MIDDLE GRADE STUDENTS' INTERPRETATIONS OF CARTESIAN AXES LABELS

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This paper examines some ways middle schoolers interpreted pre-made Cartesian graph axes labels and how this thinking interacted with their constructions of reference frames and coordinate systems. We illustrate several examples of students' interpretations along with detailed analyses of their thinking. We argue that axes labels can hold many unique meanings despite their simplicity on the surface, leading us to re-consider task design and how to anticipate and acknowledge the power of student thinking within such tasks.

Keywords: Mathematical Representations, Cognition, Middle School Education

Graphical representations are widespread in STEM fields and in everyday society (Costa, 2020; Kwon et al., 2021), necessitating students to develop meanings for graphs that enable them to interpret graphical information. Graphs contain many features students must interpret. Kosslyn (1989) classified the general "constituents" featured in a graph to include the framework (e.g., axes), the specifier (e.g., points), the background (imagery behind the display), and the labels (e.g., words, numbers, pictures). Specifically, graph labels are used to "provide interpretation" of either the framework or specifier (p. 189). Kosslyn summarized label properties that may impact reader accessibility in capturing intended interpretations, such as font styles and placement.

Given that labels serve a purpose of aiding graph interpretation and that the literature has documented graph interpretation as non-intuitive (e.g., Clement, 1989; Frank, 2016; Lai et al., 2016), it is worthwhile to consider how label design choices impact students' constructed meanings towards a graph. Additionally, students may use personal meanings in conjunction with given labels that may not align with designer intentions. Understanding what these personal meanings could consist of can help inform task design, along with anticipations of potential student thinking researchers and teachers can leverage when students are interpreting graph labels. In this paper, we address the question, "In what ways do sixth-grade students interpret and use pre-constructed axes labels?", where we view axes labels to encompass text inscriptions on the horizontal and vertical axis (e.g., words, numerical values). We discuss how such student thinking may provide insight for task design and considerations of student reasoning.

Relevant Literature and Theoretical Underpinnings

In this section, we review the literature regarding the complexities surrounding student interpretations of graph features, with our focus on axes labels in mind. We also discuss the theoretical underpinnings that guide our study.

Student Thinking Towards Related Graph Features

Research has shown that interpreting graph features is complex for students. Boote and Boote (2017) noted interpreters of graphs are required to digest copious amounts of information quickly, tasking students to "parse information needed to interpret" the graph into "visual chunks that are goal-directed" (p. 459). When balancing various features, Whitacre and Saul (2016) found that students often over-relied on textual captions to make conclusions. Although captions are outside the graph unlike axes labels within the graph, we note that students still must make sense of multiple representational formats (textual and graphical) simultaneously when thinking about axes labels. More specifically, axes as features could interfere with student interpretation of their labels. Axes can simultaneously contain several meanings that could seem conflicting as students make sense of graphs. First, axes can be viewed as empty or dense (Klein et al., 1998). Second, axes may (and are often intended to) convey both an ordering and a measurement (Herscovics, 1989), holding the "dual sense of directionality in the Cartesian space" (Leinhardt et al., 1990, p. 54). Third, researchers have identified student difficulties with creating qualitative and quantitative axes scales (Condon et al., 2024; Mevarech & Kramarsky, 1997).

Students have identified axes labels as essential to communicating information. For example, Mhlolo's (2015) study indicated that students added labels as they attempted to convey information in their graphs to others. Given that students see labels as important for communicating and often see smaller design choices as not arbitrary in comparison to mathematicians (Wagner, 1981), we argue that the design of axes labels can have more impact on graph interpretation than we may intend or are aware of.

Theoretical Perspectives Towards Meanings of Representations

We adopt the radical constructivist learning perspectives of von Glasersfeld (1982; 1987), including that representations, such as labels, do not carry internal meanings on their own. Even though someone may create a representation, those who later encounter the representation must construct their own meanings for the representation. These meanings may or may not be compatible with the original creator's; rather, meanings are constructed in alignment with what has been viable to the interpreter. We emphasize that students' constructions of these meanings are reasonable relative to their mathematical realities. In line with this perspective, we view graph labels as not holding meaning at the onset. Hence, when interpreting a label, students necessarily rely on their current meanings, such as those grounded in prior knowledge and contextual experience with labels, schooling conventions, and other possible resources such as in early encounters with drawing (diSessa et al., 1991; Sherin, 2000). In this sense, we closely examined unanticipated ways students interpreted graph labels to better understand how pre-made labels may interact with students' meanings that make sense to them in the moment. We wanted to analyze more than when meanings were not aligned with design intentions; rather, we wanted to learn more about what strategies and thinking may have built these interpretations.

