$D_u : W^{k,2,\delta}(u^*T\hat{Y}) \to W^{k-1,2,\delta}\left(u^*T\hat{Y} \otimes_{\mathbb{C}} \Omega^{0,1}(\mathbb{R} \times S^1) \right),$$

of u is defined as follows. Given any symmetric connection ∇ on $T\hat{Y}$, the linearized operator takes the form [Wen15, Section 2.1]

(4.13) $$D_u w = \nabla w + J_f(u) \circ \nabla w \circ j + (\nabla_w J_f) \cdot du \circ j$$

- A J_f-holomorphic curve u is *transversely cut-out* if the linearized operator D_u is surjective.

Proposition 4.19. *Consider Setup 4.1. The linearized Cauchy-Riemann operator at u_η is surjective.*

Our strategy for proving Proposition 4.19 is to show that the linearized operator splits into a direct sum of real-linear Cauchy–Riemann operators on complex line bundles. Then, we apply the automatic transversality results from [Wen10] to deduce surjectivity.

Recall the contact flag $Y_3 \subset \cdots \subset Y_{2n-1} = Y$. It defines a splitting

(4.14) $$\xi = \xi_3 \oplus \cdots \oplus \xi_{2n-1}$$

of the contact distribution into two-dimensional sub-bundles as follows. The bundle ξ_3 is defined to be the contact structure of Y_3. For $i > 1$, the bundle ξ_{2i+1} is defined to be the symplectic complement of TY_{2i-1} in the contact structure of Y_{2i+1}. Denoting by V the span of ∂_r and R, we have a splitting

$$T\hat{Y}_{2i+1} = V \oplus \left(\bigoplus_{k=1}^{i} \xi_{2k+1} \right), \quad i = 1, \ldots, n-1.$$

Moreover, it follows from the assumptions in Setup 4.1 that J_f preserves this splitting. For any $i = 1, \ldots, n-1$, we write D_i for the linearized operator of u_η, considered as a J_f-holomorphic curve in $\hat{Y}_{2i+1}$. We will also denote the full linearized operator D_{n-1} by D.

Lemma 4.20 (Decomposing the linearized operator). *For any $2 \leqslant i \leqslant n+1$ there exists a Fredholm real-linear Cauchy–Riemann operator L_i on the bundle ξ_{2i+1} such that the operator D_i has the matrix form*

$$\begin{pmatrix} D_{i-1} & 0 \\ 0 & L_i \end{pmatrix}$$

with respect to the splitting $T\hat{Y}_{2i+1} = T\hat{Y}_{2i-1} \oplus \xi_{2i+1}$.

The linearized operator can be written in terms of a symmetric connection on $T\hat{Y}$, as in (4.13). It will be convenient, for the purposes of computation, to use the Levi–Civita connection of a metric which is well-adapted to the contact flag.

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[5]The domain of the linearized operator considered by Pardon in [Par15] is slightly bigger. Since our aim here is to prove surjectivity this does not cause any issue.

Lemma 4.21 (Choice of connection). *There exist symmetric connections ∇^i on $T\hat{Y}_{2i+1}$, such that for any v tangent to the hypersurface Y_{2i-1}, the operator ∇^i_v can be written as*

$$(4.15) \qquad \nabla^i_v = \begin{pmatrix} \nabla^{i-1}_v & 0 \\ 0 & \nabla'_v \end{pmatrix}$$

with respect to the splitting $T\hat{Y}_{2i+1} = T\hat{Y}_{2i-1} \oplus \xi_{2i+1}$.

Proof. Let h_1 be the metric on Y_3 equal to the one induced by J_f. Define for each $i \in \{2, \ldots, n-1\}$ a metric h_i on Y_{2i+1} which satisfies the following conditions:

- The submanifold Y_{2i-1} is totally geodesic with respect to h_i.
- The splitting

$$TY_{2i+1}|_{Y_{2i-1}} = TY_{2i-1} \oplus \xi_{2i+1}$$

   is h_i-orthogonal.

The metrics h_i can be constructed inductively by repeated applications of the following standard trick. Fix any Riemannian metric h on Y_{2i+1} which restricts to h_{i-1} on Y_{2i-1} and makes ξ_{2i+1} orthogonal to TY_{2i+1}. Use the restriction of the exponential map to ξ_{2i+1} to define a tubular neighborhood U of Y_{2i-1} in Y_{2i+1}. The involution $v \mapsto -v$ on ξ_{2i+1} then defines a smooth involution $\iota : U \to U$ that fixes TY_{2i-1}. Set $h_i := \frac{1}{2}(h + \iota^* h)$ on U and extend it outwards. Then ι is an isometry for h_i. Given any $(p, w) \in TY_{2i-1}$, the corresponding geodesic is mapped under ι to a geodesic corresponding to $(\iota(p), \iota(w)) = (p, w)$. In other words this geodesic is preserved by ι and this is in Y_{2i-1}.

For each $i \in \{2, \ldots, n-1\}$, let $\hat{h}_i$ denote the cylindrical metric over h_i on the cylinder $\hat{Y}_{2i+1}$. Let ∇^i denote the Levi–Civita connection on $\hat{Y}_{2i+1}$ with respect to $\hat{h}_i$. By definition, ∇^i is symmetric. The splitting (4.15) follows from the fact that $\hat{Y}_{2i-1}$ is totally geodesic with respect to $\hat{h}_i$ and hence the $\hat{h}_i$ parallel transport preserves the orthogonal direct sum decomposition $TY_{2i+1}|_{Y_{2i-1}} = TY_{2i-1} \oplus \xi_{2i+1}$. $\qquad\square$

The next lemma states that the asymptotic operators of D, the linearized operator of u_η in $\hat{Y}$, respect the direct sum decomposition induced by the contact flag, and that the pieces in the contact direction are non-degenerate.

Lemma 4.22. *The asymptotic operators $A_\pm$ of D respect the splitting (4.14). Moreover, their restrictions to $\gamma_\pm^* \xi_{2j+1}$ are non-degenerate.*

Proof. Consider the connections from Lemma 4.21, and write $\nabla := \nabla^{n-1}$ for the connection on the total manifold $\hat{Y}$. Following [Wen10], the asymptotic operators $A_\pm$ are defined by $\lim_{s \to \pm\infty} D(\partial_s) = \nabla_s - A_\pm$. Evaluating (4.13) as $s \to \pm\infty$ we obtain

$$\lim_{s \to \pm\infty} D(\partial_s) = \lim_{s \to \pm\infty} \nabla_s + J_f \nabla_t + (\nabla J_f)\partial_t u$$

$$= \nabla_s + J_f \nabla_{T \cdot R_f} + (\nabla J_f)T \cdot R_f = \nabla_s + T J_f \nabla_{R_f} + T\nabla(J_f R_f) - T J_f \nabla R_f$$

$$= \nabla_s + T \cdot J_f(\nabla_{R_f} - \nabla R_f) + T\nabla(J_f R_f).$$

Notice that $J_f R_f = \partial_r$ is covariantly constant, since the metric defining ∇ is cylindrical. Therefore the asymptotic operators are

$$(4.16) \qquad A_\pm = -T \cdot J_f(\nabla_{R_f} - \nabla R_f) + T\nabla(J_f R_f) = -T \cdot J_f[R_f, -].$$

Since J_f and the flow of R_f preserve the splitting, the operator $J_f[R_f, -]$ does as well. Denoting by $A_\pm^i$ the restrictions of $A_\pm$ to $\gamma_\pm^* \xi_{2i+1}$, it remains to show that they are non-degenerate. Notice that the operators $A_\pm^i$ coincide with the restriction to $\gamma_\pm^* \xi_{2i+1}$ of the linearization of the flow φ_t^f of R_f. Let $\tau_\pm$ be trivializations of ξ_{2i+1} along $\gamma_\pm$ that are invariant under the periodic Reeb flow φ_t, as in 4.10. As shown in the proof of Lemma 4.17, the linearization of the flow φ_t^f is conjugate

to a path of matrices $d\phi_t$ that solves the ODE $d\dot\phi_t = -J_0 \cdot \text{Hess}(\varepsilon f)d\phi_t$. Note that the Hessian of f is degenerate in the direction of the Reeb vector field R, since f is R-invariant. However, by the Morse-Bott condition, the restriction of the Hessian to ξ_{2i+1}, is non-degenerate. When ε is sufficiently small, this implies that $\{d\phi_t\}_{t\in[0,T]}$ is non-degenerate as well. $\qquad\square$

We are now ready to show that the linearized operator splits into a direct sum.

Proof of Lemma 4.20. Let ∇^i be the connection from Lemma 4.21, and write D_i as

$$D_i = \underbrace{\nabla^i + J_f \circ \nabla^i \circ j}_{(i)} + \underbrace{(\nabla^i J_f) \cdot du_\eta \circ j}_{(ii)}.$$

The splitting of the first part, (i), follows immediately from the splitting of ∇^i and the fact that J_f preserves the decomposition. Indeed, since $\partial_s u_\eta$ and $\partial_t u_\eta$ lie in the subspace $T\hat{Y}_{2i-1}$, Lemma 4.21 guarantees that $\nabla^i_{\partial_s u_\eta} = \nabla^{i-1}_{\partial_s u_\eta} \oplus \nabla'_{\partial_s u_\eta}$, and the same for $\partial_t u_\eta$. Therefore,

$$(i) = \begin{pmatrix} \nabla^{i-1} + J_f(u_\eta)\nabla^{i-1} \circ j & 0 \\ 0 & \nabla' + J_f(u_\eta)\nabla' \circ j. \end{pmatrix}.$$

Let us rewrite the second part of D_i applied to ∂_s (the computation for ∂_t is completely analogous):

$$\begin{aligned}
(ii)(\partial_s) &= (\nabla^i J_f) \cdot du_\eta \circ j\partial_s = (\nabla^i J_f) \cdot \partial_t u_\eta \\
&= (\nabla^i J_f) \cdot T \cdot R_f = T \cdot \left(\nabla^i(J_f R_f) - J_f \nabla^i R_f\right) \\
&= T \cdot \left(\nabla^i(-\partial_r) - J_f \nabla^i R_f\right) \\
(4.17) &= T \cdot \left(0 - J_f \nabla^i R_f\right) = -TJ_f \nabla^i R_f.
\end{aligned}$$

where, in the second and third lines, we used the fact that $\partial_t u_\eta = T \cdot R_f$ by construction, and that $R_f = -J_f \partial_r$ respectively. In the last row we used the fact that the metric $\hat{h}^i$ is cylindrical in the r-direction, which implies that the vector field ∂_r is covariantly constant. We now evaluate $(ii)(\partial_s) = -TJ_f \nabla^i R_f$ on a tangent vector field v and split into cases with respect to the decomposition $TY_{2i+1}|_{Y_{2i-1}} = TY_{2i-1} \oplus \xi_{2i+1}$:

- $v \in \Gamma(u_\eta^* TY_{2i-1})$: Using Lemma 4.21 and the fact that R_f is tangent to $\hat{Y}_{2i-1}$ as well, we obtain $\nabla^i_v R_f = \nabla^{i-1}_v R_f$. Therefore $(ii)(\partial_s)v = -TJ_f \nabla^{i-1}_v R_f$.
- $v \in \Gamma(u_\eta^* \xi_{2i+1})$: Extend v into a vector field $\tilde{v}$ defined on a neighborhood of the image of u_η in $\hat{Y}_{2i+1}$. Then, along the hypersurface $\hat{Y}_{2i-1}$ we have

$$\begin{aligned}
(ii)(\partial_s)v &= -TJ_f \nabla^i_v R_f = -TJ_f \left(\nabla^i_{R_f}\tilde{v} + [\tilde{v}, R_f]\right) \\
&= TJ_f \left([R_f, \tilde{v}] - \nabla^i_{R_f}\tilde{v}\right) = TJ_f \left([R_f, v] - \nabla^i_{R_f}v\right) \\
&\overset{\text{Lemma 4.21}}{=} TJ_f \left([R_f, v] - \nabla'_{R_f}v\right) = J_f[\partial_t u_\eta, v] - J_f \nabla'_{\partial_t u_\eta}v \\
&= J_f[du_\eta \circ j(\partial_s), v] - J_f \nabla' \circ j(\partial_s)(v).
\end{aligned}$$

Above, we used the fact that the connection ∇^i is symmetric and that R_f is tangent to the hypersurface $\hat{Y}_{2i-1}$.

Overall we conclude that the second part of D_i decomposes as

$$(ii) = \begin{pmatrix} (\nabla^{i-1}J_f) \cdot du_\eta \circ j & 0 \\ 0 & J_f[du_\eta \circ j, -] - J_f \nabla' \circ j \end{pmatrix}.$$

Summing (i) and (ii) we obtain a decomposition of D_i:

$$D_i = \begin{pmatrix} D_{i-1} & 0 \\ 0 & \nabla' + J_f[du_\eta \circ j, -] \end{pmatrix}.$$

A simple computation shows that $L_i := \nabla' + J_f[du_\eta \circ j, -]$ is a Cauchy-Riemann operator. Moreover, since $\partial_t u_\eta = TR_f$, the asymptotic operators of L_i are indeed the restrictions of A_+ to $\gamma^*_\pm \xi_{2i+1}$. By Lemma 4.22, they are non-degenerate. This implies that L_i is a Fredholm operator (e.g. [Wen10, Section 2.1]), and thus concludes the proof.   □

Having established the splitting of the linearized operators D_i, we are ready to prove surjectivity.

Proof of Proposition 4.19. We will prove by induction that the Cauchy-Riemann operators D_i are surjective for all i. This will use the decomposition from Lemma 4.20, as well as automatic transversality [Wen10] for D_1 and L_i. Starting with the base case of the induction, D_1 is the linearized operator at the (non-constant) curve u_η inside the 4-dimensional manifold $\hat{Y}_3$. Theorem 1 from [Wen10] states that[6], in a 4-dimensional manifold, an immersed pseudoholomorphic curve is regular if

$$(4.18) \qquad \operatorname{ind}(u) > \frac{1}{2}(\operatorname{ind}(u) - 2 + 2g + \#\Gamma_0 + \#\pi_0(\partial\Sigma)),$$

where:

- Σ is the domain of the curve,
- g is the genus of Σ,
- Γ_0 is the set of ends of u with even CZ index.

In our case, $\Sigma = \mathbb{R} \times S^1$, and hence $g = 0$ and $\#\pi_0(\Sigma) = 0$. Moreover, $\#\Gamma_0 \leqslant 2$ since u_η has only two ends. Therefore, condition (4.18) holds if $\operatorname{ind}(D_1) > 0$. The latter inequality obviously holds in our case (in fact, $\operatorname{ind}(D_1) = \dim\left(\hat{Y}_3\right) - 2 = 2$).

