Operationalizing and Assessing Quantitative Reasoning in Introductory Physics

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1 Introduction

Learning to symbolize and reason about the covarying relationships between abstract quantities, while being introduced to over 100 new physical quantities, characterizes a typical student's experience in an introductory algebra-based or calculus-based physics course. Students who are enrolled in a physics course at the upper high school or early university levels are typically also enrolled in a course in algebra 2 (functions, equations and inequalities, logarithmic and exponential relationships, and polynomial equations), precalculus, or calculus. There is an opportunity for mathematics instruction to help enrich students' experiences mathematizing in physics contexts, and for physics instruction to help students develop better conceptual understanding of the mathematics that they use. This chapter seeks to make connections between the mathematics and physics worlds, inspiring instruction that can result in a deeper understanding and appreciation of the mathematical nuances of the symbolic models that describe the physical world. What follows is written to help bridge these two instructional worlds.

Quantitative literacy (QL) is the ability to adequately use elementary mathematical tools to interpret and manipulate quantitative data and ideas that arise in individuals' private, civic, and work lives (Gillman, 2005). We also note that quantitative literacy requires an *inclination* to describe real-world phenomena mathematically. Quantitatively literate individuals recognize the value in considering mathematics as a way to understand and reason about real-life situations. In this chapter we consider Physics

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Quantitative Literacy (PQL), i.e., quantitative literacy in the context of (introductory) physics, and argue that it is a pedagogically and intellectually fertile actualization of QL.

Introductory physics uses familiar mathematics in distinct ways to describe the world and make meaning. To an expert, a physics equation “tells the story” of an interaction or process. Quantitative modeling, in which patterns are expressed using mathematical functions that relate physical quantities to each other, is the backbone of PQL. For example, when reading the equation,

$$x(t) = +20\,\mathrm{m} + (-3\,\mathrm{m/s})t + \frac{1}{2}\left(-9.8\,\mathrm{m/s^2}\right)t^2$$

an expert may quickly construct a mental story of how the position of a projectile varies with time, starting 20 m above the ground and launched with a speed of 3 m/s vertically downward. The one-dimensional coordinate system is determined in this case by the physical fact that the acceleration due to gravity points downward, toward the earth. Part of the challenge of learning physics is developing the ability to decode symbolic representations in this manner.

While the ability to describe the physical world quantitatively as exemplified above is a goal of introductory physics courses, little has been done to determine specific, assessable learning objectives related to PQL. This may be, in part, due to a lack of self-awareness on the part of instructors about what PQL entails, and how they, as experts, reason quantitatively in contexts of introductory physics. There is a growing body of literature that seeks to better clarify what PQL entails in introductory physics (Bajracharya et al., 2012; Boudreaux et al., 2020; Eichenlaub & Redish, 2019; Eriksson et al., 2018; Hayes & Wittmann, 2010; Huynh & Sayre, 2018; Redish, 2021; Torigoe & Gladding, 2011; White Brahmia et al., 2020, 2021). This section builds on that prior work. In order to frame improving quantitative literacy in a physics instructional context, we first operationalize *physics quantitative literacy* (*PQL*) in Sect. 2. Next, in Sect. 3, we outline introductory physics learning objectives that can help instructors meet the broad goal of developing students’ PQL, and suggest areas of overlap with concurrent mathematics courses. Lastly, in Sect. 4, we describe an assessment instrument we’ve developed to help instructors determine whether or not their instructional methods are helping students meet the PQL learning objectives.

2 Operationalizing Physics Quantitative Literacy

PQL relies on a blend of conceptual and procedural mathematics and physics content to formulate and apply quantitative models to describe the physical world. Figure 1 shows a visual representation of the process of quantitative modeling in physics, beginning with observations that can lead to creation of *base quantities*. We define base quantities, such as time, position, and change in position, as those that can be created from observations and a single type of measurement. Quantitative modeling

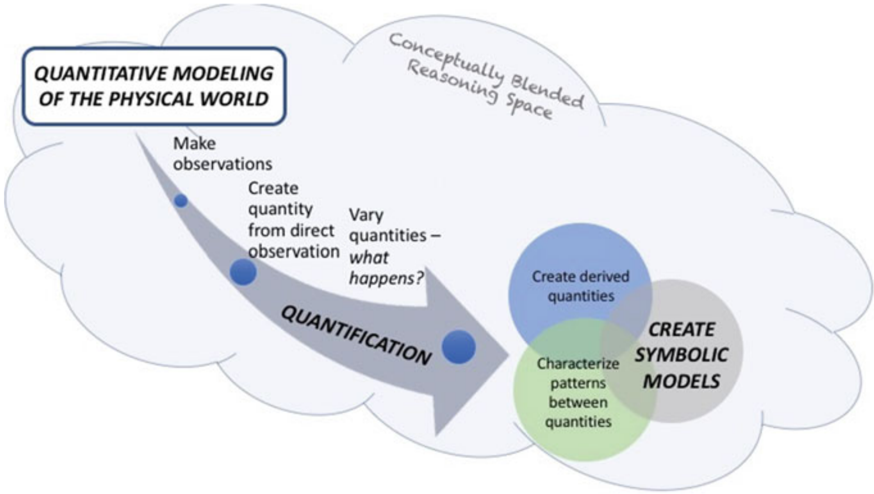


Fig. 1 Quantitative modeling in physics

continues with the exploration of these quantities and their relationships to each other, resulting in derived or composite quantities (such as velocity or speed), established relationships between quantities, and more formal symbolic models. In this section, we operationalize PQL by describing quantitative modeling as outlined in Fig. 1—the process of mathematizing the physical world.

The equation $x(t) = +20\,\text{m} + (-3\,\text{m/s})t + \frac{1}{2}\left(-9.8\,\text{m/s}^2\right)t^2$ in the introductory vignette above is an instantiation of the kinematics equation $x(t) = x_o + v_o t + \frac{1}{2}at^2$, which describes the time-dependent position of an object moving with constant acceleration a and initial velocity v_o from initial position x_o. It is introduced in the first week of almost all college-level introductory physics courses. This equation is a result of the quantitative modeling process in Fig. 1. The first step in the process is observations leading to the creation of base quantities position and time. These quantities appear in the kinematics equation as variables (x and t) and a parameter (initial position x_o). Consideration of how the base quantities vary and covary leads to the derived quantities of velocity and acceleration, which appear as parameters v_o and a. It also leads to characterizations of patterns between the quantities: position can be described as a function of time (x can be expressed as $x(t)$) and depends on the "accumulation" of displacement due to motion characterized by initial velocity v_o and acceleration a. The result is a symbolic model, the general kinematics equation $x(t) = x_o + v_o t + \frac{1}{2}at^2$.

As Fig. 1 depicts, quantitative modeling occurs in a conceptually blended mental space. Quantitative modeling in physics is not simply "doing mathematics with physics quantities." It requires a novel combination of mathematical and physical reasoning. Conceptual blending theory (CBT) (Fauconnier & Turner, 2002) provides a framework for characterizing this combination. Fauconnier and Turner describe a cognitive process in which a unique mental space is formed from two (or more) separate mental spaces. The blended space can be thought of as a product of the input spaces, rather than a separable sum. According to CBT, development of expert mathematization in physics would occur not through a simple addition of new elements (physics quantities) to an existing cognitive structure (arithmetic), but rather through the creation of a new and independent cognitive space. This space, in which creative, quantitative analysis of physical phenomena can occur, involves a continuous interdependence of thinking about the mathematical and physical worlds. Development of PQL involves the creation of a new cognitive space that depends on both mathematical and physical reasoning, but is not a simple, separable sum of these two spaces.