Thinking Within Coordinate Systems and Reference Frames

We define a graphical representation as a spatial depiction of quantities (Thompson, 2011) to mathematize some phenomena. Here, a graphical representation consists of three layers: reference frames (RFs), a coordinate system (CS; e.g., Cartesian plane), and a graph. RFs are mental structures used to gauge the relative extents of various attributes in the phenomenon being depicted (Levinson, 2003; Lee, 2017; Joshua et al., 2015). Thinking within RFs entails attending to and establishing reference points, directionality, and having an idea of what and how to measure the quantities being depicted (Joshua et al., 2015; Lee et al., 2019). CSs are the geometric embodiment of the RFs (e.g., axes) that allow an individual to systematically express

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and coordinate RFs. Finally, a graph is a collection of points depicted upon the underlying CS. The ways of thinking about a graph fundamentally depends on the RFs and CSs upon which they are created. Axes can constitute a CS and would reflect the mental RFs an individual establishes.

We have distinguished between two ways a CS could be used: spatial or quantitative (Lee et al., 2020). Constructing a spatial CS involves an individual organizing a space by (mentally) overlaying a CS onto some physical or imagined space being represented where objects within that space are tagged with coordinates. A quantitative CS is used to coordinate sets of quantities by constructing a geometrical representation of the product of measure spaces. Constructing a quantitative CS involves an individual abstracting quantities from the space in which a phenomenon occurs and projecting them onto a new space, different from the one in which the quantities were originally conceived. Students may interpret a given graph differently and so labels may be leveraged to understand the CS in different ways. For example, a spatial interpretation of a CS may use labels as landmarks or a description of the situation while a quantitative interpretation of a CS may use labels to orient quantities.

Methods

We present data from two projects, Project A and Project B, both aimed to examine middle school students' graphing activities and meanings. The data presented here come from clinical interviews (Clement, 2000; Ginsburg, 1997). All sessions were video-recorded, and student work was digitized. An Interviewer Researcher (IR) conducted interviews (the IR varied across sessions), and most sessions also had a Witness Researcher (WR) present.

Tasks and Participants

The participants focused on in this paper were chosen because their label interpretations differed from the intended design of one of the following two tasks. The Deep Sea Diver (DSD) task was used in Project A, and the Family Frenzy (FF) task was used in Project B. In DSD (see Figure 1(a)), students used the given graph to locate points with higher and lower temperatures and speeds. The FF task (see Figure 1(b)) was implemented via Desmos and requested that students drag or sketch objects onto the CS to depict the heights and ages of given images of families. The FF task was modified from Swan's (1985) Bus Queue task.

Each task has axes labels that we intended to allude to properties of their CSs. DSD had labels such as "Swimming Speed", which were intended to convey which axis was meant to orient each quantity. There were also labels to the far sides of each axis meant to provide quantity directionality (e.g., "Slower" or "Colder"). We intentionally shifted from the conventional directionality of quantities in Cartesian space. FF included the label indications of the quantities, "Height" and "Age". From a design perspective, we see a similarity in that both graphs label the quantity intended to be represented on each axis (i.e., labeling the speed and temperature in DSD and labeling height and age in FF). The different placement of those labels across the tasks (centered vs. to the side) was not an intentional consideration. Design considerations in both cases included how spatial and quantitative CSs and/or RFs may be leveraged in combination with the provided labels. For instance, height being on the vertical axis for the first prompt in Figure 1(b) could be a natural RF for students, matching the spatial context of height in real life (e.g., Bjuland et al., 2008). However, we anticipated switching height to the horizontal axis for the next prompt and introducing non-normative directionality in the DSD task may uncover other types of thinking (e.g., Moore et al., 2019). We also kept the tasks numerically open-ended (e.g., we did not add numbers on the axes) to examine closer how students may construct their own meanings.