Having established the base case of the induction we move on to the induction step. We assume that D_{i-1} is surjective for some i and show surjectivity of D_i. By the decomposition of D_i given in Lemma 4.20, this is equivalent to showing that the real-linear Cauchy-Riemann operator L_i is surjective. This is an operator on sections of a complex line bundle. Proposition 2.2 from [Wen10] states that[7] given a complex line bundle $E \to \Sigma$, a Cauchy-Riemann Fredholm operator $L : W^{1,2}(E) \to L^2(E \otimes_\mathbb{C} \Omega^{0,1}(\mathbb{R} \times S^1)$ is surjective if

$$(4.19) \qquad \operatorname{ind}(L) \geqslant 0 \quad \text{and} \quad \operatorname{ind}(L) > \frac{1}{2}(\operatorname{ind}(L) - 2 + 2g + \#\Gamma_0 + \#\pi_0(\partial\Sigma)),$$

where:

- g is the genus of Σ,
- Γ_0 is the set of ends for which the CZ index of the asymptotic operator is even.

As before, in our case $g = 0$ and $\#\pi_0(\partial\Sigma) = 0$. Therefore the second condition in (4.19) amounts to $\operatorname{ind}(L) > -2 + \#\Gamma_0$. Since the domain Σ in our case is a cylinder, $\#\Gamma_0 \leqslant 2$, and we are left with the requirement $\operatorname{ind}(L) > 0$. A simple computation shows that $\operatorname{ind}(L_i) = 2$ for all i. This essentially follows from the direct sum decomposition given in Lemma 4.20, the additivity of the Fredholm index, and the fact that $\operatorname{ind}(D_i) = 2i$. We therefore conclude that L_i is surjective, and this completes the proof.   □

Remark 4.23. In the above proof we showed that L_i is surjective as an operator between the spaces $W^{1,p}$ and L^p. Note that this implies surjectivity of the corresponding operator between $W^{k,2}$ and $W^{k-1,2}$, by elliptic regularity. Indeed, the cokernel of L_i is the kernel of its formal adjoint operator. Applying elliptic regularity to the formal adjoint, we find that its kernel is independent of k and p.

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[6]We only state a special case of Wendl's theorem, adapted to our notations.

[7]Again, we state a special case adapted to our notations.

4.3.2. *The evaluation map is a submersion.* Let z, η, u_η be as given in Section 4.1. We showed in Proposition 4.19 that the linearized operator

$$D : W^{k,2,\delta}(u_\eta^* T\hat{Y}) \to W^{k-1,2,\delta}\big(u^* T\hat{Y} \otimes_{\mathbb{C}} \Omega^{0,1}(\mathbb{R} \times S^1)\big)$$

is surjective. This implies that the moduli space $\mathcal{M}(\hat{Y}, J_f; \gamma_+, \gamma_-)$ of J_f-holomorphic cylinders near u_η admits the structure of a smooth manifold of dimension $\mathrm{ind}(D)$ near u_η, with tangent space at u_η identified with $\ker(D) \oplus \mathbb{R} \cdot \partial_r$. The extra factor comes from the fact that we are *not* quotienting by the translation action; recall that the operator D controls the deformations of the quotient moduli space $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-) = \mathcal{M}(\hat{Y}, J_f; \gamma_+, \gamma_-)/\mathbb{R}$.

Our ultimate goal is to show that the point-constrained moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is transversely cut-out at u_η. The point constrained moduli space is defined as the inverse image $\mathrm{ev}^{-1}(0, z)$, where ev is the evaluation map

$$\mathrm{ev} : \mathcal{M}(\hat{Y}, J_f; \gamma_+, \gamma_-) \times \mathbb{R}_s \times S^1_t \to \hat{Y}, \qquad (u, s, t) \mapsto u(s, t).$$

It is transversely cut-out at u_η if and only if ev is a submersion at the point $(u_\eta, 0, 0)$. The linearization of ev at $(u_\eta, 0, 0)$, which we write as D_{ev}, is the linear map

$$D_{\mathrm{ev}} : \ker(D) \oplus \mathbb{R} \oplus T_{(0,0)}(\mathbb{R} \times S^1) \to T_{(0,z)}\hat{Y}$$

defined by

(4.20) $$(V, c, v) \mapsto V(0, 0) + c \cdot \partial_r + (\partial_v u_\eta)(0, 0).$$

In order to show that the moduli space $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ is cut-out transversely at u_η, it remains to prove the following proposition.

Proposition 4.24. *The map D_{ev} is surjective.*

The proof of Proposition 4.24 uses the surjectivity of the linearized evaluation map on Morse flow lines, as stated in the following lemma.

Lemma 4.25. *Let $Morse(\gamma_+, \gamma_-)$ be the space of Morse flow lines from γ_+ to γ_-. Then it is a smooth manifold and, for every $x \in Morse(\gamma_+, \gamma_-)$ the map*

$$D_{ev}^{Morse} : T_x Morse(\gamma_+, \gamma_-) \to T_{x(0)}Y, \quad \textit{defined by} \quad V \mapsto V(0)$$

is surjective.

Proof. The space $Morse(\gamma_+, \gamma_-)$ is composed of all smooth maps $x : \mathbb{R} \to Y$ such that $\dot{x}(s) = \nabla_J f(x(s))$, which limit to points in γ_+ and γ_- respectively as $s \to \pm\infty$. Consider the natural evaluation map

$$\mathrm{ev}_{Morse} : Morse(\gamma_+, \gamma_-) \to Y$$

sending a flow line x to the point $x(0)$. The image of $Morse(\gamma_+, \gamma_-)$ under this evaluation map is the intersection of the unstable manifold of γ_+ and the stable manifold of γ_-. This intersection is a smooth open submanifold of Y. This follows from the fact that the unstable manifold of γ_+ and the stable manifold of γ_- are both smooth open submanifolds of maximal dimension in Y, so they must intersect transversely. By existence and uniqueness of ODEs, this implies that the space $Morse(\gamma_+, \gamma_-)$ is a smooth manifold and that the evaluation map is a diffeomorphism onto an open subset of Y.

The tangent space $T_x Morse(\gamma_+, \gamma_-)$ at any flow line x is a $2n - 1$-dimensional vector space of smooth vector fields V along x satisfying the linearized Morse flow line equation

$$\nabla_s V - \mathrm{Hess}_J f(V) = 0,$$

where ∇ denotes the Levi–Civita connection of the metric g_J and $\mathrm{Hess}_J(f)$ is the Hessian of f with respect to this metric. The linearization of evaluation map at x is the operator

$$D_{\mathrm{ev}}^{\mathrm{Morse}} : T_x \mathrm{Morse}(\gamma_+, \gamma_-) \to T_{x(0)}Y, \qquad V \mapsto V(0).$$

Since $\mathrm{ev}_{\mathrm{Morse}}$ is a diffeomorphism, its linearization is surjective. More explicitly, by existence of solutions to ODEs, for any point x_0' in a neighborhood of $x(0)$, there is a flow line x' such that $x'(0) = x_0'$. □

Proof of Proposition 4.24. There is a natural parameterization of a set of cylinders near u_η in $\mathcal{M}(\hat{Y}, J_f; \gamma_+, \gamma_-)$ with a neighborhood of η in the space of Morse flow lines from γ_+ to γ_-. Lemma 4.25 states that the corresponding evaluation map, which sends a flow line to its image at the point $0 \in \mathbb{R}$, is a submersion. We will show that the linearization of this evaluation map at η factors through D_{ev}.

In Section 4.1 we explained that every Morse flow line x can be lifted to a J_f holomorphic curve given by

$$u_x(s,t) := (a(s), \varphi_{T \cdot t}(x(\varepsilon T s))), \qquad \text{where} \qquad \dot{a}(s) = T \cdot e^{\varepsilon f(x(\varepsilon T s))}, \quad a(0) = 0.$$

Consider the lift map $L : \mathrm{Morse}(\gamma_+, \gamma_-) \to \mathcal{M}(\gamma_+, \gamma_-; J_f)$ that sends a flow line x to its lift u_x. The space $\mathcal{M}(\hat{Y}, J_f; \gamma_+, \gamma_-)$ is cut out transversely near u_η by Proposition 4.19. Therefore we can consider the linearization of L at η,

$$T_\eta L : T_\eta \mathrm{Morse}(\gamma_+, \gamma_-) \to \ker(D) \oplus \mathbb{R},$$

which sends a vector field $V \in T_\eta \mathrm{Morse}(\gamma_+, \gamma_-)$ to the vector field

$$T_\eta V(s,t) := \int_0^s \varepsilon df(V(\varepsilon T s')) \cdot \dot{a}(s') ds' \cdot \partial_r + d\varphi_{T \cdot t} \cdot V(\varepsilon T s)$$

in $\ker(D)$. Denoting by $D\Pi$ the projection $T_{(0,z)}\hat{Y} \to T_z Y$, we claim that

$$(4.21) \qquad\qquad D_{\mathrm{ev}}^{\mathrm{Morse}} = D\Pi \circ D_{\mathrm{ev}} \circ T_\eta L.$$

Indeed, given $V \in T_\eta \mathrm{Morse}(\gamma_+, \gamma_-)$, the operator $D_{\mathrm{ev}}^{\mathrm{Morse}}$ sends it to $V(0)$. On the other hand, since the coefficient of ∂_r in $T_\eta V$ vanishes when $s = 0$, the projection $D\Pi \circ D_{\mathrm{ev}}$ sends $T_\eta V(0,0)$ to $V(0)$ as well.

Now we use (4.21) to conclude the proposition. By Lemma 4.25, the operator $D_{\mathrm{ev}}^{\mathrm{Morse}}$ is surjective. This implies that the map $D\Pi \circ D_{\mathrm{ev}}$ must be surjective, so by definition of $D\Pi$ we deduce

$$\mathrm{Span}(\partial_r) + \mathrm{Im}(D_{\mathrm{ev}}) = \mathrm{Im}(D\Pi \circ D_{\mathrm{ev}}) = T_{(0,z)}\hat{Y}.$$

Recalling (4.20), we see that $\mathrm{Span}(\partial_r) \subset \mathrm{Im}(D_{\mathrm{ev}})$. Therefore, $\mathrm{Im}(D_{\mathrm{ev}}) = \mathrm{Span}(\partial_r) + \mathrm{Im}(D_{\mathrm{ev}}) = T_{(0,z)}\hat{Y}$, i.e., D_{ev} is surjective. □

4.4. **Relation to abstract constraints.** Thus far in §4, we have counted points in moduli spaces of cylinders with a point constraint. To conclude this section, we relate this count to a point count in the moduli spaces involved in the U-maps constructed in §3.2.

We continue to work with the objects and notation described in Setup 4.1. We can also work with the weaker hypothesis discussed in Remark 4.3.

Let $X : Y \to Y$ be the trivial cobordism $[-\delta, \delta]_s \times Y$ equipped with the Liouville form

$$\lambda = e^s \alpha_f = e^{s + \varepsilon f} \alpha$$

and let $E \subset X$ be an embedded, irrational ellipsoid that is the image of an embedding

$$\iota : E(b_1, \ldots, b_n) \to X \qquad\qquad \text{such that} \qquad \iota(0) = 0 \times z \in [-\delta, \delta] \times Y$$

The boundary ∂E has a shortest simple orbit κ. Let J' be a compatible almost complex structure on $X \backslash E$ and fix the shorthand notation

$$\mathcal{M}_{X \backslash E} := \mathcal{M}_{0,A}(X \backslash E, J'; \gamma_+, \Gamma_-), \qquad \text{where} \qquad \Gamma_- = (\gamma_-, \kappa).$$

This is the moduli space of genus 0 J-holomorphic curves in the cobordism $X \backslash E : Y \to Y \cup \partial E$ asymptotic to γ_+ at $+\infty$ and $\gamma_- \cup \kappa$ at $-\infty$. Moreover, let J'' be an almost complex structure on E and fix the shorthand notation

$$\mathcal{M}_E := \mathrm{ev}^{-1}(0) \subset \mathcal{M}_{0,A,1}(E, J''; \kappa, \varnothing).$$

This is the moduli space of J''-holomorphic planes in E that pass through $0 \in E$ and that are asymptotic at $+\infty$ to κ. The goal for the rest of the section is to prove the following result.

Proposition 4.26. *Assume that the period of $\gamma_\pm$ is equal to the minimal period of the Reeb flow of α. There exists a choice of complex structures J' on $X \backslash E$ and J'' on E such that the moduli spaces*

$$\mathcal{M}_{X \backslash E} \qquad \text{and} \qquad \mathcal{M}_E$$

are transversely cut out, 0-dimensional and equal to their compactifications (i.e. there are no buildings in their compactifications). Furthermore, they satisfy

(4.22) $$\#\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) = \#\mathcal{M}_{X \backslash E} \cdot \#\mathcal{M}_E \quad \mod 2$$

Remark 4.27. In fact, the equality (4.22) should be true over $\mathbb{Q}$ for any choice of complex structures, as a virtual count of points in the framework of Pardon [Par15, Par16]. This version of this result is proven by Siegel [Sie19], using J-holomorphic cascades and the Morse-Bott formalism.

In the spirit of the other results of this paper and for the sake of completeness, we provide a proof that uses only transversely cut out holomorphic curves and avoids Morse-Bott theory.

Our strategy to prove Proposition 4.26 is quite standard. First, we choose a complex structure satisfying a number of regularity hypotheses for somewhere injective curves. Second, we show the desired compactness results for $\mathcal{M}_{X \backslash E}$ and $\mathcal{M}_E$ for the chosen complex structures. Last, we use a parametric moduli space to construct a topological cobordism

$$\widetilde{\mathcal{M}} \qquad \text{from} \qquad \overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) \qquad \text{to} \qquad \mathcal{M}_{X \backslash E} \times \mathcal{M}_E.$$

The existence of such a cobordism proves the desired result.

To begin the argument, we fix almost complex structures such that the relevant moduli spaces of somewhere injective curves are sufficiently regular. We are permitted to do this by standard generic transversality results (reviewed in §2.2.10, see Proposition 2.5).