The remainder of this section uses Fig. 1 as a guide to fully operationalize PQL. Section 2.1 details quantitative modeling, of which quantification is a foundation. In Sect. 2.2, we discuss in detail two facets of quantitative modeling that are particularly important in the contexts of introductory-level physics: reasoning about sign and signed quantities; and covariational reasoning with quantities. While reasoning about sign and covariational reasoning have been well-researched by the mathematics education community, recent work by the authors and their collaborators suggest these modes of reasoning as used in physics contexts by physics experts are distinct from the analogous modes in mathematical contexts (White Brahmia et al., 2020). Characterization of these types of reasoning with physics quantities is necessary to understand quantification and quantitative modeling in physics courses, especially for developing assessable learning objectives.

2.1 Quantitative Modeling in Physics

Quantification is a facet of quantitative modeling, and generates the building blocks for the mathematical descriptions involved in quantitative modeling. Thompson defines quantification as “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute’s measure entails a proportional relationship... with its unit” (Thompson, 2011, p. 37). For example, a bus’s motion can be quantified by a velocity (combining the mathematical objects of ratio and vector) relative to the ground. Thompson considers quantification to be “a root of mathematical thinking,” and argues that learners develop their mathematics from reasoning about quantities. In work involving middle school algebra students, Ellis (2007) claims that modes of mathematical structural reasoning are more likely to develop when students practice with quantities that are composed of other quantities through multiplication or division, rather than the strictly numerical patterns and algorithms common to school mathematics. Ellis claims it is precisely these kinds of quantities that help develop students’ abilities to create powerful generalizations. White Brahmia (2019) argues that quantification is the overlooked first step in the modeling process in physics instruction.

Quantification in introductory-level physics courses is typically not generative. Students are rarely asked to create new quantities to describe attributes. Instead, quantification in introductory-level physics courses is focused on the understanding and use of introduced quantities to describe processes and physical phenomena. Students are asked to participate in quantitative modeling with already-defined physical quantities.

Just as conceptual understanding of mathematical operations enriches cognition, so too does understanding the meaning and calculation of introduced quantities. Consider the two common framings of division as the process of *sharing* or *segmenting,* as described by Thompson and Saldanha (2003). Sharing is the partitioning of a number into some number of equal-sized portions (e.g., $\frac{12}{3} = 4$ shares in each of 3 portions). Segmenting is portioning out a number in groups of a given size (e.g., $\frac{12}{4} = 3$ portions of size 4). Thompson and Saldanha (2003) demonstrate that "operational understanding of division entails a conceptual isomorphism between" sharing and segmenting. These framings are productive in the context of numbers and can help new learners to visualize the meaning of division. Moreover, they are productive for students in many "real-life" scenarios. Contrast, however, this conceptual understanding of ratio and division with the construction of velocity as a vector quantity. Velocity can be understood by framing division as an operation which relates (Thompson et al., 2014) a change in position, which is a vector, to a time interval, which is a scalar, and produces a quotient entirely different from the dividend and the divisor. Velocity as the vector rate of change of position has its own physical meaning. Thompson et al. (2014) argue that understanding of a ratio quantity created by comparing two quantities of different natures is equivalent to understanding "relative magnitude" and note "high-level scientific reasoning that involves physical quantities typically involves conceiving of relative magnitudes." In our experience, many students coming out of mathematics courses lack this understanding. We also find that it is uncommon for physics instructors to make explicit this difference when introducing velocity—that division is now performed for a different reason than it was when calculating, for example, the duration of a process that takes one-fourth as long as another, $\frac{22\,\text{s}}{4} = 5.5\,\text{s}$.

We note that PQL includes an *inclination* or habit-of-mind to quantify or create quantitative models, hereafter referred to as "models." The modeling shown in Fig. 1 begins with observations of the world, which may lead to quantification for individuals with high QL. Ability to think mathematically is not enough; it must be accompanied by a recognition that the physical world can be described quantitatively, and an inclination to develop and understand the model.

Observation and quantification are crucial first steps in developing models in physics. Modeling can also result in novel composite quantities. Acceleration is one such composite physical quantity: $\vec{\text{a}}$ is the ratio of a change in velocity $\Delta\vec{v}$ and an interval of time Δt. The creation of acceleration as a quantity is a result of a quantitative model: Galileo famously wrestled with the mathematical decision of whether to describe accelerated motion with a ratio of change in velocity to distance traveled or change in velocity to elapsed time. His choice of the latter led to the formal concept of acceleration, a foundation for the subsequent Newtonian synthesis. The quantitative modeling demonstrated in the introductory vignette involves both a procedural and conceptual mastery of the prerequisite mathematics (Redish & Kuo, 2015; Thompson, 2011). Gray and Tall (1994) describe this combination of procedural and conceptual mastery in mathematical contexts as *proceptual* understanding and name it as a target learning goal for mathematics courses. Gray and Tall (1994) highlight the distinction between procedural efficiency and conceptual understanding, explaining that "the symbol $\frac{3}{4}$ stands for both the process of division and the concept of fraction." In the terms of Thompson and Saldanha (2003), proceptual understanding involves both the conceptualization of fraction, and the conceptualization and action of division.

We argue that quantitative modeling also requires proceptual understanding of physics quantities themselves. Consider the quantity *average velocity*, $\vec{v}_{av} = \frac{\Delta\vec{x}}{\Delta t}$. A physics student with a proceptual understanding of velocity would be procedurally proficient at determining an object's average velocity by dividing its displacement by the elapsed time, as well as understand conceptually that the ratio is a quantity unto itself, $\vec{v}_{av}$, with its own properties and meaning.

We also argue that to succeed in physics courses, it may not be enough to understand the mathematics as taught in mathematics courses. In introductory physics, "flexibility" with mathematics is expected of students—they are expected to understand and apply mathematics in ways that are different than they may have been taught in prior mathematics courses. This flexibility is a hallmark of expert-like reasoning in physics (Sherin, 2001; Vlassis, 2004). A physics expert is able to distinguish between a negative sign used to indicate the type of electric charge in surplus in a given system (Olsho et al., 2021), and one used to indicate the direction of a component of an electric field relative to an assigned coordinate system (White Brahmia et al., 2020); a product may indicate an increase or accumulation of a quantity, or the creation of a new quantity. Physics experts readily interpret these aspects of the mathematization of physical systems (Fig. 2).

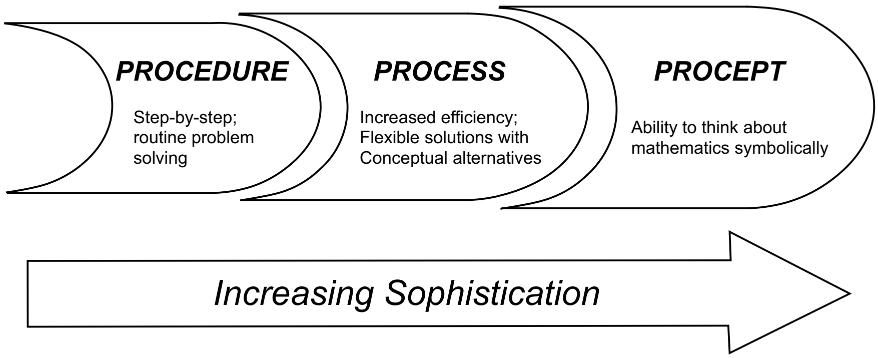


Fig. 2 Proceptual development, adapted from Tall (2008)

Familiarity with multiple representations is a foundation for modeling in physics (Brewe, 2008). This familiarity facilitates the expert-like habit of seeking coherence between varied representations of quantities and relationships. Because quantitative modeling requires proceptual understanding of the mathematics used to relate physics quantities, as well as familiarity with the physics quantities themselves, we suggest that students must have some experience with the multiple representations taught in mathematics and physics courses (e.g., symbolic, graphical, and diagrammatic). The ability to think abstractly about physics quantities allows for greater understanding of the physical phenomena or qualities that the quantities represent; for example, students are able to consider the meaning of the quantity "electric potential" beyond its algebraic representation. Familiarity with multiple representations and physics quantities also allows students to make useful generalizations about quantities—they are able to consider similarities and differences between disparate vector quantities such as electric field and acceleration, or scalar quantities such as mass and charge. A proceptual understanding of the mathematics may help develop a deeper understanding of the physics quantities, which can, in turn, deepen understanding of the mathematics (Sealey & Thompson, 2016).