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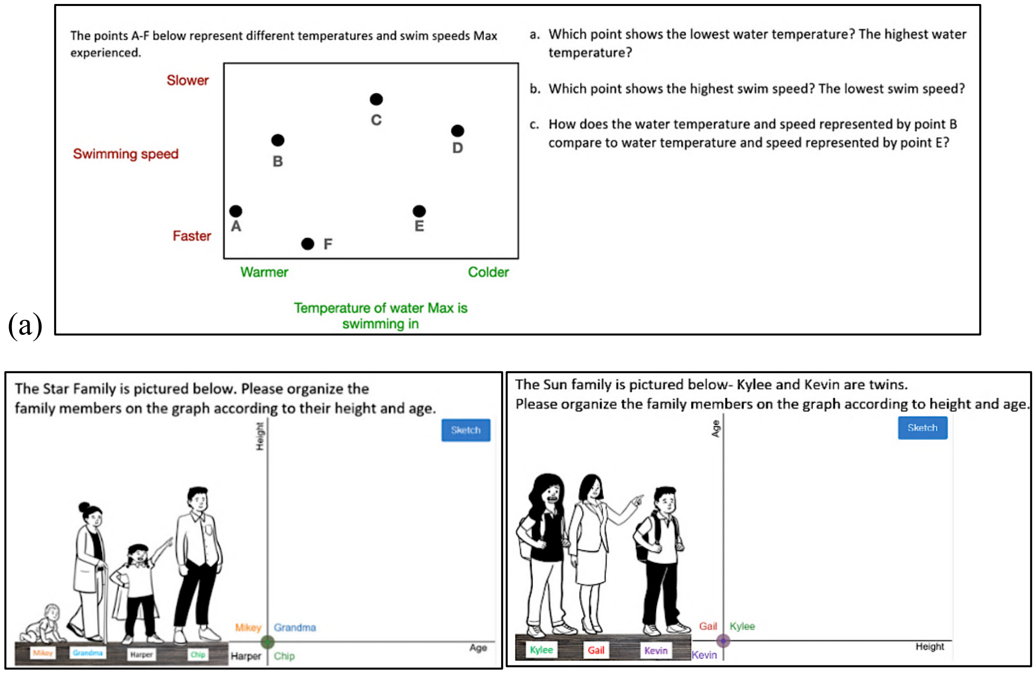
Figure 1: Tasks: (a) Deep Sea Diver and (b) Family Frenzy



(a) (b)

Though we acknowledge the types of thinking towards labels in the tasks may be similar, due to the differing nature of the tasks, we present the main results from each task separately to ease the reader. For DSD, we provide results from three sixth graders in Project A: Danielle, Jamal, and Mari. We also examine one sixth grader's, Maverick's, work with FF from Project B. Note that Project A students worked with the DSD context in multiple tasks, where they also explored a spatial task within the ocean. Danielle completed the spatial task first, while Jamal and Mari did the spatial task after the task in Figure 1(a). We provide this information because of the contextual knowledge the students leveraged regarding the ocean (see Results).

Data Analysis

To build a model of the students' current meanings towards labels in the tasks, we first created a thick description of each students' activity with the task (Geertz, 1973). Thick descriptions included all relevant activity while working on the task, such as timestamps, movements and writing with accompanying screenshots, and direct student quotes. After we drafted thick descriptions, we used conceptual analysis (von Glasersfeld & Steffe, 1991; Thompson 2008) to build a model for how students interpreted the labels and what meanings they leveraged to do this, such as their defined RFs and CSs. As we did this, we took notes and built a larger characterization summary, continuing to re-examine previous parts of our notes to determine if our characterization was supported, if students shifted over time in the episode, or if original interpretations should be reconsidered. Each description was analyzed by pairs of authors from the project teams. After the initial characterization for each student was built, four of the authors met to re-watch relevant videos and continue to discuss the characterization, engaging in the re-examination process described above until consensus was reached.

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Results

Deep Sea Diver

The students explored the DSD CS in different ways, attending to the labels in conjunction with their individually developed RFs.

Labels relating to context. All three students initially alluded to using contextual, spatial ideas about the ocean to guide their thinking. That is, a point higher in space should be warmer since it would be closer to the Sun's heat. After talking through their reasoning and the labels further, Jamal and Danielle switched to strategies which will be discussed in the next sections. Mari, however, re-defined the spatial context to correspond with the given labels.