Setup 4.28. For the rest of the section, fix compatible almost complex structures

$$J' \text{ on } X \backslash E \qquad \text{and} \qquad J'' \text{ on } E$$

satisfying the following properties:
  (i) the moduli spaces of finite energy, somewhere injective curves

$$\mathcal{M}^{\mathrm{i}}_{g,A,m}(X \backslash E, J'; \Gamma_+, \Gamma_-) \qquad \text{and} \qquad \mathcal{M}^{\mathrm{i}}_{h,b,l}(E, J''; \Xi_+, \Xi_-) \qquad \text{are regular,}$$

  (ii) the evaluation maps

$$\mathrm{ev} : \mathcal{M}^{\mathrm{i}}_{h,b,1}(E, J''; \Xi_+, \Xi_-) \to \hat{X} \qquad \text{are transverse to } z.$$

Given J' and J'' as in Setup 4.28, there is a compactified moduli space that we denote by

$$\overline{\mathcal{M}}_{X \backslash E \sqcup E}$$

consisting of genus 0 buildings $\bar{u}$ with the following levels.
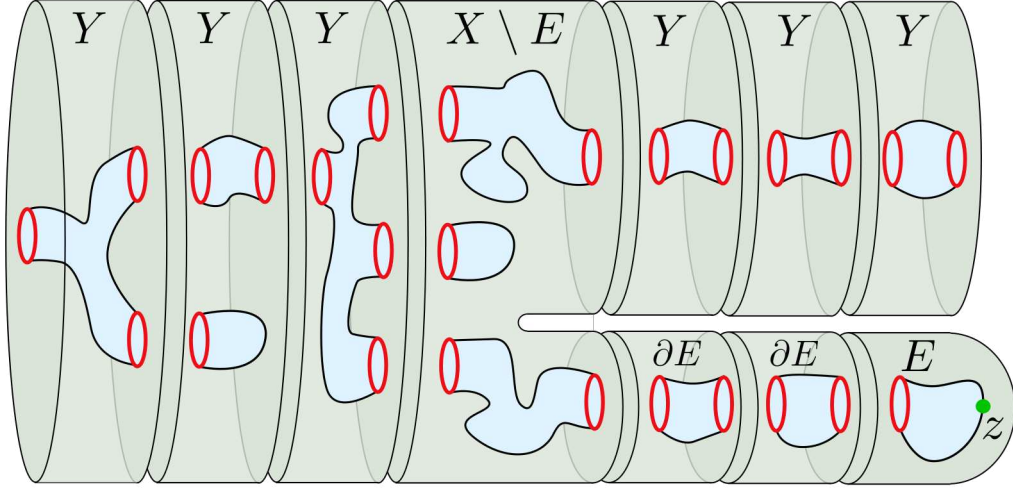 • A sequence of levels $u_1^+, \dots, u_k^+$ in $\hat{Y}$.

FIGURE 10. A possible building in the moduli space $\overline{\mathcal{M}}_{X\backslash E \sqcup E}$. We will show in Lemma 4.29 that (generically) these buildings must be much simpler than this.

- A single level u in the cobordism $\widehat{X\backslash E}$ of genus 0 with one positive end.
- A sequence of levels $u_1^-, \ldots, u_l^-$ in $\hat{Y}$.
- A sequence of levels $v_1, \ldots, v_m$ in $\widehat{\partial E}$.
- A single level v in $\hat{E}$.

where the positive ends and negative ends of adjacent levels match in the usual way. A depiction of a possible building $\bar{u}$ is depicted in Figure 10.

We now show that (under our hypotheses) the only buildings appearing in $\overline{\mathcal{M}}_{X\backslash E \sqcup E}$ are those of the simplest possible form. We will require the following lemma.

Lemma 4.29. *Consider $\varepsilon > 0$ from Setup 4.1 and let J' as in Setup 4.28. Let ζ_+ and ζ_- be orbits of Y satisfying*

$$(4.23) \qquad\qquad \mathcal{A}(\gamma_+) \geqslant \mathcal{A}(\zeta_+) \geqslant \mathcal{A}(\zeta_-) \geqslant \mathcal{A}(\gamma_-)$$

Finally, let u be a finite energy, connected J'-holomorphic map in $\widehat{X\backslash E}$ of genus 0 such that

$$(4.24) \qquad\qquad u \to \zeta_+ \text{ at } +\infty \qquad \text{and} \qquad u \to \zeta_- \cup \Xi \text{ at } -\infty$$

where Ξ is a non-empty orbit sequence in $Y \cup \partial E$. Then for sufficiently small ε, we must have

$$\zeta_+ = \gamma_+ \qquad \zeta_- = \gamma_- \quad \text{and} \quad \Xi = \kappa$$

Proof. First, factor u as a branched cover ψ and a somewhere injective map v,

$$u : \Sigma' \xrightarrow{\psi} \Sigma \xrightarrow{v} \hat{Y}.$$

Here v has a positive and negative ends asymptotic to orbits

$$\beta_\pm \text{ covered by } \zeta_\pm \qquad \text{and} \qquad \Xi' \text{ covered by } \Xi.$$

It suffices to show that $\beta_\pm = \gamma_\pm$ and $\Xi' = \kappa$. Note that this will imply that $u = v$.

First, note that v is genus 0. Indeed, we can compactify ψ to a branched cover $S^2 \to \bar{\Sigma}$ where $\bar{\Sigma}$ is a closed surface given by compactifying Σ along the punctures. Any closed surface with a non-trivial branched cover from S^2 is genus 0.

Next, note that γ_+ and γ_- are orbits corresponding to perturbations of Morse-Bott orbits of period equal to the period T of the Reeb flow of α_f. Thus, using the action bounds (4.23), we

have

$$\mathcal{A}(\beta_+) - \mathcal{A}(\beta_-) = \frac{\mathcal{A}(\zeta_+) - \mathcal{A}(\zeta_-)}{\deg(\psi)} \leqslant \frac{\mathcal{A}(\gamma_+) - \mathcal{A}(\gamma_-)}{\deg(\psi)} = O(\varepsilon).$$

Thus, the orbit set Ξ' cannot contain any orbits in Y for small ε, otherwise the action of the orbit set $\beta_- \cup \Xi'$ would be larger than that of β_+. Moreover, any pair of orbits of α_f satisfying

$$\mathcal{A}(\beta_+) - \mathcal{A}(\beta_-) \leqslant O(\varepsilon) \qquad \text{and} \qquad \mathcal{A}(\beta_-) \leqslant \mathcal{A}(\beta_+) \leqslant \mathcal{A}(\gamma_+) \simeq T + O(\epsilon)$$

must be perturbations of Morse-Bott orbits in the same Morse-Bott family S_L for sufficiently small ε. In particular, $\beta_\pm$ correspond to pairs $(S_L, p_\pm)$, where $L \leqslant T$ (where T is the period of the flow of E) and $p_\pm$ are critical circles of f. By Lemma 4.7, the Conley-Zehnder indices $CZ(\beta_\pm, \tau_\pm)$ in any trivialization $\tau_\pm$ of ξ over $\beta_\pm$ is given by

$$CZ(\beta_\pm, \tau_\pm) = RS(\beta_\pm, \tau_\pm) - \dim(S_L)/2 + \mathrm{ind}_{Morse}^{S_L}(f; p_\pm)$$

where $RS(\beta_\pm, \tau_\pm)$ is the Robbin-Salamon index of the unperturbed orbit and $\mathrm{ind}_{Morse}^{S_L}(f; p_\pm)$ denotes the Morse-Bott index of the critical circle $p_\pm$ in the tangent directions to S_L. In particular, let Σ be a cylinder connecting β_+ and β_- formed by a 1-parameter family of orbits of period L connecting β_+ and β_-, and let τ be a trivialization of ξ over Σ. Then

$$CZ(\beta_+, \tau) - CZ(\beta_-, \tau) \leqslant 2n - 2,$$

Since $H_2(Y; \mathbb{Q}) = 0$ by hypothesis, there is a well-defined difference of SFT grading between β_+ and β_-, and in those terms the above inequality states that

$$|\beta_+| - |\beta_-| \leqslant 2n - 2$$

Equality can occur if and only if $\dim(S_L) = 2n - 2$, p_+ is the maximum circle of f and p_- is the minimum of f. Since $L \leqslant T$ and T is the period of the flow of α_f, this only if $S_L = S_T$, and $\beta_\pm = \gamma_\pm$.

Finally, note that the Fredholm index of the holomorphic curve v is given in terms of relative SFT gradings by

$$\mathrm{ind}(v) = |\beta_+| - |\beta_-| - |\Xi'| \leqslant 2n - 2 - |\Xi'|.$$

Here τ is a trivialization of ξ over v. By the calculation in Example 2.16, we know that

$$|\Xi'| = n - 3 + \sum_{\gamma \in \Gamma} CZ(\gamma) \geqslant 2n - 2,$$

since every orbit γ on the boundary of an ellipsoid had $CZ(\gamma) \geqslant n + 1$. Moreover, equality holds if and only if Ξ' is the length 1 sequence of the minimum action orbit κ. Overall we conclude that

$$\mathrm{ind}(v) \leqslant 0$$

with equality if and only if

$$u = v, \qquad \beta_+ = \gamma_+, \qquad \beta_- = \gamma_- \qquad \text{and} \qquad \Gamma = \kappa.$$

By hypothesis, every somewhere injective curve is non-negative index. Therefore $\mathrm{ind}(v) = 0$ and thus it coincides with u, and their ends coincide with γ_+ and $\gamma_- \cup \kappa$ as required. □

Remark 4.30. The reasoning in Lemma 4.29 relies on Setup 4.1(vi). In particular, we use the fact that γ_+ and γ_- are in the *minimal period* Morse-Bott family of orbits of dimension $2n - 2$.

Lemma 4.31. *Let $\varepsilon > 0$ in Setup 4.1 be sufficiently small and choose J', J'' as in Setup 4.28. Then*

(i) *The moduli spaces $\mathcal{M}_{X \setminus E}$ and $\mathcal{M}_E$ are compact, regular and 0-dimensional.*

(ii) *We have an equality of moduli spaces*

$$\overline{\mathcal{M}}_{X \setminus E \sqcup E} = \mathcal{M}_{X \setminus E} \times \mathcal{M}_E$$

Proof. The regularity and dimension of both moduli spaces follows from our choice of J' and J''. We now argue for the compactness of $\mathcal{M}_{X\backslash E}$ and $\mathcal{M}_E$, and the equality (ii).

First, to show that $\mathcal{M}_{X\backslash E}$ is compact, consider an arbitrary building

$$\bar{u} \in \overline{\mathcal{M}}(\gamma_+, \gamma_- \cup \kappa; J') = \overline{\mathcal{M}}_{0,A}(X\backslash E, J'; \gamma_+, \gamma_-)$$

Denote by u the level of $\bar{u}$ in $X\backslash E$. Then, u is genus 0 and has one positive end asymptotic to an orbit ζ_+ with $\mathcal{A}(\zeta_+) \leqslant \mathcal{A}(\gamma_+)$, a negative end ζ_- satisfying $\mathcal{A}(\zeta_-) \geqslant \mathcal{A}(\gamma_-)$ and some other negative ends Ξ. Thus by Lemma 4.29, we have

$$\zeta_+ = \gamma_+, \qquad \zeta_- = \gamma_-, \qquad \text{and} \qquad \Xi = \kappa.$$

This implies that $\bar{u} = u$ and so $\mathcal{M}_{X\backslash E}$ is compact.

Next, to show that $\mathcal{M}_E$ is compact, let $\bar{v}$ be a building in the compactification $\overline{\mathcal{M}}(\kappa; z, J'')$, then there is a bottom level

$$v : \Sigma \to \hat{E} \qquad \text{with} \qquad z \in \text{Im}(v)$$

Since the whole building is genus 0, so is v, and v has no negative ends since E is a cobordism to the empty set. Moreover, the action of the positive ends Ξ_+ must be less than $\mathcal{A}(\kappa)$. Since κ has the minimum action over all orbit sets, we must have $\Xi_+ = \kappa$, so

$$v \in \mathcal{M}_E \qquad \text{and} \qquad v = \bar{v}$$

Finally, to show (ii), let $\bar{u}$ be a building in the compactified moduli space $\overline{\mathcal{M}}_{X\backslash E \sqcup E}$. Then the level u in $X\backslash E$ satisfying the hypotheses of Lemma 4.29, and so we have

$$u \in \mathcal{M}_{X\backslash E}$$

The level v of $\bar{u}$ in E must thus have positive ends of action bounded by the actions of the negative end κ of u. This is only possible if

$$v \in \mathcal{M}_E$$

The only remaining levels in $\bar{u}$ must be trivial for action reasons, so $\bar{u}$ consists only of the levels u and v. This proves the result. $\square$

Given a choice of J' and J'', there is a family $J_{[1,\infty)}$ of almost complex structures on X,

$$J'\#_R J'' \qquad \text{for} \qquad R \in [1, \infty).$$

This family is acquired by identifying X with the space

$$X \simeq X\backslash E \cup_{\partial E} [0, R] \times \partial E \cup_{\partial E} E$$

and setting $J'\#_R J''$ be equal to J' on $X\backslash E$, J'' on E and letting $J'\#_R J''$ be translation invariant on $[0, R] \times \partial E$ in the $[0, R]$-direction. This family has associated parametric moduli space of genus 0 curves from γ_+ to γ_-, i.e. the space

$$\mathcal{M}_{0,A,1}(X, J_{[1,\infty)}; \gamma_+, \gamma_-).$$

This map has an evaluation map ev to $\hat{X}$, and we now need to consider the parametric moduli space of cylinders passing through z. We adopt the shorthand notation

$$\widetilde{\mathcal{M}}_{[1,\infty)} := \text{ev}^{-1}(z) \subset \mathcal{M}_{0,A,1}(X, J_{[1,\infty)}; \gamma_+, \gamma_-).$$

See §2.2.9 for a discussion of parametric moduli spaces. By the appropriate version of SFT compactness (cf. [BEH+03, Prop. 10.6]), given a sequence of elements

$$(R_i, u_i) \in \widetilde{\mathcal{M}}_{[1,\infty)} \qquad \text{with} \qquad R_i \to \infty$$

there is a limiting building

$$\bar{u} \in \overline{\mathcal{M}}_{X\backslash E \sqcup E} \overset{Lemma\ 4.31}{=} \mathcal{M}_{X\backslash E} \times \mathcal{M}_E$$

that u_i converges to with respect to the BEHWZ topology in [BEH$^+$03]. Since $\mathcal{M}_{X \setminus E}$ and $\mathcal{M}_E$ consist entirely of regular curves (by Setup 4.28), there is a gluing map

$$\text{glue} : \mathcal{M}_{X \setminus E} \times \mathcal{M}_E \times (R, \infty) \to \widetilde{\mathcal{M}}_{[1,\infty)}$$

that is a homeomorphism for sufficiently large R. The construction of this gluing map is carried out, for example, in [Par15, §5][8].

Lemma 4.32. *There exists a $[0, \infty)$-family of compatible almost complex structures on X, denoted*

$$J_{[0,\infty)} = \{J_R\}_{R \in [0,\infty)}$$

that has the following properties.

    (i) *$J_R = J_f$ for R near 0 and $J_R = J' \#_R J$ for R sufficiently large.*
    (ii) *The parametric moduli space*

$$\widetilde{\mathcal{M}}_{[0,\infty)} := \text{ev}^{-1}(z) \subset \mathcal{M}_{0,A,1}(X, J_{[0,\infty)}; \gamma_+, \gamma_-)$$

       is a 1-manifold with boundary $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$.
    (iii) *The natural projection map to the parameter space*

$$\pi : \widetilde{\mathcal{M}}_{[0,\infty)} \to [0, \infty)$$

       is proper.