Success in a physics course requires conceptualizing models that were generated by someone else; moreover, students are expected to understand the symbolizing of quantity and covariational relationships between quantities as if they created the models themselves. This depth of understanding involves recognizing patterns and decoding symbolic models. In the following section, we explicate these cognitive activities by focusing on two areas of reasoning central to the quantitative models featured in introductory-level physics.

2.2 *Facets of Quantitative Reasoning in Introductory Physics*

In this section, we discuss two facets of quantitative modeling that are of particular importance in introductory physics: reasoning about sign and signed quantities; and covariational reasoning, including reasoning about compound quantities. As discussed earlier, reasoning about sign is of particular importance to quantification of base quantities in physics, while covariational reasoning plays a substantial role in development of quantitative models and quantification of composite or derived quantities.

2.2.1 Reasoning About Sign and Signed Quantities in Physics

Negative integers represent a more cognitively difficult mathematical object than positive integers do for pre-college mathematics students (Bishop et al., 2014). Mathematics education researchers have isolated a variety of "natures of negativity" fundamental to algebraic reasoning in the context of high school algebra—the many meanings of the negative sign that must be distinguished and understood for students to develop understanding (Gallardo & Rojano, 1994; Nunes, 1993; Thompson & Dreyfus, 1988; Vlassis, 2004). These various meanings of the negative sign, which will be discussed in greater detail below, form the foundation for scientific quantification, where the mathematical properties of negative numbers are well-suited to represent natural processes and quantities. Recognition that the negative sign has different meanings in different contexts, and correct interpretation of the meaning of a negative sign in a given context—called "flexibility" with negativity by mathematics education researcher Vlassis (2004)—is a known challenge in mathematics education. There is mounting evidence that reasoning about negative quantity poses a significant hurdle for physics students at the introductory level and beyond (Bajracharya et al., 2012; Ceuppens et al., 2019; Eriksson et al., 2018; Hayes & Wittmann, 2010; Huynh & Sayre, 2018; White Brahmia et al., 2020).

In physics, as in mathematics, it is convention that an unsigned quantity is a positive quantity (e.g., "5 μC" is taken to mean a charge of $+\,5\ \mu$C). While research indicates that students are not facile at interpreting the meaning of negative signs specifically, we suggest that it is the presence of an explicit sign associated with a quantity that results in the difficulty. Indeed, physics education researchers report that a majority of students enrolled in a calculus-based physics course struggled to make meaning of negative *and* positive quantities in spite of completing Calculus I and more advanced courses in mathematics (White Brahmia & Boudreaux, 2016, 2017). In our discussion below, we focus on negativity and use of the negative sign (as by convention, that is the context in which use of an explicit sign is necessary), but suggest the applicability to sign and signed quantities more generally.

Flexibility with negativity and interpretation of the negative sign in different physics contexts plays an important role in both quantification specifically and quantitative modeling more generally. Sherin's (2001) "symbolic forms" were developed to explain how successful physics students interpret and create equations. Sherin suggested that students associate symbolic patterns with physical and mathematical meaning. Work by mathematics and science education researchers has expanded Sherin's original list of symbolic forms (Dorko & Speer, 2015; Rodrigues et al., 2019; White Brahmia, 2019). While mathematics education researchers identified a "measurement" symbolic form as consisting of magnitude, units, and exponent (Dorko & Speer, 2015), research in physics contexts suggests a "quantity" symbolic form consisting of *sign,* value, and units, where the sign carries physical meaning related to the specific quantity (White Brahmia, 2019). These two symbolic forms are shown in Fig. 3.

The difference between the symbolic forms speaks to the importance of sign when considering physics quantities. Quantities representing change, such as $\Delta v = v_{\text{final}} - v_{\text{inital}}$ (i.e., change in speed), are fundamental to introductory-level physics but are discussed less in mathematics course. The "quantity" symbolic form includes the expectation of a sign associated with each quantity, which in the case of Δv informs whether the speed is increasing or decreasing. Expressed using the "measurement" symbolic form, Δv would only consist of magnitude and units, and omits important information about the nature of the change. The inclusion of sign allows for a more complete description of an object's motion.

(a) Creating Quantity: Measurement (Dorko & Speer, 2015)

(b) Quantity (White Brahmia, 2019)

Fig. 3 Symbolic forms relevant to physics quantities

The meaning of the sign is of particular significance for scalar quantities, where the meaning may be consistent with mathematical conventions (comparison to a reference or zero, such as for temperature) or could be part of a model or physics convention (for example, heat Q is negative for a system when thermal energy is transferred out of that system). Introductory-level physics students also see a use of sign that is idiosyncratic to physics: sign as an indication of type, as with electric charge, where the sign of the net charge on an object indicates the *type* of charge (positive or negative) in surplus on the object (Olsho et al., 2021).

Sign plays an important role for vector quantities as well. Vector quantities are always interpreted geometrically (i.e., having a magnitude and a direction) in college-level physics courses; a negative sign associated with a vector or vector component thus indicates its direction, either relative to a defined coordinate system ($F_x = -3\text{N}$) or to another vector, as in the quantitative statement of Newton's Third Law ($\vec{F}_{12} = -\vec{F}_{21}$).

For quantitative modeling more generally, students must consider the meaning of negative (and positive) signs when they are used to model physical relationships or processes, or to compare or combine quantities. In these cases, positive and negative signs can be used to describe how quantities relate to each other, or as part of the operations of addition and subtraction—divergent uses of the same symbols. Students are introduced to expressions that relate quantities that oppose or are opposite to each other. Even when used to indicate the operation of subtraction, the negative sign has varied meanings in physics contexts. To describe the many meanings of the negative signs in the contexts of introductory-level physics, White Brahmia, et al. (2020) developed a framework of the natures of negativity in introductory physics, based on an analogous framework in the context of algebra (Vlassis, 2004). An abbreviated version of the physics framework is shown in Table 1. The framework outlines three uses or facets of the negative sign in physics: as associated with a single *quantity*; as used describe a *relationship* between multiple quantities; and as used to denote the *operation* of subtraction. As seen in Table 1, each of these facets is itself multifaceted, which is an indication of the many nuances of negativity in physics contexts.

Table 1 Abbreviated version of the framework of natures of negativity in introductory physics (White Brahmia et al., 2020)

(Q) Quantity	(R) Relationship	(O) Operation
1. Scalar	**1. Opposes**	**1. Removal (physical)**
a. Type (charge only)	**2. Opposite**	**2. Difference (temporal)**
b. Change, rate of change	**3. Relative Orientation**	**4. Removal (modeling)**
c. Comparison to reference	**4. Negative exponents**	**3. Difference (other)**
d. Models, convention		
2. Vector component		

Boldface indicates a facet of a main nature of negativity

Use of the negative sign to convey physical meaning is a basis of quantitative modeling. Even at the college introductory level, combinations of positive and negative signs are necessary to model processes and relationships. Further, the negative sign associated with a given quantity can have multiple correct interpretations. For example, when a force does negative work on a system, it can be interpreted as an indication that the force acts to decrease the mechanical energy of the system. The negative sign also indicates that the force is applied in a direction opposite to the direction of the displacement of the system. White Brahmia and Boudreaux (2017) found that students who understood that a force does negative work on a system when applied in a direction opposite to the system's displacement were more likely to understand that a net negative work is associated with a decrease in the system's energy. The researchers interpreted this result as an indication that a mathematical understanding about the scalar product catalyzed a more robust understanding about the change in system energy. This is an example of how understanding positive and negative signs is associated with more complete understanding of physics quantities, and the quantities' meanings within physics models (White Brahmia, 2019).