Mari chose Point F to have the lowest water temperature, noticing the Colder label being on the bottom of the graph, "it says down here, you get colder," also using reasoning of the Sun's effects, "the more you get down, you get colder." She then continued the remaining parts of the task viewing the graph space as the ocean. For example, she selected C as the warmest point and associated slower swim speeds with colder water temperatures at lower parts of the ocean. When Mari was asked how the axes labels related to the problem, she re-oriented her contextual interpretation of the ocean to match the speed labels on the vertical axis. She maintained the Sun's impact on temperature but re-defined how the amount of water in each part would impact speed, saying at the top it would be slower because "it has less water," and faster at the bottom because, "there's, like, a lot more water, so they have a lot of room to swim."

Through Mari's reasoning about the water temperature, we infer that she viewed the graph as a spatial CS representing the ocean and leveraged related contextual knowledge. Mari relied on spatial reference objects for temperature by using the top and bottom of the ocean to orient herself and adjusted her original interpretation of the contextual situation (colder in the ocean would be slower speed) to match the given quantitative labels for speed (more water at the bottom would be faster speed). Although we intended the task to involve quantities disembedded from the ocean context, we note that Mari was still able to use the given labels to work with her own reasoning. She did this by initially pointing out the "Colder" label, which we anticipated to be used as on the right side of a disembedded horizontal axis. However, Mari used it as a reference to align with colder temperature at the bottom of the ocean since it was labeled on the bottom of the graph. Additionally, she matched the "Swimming Speed" labels with her original spatial interpretations by defining a contextual reference related to amount of water.

Labels to estimate proximity. Both Danielle and Jamal used labels as reference objects to estimate the proximity from the points to the labels to decide the extent of points' quantities. As Danielle moved away from estimating temperature based on the ocean's context, she started to use the labels "Warmer" and "Colder" as reference objects that indicated either more warm or cold when the point was closer to the corresponding word. She estimated proximity by gesturing diagonally from a given point to the closest word. For example, when reasoning about why she changed her original answer for the coldest temperature from Point F (spatially the lowest point) to Point E, she explained that F was "by the Warmer" and E was "by the Colder," gesturing diagonally back and forth between each point and corresponding label.

Jamal also estimated proximity from points to a label, but he did this for the "Swimming Speed" label in conjunction with the vertical axis. When asked which points would show the fastest and slowest speeds, Jamal pointed to the "Swimming Speed" label, then placed his palm sideways along the graph, "it would probably go this way, horizontal," then traced his palm out to the right end of the paper. After this, he used the vertical axis as a reference object to indicate faster speeds, estimating proximity to that axis. When comparing B and D, Jamal indicated B as

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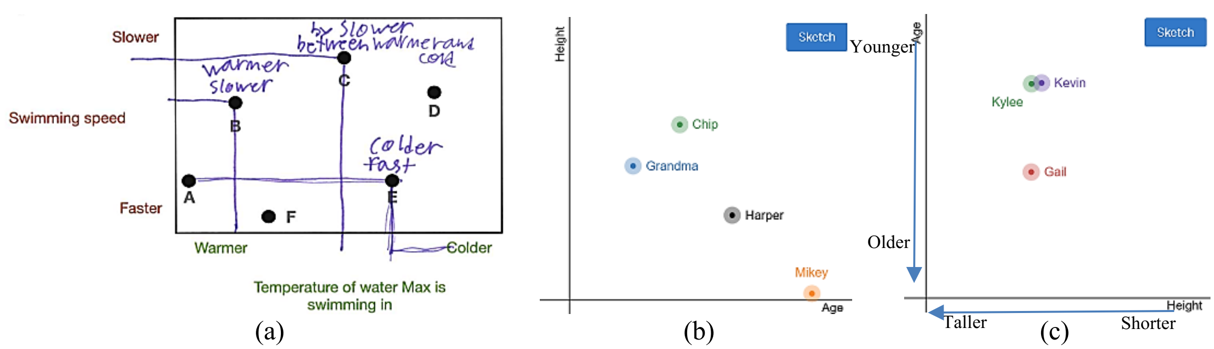
faster, “because it’s closer to this side […] and it’s right where the Swimming Speed is,” placing his hand horizontally over Point B to the Swimming Speed label. He also determined that Point A would have the fastest speed, since again, “it’s closer to Swimming Speed.” Here, Danielle and Jamal used labels as orienting pieces of axes instead of axes as a whole.