Proof. Let $\{J_R\}_{R \in [0,\infty)}$ be a 1-parameter family of compatible almost complex structures on X satisfying (i). We use $\mathcal{M}_R$ to fiber of π at R, i.e.

$$\mathcal{M}_R := \{u \ : \ (R, u) \in \widetilde{\mathcal{M}}_{[0,\infty)}\}$$

We also let $\mathcal{M}_R^i$ denote the subset of somewhere injective curves in $\mathcal{M}_R$ and $\overline{\mathcal{M}}_R$ denote the BEHWZ compactification. All of the holomorphic curves in the moduli spaces

$$\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) \qquad \text{and} \qquad \mathcal{M}_R \text{ for large } R > 0$$

consist entirely of regular curves (and no buildings) by construction of J' and J'', and Proposition 4.4. As regularity is an open condition, we automatically know that $\mathcal{M}_R$ is compact and regular (in the usual sense, not parametrically) outside of $[a, b] \subset [0, \infty)$ for some $0 < a < b < \infty$. Using Proposition 2.5, we perturb $\{J_R\}_{R \in [0,\infty)}$ on $(a - \delta, b + \delta)$ for small δ so that the moduli spaces

$$\mathcal{M}_{g,A,m}^i(X, J_{[a,b]}; \Gamma_+, \Gamma_-)$$

are parametrically regular and the natural evaluation maps

$$\text{ev} : \mathcal{M}_{g,A,1}^i(X, J_{[a,b]}; \Gamma_+, \Gamma_-) \to \hat{X}$$

are transverse to z. We now claim that under these hypotheses, we have

(4.25) $$\mathcal{M}_R^i = \mathcal{M}_R = \overline{\mathcal{M}}_R \qquad \text{for any } R \in [a, b].$$

This implies that the space $\widetilde{\mathcal{M}}_{[0,\infty)}$ is a compact 1-manifold, implying (ii) and (iii).

To prove (4.25), we consider a J_R-holomorphic building

$$\bar{u} \in \overline{\mathcal{M}}_R.$$

Let u be the level of $\bar{u}$ that contains z and let v be the underlying simple curve. By identical reasoning to Lemma 4.29, v must be a holomorphic cylinder in $\hat{X}$ of area $O(\varepsilon)$. For small ε, v must therefore must be asymptotic to closed orbits $\beta_\pm$ that are Morse perturbations of orbits in the same Morse-Bott family S_L. In particular, the Fredholm index of v satisfies

$$\text{ind}(v) \leqslant |\beta_+| - |\beta_-| \leqslant 2n - 2,$$

---

[8]Note that the gluing result in [Par15] is much more general than the one required here. In particular, we only require the gluing map for a specific stratum of the moduli space denoted $\mathcal{M}_{\text{IV,reg}}$ in [Par15].

with equality if and only if

$$(4.26) \qquad v = u, \qquad \beta_+ = \gamma_+ \quad \text{and} \quad \beta_- = \gamma_-.$$

Here $|\beta_+| - |\beta_-|$ again denotes the relative grading of β_+ and β_-.

Thus consider the case where $\text{ind}(v) < 2n - 2$. Then the parametric moduli space containing v with a marked point added,

$$\mathcal{M}^{\text{i}}_{0,A,1}(X, J_{[a,b]}; \beta_+, \beta_-),$$

is a manifold of dimension less than $2n$. By the Sard-Smale theorem, the image of the (smooth) evaluation map

$$\text{ev} : \mathcal{M}^{\text{i}}_{0,A,1}(X, J_{[a,b]}; \beta_+, \beta_-) \to \hat{X}$$

is of the first category (i.e. a countable union of nowhere dense sets) and has dense complement. In particular, there are points arbitrarily close to z that are not in the image of ev. Therefore, after pulling back $J_{[a,b]}$ by a $[a, b]$-family of small symplectomorphisms supported near z, we can assume that

$$z \notin \text{ev}\left(\mathcal{M}^{\text{i}}_{0,A,1}(X, J_{[a,b]}; \beta_+, \beta_-)\right)$$

for any β_+, β_- satisfying the hypotheses above. In particular, after modifying $J_{[a,b]}$, we may assume that v does not contain z unless $\text{ind}(v) = 2n - 2$, which implies (4.26). This concludes the proof. $\qquad\square$

We can now conclude this section with a proof of Proposition 4.26.

Proof. The compactness, transversality and dimension of the moduli spaces

$$\mathcal{M}_{X \setminus E} \qquad \text{and} \qquad \mathcal{M}_E$$

are demonstrated in Lemma 4.31. To prove (4.22), we simply note that for a choice of $J_{[0,\infty)}$ as in Lemma 4.32, the parametric moduli space

$$\widetilde{\mathcal{M}} := \pi^{-1}([0, R]) \subset \widetilde{M}_{[0,\infty)y}$$

is a cobordism from $\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z)$ to $\mathcal{M}_{X \setminus E} \times \mathcal{M}_E$ as long as $a > 0$ is close to 0 and b is sufficiently large. $\qquad\square$

5. VANISHING SPECTRAL GAP FOR ELLIPSOIDS

Our main goal for this section is to prove Theorem 6, which states that the strong closing property holds for all ellipsoids. We will show that the spectral gap vanishes for ellipsoids and therefore, the strong closing property will follow from Theorem 3.19.

Following the strategy discussed in §1.3, we proceed in two steps. First, in Section 5.1 we consider ellipsoids $E(m) = E(m_1, \ldots, m_n)$ where $m_j \in \mathbb{N}$. The Reeb flow is periodic on the boundary $\partial E(m)$, so we may apply the results of Section 4. Specifically, we use Proposition 4.4 and Proposition 4.26 to show that certain coefficients of the map U_{P_0}, defined in Example 3.3, are non-zero and deduce that the spectral gap of a certain class vanishes for such ellipsoids. Second, in Section 5.2, we use the vanishing of spectral gaps for integer ellipsoids to show that the total spectral gap vanishes for irrational ellipsoids as well.

Notation 5.1. We fix the following notation for the rest of the section. Given an n-tuple of positive numbers $a := (a_1, \ldots, a_n)$, consider the ellipsoid $E(a)$.

- We denote by $\{M_k\}_{k \in \mathbb{N}}$, or $\{M_k^a\}_{k \in \mathbb{N}}$, the sequence obtained by reordering the union $a_1 \cdot \mathbb{N} \cup \cdots \cup a_n \cdot \mathbb{N}$ to be a non-decreasing sequence with repetitions.
- We let γ_k, or γ_k^a denote the k-th periodic Reeb orbit ordered by action. Here we introduce Morse-Bott perturbations if the Reeb flow on $\partial E(a)$ is degenerate.

- We let x_k denote the generator of $CH(\partial E(a))$ corresponding to γ_k (see Example 2.16). Note that

$$|x_k| = 2n - 2 + 2k \qquad \text{and} \qquad \mathcal{A}(\gamma_k) = M_k^a$$

5.1. **Rational Ellipsoids.** Consider an integer ellipsoid $E(m) := E(m_1, \ldots, m_n)$ where $m_j \in \mathbb{N}$ for $j = 1, \ldots, n$. Denote the least common multiple of $m_1, \ldots, m_n$ by

$$T := \mathrm{lcm}(m_1, \ldots, m_n)$$

The Reeb flow on $\partial E(m)$ is periodic of period T. Since m_j divides T for all $j = 1, \ldots, n$, the common multiple T appears in the sequence $\{M_k^m\}_k$ exactly n times. We denote the first index in which T appears in M_k^m by

$$k_T := \min\{k \in \mathbb{N} : M_k^m = T\}.$$

The main goal of this subsection is to prove the following result.

Theorem 5.2. *Let $E(m)$ be an integer ellipsoid as above, and let $T := \mathrm{lcm}(m_1, \ldots, m_n)$. Then*

$$\langle U_{P_0} x_{k_T + n - 1}, x_{k_T} \rangle \neq 0.$$

Here P_0 is the tangency abstract constraint from Example 3.3.

As mentioned above, the least common multiple T of $m_1, \ldots, m_n$ occurs with multiplicity n in the sequence M_k^m. Therefore, the action $M_{k_T + n - 1}$ of $x_{k_T + n - 1}$ and the action M_{k_T} of x_{k_T} are both equal to T. This implies the following corollary of Theorem 5.2.

Corollary 5.3. *(Theorem 7) Let $E(m)$ be an integer ellipsoid. Then*

$$\mathfrak{s}_{U\sigma}(\partial E, \lambda|_{\partial E}) = \mathfrak{s}_\sigma(\partial E, \lambda|_{\partial E}) = T \qquad \text{where} \qquad U = U_{P_0} \text{ and } \sigma = x_{k_T + n - 1}.$$

Remark 5.4 (Generalizations). Theorem 5.2 and Corollary 5.3 can be generalized in two directions.

(i) If $E(a)$ is a rational ellipsoid, i.e. an ellipsoid where

$$a_i/a_j \in \mathbb{Q} \text{ for any } i, j, \qquad \text{or equivalently} \qquad c \cdot a = m \in \mathbb{Z}^n \text{ for some } c > 0.$$

In this case, the contact form on $\partial E(a)$ is simply a scaling of the one on $\partial E(m)$. Moreover, there is an equivalence of filtered groups

$$CH^L(\partial E(a)) = CH^{c \cdot L}(\partial E(c \cdot a))$$

that commutes with all (constrained) cobordism maps. Thus Theorem 5.2 and Corollary 5.3 generalize immediately to this case.

(ii) Any positive integer multiple q of the period T similarly corresponds to n classes

$$x_{k_q}, \ldots, x_{k_q + n - 1} \in CH(\partial E(m))$$

of the same action. Our proof of Theorem 5.2 almost generalizes to show that

$$\langle U_{P_0} x_{k_T + n - 1}, x_{k_T} \rangle \neq 0.$$

However, the proof uses Proposition 4.26, which we only show for the case of $q = T$. We believe that the discussion in [Sie19, §5.5], rigorously carried out using the VFC methods of [Par15], should suffice to generalize the moduli space correspondence in Proposition 4.26, but we do not carry out this analysis here.

Proof of Theorem 5.2. The result follows from combining Proposition 4.4 and Proposition 4.26, which together assert that the moduli space count corresponding to the coefficient of x_{k_T} in $U_{P_0} x_{k_T + n - 1}$ is non-zero.

Step 1 - Setup. We start by showing that our setting satisfies the assumptions of Propositions 4.4 and 4.26, which are stated in Setup 4.1. Let (Y, α) be the boundary of the integer ellipsoid

$E := E(m)$ with its standard contact form. The Reeb flow associated with this contact form is given by

$$(z_1, \ldots, z_n) \mapsto (e^{2\pi i t/m_1} z_1, \ldots, e^{2\pi i t/m_n} z_n)$$

and has period T. Consider the function $f : Y \to \mathbb{R}$ given by the restriction to ∂E of the map

$$(5.1) \qquad \tilde{f} : \mathbb{C}^n \to \mathbb{C}^n, \quad z = (z_1, \ldots, z_n) \mapsto \frac{\sum_{j=1}^n (j-1)|z_j|^2}{\sum_{j=1}^n |z_j|^2}.$$

The function f is invariant under the $\mathbb{T}^n$-action on $\mathbb{C}^n$ and thus is invariant under the Reeb flow. Its critical circles are precisely the intersections of ∂E with the complex axes

$$\gamma_i := \partial E \cap \mathbb{C}_i = \left\{ \left(0, \ldots, \sqrt{\frac{m_i}{\pi}} e^{2\pi i t/m_i}, 0, \ldots, 0\right) : t \in [0, m_i] \right\}.$$

The function f is Morse–Bott. Indeed, the restriction of its Hessian to the normal to γ_i is non-degenerate, can be seen as follows. Along γ_i the Hessian of the map $\tilde{f}$ is given by

$$(5.2) \qquad \mathrm{Hess}(\tilde{f})|_{\gamma_i} = \frac{\pi}{m_i} \cdot \mathrm{diag}(1-i, 1-i, 2-i, 2-i, \ldots, n-i, n-i),$$

which is degenerate only on the i-th $\mathbb{C}$ factor. The tangent space to ∂E along γ_i is given by

$$T\partial E|_{\gamma_i} = \mathbb{C}^{i-1} \oplus \left\langle R|_{\gamma_i} \right\rangle \oplus \mathbb{C}^{n-i},$$

where R is the Reeb vector field. Therefore, the Hessian of f along γ_i is degenerate only in the Reeb direction which lies in the tangent space of γ_i, so f is Morse-Bott with index

$$\mathrm{ind}(f, \gamma_i) = 2(i-1) \qquad \text{at the critical circle } \gamma_i.$$

The orbits γ_+ and γ_- are the period T iterate of γ_1 and γ_n, i.e.

$$\gamma_+ := \gamma_1^{T/m_1} \qquad \text{and} \qquad \gamma_- := \gamma_n^{T/m_n}.$$

The contact flag is given by the following sequence $Y_3 \subset \cdots \subset Y_{2n-1} = Y$ of nested ellipsoids:

$$Y_{2j-1} := \partial E \cap \left(\mathbb{C}^{j-1} \times \{0\}^{n-j} \times \mathbb{C}\right), \qquad j = 2, \ldots, n.$$

Since $Y_{2j-1} \simeq \partial E(a_1, \ldots, a_{j-1}, a_n) \subset \mathbb{C}^j$, we have $H_2(Y_{2j-1}) = 0$. Moreover, Y_{2j-1} is invariant under the periodic Reeb flow and the gradient flow of f. The intersection $\xi_{2j-1} := \xi \cap TY_{2j-1}$ is a J-invariant contact structure on Y_{2j-1}. The formula (5.2) implies that the restriction of f to Y_{2j-1} is also Morse–Bott. In addition, for i and j such that $\gamma_i \subset Y_{2j-1}$ (that is, either $i = n$ or $i < j$), the restriction of $\mathrm{Hess}(f)|_{\gamma_i}$ to the symplectic orthogonal of ξ_{2j-1} in ξ_{2j+1} is equal to $\mathrm{diag}(j-i, j-i)$, which is positive definite unless $i = n$.

Finally, take z be any point in $Y_3 \backslash \{\gamma_- \cup \gamma_+\}$. Since γ_+ and γ_- are the only critical circles in Y_3, z lies in the intersection of the stable manifold of γ_- and the unstable manifold of γ_+ which concludes the assumptions stated in Setup 4.1.

