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Conformal Invariance of Clifford Monogenic Functions in the Indefinite Signature Case

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Abstract

We extend constructions of classical Clifford analysis to the case of indefinite non-degenerate quadratic forms. Clifford analogues of complex holomorphic functions—called monogenic functions—are defined by means of the Dirac operators that factor a certain wave operator. One of the fundamental features of quaternionic analysis is the invariance of quaternionic analogues of holomorphic function under conformal (or Möbius) transformations. A similar invariance property is known to hold in the context of Clifford algebras associated to positive definite quadratic forms. We generalize these results to the case of Clifford algebras associated to all non-degenerate quadratic forms. This result puts the indefinite signature case on the same footing as the classical positive definite case.

Keywords Conformal transformations on $\mathbb{R}^{p,q}$ · Möbius transformations · Vahlen matrices · Monogenic functions · Clifford analysis · Clifford algebras · Conformal compactification

1 Introduction

Many results of complex analysis have analogues in quaternionic analysis. In particular, there are analogues of complex holomorphic functions called (left and right) regular functions. Some of the most fundamental features of quaternionic analysis are the quaternionic analogue of Cauchy's integral formula (usually referred to as Cauchy-Fueter formula) and the invariance of quaternionic regular functions under conformal

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(or Möbius) transformations. For modern introductions to quaternionic analysis see, for example, [9, 21].

Complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ are special cases of Clifford algebras. (For elementary introductions to Clifford algebras see, for example, [7, 11].) There is a further extension of complex and quaternionic analysis called Clifford analysis. For Clifford algebras associated to positive definite quadratic forms on real vector spaces, Clifford analysis is very similar to complex and quaternionic analysis (see, for example, [3, 10, 12] and references therein). Furthermore, Ryan has initiated the study of Clifford analysis in the setting of complex Clifford algebras [17].

Let $\mathrm{Cl}(V)$ be the universal Clifford algebra associated to a real vector space V with non-degenerate quadratic form Q. The Dirac operator D on V is introduced by means of factoring the wave operator associated to Q. Then the Clifford monogenic functions $f : V \to \mathrm{Cl}(V)$ are defined as those satisfying $Df = 0$. An analogue of Cauchy's integral formula for such functions was established in [15]. In this paper we focus on another important feature of Clifford analysis—conformal invariance of monogenic functions. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix producing a conformal (or Möbius) transformation on V

$$x \mapsto (ax + b)(cx + d)^{-1}, \tag{1}$$

and $f : V \to \mathrm{Cl}(V)$ is a monogenic function, then it is known in the case of positive definite quadratic forms Q that

$$J_A(x) f(Ax) = \frac{(cx + d)^{-1}}{|(x\tilde{c} + \tilde{d})(cx + d)|^{n/2-1}} f(Ax) \tag{2}$$

is also monogenic [4, 18]. We extend this result to all non-degenerate quadratic forms Q having arbitrary signatures. While the formula in the mixed signature case visually appears the same, its meaning is slightly different. First of all, the transformation (1) takes place on the conformal closure of the vector space V, and it is different from the one-point compactification of V. Secondly, one needs to revisit the definition of the monogenic functions in this context and choose the "right" Dirac operator D. The choice of signs in our definition of the operator D is compatible with [5, 6, 15]. Furthermore, we argue that this is the most natural choice of the Dirac operator that makes the results valid, while other choices would not work. Interestingly, Bureš-Souček [5] arrive at the same Dirac operator from a different perspective—invariance under the action of the group $\mathrm{Spin}(p, q)$.

These two features of Clifford analysis with indefinite signature—the analogue of Cauchy's integral formula and the conformal invariance of monogenic functions put the indefinite signature case on the same footing as the classical positive definite case.