Labels as a reference object for projections. In addition to considering proximity to the labels, Danielle and Jamal used projections between the points and the axes by gauging perpendicular distances between points and corresponding locations on the axes.

Danielle often projected points in a straight line either down to the horizontal axis or left to the vertical axis and then estimated the quantity by considering which label on the axis this projection landed closest with. For instance, she identified C as the lowest swim speed, gesturing horizontally from the point to the axis back and forth, tapping the axis and labels as she located C as “in the middle of swimming speed and slower.” We interpret that Danielle imagined the projection of C horizontally to the vertical axis landing between the labels of “Swimming Speed” and “Slower” (see horizontal line from C in Figure 2(a)). With temperatures, Danielle projected points to the horizontal axis and then gauged proximity to the labels. After again showing E’s diagonal distance to the label “Colder” to clarify why it was the coldest, the IR asked Danielle if she could show another cold point. She chose D, gesturing vertically from the point down to the axis, and then gesturing over to the “Colder” label, “it’s *almost* by the cold.” Here, Danielle used the projection from the point to the resulting location on the axis to determine the amount, illustrated by her markings in Figure 2(a) as she discussed some other points.

Figure 2: Some of Danielle’s Projections (a) and Maverick’s Placement for the First (b) and Second (c) Prompts with Added Directionality Inscriptions



While Danielle projected from points to the axes, Jamal projected the labels into the graph space to consider where the points were in relation to the labels, “Warmer” and “Colder”. When reasoning about the highest and lowest temperatures, he referred to the “Warmer” label, “this would probably go up,” gesturing from the label up into the graph and then repeating the gesture with the “Colder” label. He identified A as the warmest and D as the coldest, based on the “sides” of the graph they were on in relation to these projections. Hence, Jamal used the labels of “Warmer” and “Colder” projected up into the space to differentiate the temperature extents of points. We note that in these cases, Jamal and Danielle were making sense of the labels in ways we expected by task design, albeit in slightly different ways (Danielle projecting from points to axes and Jamal projecting from axis to points’ regions). We note that these perpendicular projections differ from the gauging of distance in the prior section. The projections could evidence that students reasoned about the extent of a quantity by utilizing the length along axes, establishing a quantitative RF (e.g., temperature increases moving right). Alternatively, gauging

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how close a point is to a label is reminiscent of establishing a reference object to indicate the extent of a quantity in a more qualitative sense.

Family Frenzy

Instead of defining directionality in CSs conventionally (i.e., quantities increase as they get further away from the origin), Maverick chose to define directionality a different way when spatial directionality was not preserved in the given CS (i.e., height not on the vertical axis). Maverick reasoned that quantities increased moving away from their labels.

For the first prompt, with height on the vertical axis, Maverick eventually produced the configuration in Figure 2(b). He explained his choices for height by pointing to different family members' heads in the image and dragging his finger horizontally to the point placed in the graph. Maverick indicated leveraging the top of the family members' vertical span in the image as a spatial reference object for where to orient the point in the graph. This reference object related to the endpoint of the vertical span aligned with the Height label on the vertical axis, implicitly defining height to increase going up. When the IR asked why the graph showed Grandma's point was older, Maverick placed the cursor directly below Grandma's point on the horizontal axis, "Because on this line, she's farther than Chip." Maverick summarized that height increased going up and that age gets younger from left to right. Maverick had established a quantitative directionality for age on the horizontal axis increasing from right to left while also defining height to increase upwards using both spatial directionality from the image and the distinction from the Height label on the vertical axis. This is evidenced by his motioning between the image and graph while also attending to the labels in his reasoning.