Step 2 - Applying §4. Now we apply the results of §4 to show that the coefficient $\left\langle U_{P_0} x_{k_T+n-1}, x_{k_T} \right\rangle$ does not vanish. First, notice that the contact form α_f on Y has generators only in even gradings. Indeed, this follows from Lemma 4.8 together with the fact that the Morse indices of f on any Morse–Bott family are even. This is due to the fact that the Morse–Bott families are products of some of the $\mathbb{C}$ factor, and the Hessian of f is a scalar matrix on each $\mathbb{C}$ factor. We conclude that the differential vanishes and $CH(S^{2n-1}, \xi_{std}) \cong A(Y, \alpha_f)$. Consider the trivial exact cobordism

$$X = [-\delta, \delta] \times Y : (Y, e^\delta \alpha_f) \to (Y, e^{-\delta} \alpha_f).$$

We let $W \subset X$ be an embedded irrational ellipsoid with minimal Reeb orbit κ. By Proposition 4.4 and 4.26, for any $\varepsilon > 0$ sufficiently small as in Setup 4.1, we can choose a compatible complex structure J on $X \backslash W$ and $\delta > 0$ (as in §4.4) such that the moduli space

$$\overline{\mathcal{M}}_{0,A}(\gamma_+, \gamma_- \cup \kappa) = \mathcal{M}_{0,A}(\gamma_+, \gamma_- \cup \kappa)$$

is regular, contains no buildings and has point count

$$\#\overline{\mathcal{M}}_{0,A}(\gamma_+, \gamma_- \cup \kappa) = \#\overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) = 1 \mod 2.$$

Here $A \in S(X\backslash W; \gamma_+, \gamma_- \cup \kappa)$ is the unique homology class from γ_+ to $\gamma_- \cup \kappa$ in $X\backslash W$. On the otherhand, the abstract constraint P_0 is the dual constraint (see Example 3.5) of the generator κ of $CH(W)$. Thus by Lemma 3.10, we know that

$$\langle U_{P_0} x_{k_T+n-1}, x_{k_T} \rangle = 1 \mod 2.$$

In particular, $\langle U_{P_0} x_{k_T+n-1}, x_{k_T} \rangle$ is non-zero. $\qquad\square$

5.2. **Irrational Ellipsoids.** We now use the non-vanishing of coefficients of U_{P_0} for integer ellipsoids, established in Theorem 5.2, to conclude that the spectral gap vanishes for all ellipsoids.

Theorem 5.5. *For any ellipsoid $E(a) := E(a_1, \ldots, a_n)$, we have $gap(\partial E(a)) = 0$.*

We begin by giving a very simple proof using the rational ellipsoid case (Corollary 5.3 and Remark 5.4) and Proposition 3.22. We start with the following approximation property for ellipsoids. This uses Dirichlet's approximation theorem.

Lemma 5.6. *Let $E(a)$ be any ellipsoid and fix $\varepsilon > 0$. There exists an ellipsoid $E(r) = E(r_1, \ldots, r_n)$ with rationally dependent entries such that*

$$(5.3) \qquad E(r) \subseteq E(a) \subset \sqrt{1 + \frac{\varepsilon}{T}} \cdot E(r)$$

Here T denotes the period of the Reeb flow on $\partial E(r)$.

Proof. We apply the simultaneous version of Dirichlet's approximation theorem to the sequence

$$\frac{1}{a_1}, \ldots, \frac{1}{a_n}.$$

This result states that, for any natural number N, there exist integers $q_1, \ldots, q_n$, and T' such that $T' \leqslant N$ and for all $i = 1, \ldots, n$,

$$\left| \frac{1}{a_i} - \frac{q_i}{T'} \right| \leqslant \frac{1}{T' \cdot N^{1/n}}, \quad \text{or equivalently,} \quad |a_i q_i - T'| \leqslant \frac{a_i}{N^{1/n}}.$$

For $\varepsilon > 0$ fixed as in the lemma statement, choose N so that

$$N > \left(2\frac{\max\{a_1, \ldots, a_n\}}{\varepsilon} \right)^n$$

Under this hypothesis on N, we have the following bound for each i,

$$(5.4) \qquad |a_i q_i - T'| < \varepsilon/2.$$

Denote $T := T' - \varepsilon/2$ and $r_i := \frac{T}{q_i}$, and consider

$$E(r) := E(r_1, \ldots, r_n) = E\left(\frac{T}{q_1}, \ldots, \frac{T}{q_n} \right).$$

Clearly, the entries of $E(r)$ are rationally dependent. To check that $E(r) \subset E(a)$, we note that

$$a_i \overset{(5.4)}{>} \frac{T'}{q_i} - \frac{\varepsilon}{2q_i} = \frac{T' - \varepsilon/2}{q_i} = r_i.$$

To check the second inclusion in (5.3), we note that

$$a_i \overset{(5.4)}{<} \frac{T'}{q_i} + \frac{\varepsilon}{2q_i} = \frac{T' + \varepsilon/2}{q_i} = \frac{T + \varepsilon}{q_i} = (1 + \frac{\varepsilon}{T}) \cdot \frac{T}{q_i} = (1 + \frac{\varepsilon}{T}) \cdot r_i.$$

Therefore, we find that

$$E(a) \subset \sqrt{1 + \frac{\varepsilon}{T}} \cdot E(r) = E\left((1 + \frac{\varepsilon}{T}) \cdot r \right)$$

Note that T as above is a positive integer satisfying $T/r_i = q_i$ for all $i = 1, \ldots, n$, but it is not necessarily the smallest positive integer with this property (i.e. the period of the Reeb flow on $\partial E(r)$). However, for any $T_0 < T$ we have

$$\sqrt{1 + \frac{\varepsilon}{T}} \cdot E(r) \subset \sqrt{1 + \frac{\varepsilon}{T_0}} \cdot E(r)$$

and so the assertion of the lemma still holds.   □

Having established a good enough approximation of a general ellipsoid by a rational one, we are ready to prove that the spectral gap vanishes for all ellipsoids.

Proof of Theorem 5.5. Let $E(a)$ be any ellipsoid. By Lemma 5.6, for each $\varepsilon > 0$ there exists an ellipsoid $E(r(\varepsilon))$ with rationally dependent entries $r_i(\varepsilon)$ such that

$$E(r(\varepsilon)) \subseteq E(a) \subset \sqrt{1 + \frac{\varepsilon}{T(\varepsilon)}} \cdot E(r(\varepsilon))$$

Here $T(\varepsilon)$ is the period of the Reeb flow on $\partial E(r(\varepsilon))$. This implies that the contact forms satisfy

$$\lambda_{\mathrm{std}}|_{\partial E(r(\epsilon))} \leqslant \lambda_{\mathrm{std}}|_{\partial E(a)} \leqslant (1 + \frac{\varepsilon}{T}) \cdot \lambda_{\mathrm{std}}|_{\partial E(r)}.$$

Here the contact forms on $\partial E(r(\epsilon))$ and $\partial E(a)$ are identified with contact forms on the sphere, and the order on contact forms is as in Notation 3.21. By Corollary 5.3 and Remark 5.4, there is a class

$$\sigma(\varepsilon) \in CH(\partial E(r(\varepsilon))) \qquad \text{with} \qquad \mathfrak{s}_{U_{P_0}\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = \mathfrak{s}_{\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = T(\varepsilon)$$

In particular, the spectral gap of $\partial E(r(\varepsilon))$ satisfies

$$\mathrm{gap}_{\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = 0$$

The result is now an immediate consequence of Proposition 3.22, which asserts that

$$\mathrm{gap}(E(a)) \leqslant \lim_{\varepsilon \to 0} \mathrm{gap}_{\sigma(\varepsilon)}(E(r(\varepsilon))) = 0 \qquad\qquad\qquad □$$

5.3. **Structure Of The U-Map.** In [Iri22], Irie stated a second conjecture, which claims that certain sequence of coefficients of the U_{P_0} map do not vanish for ellipsoids. Our methods suffice to prove a related (but weaker) structure result of this type.

Theorem 5.7. *Let $E(a)$ be an irrational ellipsoid. There exists a sequence $k(i) \xrightarrow[i \to \infty]{} \infty$ such that*

$$(5.5) \qquad \langle U_{P_0} x_{k(i)+n-1}, x_{k(i)} \rangle \neq 0 \qquad \text{and} \qquad \mathfrak{s}_{x_{k(i)+n-1}}(\partial E(a)) - \mathfrak{s}_{x_{k(i)}}(\partial E(a)) \xrightarrow[i \to \infty]{} 0.$$

This statement can compared to [Iri22, Conj. 5.1][9]. Theorem 5.7 implies Theorem 5.5 in the irrational case, and also uses the approximation of irrational ellipsoids by rational ones, given in Lemma 5.6.

Proof. We will show that, for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$(5.6) \qquad\qquad \langle U_{P_0} x_{k+n-1}, x_k \rangle \neq 0 \qquad \text{and} \qquad \mathfrak{s}_{x_{k+n-1}}(E(a)) - \mathfrak{s}_{x_k}(E(a)) \leqslant \varepsilon.$$

Since $E(a)$ is irrational, the spectral invariants $\mathfrak{s}_{x_k}(\partial E(a)) = M_k^a$ are strictly increasing in k. Therefore, for any fixed k, the difference in (5.6) is non-zero. This implies that we can construct a sequence of distinct $k(i)$ diverging to ∞ satisfying (5.5).

Thus, fix $\varepsilon > 0$. Let $E(r)$ be the ellipsoid with rationally dependent entries from Lemma 5.6, and let T be the period of the Reeb flow on $\partial E(r)$. Our proof consists of three steps. First, we study the cobordism map

$$\Phi : A(\partial E((1 + \frac{\varepsilon}{T}) \cdot r)) \to A(\partial E(a)) \qquad \text{of the cobordism map} \qquad X = E((1 + \frac{\varepsilon}{T}) \cdot r)) \backslash E(a)$$

---

[9]However, we warn the reader that there is a sizeable difference in our notation from [Iri22].

Next, we use the information we derive on Φ to compare the U_{P_0}-maps for the two ellipsoids, apply Theorem 5.2 to $E((1 + \frac{\varepsilon}{T}) \cdot r))$ and deduce the non-vanishing of a certain coefficient in the U_{P_0}-map of $E(a)$. Finally, we compare the actions of the relevant orbits in both ellipsoids and conclude that the difference in (5.6) is bounded by ε.

We let $\{y_k\}_k$ denote the generators of $A(\partial E(r))$. Note that $A(\partial E((1 + \frac{\varepsilon}{T})r)) = A(\partial E(r))$ where the action is scaled by $1 + \frac{\varepsilon}{T}$. We will only use $\mathcal{A}(y_k)$ to denote the action with respect to $\partial E(r)$. We also let k_T be the first index in which T appears in the sequence $\{M_k^r\}_k$.

Step 1 - Cobordism Map. The goal of this step is to show that the image of y_{k_T+n-1} under Φ does not contain $x_1 \cdot x_{k_T}$ for sufficiently small ε. That is

$$(5.7) \qquad \left\langle \Phi(y_{k_T+n-1}), x_1 \cdot x_{k_T} \right\rangle = 0 \qquad \text{for sufficiently small } \varepsilon > 0.$$

Suppose otherwise. Then since the map Φ decreases action and $\mathcal{A}(x_1) = M_1^a = a_1$, we have

$$(5.8) \qquad (1 + \frac{\varepsilon}{T}) \cdot M_{k_T+n-1}^r = (1 + \frac{\varepsilon}{T}) \cdot \mathcal{A}(y_{k_T+n-1}) \geqslant \mathcal{A}(x_1) + \mathcal{A}(x_{k_T}) = a_1 + M_{k_T}^a.$$

On the other hand, by the definition of k_T and Lemma 5.6, we have

$$M_{k_T+n-1}^r = M_{k_T}^r = T \qquad \text{and} \qquad M_k^r \leqslant M_k^a \leqslant (1 + \frac{\varepsilon}{T}) \cdot M_k^r \quad \text{for all } k.$$

We can then calculate that

$$(1 + \frac{\varepsilon}{T}) \cdot M_{k_T+n-1}^r - M_{k_T}^a = (1 + \frac{\varepsilon}{T}) \cdot T - M_{k_T}^a \leqslant (1 + \frac{\varepsilon}{T}) \cdot T - M_{k_T}^r = (1 + \frac{\varepsilon}{T}) \cdot T - T = \varepsilon.$$

When ε is smaller than a_1, this contradicts (5.8) and thus proves (5.7).

Step 2 - U-map. The goal of this step is to prove that

$$(5.9) \qquad \left\langle U_{P_0}(x_{k_T+n-1}), x_{k_T} \right\rangle \neq 0.$$

To prove this, we consider the commutative diagram

$$
\begin{CD}
A(\partial E((1 + \frac{\varepsilon}{T}) \cdot r)) @>{U_{P_0}}>> A(\partial E((1 + \frac{\varepsilon}{T}) \cdot r)) \\
@VV{\Phi}V @VV{\Phi}V \\
A(\partial E(a)) @>{U_{P_0}}>> A(\partial E(a))
\end{CD}
$$

We need three observations about the terms in this diagram. First, by Theorem 5.2 and Remark 5.4, we know that that

$$(5.10) \qquad c_1 := \left\langle U_{P_0}(y_{k_T+n-1}), x_{k_T} \right\rangle \neq 0.$$

Moreover, since Φ is a $\mathbb{Z}$-graded isomorphism of algebras that decreases word length (by Lemma 2.19), we must have

$$(5.11) \qquad c_2 := \left\langle \Phi(y_k), x_k \right\rangle \neq 0,$$

Finally, by (5.7) we know that $\Phi(y_{k_T+n-1})$ has a zero $x_1 \cdot x_{k_T}$-coefficient. In other words

$$(5.12) \qquad \left\langle \Phi(y_{k_T+n-1}), x_1 \cdot x_{k_T} \right\rangle = 0.$$

Combining the equations (5.10-5.11), we can thus conclude that

$$\left\langle U_{P_0} \circ \Phi(y_{k_T+n-1}), x_{k_T} \right\rangle = \left\langle \Phi \circ U_{P_0}(y_{k_T+n-1}), x_{k_T} \right\rangle = c_1 \cdot \left\langle \Phi(y_{k_T}), x_{k_T} \right\rangle = c_1 \cdot c_2 \neq 0.$$

We now claim that (5.12) implies the desired formula (5.9). Indeed, consider the element

$$h := \left\langle \Phi(y_{k_T+n-1}), x_{k_T+n-1} \right\rangle \cdot x_{k_T+n-1} - \Phi(y_{k_T+n-1})$$

Since Φ preserves grading and respects word length, h is a linear combination of monomials $x_{i_1} \ldots x_{i_k}$ of length $k \geqslant 2$. Furthermore, $\left\langle h, x_1 \cdot x_{k_T} \right\rangle = 0$ by (5.12). Since U_{P_0} satisfies the Leibniz rule (Lemma 3.12), the only terms of word-length greater than 1 that can be mapped to x_{k_T} are of the form $x_j \cdot x_{k_T}$, for j such that $U_{P_0}(x_j)$ is a multiple of the unit. Since the U_{P_0} decreases degree

by $2n - 2$, the only elements that could be mapped to a constant have to be of degree $2n - 2$ and hence a multiple of x_1. Over all, $x_1 \cdot x_{k_T}$ is the only term of word-length greater than 1 that can be mapped to x_{k_T}. As a consequence, we conclude that $\langle U_{P_0}(h), x_{k_T} \rangle = 0$. Therefore,

$$\langle \Phi(y_{k_T+n-1}), x_{k_T+n-1} \rangle \cdot \langle U_{P_0}(x_{k_T+n-1}), x_{k_T} \rangle = \left(\langle U_{P_0} \circ \Phi(y_{k_T+n-1}), x_{k_T} \rangle + \langle U_{P_0}(h), x_{k_T} \rangle \right)$$
$$= \langle U_{P_0} \circ \Phi(y_{k_T+n-1}), x_{k_T} \rangle \neq 0.$$

Since $\langle \Phi(y_{k_T+n-1}), x_{k_T+n-1} \rangle \neq 0$, this implies (5.9).