2.2.2 Covariational Reasoning in Physics

Covariational reasoning, "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354) has been shown to be strongly associated with student success in calculus by mathematics education researchers (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1994). Physics covariational reasoning plays a substantial role in physics quantitative modeling. It involves finding the relationship between quantities, and representing that relationship symbolically. These are both key facets of quantitative modeling as depicted in Fig. 1. In college-level introductory physics courses, students are routinely asked to describe how quantities relate to each other, and how a change in one quantity affects another quantity.

However, few studies by physics education researchers have explored how covariational reasoning is used in introductory physics contexts. A study reported on by Zimmerman et al. (2020a, 2020b) suggests that covariational reasoning in physics graduate students ("experts" in introductory physics contexts) differs in some ways from that in mathematics graduate students, as reported by Hobson and Moore (2017). In particular, physics experts display a number of specific behaviors—one of which will be described in the paragraphs below—that allow them to consider the relationship between two variables while reducing or even eliminating the formal, novel covariational reasoning seen in mathematics experts in similar contexts. These behaviors allow for physicists to engage in reasoning about the quantities themselves, as well as the relationship between the quantities, in a way that is not typically necessary in mathematical contexts. For this reason, we call the covariational reasoning done by physics experts "covariational reasoning with quantities" or simply "physics covariational reasoning."

Zimmerman et al. (2020a, 2020b) have identified a number of behavior in physics experts that seem to facilitate covariational reasoning. In what follows we focus on a particular instance of the overarching physics expert behavior which Zimmerman et al. (2020a, 2020b) call "compiled relationships": the use and creation of defined relationships between two quantities that may or may not be in the problem statement in order to help address the relationship between two quantities in the specified task. We suggest that this is a cognitive activity that is distinct in physics covariational reasoning, and that it allows for greater focus on the meaning of physics quantities. The use of compiled relationships as an expert behavior relies on the fact that there are relatively few functions that make up the models encountered in a college-level physics course—most involve linear or inverse relationships, basic trigonometric functions, simple quadratics, or exponential decay. Most physical contexts at this level can be mathematized with just this handful of functions, with which expert physicists become very familiar. Therefore, physics experts come to expect one of these common functions, and readily mathematize tasks that involve novel covariational reasoning for mathematics experts—for whom any function is possible.

The behavior encompassed by the compiled relationships category has several facets. Here, we define a facet which we call "automatic mathematization" (Zimmerman et al., 2020a, 2020b) which illustrates a key difference between the way physics experts and mathematics experts approach quantitative modeling tasks that involve covariational reasoning. *Automatic mathematization* is the almost-immediate, automatic assignment of a known functional relationship between quantities. This mathematization is typically guided by the physics and draws on well-tested models of nature. It may be as simple as a learned rule such as "force decreases as $\frac{1}{r^2}$" or more complex, requiring identification of a physical phenomenon in a particular context and then mathematizing. An example of the latter was seen in interviews with physics graduate students who were asked to draw a graph relating intensity of light in liquid as a function of the depth of the water. Several of the interviewees recognized that a decrease in light intensity with increasing distance from the light source was due to the physical phenomenon of scattering. These graduate students then assumed that the intensity would therefore decrease exponentially with increasing distance from the source, connecting the physical phenomenon of scattering to the function $f(x) = e^{-x}$. Zimmerman et al. (2020a, 2020b) report on several other physics-specific expert behaviors that were not reported on in the studies of mathematics graduate students by Hobson and Moore (2017), Moore (2014). They conclude that physics covariational reasoning is built on a proceptual understanding of quantities themselves, and a handful of functions. The physics expert behaviors described above—and others—allow physicists to make sense of the quantities, through their physical interpretation, and the mathematical relationships between quantities simultaneously. We believe that this blended sensemaking is characteristic of physics covariational reasoning, and therefore, of quantitative modeling in physics.

In this section, we have described our work exploring how experts reason quantitatively. In Fig. 1 we outline the reasoning that goes into generating and interpreting symbolic models in physics. The quantitative modeling demonstrated in the vignette in the introduction exemplifies this reasoning process, where the position and time are quantities that emerge from direct observation and the velocity and acceleration are *derived* quantities that characterize the motion. Unlike "measures" in mathematics, physics quantities typically include a sign that carries its own important meaning. The covariational relationship between quantities is symbolized in the kinematics equation shown.

By identifying these sophisticated reasoning patterns, we create targets for assessable PQL-related learning objectives—discussed in the next section—for students enrolled in introductory physics courses.

3 Assessable PQL Learning Objectives

Having operationalized PQL in the previous section through frameworks that characterize expert reasoning, in this section we describe the development of assessable PQL learning objectives for the college-level introductory physics course, using expert PQL as a target. We note that explicit PQL learning objectives in introductory physics are uncommon, largely because the kind of reasoning outlined in the previous section is assumed by most physics instructors to be developed in the prerequisite mathematics courses. There is a gap between physics and mathematics instruction that this work seeks to help close.

Developing learning objectives (LOs) that can help guide instructional efforts toward effective development of PQL builds on the sustained and productive department-wide efforts developing undergraduate physics course learning objectives (Chasteen et al., 2011). In this section we discuss evidence-based PQL learning objectives, and in the next, an example of an assessment instrument that can be used to assess the effectiveness of instruction at meeting some of these objectives.

3.1 Methodology

The methodology we describe here for developing LOs discusses the overall development process, and also includes LOs that are not associated with PQL. The remainder of the chapter focuses specifically on the subset of LOs associated with developing PQL.

At the outset, we recognized that effective LOs articulate values shared by a broad group of instructors. Our first step in creating a succinct set of assessable learning objectives for the introductory physics sequence involved consolidating the outcomes of prior systematic efforts by the physics education research community, representing hundreds of the researcher's hours spent collaborating with departmental colleagues. Past department-level efforts in the United States have focused mainly on courses beyond the introductory level, which rely on a proceptual understanding of calculus. PQL at the introductory level helps build the foundation for the calculus-thinking that underpins modeling in physics; we approached this project through the lens of conceptually understanding the mathematical foundations of algebraic physics models.

In order to develop a set of LOs that are broadly appealing and recognizable to most instructors, we started with the existing LOs from a variety of widely respected sources.[1] We conducted a card-sorting task with those LOs, and supplemented the results where appropriate. Learning scientists have used card-sorting tasks to investigate mental organization of disciplinary knowledge (Chi et al., 1981; Schoenfeld & Herrmann, 1982). Experts are given cards showing various content with no pre-established groupings. They are then asked to sort the cards into groups that they feel make the most sense, and describe each group. The first author (SWB), a physics education research postdoc (whose dissertation specialization was surface science), and a senior astrophysics graduate student with extensive teaching and curriculum development experience, employed a card-sorting task with learning objectives that span the introductory physics course. On each card was a single objective. The researchers independently sorted the objectives into groups, then discussed their groups, and modified their sortings until they reached agreement.

The overall structure of the resulting learning objectives is hierarchical (see Fig. 4) and includes a novel level not seen in other efforts—*sequence-level* objectives that span the entire introductory physics sequence. The sequence level includes a limited number of LOs that blend the professional science practices and physics habits-of-mind characteristics of high-functioning STEM professionals. We recognize that this level of learning takes a long time and may not be measurable over the course of one term. It is mainly at the sequence level that we include objectives designed to develop skills that are strongly associated with PQL. Developing a proceptual understanding

[1] We looked to the high-quality practices of NGSS and the College Board, which have been carefully crafted over several years, for guidance in developing our own learning objectives at the sequence level (NSTA; College Board, 2020). They created the individual sequence-level LOs used in the sorting task by gathering the LOs from the multiple sources (Beichner, 2011; Etkina et al., 2006; Kozminski et al., 2014; LGBT + Physicists, 2013; SEI).