For the second prompt where age and height labels were switched, Maverick started by adjusting points according to again age increasing right to left and height increasing going up. He then paused and moved the points so that Gail's point was the lowest point (Figure 2(c)). Explaining his change, Maverick said, "I'm thinking about… these are switched," motioning to the axes. After expressing confusion for a moment, he pointed to the Height label on the right side of the horizontal axis, "this is the height," moving his finger to the left along the axis, "so if you go up, it's higher." IR asked about age; Maverick explained he placed the points in the vertical span because he thought "if you go down, it gets higher." When asked about height, Maverick explained he perceived Kylee as tallest. Explaining his placement of her to the left of Kevin, he dragged his finger to the left along the Height axis from the label, "because if you go this way on the height, it makes them taller." In short, Maverick established both axes as increasing as they became further from the axis label (see the arrows added in Figure 2(c)).

Maverick reflected on the task, "you can't tell that if you move it down, it gets older, or if you move it up, it gets older," tracing along the age axis. He then said he made a choice based on the labels, "the age starts here," pointing to the Age label at the top of the age axis and provided the same reasoning for height. The IR asked Maverick if he was referring to the words on the axes, and he agreed. We observe that Maverick noticed the graph did not have an explicit marking of direction ("you can't tell"). Therefore, he defined the labels as reference objects to orient where the quantity started when there was no other information. However, in the first prompt when the Height axis was defined to be aligned with the spatial image, he leveraged this information. Maverick relied on a spatial RF (the vertical span of the image) when that RF was compatible with the labeled axes, and he switched to a re-defined quantitative directionality using axes labels as a starting point when this was not compatible with the labeled CS (e.g., when height was no longer vertical, the word was used as a reference for the shortest height).

Table 1 presents a summary of the discussed strategies.

Table 1: Summary of Student Label Interpretations

Task	Type of Interpretation	Student(s) and Summary
DSD	Context: connecting labels to the ocean	• Mari: re-defined spatial RFs to align with given quantitative labels
	Proximity: gauging near-ness between points and labels	• Danielle: proximity to temperature labels • Jamal: proximity to vertical axis and its labels
	Projections: evaluating distances between points and locations on axes	• Danielle: projections from points to axes labels • Jamal: projections up from temperature labels
FF	Directionality: defining labels as starting points for directionality	• Maverick: defined quantities to increase as they move away from their labels

Discussion & Limitations

In this paper, we illustrated unique ways that students interpreted axes labels, including using them as indicators to fit a spatial context (Mari), for gauging distance and projections (Jamal, Danielle), and for directionality (Maverick). We observed students drawing on both spatial and non-normative quantitative RFs as they found fit. We contend that even though many interpretations were different than intended by design, these interpretations are reasonable and could be leveraged in instruction. For one, students are often introduced to bar charts and maps earlier than Cartesian graphs in their schooling (Leinhardt et al., 1990; Mhlolo, 2015). It is worth noting that the labels on bar charts are often categorical, indicating on the horizontal axis qualitatively where to look for information, and that labels on maps indicate where a location is. We argue that switching from such meanings of labels to labels in Cartesian, quantitative graphs could interfere with students using labels as indications of more of something, as Danielle and Jamal did, or where to start a quantity, as Maverick did, based on their possible prior learning experiences. Further, Lee et al. (2020) outlined how spatial constructions of CSs such as Mari's are productive, as students still establish meanings to mathematize the situation, and the thinking patterns establishing spatial and quantitative CSs may entail similar processes.

We argue it is important to consider the specific situation involved as students grapple with graph features such as labels, as situational factors could involve the exact setting and task (e.g., design features), in addition to past experiences (Leinhardt et al., 1990). It is worthwhile to acknowledge what these labels may mean to students as we design and implement tasks, as many of the observed interpretations were unexpected to us. We hope that sharing our results can "Educate for Change" by highlighting strengths of unexpected strategies. Because we value the power of student thinking instead of making assumptions that intended meanings will be re-constructed (von Glasersfeld, 1982), we were able to identify novel ways of thinking. These ways of reasoning are important to be aware of as we design tasks that are intended to support students in developing graph meanings, especially when the literature has found that meanings are not easily developed. We acknowledge that the graphs we examined are limited, and we predict that other types of graphs (e.g., graphs with numerical labels, non-Cartesian, spatial) would offer additional unique interpretations. Moreover, our small sample size and the lack of data specific to these students' educational backgrounds limit us from making specific connections between interpretations. We call for continued research on how we may leverage student thinking as they explore the communication that graph labels may convey to them.

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