Step 3 - action comparison. Now choose $k = k_T$. We proved the U-map formula in (5.6) in Step 2, and we conclude by showing that the difference in (5.6) is bounded by ε. First, note that by definition

(5.13) $$\mathfrak{s}_{x_{k+n-1}}(\partial E(a)) - \mathfrak{s}_{x_k}(\partial E(a)) = M_{k+n-1}^a - M_k^a.$$

Recalling that $M_k^r \leqslant M_k^a \leqslant (1 + \frac{\varepsilon}{T}) \cdot M_k^r$ for all k, we have

$$\mathfrak{s}_{x_{k+n-1}}(\partial E(a)) - \mathfrak{s}_{x_k}(\partial E(a)) \leqslant (1 + \frac{\varepsilon}{T}) M_{k+n-1}^r - M_k^r = (1 + \frac{\varepsilon}{T}) \cdot T - T = \varepsilon.$$

This proves the desired bound, and concludes the proof. □

Remark 5.8 (Generalizations). Theorem 5.7 can be extended to rational ellipsoids given a more general correspondence result than Proposition 4.26, such as in [Sie19]. This is due to the fact that our assumption that the period of the orbits $\gamma_{\pm}$ is *equal* to the minimal period of the flow, rather than is *divisible* by this minimal period, was used only in Section 4.4. Given a more general correspondence result, one would be able to prove Theorem 5.2 for any T that is divisible by $\text{lcm}(m_1, \ldots, m_n)$, and hence conclude the assertion of Theorem 5.7 for rational ellipsoids.

6. Closing lemma for a Reeb flow that is not nearly integrable.

In this section, we apply the results of §3 and §4 to prove the strong closing property for the family of non-toric contact forms α_a discussed in §1.4.

6.1. **Contact Manifolds With Torus Actions.** In order to study our family of examples, we will need a few results about contact manifolds with torus actions.

6.1.1. *Lifting Torus Actions.* Recall that a contact manifold Y with contact form α is *periodic* if the Reeb flow is periodic. In this case, the quotient $X = Y/S^1$ is a symplectic orbifold.

Here we investigate the lifting of toric structures from the symplectic orbifold X to Y via the projection $\pi : Y \to X$. We require the following lemma.

Lemma 6.1. *Let (Y, α) be a periodic contact manifold with symplectic orbifold quotient X. Then the bijection between Reeb-invariant smooth functions on Y and smooth functions on X*

$$C^{\infty}(X; \mathbb{R}) \simeq C_R^{\infty}(Y; \mathbb{R}) \qquad \text{given by} \qquad H \mapsto \tilde{H} = H \circ \pi$$

has the following properties.

(a) *(Vector-fields) The Hamiltonian vector-field X_H of H on X and the contact Hamiltonian vector-field $V_{\tilde{H}}$ of $\tilde{H}$ are related by*

$$V_{\tilde{H}} = \tilde{H} \cdot R + (X_H)_{\xi},$$

where $(X_H)_{\xi}$ denotes the lift of X_H to a vector-field tangent to $\xi = \ker(\alpha)$. In particular

$$\mathcal{L}_{V_{\tilde{H}}} \alpha = 0.$$

(b) *(Commutator) If F and H are two Hamiltonians on X, then*

$$[V_{\tilde{F}}, V_{\tilde{H}}] = \{F, H\} \cdot R + [X_F, X_H]_{\xi}.$$

(c) *(Periodic) If X_H is periodic with period 1 and H has a critical point p with $H(p) \in \mathbb{Z}$, then V_H is periodic of period 1.*

Proof. For (a), we note that the contact Hamiltonian vector-field $V_{\tilde{H}}$ of $\tilde{H}$ satisfies

$$\alpha(V_{\tilde{H}}) = \tilde{H} \qquad \text{and} \qquad d\alpha(V_{\tilde{H}}, -) + d\tilde{H} = d\tilde{H}(R) \cdot \alpha = 0$$

The first identity says that the R-component of $V_{\tilde{H}}$ is $\tilde{H} \cdot R$ and the second identity says that the ξ-component of $V_{\tilde{H}}$ is X_H. For (b), note that

$$\alpha([V_{\tilde{F}}, V_{\tilde{H}}]) = V_{\tilde{F}}(\alpha(V_{\tilde{H}})) - (\mathcal{L}_{V_{\tilde{F}}}\alpha)(V_{\tilde{H}}) = d\tilde{H}(V_{\tilde{F}}) = dH(X_F) = \{F, H\}$$

Moreover, we have

$$d\alpha([X_F, X_H]_\xi, -) = d\alpha([V_{\tilde{F}}, V_{\tilde{H}}], -) = -d\{F, H\}$$

These two identities imply that the R-components and ξ-components of the two vector-fields in (b) agree. Finally, to show (c), let H be a Hamiltonian on X with $H(p) \in \mathbb{Z}$ at some critical point p, and suppose that X_H is periodic of period 1. The contact Hamiltonian vector field of any constant Hamiltonian C is $C \cdot R$. Therefore, the flow generated by the constant Hamiltonian $C = H(p) \in \mathbb{Z}$ is 1-periodic. As a consequence, it is enough to show that the flow of $H' = H - H(p)$ is 1-periodic. Notice that $H'(p) = 0$ and let q be a non-critical point of $X_{H'}$. Choose an arc η from p to q and let $f : D \to X$ be the disk acquired by flowing η by $X_{H'}$ for time 1. The area of this disk is

$$\int_D \omega = -\int_\eta dH' = -(H'(q) - H'(p)) = -H'(q)$$

Thus the flow of the horizontal lift $(X_{H'})_\xi$ of $X_{H'}$ over γ carries a point y with $\pi(y) = q$ to the time $-H'(q)$ Reeb flow of y. On the other hand

$$V_{\tilde{H}'} = \tilde{H}' \cdot R + (X_{H'})_\xi$$

The flow of $\tilde{H}' \cdot R$ flows y to the time $H'(q)$ Reeb flow of y. Therefore, the flow by $V_{\tilde{H}'}$ carries y to itself after time 1. □

Remark 6.2. Note that if $H : X \to \mathbb{R}$ generates a circle action on a symplectic orbifold (X, ω) and ω is rational (i.e. $[\omega] \in H^2(X; \mathbb{Q})$) then by the same calculation as in the proof of Lemma 6.1(c) above, we have

$$H(q) \in \mathbb{Q} \qquad \text{for any critical point } q \text{ of } H$$

This hypothesis holds, for instance, when $X = Y/S^1$ is the orbifold quotient of a contact manifold with periodic Reeb flow.

An immediate consequence of Lemma 6.1 is the following lifting property for torus actions.

Lemma 6.3. *Let (Y, α) be a closed periodic contact manifold with symplectic orbifold quotient X. Then any Hamiltonian $\mathbb{T}^k$-action on X lifts to a Hamiltonian $\mathbb{T}^k$-action on Y preserving α. Conversely, any Hamiltonian $\mathbb{T}^k$-action on Y preserving α descends to a unique Hamiltonian action on X.*

Proof. Let μ be the moment map for a Hamiltonian $\mathbb{T}^k$-action on X with components $\mu_1, \dots, \mu_k$. After translating the moment map, we can assume that $\mu = 0$ at some critical point p of μ. Consider the map

$$\tilde{\mu} = \mu \circ \pi : Y \to \mathbb{R}^k$$

By Lemma 6.1(c), the components $\tilde{\mu}_i$ each generate an S^1-action that commute by Lemma 6.1(b). Thus $\tilde{\mu}$ is a moment map for a $\mathbb{T}^k$-action generated by contact Hamiltonians. The other direction of the lemma is clear. □

We say that a contact $(2n+1)$-manifold Y (respectively, a symplectic $2n$-manifold X) is *integrable* if there is a Hamiltonian $\mathbb{T}^{2n+1}$-action on Y (respectively, $\mathbb{T}^{2n}$-action on X) that is free on a dense open set. Lemma 6.3 implies the following.

Corollary 6.4. *The circle action generated by the Reeb flow on a periodic contact $(2n + 1)$-manifold Y extends to an integrable system on Y if and only if the symplectic orbifold quotient $X = Y/S^1$ is integrable.*

6.1.2. *Reeb Flows On Spaces With $\mathbb{T}^2$-Actions.* Let (Y, ξ) be a contact manifold with contact form $\alpha = \alpha_1$ and Reeb vector-field $R = R_1$ generating an S^1-action. Let $X = Y/S^1$ be the symplectic orbifold quotient, and consider a Morse function

$$H : X \to \mathbb{R}$$

generating an S^1-action on X and with minimum value 1. Then the lift

$$R_2 = V_{\widetilde{H}} = \widetilde{H} \cdot R + (X_H)_\xi$$

is a Reeb vector-field with contact form $\alpha_2 = \alpha/\widetilde{H}$. In light of Lemma 6.1, R_1 and R_2 generate a $\mathbb{T}^2$-action by Reeb vector-fields on (Y, ξ). For each $a = (a_1, a_2)$ with $a_i > 0$, we have a Reeb vector-field

$$R_a = a_1 R_1 + a_2 R_2 \quad \text{with contact form} \quad \alpha_a = \frac{\alpha}{a_1 + a_2 \widetilde{H}}.$$

Note that our assumption that H is Morse implies that the above $\mathbb{T}^2$ action is effective.

There are essentially two dynamical cases for the Reeb vector-field R_a: the periodic case and the non-periodic case. In order to describe the periodic case, we require the following generalization of the least common multiple.

Definition 6.5. The *(generalized) least common multiple* of two real numbers $s_1, s_2 > 0$ is given by

$$\operatorname{lcm}(s_1, s_2) := \inf \left\{ c \cdot q_1 \cdot q_2 \ : \ s_1 = c \cdot q_1 \text{ and } s_2 = c \cdot q_2 \text{ for } c \in \mathbb{R} \text{ and } q_1, q_2 \in \mathbb{Z} \right\},$$

Here the infimum of an empty set is defined to be $+\infty$.

Note that $\operatorname{lcm}(s_1, s_2)$ is finite if and only if s_1 and s_2 are rationally dependent. Moreover, if $a_1, a_2 \in \mathbb{N}$ then this definition coincides with the standard least common multiple.

Lemma 6.6. *If a_1 and a_2 are rationally dependent (i.e. $a_1/a_2 \in \mathbb{Q}$), then R_a is periodic of period*

$$T = \operatorname{lcm}(1/a_1, 1/a_2)$$

Moreover, the Reeb flow is Morse-Bott and the simple closed orbits come in two types.

    (a) Orbits of period T, coming in Morse-Bott families S of dimension $\dim(X) = \dim(Y) - 1$.
    (b) Isolated orbits whose projection to X is a critical point x of H, with period $(a_1 + a_2 H(x))^{-1}$.

Proof. We start by showing that R_a is periodic of the claimed period. Let φ^t denote the flow of R_a, which we can write as

$$\varphi^t = \varphi_1^{a_1 t} \circ \varphi_2^{a_2 t}$$

Here φ_i^t is the flow of the Reeb vector-field R_i, for $i = 1, 2$. We claim that $\varphi^L = \operatorname{Id}$ if and only if $L = c q_1 q_2$ for a real number $c > 0$ that satisfies

$$\frac{1}{a_1} = c q_1 \quad \text{and} \quad \frac{1}{a_2} = c q_2$$

Indeed, if $L = c q_1 q_2$ is of the stated form, then

$$a_1 L = c q_1 q_2 a_1 = q_2 \qquad a_2 L = c q_1 q_2 a_2 = q_1$$

Therefore $\varphi_1^{a_1 L} = \varphi_2^{a_2 L} = \operatorname{Id}$ since φ_1 and φ_2 are 1-periodic. Conversely, recall that the $\mathbb{T}^2$ action generated by R_1 and R_2 is effective. Therefore, there exists a free orbit of the torus action on Y, and thus a point p where

$$\varphi_i^L(p) = p \quad \text{if and only if} \quad L \in \mathbb{Z}$$

It follows that $\varphi^L(p) = p$ if and only if $a_1 L$ and $a_2 L$ are integers, i.e. if is of the form $L = c q_1 q_2$ stated above. This confirms the claim, and it now follows from Definition 6.5 that φ^t is periodic with period $T = \operatorname{lcm}(1/a_1, 1/a_2)$.