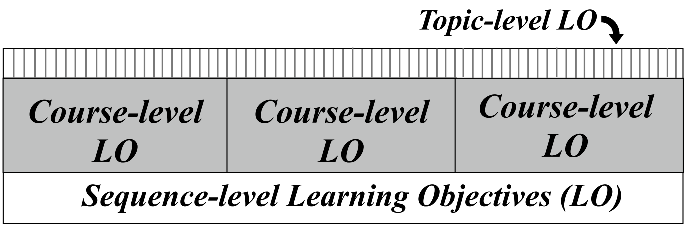


Fig. 4 Hierarchical structure of LOs

of models in physics also happens slowly, over the period of multiple sequential courses.

The course-level objectives include 10–15 overarching content themes that are specific to that course. Lastly, unit-level objectives, often thought of as the specific content, correspond in duration to a typical chapter in a college course.

The introductory physics sequence-level learning objectives resulting from the card-sorting task are listed in Table 2. The resulting consensus includes three themes around which similar LOs clustered: physics habits of mind, understanding models and limits, and professionalism and workplace skills.

Note that there is no separation between lab and lecture course objectives. While some objectives lend themselves better to the lab, there is considerable cognitive overlap. There is compelling evidence that it takes both laboratory and lecture/recitation experiences for these learning objectives to be met. Labs which emphasize following instructions and the development of technical laboratory skills miss an opportunity to help students develop the ability to design ways to answer scientific questions (Canright & White Brahmia, 2021; Etkina, 2015).

We share an example that is ubiquitous in physics: the inverse-square covariational relationship, which is central to many physics models (e.g., Coulomb's law, Newton's Law of Gravitation, light and sound intensity). The following example shows the learning objectives that are part of developing reasoning associated with Coulomb's law, a $\frac{1}{r^2}$ force, and the associated field in an electromagnetism course (typically the second term in an introductory sequence), and demonstrates how the levels shown in Fig. 4 differ:

- The relevant unit-level LOs include:
  - **Analogy to Gravitation**: Use Newton's 3rd law to reason about the force vector direction along a line connecting the two interacting objects; Use the $\frac{1}{r^2}$ structure of the gravitational and electrical forces to reason covariationally about similarities in the interactions between massive objects and between charged objects.
  - **Coulomb's Law**: State Coulomb's Law in equation form and explain the covariational relationship between the electrostatic force and (1) the magnitude of the charges, and (2) the separation of the charges
- The relevant course-level LOs include:

Table 2 Sequence-level learning objectives

HM: physics habits of mind
HM-1. Translation between physical and symbolic world: develop the inclination and ability to translate between the physical and symbolic worlds in an effort to quantitatively reason about how nature works
HM-2. Reasoning with physical quantities: reason abstractly and quantitatively with new scalar and vector quantities: make physical sense of the quantities and mastering their mathematical structures
HM-3. Multiple representations: create and translate between multiple representations of the same concept (e.g., text, equations, graphs, diagrams)
HM-4. Problem articulation: articulate what it is that needs to be solved in a particular problem, what is known and represent them using a non-verbal representation
HM-5. Perseverance: recognize that wrong turns are valuable in learning the material, recover from mistakes, and persisting in working to the solution even when there is no clear path to the endpoint
HM-6. Sensemaking with quantity: effectively use unit reasoning, vector and scalar natures and limiting cases to make sense of answers
HM-7. Order of magnitude and reasonableness: anticipate the order of magnitude to judge the reasonableness of measurements and calculations
HM-8. Reasoning based on mathematical structure: look for and make use of patterns associated with mathematical structure to reason across contexts and scale
HM-9. Recognizing uncertainties: be able to recognize that all measured quantities have inherent uncertainties
ML: understanding models and their limits in physics
ML-1. Making observations: form a scientific question, design and carry out experiments to look for patterns
ML-2. Developing a model: analyze and interpret data while attending to uncertainty in measurement and construct explanations based on patterns in the data
ML-3. Reasoning with mathematical models: develop and use mathematical models and explanations, construct viable arguments, engage in argumentation from evidence and critique reasoning of others
ML-4. Model limitation: articulate assumptions made when applying a model, and the range over which a particular model is a valid description of nature
ML-5. Model testing: design an experiment to test the model and make a prediction of the outcome based on it
ML-6. Scientific judgment: analyze and interpret data from a testing experiment while attending to uncertainty, and make a scientific judgment about the outcome
PW: professionalism and workplace
PW-1. Collective intelligence: recognizing the two features of high collective intelligence, and monitoring social climate to optimize these features (equitable speaking turns, social sensitivity)
PW-2. Collaboration: able to articulate affordances that a group brings to arriving at a creative solution, knowing what the roles are that members of effective groups t
PW-3. Inclusion: demonstrate effective communication skills in the context of a recitation or lab group that results in whole-group meaningful participation

(continued)

Table 2 (continued)

PW: professionalism and workplace
PW-4. Communicating physics: be able to communicate physics in written and oral forms
PW-5. Independent Learning: recognizing and acting on confusion: be able to articulate specifically the source of confusion and taking action to move beyond that difficulty (e.g., office hours, group study)
PW-6. Skepticism toward conclusions: recognize that scientific conclusions—whether from an outside source or from your own calculations—may be incorrect, and develop the ability to check these conclusions with simple calculations, 3rd party information, and/or common sense

The PQL-specific LOs appear in bold italic

- **Electric Force and Field**: Apply Coulomb's Law and the superposition principle to find the net force and field due to a distribution of charges
- **Sophisticated Quantities in E&M**: Distinguish between the vector and scalar nature of EM quantities and the role of $\pm$ signs

- The relevant sequence-level LOs include:
  - **ML-3: Reasoning with mathematical models**: Develop and use mathematical models and explanations, construct viable arguments, engage in argumentation from evidence.
  - **HM-8: Reasoning based on mathematical structure**: Look for and make use of patterns associated with mathematical structure to reason across contexts and scale.

In the remainder of this chapter, we focus on sequence-level objectives because PQL develops over repeated exposure, at a different rate for all students. The expectation is that by the time students have completed the introductory sequence of physics, these objectives will have been met. Sequence-level objectives in turn strongly influenced the course-level objectives, and the streamlining of the unit goals. We next look closely at the specific PQL sequence-level LOs.

3.2 Sequence-Level Learning Objectives

A subset of the sequence-level learning objectives that target PQL specifically is indicated by bold italics in Table 2. We suspect that mathematics instructors will find these familiar, and likely see overlap with their own learning objectives. We see great potential to embolden student learning, both in mathematics and in physics, if both disciplines can emphasize mathematical reasoning that is highly valued in physics. We focus here on three of the learning objectives from Table 2 to better clarify why they matter, and how they might overlap with mathematics instruction: HM-1, HM-3, and HM-6 (see Table 3).

Table 3 Sample LOs with examples from physics

HM-1	HM-3	HM-6
Translation between physical and symbolic world	Multiple representations	Sensemaking with quantity
• *Positive and negative signs* (e.g., electric charge, one dimensional velocity, displacement, acceleration) • *Summation* of conserved quantities (e.g., energy, momentum) • *Unit vectors* to represent direction of vector quantities (e.g., force, displacement, electric field)	• *Interpretation of slope and area under curve in graphs* (e.g., position vs. time, pressure vs. volume, force vs. displacement) • *Verbal interpretation of equations* (e.g., example in introduction) • *Force diagrams* to represent direction and magnitude of vector quantities (e.g., Newton's laws, statics)	• *Limiting cases* What happens in a given model for very large/small and zero values of a quantity? • *Dimensional analysis* are the units of an answer consistent? Does a model make sense in the physical world? • *Vector versus scalar reasoning* does "direction" carry meaning for a given quantity? (e.g., force, energy, momentum, time)

HM-1, **Translation between the physical and symbolic world**, is a continuous mental action of experts in physics, relying heavily on mathematical symbols to convey deep meaning. Addition and subtraction can be performed only with like quantities, and the operations carry different meaning than the integers that carry the same symbols, as was demonstrated in the introduction of this chapter.