The paper is organized as follows. Section 2 is a review of relevant topics. We start with the conformal transformations and the conformal compactification of V. Then we discuss Clifford algebras and associated groups—such as Lipschitz, pin and spin groups—and their actions on V by orthogonal linear transformations. After this we introduce the basics of Clifford analysis: the Dirac operator D and the left/right monogenic functions. We show that our definition of D is independent of the choice of basis of V (Lemma 12) and give an alternative basis-free way of defining D (Proposition 13). In Sect. 3 we introduce Vahlen matrices and explain the relation between Vahlen matrices and conformal transformations on V. The results of this section are well known, especially in the positive definite case (see, for example, [2, 8, 16, 22] and references therein), and we mostly follow [16]. In Sect. 4 we prove our main result (Theorem 36) that the function (2) is left monogenic. We also state a similar formula for right monogenic functions. We emphasize that our proof is independent of the signature of the quadratic form Q. Finally, in Appendix we discuss what would happen if the Dirac operator were defined differently and show that a different choice of signs in the definition of the operator D would result in the loss of the conformal invariance. In [15] the same choice of signs was motivated by completely different reasons.

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2 Conformal Transformations, Clifford Algebras and Monogenic Functions

In this section we briefly review the fundamentals of conformal transformations, Clifford algebras and Clifford analysis and establish our notations.

2.1 The Conformal Compactification of $\mathbb{R}^{p,q}$

We start with a review of the conformal compactification $N(V)$ of a vector space $V \simeq \mathbb{R}^{p,q}$ and the action of the indefinite orthogonal group $\mathrm{O}(V \oplus \mathbb{R}^{1,1}) \simeq \mathrm{O}(p+1, q+1)$ on $N(V)$ by conformal transformations. The reader may wish to refer to, for example, [13, 20] for more detailed expositions of the results.

Let V be an n-dimensional real vector space, and Q a quadratic form on V. Corresponding to Q, there is a symmetric bilinear form B on V. The form Q can be diagonalized: there exist an orthogonal basis $\{e_1, \dots, e_n\}$ of (V, Q) and integers p, q such that

$$Q(e_j) = \begin{cases} 1 & 1 \le j \le p; \\ -1 & p+1 \le j \le p+q; \\ 0 & p+q < j \le n. \end{cases} \tag{3}$$

By the Sylvester's Law of Inertia, the numbers p and q are independent of the basis chosen. The ordered pair (p, q) is called the signature of (V, Q). From this point on we restrict our attention to non-degenerate quadratic forms Q, in which case $p + q = n$.

If Q is such a form with $q = 0$ or $p = 0$, then it is positive or negative definite respectively.

Associated with the quadratic form is the indefinite orthogonal group consisting of all invertible linear transformations of V that preserve Q:

$$\mathrm{O}(V) = \big\{T \in \mathrm{GL}(V);\ Q(Tv) = Q(v) \text{ for all } v \in V\big\}.$$

We frequently identify (V, Q) with the generalized Minkowski space $\mathbb{R}^{p,q}$, which is the real vector space $\mathbb{R}^{p+q}$ equipped with indefinite quadratic form

$$Q(u) = (x_1)^2 + \cdots + (x_p)^2 - (x_{p+1})^2 - \cdots - (x_{p+q})^2.$$

The symmetric bilinear form associated to Q is

$$\begin{aligned} B(u, v) &= \tfrac{1}{2}\big[Q(u+v) - Q(u) - Q(v)\big] \\ &= x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}. \end{aligned}$$

In this context, it is common to write $\mathrm{O}(p, q)$ for the indefinite orthogonal group $\mathrm{O}(\mathbb{R}^{p,q})$.

By a *conformal transformation* of (V, Q), we mean a smooth mapping $\varphi : U \to V$, where $U \subset V$ is a non-empty connected open subset, such that the pull-back

$$\varphi^* B(u, v) = \Omega^2 \cdot B(u, v) \tag{4}$$

for some smooth function $\Omega : U \to (0, \infty)$. (Note that we do not require conformal transformations to be orientation preserving.)

The *conformal compactification* $N(V)$ of V is constructed using an ambient vector space $V \oplus \mathbb{R}^{1,1}$. The 2-dimensional space $\mathbb{R}^{1,1}$ has a standard basis $\{e_-, e_+\}$, and the quadratic form Q on V naturally extends to a form $\hat{Q}$ on $V \oplus \mathbb{R}^{1,1}$ so that

$$\hat{Q}(v + x_- e_- + x_+ e_+) = Q(v) - (x_-)^2 + (x_+)^2.$$

By definition, $N(V)$ is a quadric in the real projective space:

$$N(V) = \big\{[\xi] \in P(V \oplus \mathbb{R}^{1,1});\ \hat{Q}(\xi) = 0\big\}.$$