Next, we show that the orbits of R_a come in the types discussed above. First, we write

$$R_a = a_1 R_1 + a_2 R_2 = (a_1 + a_2 \cdot \widetilde{H}) R_1 + a_2 (X_H)_\xi$$

where $(X_H)_\xi$ is the horizontal lift of the Hamiltonian vector-field of H on X. Now consider a point $p \in Y$ and the orbit γ through it. If $\pi(p) \in X$ is a critical point of H, then $X_H = 0$ and γ is simply an orbit of $\left(a_1 + a_2 \widetilde{H}(\pi(p))\right) R_1$, falling into type (b). If $\pi(p)$ is not a critical point of H, then the orbit of p under the $\mathbb{T}^2$-action generated by R_1 and R_2 is free, denote this orbit by Σ. Under the identification $\Sigma \simeq \mathbb{T}^2$, γ is simply a curve of rational slope. In particular, its period is $\text{lcm}(1/a_1, 1/a_2)$, which is the period of R_a. $\qquad \square$

Lemma 6.7. *If a_1 and a_2 are rationally independent (i.e a_1/a_2 is irrational), then*

(a) *The simple orbits γ of α_a each project to a critical point of H on X.*

(b) *Every Reeb orbit γ is non-degenerate, elliptic and has even SFT grading i.e.*

$$|\gamma| = 0 \mod 2.$$

Proof. (a) follows from an identical analysis to the proof of Lemma 6.6. To see (b), choose a critical point x of H and let γ be a closed Reeb orbit with $\pi(\gamma) = x$. The period of γ is given by

$$\frac{k}{a_1 + a_2 \cdot H(x)} \quad \text{for} \quad k \in \mathbb{Z}_+.$$

Assume, first, that X is a manifold at x (i.e. an orbifold with no isotropy). Given a point $p \in \gamma$, the contact structure ξ_p is identified with $T_x X$ via the differential $d\pi : \xi \to TX$. Moreover, the restriction $d\varphi^t_{R_1}|_\xi$ of the linearized Reeb flow of R_a to ξ can be identified with the linearized Hamiltonian flow with a time reparametrization, $d\varphi^{a_2 \cdot t}_H$. Therefore, the linearized Reeb flow at time $\frac{k}{a_1 + a_2 \cdot H(x)}$ can be identified with the map

$$d\varphi^L_H(x) : T_x X \to T_x X \quad \text{where} \quad L = \frac{k a_2}{a_1 + a_2 \cdot H(x)} = \frac{k}{a_1/a_2 + H(x)}.$$

Since H is autonomous and its flow φ^t_H is 1-periodic, the matrices $A(t) = d\varphi^t_H(x)$ for $t \in \mathbb{R}$ form a 1-periodic subgroup of the symplectic automorphism group $\text{Sp}(T_p X)$. Arguing similarly to the proof of Lemma 4.8 (i.e., pick a compatible Riemannian metric, average with respect to the path $A(t)$ and obtain an invariant compatible metric) we see that after a change of basis, A is identified with

$$A(t) = \begin{bmatrix} e^{2\pi i m_1 t} & 0 & \dots & 0 \\ 0 & e^{2\pi i m_2 t} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{2\pi i m_{n-1} t} \end{bmatrix} \qquad \text{for rationally independent integers } m_i \in \mathbb{Z}$$

It is an easy consequence of Remark 6.2 and the rational independence of a_1 and a_2 that $L \in \mathbb{R} \backslash \mathbb{Q}$. This implies that $A(L)$ (and thus the linearized Reeb flow along γ and γ itself) is non-degenerate. The Conley-Zehnder index $\mod 2$ is then

$$\text{CZ}(\gamma) = \text{CZ}(A(L)) = \sum_i \text{CZ}(e^{2\pi i m_i L}) = n - 1 \mod 2$$

Thus we find that the SFT grading is

$$(n-3) + \text{CZ}(\gamma) = 2n - 4 = 0 \mod 2 \qquad \square$$

The above lemma implies that, when a_1/a_2 is irrational, the contact dg-algebra of Y with respect to the contact form α_a has no differential. Thus we can conclude the following corollary.

Corollary 6.8. *The contact homology $CH(Y, \xi)$ satisfies*

$$CH(Y, \xi) \simeq \Lambda(\bigoplus_{i=1}^{\infty} H(X; \mathbb{Q}))$$

6.1.3. *Non-Degenerate Perturbation.* We can alternately view the non-degenerate contact forms α_a as a perturbation of the Morse-Bott contact forms of the same form. More precisely, choose $r = (r_1, r_2)$ with r_1/r_2 rational. Choose $\epsilon > 0$ and let

$$(6.1) \qquad f : Y \to \mathbb{R} \qquad \text{be given by} \qquad f = \log\left(\frac{r_1 + r_2\tilde{H}}{r_1 + (r_2 + \epsilon)\tilde{H}}\right)$$

Lemma 6.9. *The function $f : Y \to \mathbb{R}$ has the following properties.*

(a) *f converges to 0 as $\epsilon \to 0$.*
(b) *For every $b = (b_1, b_2)$, f is invariant under the Reeb flow R_b.*
(c) *f is Morse-Bott with 1-dimensional families of critical points.*
(d) *$e^f \alpha_r = \alpha_a$ where $a = (r_1, r_2 + \epsilon)$.*

Proof. Properties (a) and (d) are evident from the formula for f. To show that properties (b) and (c) hold, we start by noticing that $f = g \circ \tilde{H}$ where

$$g(s) = \frac{r_1 + r_2 \cdot s}{r_1 + (r_2 + \epsilon)s}.$$

Since $\tilde{H}$ is invariant under the flow of R_b for all $b = (b_1, b_2)$ (see Section 6.1), $f = g \circ \tilde{H}$ is invariant as well, which proves (b). To prove (c), first notice that $\tilde{H}$ is Morse-Bott with 1-dimensional families of critical points, as a lift of a Morse function to an S^1 bundle. Now, the identity $f = g \circ \tilde{H}$, together with the fact that $g'(s) < 0$ when $s > 0$, implies that f is Morse-Bott with 1-dimensional families of critical points as well, which proves (c). $\qquad \square$

6.1.4. *Periodic Approximation.* Let the contact manifold (Y, ξ) and the contact forms α_a be as in the previous sub-section. We now prove the following approximation property.

Proposition 6.10. *For any $a_1, a_2 > 0$ and any $\delta > 0$ there exist rationally dependent $r_1, r_2 > 0$ such that*

$$\alpha_r \leqslant \alpha_a \leqslant (1 + \delta/T)\alpha_r$$

where $T = \text{lcm}(r_1^{-1}, r_2^{-1})$ is the period of the Reeb flow of α_r.

This approximation property will play the roll of Lemma 5.6 in our proof of Theorem 10. We require the following arithmetic fact, which follows from an argument similar to Lemma 5.6.

Lemma 6.11. *For any pair of positive numbers $a_1, a_2 > 0$ and any $\delta > 0$ small, there exist $r_1, r_2 > 0$ with $T := \text{lcm}(r_1^{-1}, r_2^{-1}) < \infty$ such that*

$$(6.2) \qquad \qquad \frac{r_i}{1 + \delta/T} \leqslant a_i \leqslant r_i.$$

Proof. By the simultaneous version of Dirichlet approximation theorem, for each $N > 0$ there exist non-zero integers q_1, q_2 and $p < N$ such that

$$\left|a_i - \frac{q_i}{p}\right| < \frac{1}{p \cdot \sqrt{N}}, \quad \text{or} \quad \left|p - \frac{q_i}{a_i}\right| < \frac{1}{\sqrt{N}a_i}.$$

Choose N and r_i for $i = 1, 2$ so that

$$\delta/2 > \max\left\{\frac{1}{\sqrt{N}a_1}, \frac{1}{\sqrt{N}a_2}\right\} \quad \text{and} \quad r_i = \frac{q_i}{p - \delta/2}$$

Given these choices, δ and r_i satisfy the following inequalities.

$$\left|\frac{p}{q_i} - \frac{1}{a_i}\right| < \frac{1}{\sqrt{N}q_i a_i} \leqslant \frac{\delta}{2q_i} \quad \text{and thus} \quad \frac{1}{r_i} = \frac{p - \delta/2}{q_i} \leqslant \frac{1}{a_i}$$

The second inequality yields $a_i \leqslant r_i$. Moreover, note that r_1 and r_2 satisfy

$$\frac{1}{r_1} = \left(\frac{p - \delta/2}{q_1 q_2}\right) \cdot q_1 \quad \text{and} \quad \frac{1}{r_2} = \left(\frac{p - \delta/2}{q_1 q_2}\right) \cdot q_2$$

By Definition 6.5, this implies that $T := \text{lcm}(r_1^{-1}, r_2^{-1}) \leqslant p - \delta/2$. Finally, we note that

$$r_i = \frac{q_i}{p - \delta/2} = \frac{q_i}{p + \delta/2} \cdot \frac{p + \delta/2}{p - \delta/2} \leqslant a_i \cdot (1 + \frac{\delta}{p - \delta/2}) \leqslant a_i \cdot (1 + \delta/T) \quad \square$$

Proof of Proposition 6.10. Given a_1, a_2 and $\delta > 0$ let r_1, r_2 be the numbers from Lemma 6.11. By Lemma 6.6, the Reeb flow induced by α_r is periodic of minimal period $T := \text{lcm}(r_1^{-1}, r_2^{-1})$. By the assertion of Lemma 6.11

$$\frac{r_i}{1 + \delta/T} \leqslant a_i \leqslant r_i.$$

This implies the required inequality between the contact forms:

$$\alpha_r = \frac{\alpha}{r_1 + r_2 \widetilde{H}} \leqslant \frac{\alpha}{a_1 + a_2 \widetilde{H}} = \alpha_a \leqslant (1 + \delta/T) \cdot \frac{\alpha}{r_1 + r_2 \widetilde{H}} = (1 + \delta/T) \cdot \alpha_r. \quad \square$$

6.2. **A family of non-integrable contact forms.** In this section, we review the construction of the family of contact manifolds discussed in §1.4, filling in some details that were omitted in the introduction.

6.2.1. *Preliminaries.* Let us first, briefly, recall the discussion in §1.4. Consider the prequantization space

$$E \to \mathbb{C}P^1 \times \mathbb{C}P^1 \quad \text{with} \quad c_1(E) = A_1 + A_2$$

Here A_1, A_2 denote the cohomology classes Poincare dual to $[\mathbb{C}P^1 \times \text{pt}]$ and $[\text{pt} \times \mathbb{C}P^1]$. We let ξ denote the contact structure, α_1 denote the standard prequantization contact form and R_1 to denote its Reeb vector-field, which generates a 1-periodic circle action on E. We next let H denote the Hamiltonian

$$H : X \to \mathbb{R} \quad \text{given by} \quad H(x, y) = \frac{\pi}{2} \left(\frac{|x|^2}{(1 + |x|^2)} + \frac{|y|^2}{(1 + |y|^2)} \right) + 1 \quad \text{on} \quad \mathbb{C} \times \mathbb{C} \subset \mathbb{C}P^1 \times \mathbb{C}P^1.$$

By Lemma 6.1, this action lifts to a free circle action on E generated the Reeb vector-field

$$R_2 = \widetilde{H} \cdot R_1 + (X_H)_\xi \quad \text{for the contact form} \quad \alpha_2 = \frac{\alpha}{\widetilde{H}}$$

Moreover, R_2 commutes with the Reeb vector-field R_1 of α. This yields a contact $\mathbb{T}^2$-action

$$\mathbb{T}^2 \curvearrowright (E, \xi) \quad \text{generated by} \quad R_1, R_2.$$

Next, we consider the $\mathbb{Z}_4$-action on X generated by the following 4-periodic map.

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1 \quad \text{with} \quad \psi(x, y) = (\jmath(y), x)$$

Here $\jmath : \mathbb{C}P^1 \to \mathbb{C}P^1$ denotes the antipode map $[u : v] \mapsto [-u : v]$. This $\mathbb{Z}_4$-action lifts to a free action of $\mathbb{Z}_4$ on E as follows. We may view E as the unit circle bundle of the tensor product

$$L_1 \otimes L_2 \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

where L_i is the pullback (under projection to the ith factor) of the $O(1)$ line bundle over $\mathbb{C}P^1$. The antipode lifts to a unitary map $\jmath^* : L \to L$ since it descends from a unitary map of $\mathbb{C}^2$. We now define a lift of f to $L_1 \otimes L_2$ by

$$\Psi : L_1 \otimes L_2 \to L_1 \otimes L_2 \quad \text{with} \quad (x, y; u \otimes v) \mapsto (\jmath(y), x; i \cdot \jmath^* v \otimes u)$$

Here i denotes complex multiplication by i. Ψ is unitary, and thus restricts to E.

Remark 6.12. Ψ generates a free $\mathbb{Z}_4$-action on E that preserves α and α_2, and descends to the $\mathbb{Z}_4$-action generated by ψ on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

6.2.2. *Construction Of Family.* We can now define the contact manifold

$$Y := E/\mathbb{Z}_4$$

We may view Y as a prequantization space over the symplectic orbifold $X = \mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_4$ with a Hamiltonian S^1-action induced by H. By Remark 6.12, the contact structure ξ, contact forms α_1, α_2 and Reeb vector-fields R_1, R_2 all descend to Y. We continue to denote them by $\xi, \alpha_1, \alpha_2, R_1$ and R_2, and we let

$$R_a = a_1 \cdot R_1 + a_2 \cdot R_2 \quad \text{with contact forms} \quad \alpha_a$$

Lemma 6.13. *The Reeb vector-fields R_1 and R_2 commute and generate flows of period 1. In particular, they generate a contact $\mathbb{T}^2$-action on Y.*

Proof. The fact that $[R_1, R_2] = 0$ is local and so follows from the same fact on E.

To see that R_1 is periodic of period 1, note first that it is periodic of period at most 1 since it lifts to a periodic vector-field on E of period 1. To see that the period is equal to 1, note that a simple R_a-orbit γ in E that projects to a point of $\mathbb{C}P^1 \times \mathbb{C}P^1$ on which $\mathbb{Z}_4$ acts freely will project to a period 1 simple orbit γ' of R_a on Y, via the quotient map $E \to Y$.

The vector field R_2 is the lift of the period 1 Hamiltonian flow generated by the Hamiltonian H on the orbifold $X = \mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_4$, described in Lemma 6.15(c). In particular, it is period 1. $\quad\square$

We will also consider the following specific hypersurface in Y.

Definition 6.14 (diagonal hypersurface)**.** Let $Y_3 \subset Y$ be the codim-2 hypersurface defined as follows. Let $\Delta \simeq \mathbb{C}P^1$ be the diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$, and let E_3 be its inverse image under the projection $E \to \mathbb{C}P^1 \times \mathbb{C}P^1$. Then Y_3 is the quotient of E_3 by $\mathbb{Z}_4$.

We will require a number of topological properties of Y for our proof. These properties will be used to guarantee that the CZ index is defined over $\mathbb{Z}$, as well as the vanishing of certain homological intersections, see Remark 4.3. We record these properties below.

Lemma 6.15. *The contact manifold (Y, ξ) has the following topological properties.*

   (a) The Chern class $c_1(\xi)$ is torsion in $H^2(Y; \mathbb{Z})$.
   (b) $[Y_3] = 0 \in H_3(Y; \mathbb{Q})$. In particular, the intersection pairing of any $A \in H_2(Y; \mathbb{Q})$ with $[Y_3]$ is 0.
   (c) $H_2(Y_3; \mathbb{Q}) = 0$.

Proof. To prove (a), consider the contact structure ξ_E on E. We know that

$$c_1(\xi_E) = \pi^* c_1(\mathbb{C}P^1 \times \mathbb{C}P^1) = c \cdot c_1(E)$$

for a constant c. Now consider the Gysin sequence of the bundle $E \to \mathbb{C}P^1 \times \mathbb{C}P^1$.

$$\cdots \to H^0(\mathbb{C}P^1 \times \mathbb{C}P^1) \xrightarrow{c_1(E)\cup} H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\pi_*} H^1(\mathbb{C}P^1 \times \mathbb{C}P^1) \to \cdots$$

Since $c_1(E)$ and $c_1(\mathbb{C}P^1 \times \mathbb{C}P^1)$ are proportional, $c_1(\mathbb{C}P^1 \times \mathbb{C}P^1)$ is in the image of the map $c_1(E)\cup$ and kernel of π^*. Thus $c_1(\xi_E) = \pi^* c_1(\mathbb{C}P^1 \times \mathbb{C}P^1) = 0$. Now since $\mathbb{Z}_4$ acts freely on E, we know that pullback by the quotient map $\rho: E \to Y$ yields an isomorphism

$$\rho^* : H^2(Y; \mathbb{Q}) \simeq (H^2(E; \mathbb{Q}))^{\mathbb{Z}_4}$$

where the latter is the sub-space of $\mathbb{Z}_4$-invariant classes. Since $\rho^* c_1(\xi) = c_1(\xi_E) = 0$, this implies that $c_1(\xi)$ is torsion.