HM-3, **Multiple Representations**, is brought to life in the vignette at the opening of this chapter. The reliance on particular representations and the inclination to seek coherence between them is a hallmark of expert behavior around making sense of models. Equations are ubiquitous in all physics contexts. Graphical representations of position, velocity, and acceleration as a function of time are an instructional platform kinematics, bar charts are commonly used to keep track of conserved quantities, and vector diagrams are foundational in the studies of solid and fluid statics and dynamics.

HM-6, **Sensemaking with quantity**, encompasses exploring the limiting cases of single and multivariable models, using the units in a calculation both to guide and to check for sensemaking, and exploring physical-world implication of the vector or scalar nature of a quantity. As an example of the latter, multiplication and division create entirely new quantities with unique properties. Work is a scalar product of two vectors (force and displacement); it is neither force nor displacement, and not a vector. Nonetheless, students routinely conflate work and force, not differentiating between the product and a factor, or a scalar and a vector.

We have gathered a substantial amount of evidence for face validity of the sequence-level LOs. The language has been modified iteratively based on a series of interviews with faculty until the LOs reached a steady state in which they are both understood as intended and valued by instructors. Much work remains before it becomes standard practice across most institutions that undergraduate physics instruction is designed to meet evidence-based objectives, and measures of the effectiveness of instruction are based on them. In their current form, the LOs described in

Table 2 are used at our institutions with a broad set of instructors, with an associated outcome of facilitating consensus about course content, assessments, professional development, and modifications to courses.

In this section we've demonstrated the ubiquity and importance of quantification, symbolizing and modeling to physics reasoning, and provided PQL learning objectives that reflect their value to instruction. In what follows we describe an instrument that can be used to assess whether or not instruction is meeting these objectives.

4 The Physics Inventory of Quantitative Literacy

Despite the importance of physics quantitative literacy as a learning outcome in introductory physics courses, there is a dearth of instruments to assess its development. To address this need, we developed, with collaborators Smith, Boudreaux, Eaton, and Zimmerman, the *Physics Inventory of Quantitative Literacy* (PIQL), a multiple-choice reasoning inventory (White Brahmia et al., 2021). Various concept inventories, such as the *Force Concept Inventory* (Hestenes et al., 1992) and the *Force and Motion Conceptual Evaluation* (Thornton & Sokoloff, 1998) in physics, and the *Precalculus Concept Assessment* (Carlson et al., 2010) and *Calculus Concept Inventory* (Epstein, 2006) in mathematics, have raised awareness of student difficulties, leading to directed instructional interventions and improvements in curricula, and we believe that the PIQL can have an analogous impact on physics instruction. There are, however, several aspects of the PIQL that set it apart from concept inventories:

1. Instead of focusing on a single physics concept or level of mathematics, the PIQL was developed to assess facets of mathematical *reasoning* (i.e., PQL) that are important in introductory physics, and foundational to subsequent physics courses.
2. The PIQL has several "multiple-choice multiple-response" items (i.e., multiple choice questions for which there may be more than one correct answer, and for which students are asked to choose all responses that they believe are correct), which allow us to probe both conceptual mathematics and conceptual physics features of student reasoning in a given context.
3. The PIQL is designed to assess development of PQL throughout an entire introductory physics course sequence, rather than providing a measurement of concept mastery for a single course.

As the PIQL is intended to assess PQL and its development with instruction in physics, the items focus on the types of quantification and quantitative modeling that are important in introductory physics: reasoning about sign and signed quantities, and covariational reasoning. Covariational reasoning in particular is foundational to the mathematics course that is prerequisite to introductory physics courses (precalculus), and several PIQL items are adapted from items from the Precalculus Concept Assessment (Carlson et al., 2010). In addition, some PIQL items assess student reasoning about ratios and proportions; while this type of reasoning is related to covariational reasoning, we treat it as a distinct category for PIQL items. Proportional reasoning represents a domain of quantification that is particularly relevant for introductory physics, where many models involve linear relationships and many quantities are ratios of other quantities (Boudreaux et al., 2020). Reasoning about sign and signed quantities and covariational reasoning are key to quantification and quantitative modeling, as described in Sect. 2. The PIQL's focus on these facets of mathematical reasoning in physics contexts makes it an important metric for assessing whether PQL-related learning objectives are being met, particularly those in the *HM: Physics Habits of Mind* and *ML: Understanding models and their limits in physics* categories. PIQL items are not focused on procedural mathematics or calculations, which are also important in introductory physics and are well-served by meeting the mathematics prerequisites for physics. The conceptual mathematics and quantitative reasoning embodied in the PIQL are a foundation for the mathematics used in introductory physics courses at the college level, and are not typically an outcome of the prerequisite mathematics courses.

Expert-like PQL is firmly rooted in a well-formed conceptual blend of physics concepts and proceptual understanding of precalculus and algebra, as discussed in Sect. 2; therefore, novel PIQL items were developed on the theoretical foundation of Conceptual Blending Theory (Fauconnier & Turner, 2002), as well as Sherin's (2001) symbolic forms. Readers interested in the process of item develop- ment based on these theoretical frameworks should see the journal article describing the PIQL's development and validation (White Brahmia et al., 2021).

Here, we describe three items from the PIQL and relate them to the learning objectives described in Table 2, chosen to exemplify the LOs highlighted in Table 3. The first item was written to probe student understanding of sign and signed quantities, and assesses learning objective HM-1 primarily, in addition to HM- 2, HM-6, and HM-8. The second involves covariational reasoning, and assesses learning objective HM-3 primarily, along with HM-1, HM-2, HM-6, HM-8, and ML-3. The third also involves covariational reasoning, focusing on evaluation of an algebraic limit. It primarily assesses learning objective HM-6, as well as HM-1, HM-8, and ML-3.

The *Electric field* question (see Fig. 5) asks students to determine the meaning of a negative sign associated with a component of a vector quantity. In introductory physics contexts, the most useful and intuitive interpretation of a vector quantity is a geometric interpretation. Students learn that a vector is a quantity with a magnitude and a direction. Therefore, the sign associated with a vector component indicates its *direction* relative to a defined coordinate system. We find, however, that students struggle to make meaning of the sign of vector components that represent unfamiliar quantities (White Brahmia & Boudreaux, 2017). This is especially true for quantities such as electric field, and others related to electromagnetism. We believe that, for many students enrolled in college-level introductory physics courses, a lack of intuition and experience with quantities of electromagnetism, as well as unfamiliarity with the mathematical abstraction of vector fields obscures the meaning of the sign. This is despite the fact that the meaning of the sign of a vector component is understood by students in the more familiar context of mechanics. This question serves

Recall that the electric field is a vector quantity. At a location on the x-axis, the x-component of the electric field is measured to be $E_x = -10$ N/C, and the y and z-components of the field are measured to be zero.

Consider the following statements about this situation. Select the statement(s) that **must be true.** ***Choose all that apply.***

a. The source of the field is a negative charge.
b. At that location of the x-axis, the field is in the negative x-direction.
c. In this field, the motion of any charged particle is opposite to the direction of the field.
d. The magnitude of the electric field is decreasing.
e. The magnitude of the electric field at that location is 10 N/C less than it is at the origin.