Then $N(V)$ has a natural conformal structure inherited from $V \oplus \mathbb{R}^{1,1}$. More concretely, $V \oplus \mathbb{R}^{1,1}$ contains the product of unit spheres $S^p \times S^q$, and the quotient map $\varpi : V \oplus \mathbb{R}^{1,1} \setminus \{0\} \twoheadrightarrow P(V \oplus \mathbb{R}^{1,1})$ restricted to $S^p \times S^q$ induces a smooth 2-to-1 covering

$$\pi : S^p \times S^q \to N(V),$$

which is a local diffeomorphism. The product $S^p \times S^q$ has a semi-Riemannian metric coming from the product of metrics—the first factor having the standard metric from the embedding $S^p \subset \mathbb{R}^{p+1}$ as the unit sphere, and the second factor S^q having the negative of the standard metric. Then the semi-Riemannian metric on $N(V)$ is defined by declaring the covering $\pi : S^p \times S^q \to N(V)$ to be a local isometry. We also see that $N(V)$ is compact, since it is the image of a compact set $S^p \times S^q$ under a continuous map π.

The embedding map $\iota : V \hookrightarrow N(V)$ can be described as follows. We embed V into the null cone (without the origin)

$$\mathcal{N} = \left\{\xi \in V \oplus \mathbb{R}^{1,1};\ \xi \neq 0,\ \hat{Q}(\xi) = 0\right\} \tag{5}$$

via

$$\hat{\iota}(v) = v + \frac{1 + Q(v)}{2} e_- + \frac{1 - Q(v)}{2} e_+, \tag{6}$$

and compose $\hat{\iota}$ with the quotient map $V \oplus \mathbb{R}^{1,1} \backslash \{0\} \twoheadrightarrow P(V \oplus \mathbb{R}^{1,1})$.

Proposition 1 *The quadric $N(V)$ is indeed a conformal compactification of V. That is, the map $\iota : V \to N(V)$ is a conformal embedding, and the closure $\overline{\iota(V)}$ is all of $N(V)$.*

The indefinite orthogonal group $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$ acts on $V \oplus \mathbb{R}^{1,1}$ by linear transformations. This action descends to the projective space $P(V \oplus \mathbb{R}^{1,1})$ and preserves $N(V)$. Thus, for every $T \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$, we obtain a transformation on $N(V)$, which will be denoted by ψ_T.

Theorem 2 *For each $T \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$, the map $\psi_T : N(V) \to N(V)$ is a conformal transformation and a diffeomorphism. If $T, T' \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$, then $\psi_T = \psi_{T'}$ if and only if $T' = \pm T$.*

The following proposition essentially asserts that the restrictions of ψ_T to V are compositions of parallel translations, rotations, dilations and inversions.

Proposition 3 *For each $T \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$, the conformal transformation ψ_T can be written as a composition of ψ_{T_j}, for some $T_j \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$, $j = 1, \dots, k$, such that each $\iota^{-1} \circ \psi_{T_j} \circ \iota$ is one of the following types of transformations:*

1. *parallel translations $v \mapsto v + b$, where $b \in V$;*
2. *orthogonal linear transformations $v \mapsto Tv$, where $T \in \mathrm{O}(p, q)$;*
3. *dilations $v \mapsto \lambda v$, where $\lambda > 0$;*
4. *the inversion $v \mapsto v/Q(v)$.*

When $p + q > 2$, these are *all* possible conformal transformations of $N(V)$. In fact, an even stronger result is true.

Theorem 4 *Let $p + q > 2$. Every conformal transformation $\varphi : U \to V$, where $U \subset V$ is any non-empty connected open subset, can be uniquely extended to $N(V)$, i.e. there exists a unique conformal diffeomorphism $\hat{\varphi} : N(V) \to N(V)$ such that $\hat{\varphi}(\iota(v)) = \iota(\varphi(v))$ for all $v \in U$.*

Moreover, every conformal diffeomorphism $N(V) \to N(V)$ must be of the form ψ_T, for some $T \in \mathrm{O}(V \oplus \mathbb{R}^{1,1})$. Thus, the group of all conformal transformations $N(V) \to N(V)$ is isomorphic to $\mathrm{O}(V \oplus \mathbb{R}^{1,1})/\{\pm \mathrm{Id}\}$.