To prove (b), recall the construction of Y_3 from Definition 6.14. Consider the Gysin sequence

$$H_4(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Q}) \xrightarrow{c_1(E)\cap} H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Q}) \xrightarrow{\pi^*} H_3(E; \mathbb{Q}) \xrightarrow{\pi_*} H_3(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Q})$$

Now observe that $[\Delta] = \mathrm{PD}(c_1(E)) = c_1(E) \cap [\mathbb{C}P^1 \times \mathbb{C}P^1]$ and $[E_3] = \pi^*[\Delta]$. Thus by exactness of the Gysin sequence, we have

$$[E_3] = \pi^*[\Delta] = \pi^*(c_1(E) \cap [\mathbb{C}P^1 \times \mathbb{C}P^1]) = 0$$

This prove that $[E_3] = 0$. Since $[Y_3] = \rho_*[E_3]$ where $\rho : E \to Y$ is the quotient by $\mathbb{Z}_4$, this proves (b).

Finally, we prove (c). Note that by Poincare duality and the isomorphism of quotient cohomology with co-invariants, we have

$$H_2(Y_3; \mathbb{Q}) \simeq H^1(Y_3; \mathbb{Q}) \simeq (H^1(E_3; \mathbb{Q}))^{\mathbb{Z}_4} \subset H^1(E_3; \mathbb{Q})$$

It suffices to show that $H^1(E_3; \mathbb{Q}) = 0$. If we consider the Gysin sequence applies to $E_3 \to \Delta \simeq \mathbb{C}P^1$, we have

$$H^1(\mathbb{C}P^1; \mathbb{Q}) \xrightarrow{\pi^*} H^1(E_3; \mathbb{Q}) \xrightarrow{\pi_*} H^0(\mathbb{C}P^1; \mathbb{Q}) \xrightarrow{c_1(E_3)\cup} H^2(\mathbb{C}P^1; \mathbb{Q})$$

Since $H^1(\mathbb{C}P^1; \mathbb{Q}) = 0$ and $c_1(E_3)\cup$ is injective, this implies that $H^1(E_3; \mathbb{Q}) = 0$. $\quad\square$

We let (by some abuse of notation) α_1 and α_2 denote the contact forms descending from α and α_2 on E, and we denote the corresonding (commuting) Reeb vectors by R_1 and R_2, respectively. Both R_1 and R_2 are periodic.

Lemma 6.16. *Y does not admit a contact integrable structure extending the circle action generated by the Reeb vector-field R_1 of α.*

Proof. It is proven in [STW99] that B is not toric, so this is immediate from Corollary 6.4. $\quad\square$

6.3. **Proof of Theorem 10.** After studying the structure and dynamics of the contact manifold Y and the Reeb flows R_a, we are finally ready to prove that it has the strong closing property. Our first step towards a proof of Theorem 10 is to show that the assumptions of Proposition 4.4 hold for (Y, α_r) discussed above, when $r = (r_1, r_2)$ are rationally dependent. These assumptions were stated in Setup 4.1.

Lemma 6.17. *The contact manifold (Y, α_r) for rationally dependent $r = (r_1, r_2)$ satisfies the assumptions of Setup 4.1.*

Proof. In what follows we go over each of the assumptions, in the same order as stated in Setup 4.1, and explain why they hold for (Y, α_r). We use the notations from the previous subsections.

(i) Y is a contact manifold of dimension 5: See Section 6.2.

(ii) The flow of R_r is periodic: This is proved in Lemma 6.6.

(iii) An almost complex structure on ξ: Let us describe the complex structure on the contact structure of the prequantization bundle $E \to \mathbb{C}P^1 \times \mathbb{C}P^1$. It will be invariant under the $\mathbb{Z}_4$ action any thus descend to the contact structure ξ on Y.

Since E is a prequantization bundle, its contact structure is canonically isomorphic to the tangent bundle of the base, $T(\mathbb{C}P^1 \times \mathbb{C}P^1)$. Since the latter is a complex vector bundle, this defines an almost complex structure on the contact structure of E which descends to ξ. We denote this almost complex structure by J.

(iv) An invariant Morse–Bott function: Recall the function from Section 6.2:

$$H : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{R}, \qquad H([x, y]) := \frac{\pi}{2}\left(\frac{|x|^2}{(1 + |x|^2)} + \frac{|y|^2}{(1 + |y|^2)} \right) + 1.$$

The critical points of the latter function on $\mathbb{C}P^1 \times \mathbb{C}P^1$ are $[0, 0]$, $[\infty, \infty]$, $[\infty, 0]$ and $[0, \infty]$, which are the non-free points of the $\mathbb{Z}_4$ action. After the quotient by the $\mathbb{Z}_4$ action, these descend to three classes: $[0, 0]$, $[\infty, \infty]$ and $[\infty, 0] \sim [0, \infty]$. Denote by $\widetilde{H} : Y \to \mathbb{R}$ the function obtained by lifting H to E and projecting to the quotient by the

$\mathbb{Z}_4$ action. The perturbing function for us is the function $f : Y \to \mathbb{R}$ defined using $\tilde{H}$ by (6.1). By Lemma 6.9, the function f is Morse–Bott, invariant under the flow and its critical manifolds are circles. Note that Lemma 6.9 also guarantees that $e^f \alpha_r = \alpha_a$ for $a = (r_1, r_2 + \epsilon)$.

(v) No assumption to check.

(vi) No assumption to check. We take $\gamma_\pm$ to be orbits of R_r of period T, corresponding to the minimum and maximum of f. Here $T = \mathrm{lcm}\{r_1^{-1}, r_2^{-1}\}$ is the period of R_r. A simple calculation shows that the maximum of f (which is the minimum of $\tilde{H}$) lies over $[0, 0]$, and the minimum of f (which is the maximum of $\tilde{H}$) lies over $[\infty, \infty]$.

(vii) Point constraint: We need to choose a point z in the intersection of the ascending manifold of γ_- and the descending manifold of γ_+. The flow lines of f coincide with those of $\tilde{H}$ up to (orientation reversing) reparametrization. Consider the $\tilde{H}$ flow line given by $t \mapsto [e^t x_0, e^t x_0, 1]$, where $x_0 \neq 0$ is a fixed element of $\mathbb{C}P^1$ and by 1 in the last coordinate we mean 1 in a trivialisation of the bundle $E_{\{[e^t x, e^t x]\}}$. The image of this flow line connects γ_- and γ_+. Let us pick $z = ([x_0, x_0], 1)$.

(viii) Contact flag: Since Y is of dimension 5, we only need to define the codimension 2 submanifold of Y. Recall Definition 6.14 of the hypersurface $Y_3 \subset Y$.

  (a) As explained in Remark 4.3, the assumption that $H_2(Y_{2j-1}) = 0$ can be replaced by the weaker assumptions that the CZ index is defined over $\mathbb{Z}$ and that the homological intersection number of any 2-torus in Y_{2j+1} with Y_{2j-1} is zero. These are asserted by Lemma 6.15. More explicitly, item (a) of the lemma guarantee that the relative CZ index of any orbit is well defined over $\mathbb{Z}$, while (b) and (c) guarantee that the intersection number of any 2-torus in Y with Y_3 (resp., in Y_3 with any cylinder) is zero.

  (b) Y_3 is invariant under the Reeb flow: Since $R_r = r_1 R_1 + r_2 R_2$, it sufficient to show that Y_3 is invariant under both R_1 and R_2. Since Y_3 is a quotient of a restriction of the prequantization bundle, it is invariant under the flow R_1. R_2 is a linear combination of R_1 and $(X_H)_\xi$, which is the horizontal lift of the symplectic gradient of the function H defined above. Since this function is symmetric under replacing the two factors of $\mathbb{C}P^1 \times \mathbb{C}P^1$, its Hamiltonian flow preserves the diagonal Δ. As a conserquence, the quotient of the horizontal lift preserves Y_3.

  (c) $(Y_3, \xi|_{Y_3})$ is a contact manifold as is coincides with the $\mathbb{Z}_4$ quotient of the unit circle bundle $E|_\Delta \to \Delta$. For the same reason, α_{TY_3} is a contact form on Y_3.
  The diagonal Δ is invariant under the action of i as i acts simultaneously on each factor. This makes the contact planes $\xi|_{Y_3}$ also J-invariant.

  (d) $\nabla_J f$ is tangent to Y_3: Since f is a composition of a 1-variable function on $\tilde{H}$, it is enough to check that the gradient of $\tilde{H}$ is tangent to Y_3. Note that the metric defined by J the $d\alpha$ is the standard metric on $\mathbb{C}P^1 \times \mathbb{C}P^1$, lifted to E. With respect to this metric, $\nabla \tilde{H} = (x, y, 0)$ at any $([x, y], \theta) \in Y$. So, for any point $([x, x], \theta) \in Y_3$, $\nabla f = (x, x, 0)$ and so is tangent to Y_3.
  The function $f|_{Y_3}$ is Morse-Bott as f is Morse on the diagonal.

  (e) Conditions on Hessian of f: Y_3 contains two of the critical points, namely, $[0, 0]$, and $[\infty, \infty]$ corresponding to γ_+ and γ_-. So apart from γ_+ the only critical point in Y_3 is γ_-, which is a minimum and hence the Hessian is positive definite (and so is any restriction of it).

  (f) $z = ([x_0, x_0], 1) \in Y_3$.   □

Our next goal is to prove the vanishing of the spectral gap for the periodic flows corresponding to rationally dependent (a_1, a_2).

Theorem 6.18. *Suppose* $r = (r_1, r_2)$ *are rationally dependent and denote by* $T = \text{lcm}(r_1^{-1}, r_2^{-1})$ *the period of* R_r. *There exists a class* $\sigma \in CH(Y)$ *with* $\mathfrak{s}_\sigma(Y, \alpha_r) = T$ *and* $\text{gap}_\sigma(Y, \alpha_r) = 0$.

Proof. We start by describing the contact homology of Y. Consider the perturbation $\alpha_f := e^f \alpha_r$ for f defined using $\tilde{H}$, and given a positive parameter $\epsilon > 0$, by (6.1). As explained in Lemma 6.9, $\alpha_f := e^f \alpha_r = \alpha_a$ for $a = (a_1, a_2) := (r_1, r_2 + \epsilon)$. Therefore, for a generic choice of $\epsilon > 0$, α_f is non-degenerate. As explained in [Par15, §1.8], contact homology has a relative grading $\mathbb{Z}/2c_1(\xi) \cdot H_2(Y)$, which descends to an absolute $\mathbb{Z}_2$ grading. Lemma 6.7 guarantees that the $\mathbb{Z}_2$ grading of all of the orbits of $\alpha_f = \alpha_a$ is zero, and hence the differential of the contact dga with respect to α_f vanishes. Therefore, we identify the classes in $CH(Y)$ with the periodic orbits on α_a. Lemma 6.17 states that the assumptions of Proposition 4.4, as stated in Setup 4.1, hold for (Y, R_r) and the perturbing function f. Arguing as in the proof of Theorem 5.2, Proposition 4.4 together with Proposition 4.26, imply that $\langle U_{P_0} \gamma_+, \gamma_- \rangle = 1 \mod 2$, where P_0 is the tangency constraint from Example 3.3. We remind that here $\gamma_\pm$ are periodic orbits of R_r of period $T = \text{lcm}(r_1^{-1}, r_2^{-1})$, lying over the minimum and the maximum of f. Therefore, we may consider the class $\sigma := [\gamma_+] \in CH(Y)$ and compute its spectral gap:

$$\text{gap}_\sigma(Y, R_r) := \mathfrak{s}_\sigma(Y, \alpha_r) - \mathfrak{s}_{U_{P_0}\sigma}(Y, \alpha_r) \leqslant \mathfrak{s}_{[\gamma_+]}(Y, \alpha_r) - \mathfrak{s}_{[\gamma_-]}(Y, \alpha_r)$$

$$= \lim_{\epsilon \to 0} \int_{\gamma_+} e^f \alpha_r - \int_{\gamma_-} e^f \alpha_r = \int_{\gamma_+} \alpha_r - \int_{\gamma_+} \alpha_r = T - T = 0,$$

where we used the fact that $f \xrightarrow{\epsilon \to 0} 0$ and $T := \text{lcm}(r_1^{-1}, r_2^{-1})$ is the period of R_r, which coincides with the periods of $\gamma_\pm$ by our choice of these orbits. $\square$

Using Theorem 6.18 and Proposition 6.10 about approximations of irrational forms by rational ones, we can conclude the vanishing of the spectral gap for R_a for all $a = (a_1, a_2)$.

Corollary 6.19. *For any* $a = (a_1, a_2)$, $\text{gap}(Y, \alpha_a) = 0$. *In particular, it follows from Theorem 3 that* (Y, α_a) *satisfies the strong closing property.*

Proof. By Proposition 3.22, to conclude that $\text{gap}(Y, \alpha_a) = 0$ it is sufficient to show that there exists a sequence α_i and classes $\sigma_i \in CH(Y)$ such that $\alpha_i \leqslant \alpha \leqslant (1 + \epsilon_i)\alpha_i$, $\text{gap}_{\sigma_i}(Y, \alpha_i) = 0$ and $\epsilon_i \cdot \mathfrak{s}_{\sigma_i}(Y, \alpha_i) \to 0$. Applying Proposition 6.10 for $\delta = \frac{1}{i}$ we conclude that there exist $r^{(i)} = (r_1^{(i)}, r_2^{(i)})$ rationally dependent such that $\alpha_i := \alpha_{r^{(i)}}$ satisfies $\alpha_i \leqslant \alpha_a \leqslant \left(1 + \frac{1}{i \cdot T_i}\right)\alpha_i$, where $T_i := \text{lcm}(r_1^{(i)}, r_2^{(i)})$. Applying Theorem 6.18 to α_i we obtain classes $\sigma_i \in CH(Y)$ with $\mathfrak{s}_{\sigma_i}(Y, \alpha_i) = T_i$ and $\text{gap}_{\sigma_i}(Y, \alpha_i) = 0$. Since $\frac{1}{i \cdot T_i} \cdot T_i = \frac{1}{i} \to 0$, we conclude that the assumptions of Proposition 3.22 hold and $\text{gap}(Y, \alpha_a) = 0$. $\square$

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