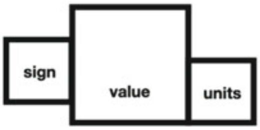


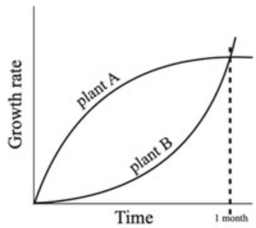
Fig. 5 *Electric field*, PIQL multiple-choice multiple-response item (top) that exemplifies a proceptual understanding of the "sign" aspect of White Brahmia's "quantity" symbolic form (bottom). The correct response is b

as an assessment of PQL-related learning objective HM-1 in particular: students are expected to translate the physical attribute of direction into a symbolic representation using sign.

The *Plant Growth* question, shown in Fig. 6, is an item that assesses students' graphical interpretation and covariational reasoning, and is based on an item from the Precalculus Concept Assessment (Carlson et al., 2010). The item features a graph with time as the independent variable, and growth rate as the dependent variable. Students are asked to compare not the growth rates of the two plants, but the amount of growth of the two plants over the period depicted in the graph. To do this, students could recognize that plant A grows at a faster rate for the entire time shown in the graph, and therefore grows more; or, a student could recognize that the area under each curve represents the accumulated growth of the associated plant. Both of these strategies require students to interpret the quantities depicted in the graph, and how they can use those quantities to compare a third, related quantity. This item is particularly relevant to sequence-level learning objectives HM-3. Students are expected to make sense of graphical representations of quantities: when the independent variable is time, and the dependent variable is a time rate of change of a given quantity, the area under the curve represents the accumulation of that quantity.

The *Fish* item, shown in Fig. 7, is also adapted from the Precalculus Concept Assessment (Carlson et al., 2010), and assesses covariational reasoning in an algebraic (rather than graphical) context. To answer, students need to determine that the expression given for $N(t)$ increases with increasing t, by recognizing that the numerator grows more quickly than the denominator. They must also recognize that answering the item requires a determination of a limit, and determine the value of the limit of the given algebraic expression. One way to determine the value of the

The graph at right represents the *growth rate* vs. *time* for two plants. Which of the following statements best describes the growth of the two plants from $t = 0$ to $t = 1$ month?



a. Plants A and B have the same amount of growth.
b. Plant A has experienced more growth than plant B.
c. Plant B has experienced more growth than plant A.
d. The graph does not provide enough information to compare the growth of the two plants.

Fig. 6 PIQL item that exemplifies the covariational reasoning used in introductory physics contexts, and understanding of graphical representations of quantitative models. The correct answer is b

The wildlife game commission released 500 fish into a lake. The function $N(t)$ defined by

$$N(t) = \frac{600t + 500}{0.5t + 1}$$

represents the approximate number of fish in the lake as a function of time (in years). Which one of the following best describes how the number of fish in the lake changes over time?

a. The number of fish gets larger each year, but does not exceed 500.
b. The number of fish gets larger each year, but does not exceed 1200.
c. The number of fish gets smaller each year, but does not get smaller than 500.
d. The number of fish gets larger each year, but does not exceed 600.
e. The number of fish gets smaller each year, but does not get smaller than 1200.

Fig. 7 *Fish*, PIQL multiple-choice single-response item, that assesses students' covariational reasoning in an algebraic context. The correct answer is b

limit of the expression is to rewrite the expression as

$$N(t) = \frac{600(t + 5/6)}{0.5(t + 2)} = 1200\frac{t + 5/6}{t + 2}.$$

As t gets large, the fraction approaches 1 from below; thus, as t increases, $N(t)$ approaches 1200 from below. We note that while the wording of the answer choices is such that less rigorous reasoning can be employed to find the correct answer, recognition of the necessity of taking a limit is central to this item. This item is well-aligned with learning objective HM-6: students must recognize the need to consider the limit of an expression for large values of t.

Interestingly, though experts categorize the items on the PIQL as primarily using proportional reasoning, reasoning about sign and signed quantities, or covariational reasoning, both exploratory and confirmatory factor analyses of student responses on the steady-state version indicated that the items on the inventory were not separable into these constructs from the students' perspective. This indicates that, from the students' perspective, the PIQL may assess a single construct (i.e., physics quantitative literacy) and that the three facets of reasoning are deeply interconnected in physics contexts for students. Student interviews as well as targeted psychometric analyses are consistent with this interpretation (White Brahmia et al., 2021). This supports our classification of the PIQL as a *reasoning* inventory, rather than a concept inventory, and that it is an appropriate metric for assessing PQL-related learning objectives, which are focused on reasoning rather than specific mathematical or physical concepts.

5 Conclusion

In this paper we define Physics Quantitative Literacy (PQL) and describe its central role in physics thinking. We operationalize PQL in the context of quantification and modeling, with a focus on covariational reasoning and reasoning about sign and signed quantities. We also demonstrate that PQL is not only central to physics learning, but has a strong overlap with concepts in algebra, and precalculus as well. We then describe our process for developing sequence-level assessable PQL learning objectives for the introductory physics sequence, and present the current version of those objectives. We note that mathematics educators are likely to see overlap with their own learning objectives for algebra and precalculus courses. It is with optimism for this synergistic potential between the disciplines that we include physics learning objectives in this chapter. Lastly, we describe an assessment instrument designed to assess some of these learning objectives, the Physics Inventory of Quantitative Literacy (PIQL), a reliable and valid reasoning inventory that assesses students' physics quantitative literacy as it develops with instruction in introductory physics courses. Results from Classical Test Theory provide evidence for its validity and reliability, and both exploratory and confirmatory factor analyses suggest that it is a single-factor instrument. We interpret the factor analysis results as an indication that the PIQL tests a single construct that we call Physics Quantitative Literacy (PQL). We presented the PIQL as a useful metric for assessing PQL-related learning objectives, and as a step toward establishing metrics for learning objectives for calculus-based introductory physics courses.

Students come to their STEM courses having succeeded in their prerequisite mathematics courses, yet they typically encounter an unfamiliar experience with the mathematics they "know." Many fail to make effective connections with their prior learning experience in order to function in the new one. There is a strong need for a proceptual facility with some of the mathematics from prerequisite courses, as relied on in introductory mathematics-based STEM courses (like physics), to be part of the students' learning progression through these courses.

We conclude by encouraging education researchers and curriculum developers from mathematics and mathematics-based disciplines, like physics, to explore the overlap between our disciplines in the work that we do. We are teaching the same students. Exploring the interface of their course-taking experiences and mutually supporting our collective learning objectives holds potential for symbiotic learning. In addition, collaboration at the interface opens possibilities of realizing new learning outcomes that may even include a more creative and generative approach in both disciplines. We consider the work in this chapter to be one step in that direction.

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References

Bajracharya, R. R., Wemyss, T. M., & Thompson, J. R. (2012). Student interpretation of the signs of definite integrals using graphical representations. *AIP Conference Proceedings*, *1413*(1), 111–114. Retrieved from https://doi.org/10.1063/1.3680006

Beichner, R. J. (2011, April). The student-centered active learning environment for undergraduate programs (SCALE-UP) project. In *APS New England section spring meeting abstracts* (p. G1-001).

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19–61.

Boudreaux, A., Kanim, S. E., Olsho, A., Brahmia, S. W., Zimmerman, C., & Smith, T. I. (2020). Toward a framework for the natures of proportional reasoning in introductory physics. In *Proceedings of the annual physics education research conference* (pp. 45–50).

Brewe, E. (2008). Modeling theory applied: Modeling instruction in introductory physics. *American Journal of Physics, 76*(12), 1155–1160.

Canright, J. P., & White Brahmia, S. (2021, August 4–5). Developing expertlike epistemologies about physics empirical discovery using virtual reality. In *Physics education research conference 2021* (pp. 75–80). Virtual conference.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying co-variational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113–145.

Ceuppens, S., Bollen, L., Deprez, J., Dehaene, W., & De Cock, M. (2019). 9th grade students' understanding and strategies when solving x (t) problems in 1D kinematics and y (x) problems in mathematics. *Physical Review Physics Education Research, 15*(1), 10101.