We emphasize that by the *group of conformal transformations of* V we mean all conformal transformations $N(V) \to N(V)$, and these are not required to be orientation preserving.

We conclude this subsection with a classification of points in the conformal compactification $N(V)$. The points in $N(V)$ are represented by the equivalence classes $[v + x_- e_- + x_+ e_+]$ in the real projective space $P(V \oplus \mathbb{R}^{1,1})$ such that $Q(v) - (x_-)^2 + (x_+)^2 = 0$, where $v \in V$, $x_-, x_+ \in \mathbb{R}$. We can distinguish between the following three classes:

- $x_- = \frac{1}{2}(1 + Q(v))$ and $x_+ = \frac{1}{2}(1 - Q(v))$. These points can be identified with vectors in V through (6).
- $Q(v) = 0$, $x_- = -\frac{1}{2}$ and $x_+ = \frac{1}{2}$. Such points represent the image of the null vectors $v \in V$ (i.e., vectors with $Q(v) = 0$) under the unique extension to $N(V)$ of the inversion map $v \mapsto v/Q(v)$.
- $Q(v) = x_- = x_+ = 0$ and $v \neq 0$. We can think of these points as the limiting points of the lines generated by the non-zero null vectors $v \in V$.

2.2 Clifford Algebras and Associated Groups

For elementary introductions to Clifford algebras see, for example, [7, 11].

Let V be a real vector space together with a non-degenerate quadratic form Q. Recall the standard construction of the universal Clifford algebra associated to (V, Q) as a quotient of the tensor algebra. We start with the tensor algebra over V,

$$\bigotimes V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots,$$

consider a set of elements of the tensor algebra

$$S = \{v \otimes v + Q(v) : v \in V\} \quad \subset \bigotimes V, \tag{7}$$

and let (S) denote the ideal of $\bigotimes V$ generated by the elements of S. Then the universal Clifford algebra $\mathrm{Cl}(V)$ associated to (V, Q) can be defined as a quotient

$$\mathrm{Cl}(V) = \bigotimes V/(S).$$

We note the sign convention in (7) chosen for the generators of the ideal (S) is not standard in the literature, but it seems to be more prevalent in the context of Clifford analysis. We fix an orthogonal basis $\{e_1, e_2, \dots, e_n\}$ of V satisfying (3), and let $e_0 \in \mathrm{Cl}(V)$ denote the multiplicative identity of the algebra. Thus $\mathrm{Cl}(V)$ is a finite-dimensional algebra over $\mathbb{R}$ generated by $e_0, e_1, \dots, e_n$. We consider subsets $A \subseteq \{1, \dots, n\}$. If A has $k > 0$ elements,

$$A = \{i_1, i_2, \dots, i_k\} \subseteq \{1, \dots, n\}$$

with $i_1 < i_1 < \cdots < i_k$, define

$$e_A = e_{i_1 i_2 \ldots i_k} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$

or, more precisely, e_A is the image of this tensor product in $\mathrm{Cl}(V)$. If A is empty, we set $e_\emptyset$ be the identity element e_0. These elements

$$\left\{ e_A : A \subseteq \{1, 2, \ldots, n\} \right\} \tag{8}$$

form a vector space basis of $\mathrm{Cl}(V)$ over $\mathbb{R}$. In particular, V can be identified with the vector subspace of $\mathrm{Cl}(V)$ spanned by $e_1, e_2, \ldots, e_n$, and we frequently do so.

If u and v are elements of $V \subset \mathrm{Cl}(V)$, then

$$uv + vu = (u + v)^2 - u^2 - v^2 = -Q(u + v) + Q(u) + Q(v) = -2B(u, v). \tag{9}$$

In particular, if u and v are orthogonal in V, $uv + vu = 0$.

If $\mathcal{A}$ is a unital algebra, by a *Clifford map* we mean an injective linear mapping $j : V \to \mathcal{A}$ such that $1 \notin j(V)$ and $j(v)^2 = -Q(v)1$ for all $v \in V$. The algebra $\mathrm{Cl}(V)$ is *universal* in the sense that each such Clifford map $j : V \to \mathcal{A}$ has a unique extension to a homomorphism of algebras $\mathrm{Cl}(V) \to \mathcal{A}$.