Chasteen, S., Perkins, K., Beale, P., Pollock, S., & Wieman, C. (2011). A thoughtful approach to instruction: Course transformation for the rest of us. *Journal of College Science Teaching, 40*(4), 24–30.

Chi, M. T., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science, 5*(2), 121–152.

College Board (2020). *AP physics 1: Course and exam description.* https://apcentral.collegeboard.org/. Accessed 2021-01-13.

Dorko, A., & Speer, N. (2015). Calculus students' understanding of area and volume units. *Investigations in Mathematics Learning, 8*(1), 23–46. https://doi.org/10.1080/24727466.2015.11790346

Eichenlaub, M., & Redish, E. (2019). Blending physical knowledge with mathematical form in physics problem solving. In *Mathematics in physics education* (pp. 127–151). Springer International Publishing. https://doi.org/10.1007/978-3-030-04627-9_6

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478. https://doi.org/10.1080/07370000701632397

Epstein, J. (2006). *The calculus concept inventory* (pp. 60–67). National STEM Assessment.

Eriksson, M., Linder, C., & Eriksson, U. (2018). Towards understanding learning challenges involving sign convention in introductory level kinematics. *Physics education research conference (PERC).*

Etkina, E. (2015). Millikan award lecture: Students of physics—Listeners, observers, or collaborative participants in physics scientific practices? *American Journal of Physics, 83*(8), 669–679. https://doi.org/10.1119/1.4923432

Etkina, E., Van Heuvelen, A., White-Brahmia, S., Brookes, D. T., Gentile, M., Murthy, S., & Warren, A. (2006). Scientific abilities and their assessment. *Physical Review Special Topics-Physics Education Research, 2*(2), 020103.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. Basic Books.

Gallardo, A., & Rojano, T. (1994). School algebra—Syntactic difficulties in the operativity. In *Proceedings of the XVI international group for the psychology of mathematics education, North American chapter* (Vol. 1, pp. 265–272).

Gillman, R. (2005). *Sigmaa on quantitative literacy introduction.* http://sigmaa.maa.org/ql/about.php

Gray, E. M. & Tall, D.O. (1994). *Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic* (Vol. 25, Tech. Rep. No. 2). Retrieved from https://www.jstor.org/stable/749505

Hayes, K., & Wittmann, M. C. (2010). The role of sign in students' modeling of scalar equations. *The Physics Teacher, 28*(4), 246–249.

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The Physics Teacher, 30*(3), 141–158.

Hobson, N. L. F., & Moore, K. C. (2017). Exploring experts' covariational reasoning. In K. Moore & P. Thompson (Eds.), *20th annual conference on research in undergraduate mathematics education* (pp. 664–672).

Huynh, T., & Sayre, E. C. (2018). *Blending mathematical and physical negativeness.* arXiv:1803.01447

Kozminski, J., Lewandowski, H., Beverly, N., Lindaas, S., Deardorff, D., Reagan, A., Dietz, R., Tagg, R., EblenZayas, M., Williams, J., & Hobbs, R. (2014). *AAPT recommendations for the undergraduate physics laboratory curriculum.* AAPT.

LGBT+ Physicists (2013). *Supporting LGBT+ physicists: A best practices guide for departments.* https://lgbtphysicists.org/files/BestPracticesGuide.pdf. Accessed January 13, 2021.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102–138.

NSTA (n.d.). *Access the next generation science standards by topic.* https://ngss.nsta.org/AccessStandardsByTopic.aspx. Accessed January 13, 2021.

Nunes, T. (1993). Learning mathematics: Perspectives from everyday life. In *Schools, mathematics, and the world of reality* (pp. 61–78).

Olsho, A., White Brahmia, S., Smith, T., & Boudreaux, A. (2021). When negative is not "less than zero": Electric charge as a signed quantity. *The Physics Teacher, 59*(4), 253–256.

Redish, E. (2021). Using math in physics: 4. Toy models. *The Physics Teacher*, *59*(9), 683–688.

Redish, E., & Kuo, E. (2015). Language of physics, language of math: Disciplinary culture and dynamic epistemology. *Science & Education, 24*(5–6), 561–590.

Rodriguez, J.-M. G., Bain, K., & Towns, M. H. (2019). Graphical forms: The adaptation of Sherin's symbolic forms for the analysis of graphical reasoning across disciplines. *International Journal of Science and Mathematics Education*, 1–17.

Saldanha, L. A., & Thompson, P. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Annual meeting of the psychology of mathematics education—North America* (Vol. 1, pp. 298–304). North Carolina State University.

Schoenfeld, A. H., & Herrmann, D. J. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 8*(5), 484.

Sealey, V., & Thompson, J. (2016). Students' interpretation and justification of "backward" definite integrals. In *Proceedings of the 19th annual conference on RUME.*

SEI. *Science education initiative at University of Colorado, Boulder*. http://www.colorado.edu/sei/fac-resources/learning-goals.html

SEI. *Carl Wieman science education initiative at the University of British Columbia.* https://cwsei.ubc.ca/resources/instructor/learninggoals

Sherin, B. L. (2001). How students understand physics equations. *Cognition and Instruction, 19*(4), 479–541.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5–24.

Thompson, P., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, 115–133.

Thompson, P. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlain, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM*^{E} (p. 33).

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics.* https://doi.org/10.1007/BF01273664

Thompson, P., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra 1, 2, 3. *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing, 4*, 1–24.

Thompson, P., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *Research companion to principles and standards for school mathematics* (pp. 95–113). National Council of Teachers of Mathematics.

Thornton, R. K., & Sokoloff, D. R. (1998). Assessing student learning of Newton's laws: The force and motion conceptual evaluation and the evaluation of active learning laboratory and lecture curricula. *American Journal of Physics, 66*(4), 338–352.

Torigoe, E. T., & Gladding, G. E. (2011). Connecting symbolic difficulties with failure in physics. *American Journal of Physics, 79*(1), 133–140.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in "negativity." *Learning and Instruction, 14*(5), 469–484. https://doi.org/10.1016/j.learninstruc.2004.06.012

White Brahmia, S., & Boudreaux, A. (2016). Exploring student understanding of the negative sign in introductory physics contexts. In *Proceedings of the 19th annual conference on research in undergraduate mathematics education.*

White Brahmia, S., & Boudreaux, A. (2017). Signed quantities: Mathematics based majors struggle to make meaning. In *Proceedings of the 20th annual conference on research in undergraduate mathematics education.*

White Brahmia, S. (2019). Quantification and its importance to modeling in introductory physics. *European Journal of Physics, 40*(4), 044001.

White Brahmia, S., Olsho, A., Smith, T. I., & Boudreaux, A. (2020). Framework for the natures of negativity in introductory physics. *Physical Review Physics Education Research, 16*(1), 010120. https://doi.org/10.1103/PhysRevPhysEducRes.16.010120

White Brahmia, S., Olsho, A., Smith, T. I., Boudreaux, A., Eaton, P., & Zimmerman, C. (2021). Physics inventory of quantitative literacy: A tool for assessing mathematical reasoning in introductory physics. *Physical Review Physics Education Research, 17*(2), 020129.

Zimmerman, C., Olsho, A., Boudreaux, A., Smith, T. I., Eaton, P., & White Brahmia, S. (2020a). Exploring student facility with "goes like" reasoning in introductory physics. In S. Wolf, M. Bennett, & B. Frank (Eds.), *Physics education research conference 2020* (pp. 605–610). Virtual conference. https://doi.org/10.1119/perc.2020.pr.Zimmerman

Zimmerman, C., Olsho, A., Loverude, M. E., & White Brahmia, S. (2020b). Identifying covariational reasoning behaviors in expert physicists in graphing tasks. In *Research in undergraduate mathematics education conference proceedings 2020*. https://doi.org/10.5642/jhummath.201602.17