There are three important involutions defined on the Clifford algebra $\mathrm{Cl}(V)$.

- The *grade involution* $\hat{}$: $\mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is induced by the Clifford map $V \to \mathrm{Cl}(V)$ defined by $x \mapsto -x$. Then, for vectors $v_1, v_2, \ldots, v_k \in V$, we have

$$v_1 \widehat{v_2 \cdots} v_k = (-1)^k v_1 v_2 \cdots v_k.$$

  Thus, in the standard basis (8), the grade involution is given by

$$\widehat{e_A} \mapsto (-1)^{|A|} e_A$$

  where $|A|$ is the cardinality of A.
- The *reversal* $\tilde{}$: $\mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is the transpose operation on the level of tensor algebra which descends to the Clifford algebra because the two sided ideal (S) generated by (7) is preserved. The transpose operation is defined on each summand $V^{\otimes k}$ of the tensor algebra $\bigotimes V$ as

$$v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1.$$

  Hence, for vectors $v_1, \ldots, v_k \in V$, we have

$$\widetilde{v_1 \cdots v_k} = v_k \cdots v_1.$$

  In the standard basis (8), the reversal is given by

$$\widetilde{e_A} = (-1)^{\frac{|A|(|A|-1)}{2}} e_A.$$

The reversal is an algebra anti-homomorphism: $\widetilde{ab} = \tilde{b}\tilde{a}$.

- The *Clifford conjugation* is defined as the composition of the grade involution and the reversal (note that these two involutions commute). We denote the Clifford conjugation by $\bar{a} = \hat{\tilde{a}}$. In the standard basis (8), the Clifford conjugation is given by

$$\overline{e_A} = (-1)^{\frac{|A|(|A|+1)}{2}} e_A.$$

Like the reversal, the Clifford conjugation is an algebra anti-homomorphism: $\overline{ab} = \bar{b}\bar{a}$.

We frequently use that the grade involution, the reversal, and the Clifford conjugation all commute with taking the multiplicative inverse in $\mathrm{Cl}(V)$:

$$\widehat{(a^{-1})} = \hat{a}^{-1}, \quad \widetilde{(a^{-1})} = \tilde{a}^{-1}, \quad \overline{(a^{-1})} = \bar{a}^{-1}, \qquad a \in \mathrm{Cl}(V).$$

Also note that a vector v is invertible as an element of $\mathrm{Cl}(V)$ if and only if it is not null, i.e., $Q(v) \neq 0$, and $v^{-1} = -v/Q(v)$.

An essential feature of Clifford algebras is that the orthogonal transformations on V can be realized via the so-called twisted adjoint action by elements of $\mathrm{Cl}(V)$. Let $v \in V$ be a vector that is not null, then every $x \in V$ splits into $x = x^{\perp} + \lambda v$ with $x^{\perp} \in V$ orthogonal to v. The map $\sigma_v : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ defined as

$$\sigma_v(x) = -vxv^{-1}$$

preserves the space V and produces a reflection in the direction of v. Indeed,

$$\sigma_v(x) = \sigma_v(x^{\perp} + \lambda v) = -vx^{\perp}v^{-1} - v(\lambda v)v^{-1} = x^{\perp} - \lambda v.$$

Combining such reflections produces various orthogonal transformations on V.

Recall the Cartan-Dieudonné Theorem (for more details see, for example, [11]).

Theorem 5 (Cartan–Dieudonné) *Let V be a vector space with non-degenerate quadratic form, then every orthogonal linear transformation on V can be expressed as the composition of at most* $\dim V$ *reflections in the direction of vectors that are not null.*

By the Cartan-Dieudonné theorem, all orthogonal linear transformations on V can be expressed using a twisted adjoint action

$$\sigma_a : x \mapsto ax\hat{a}^{-1}$$

where $a \in \mathrm{Cl}(V)$ is a product of vectors in V that are not null.

Definition 6 Let $\mathrm{Cl}^{\times}(V)$ denote the set of invertible elements in $\mathrm{Cl}(V)$. The *Lipschitz group*[1] $\Gamma(V)$ consists of elements in $\mathrm{Cl}^{\times}(V)$ that preserve vector space V under the twisted adjoint action:

$$\Gamma(V) = \{a \in \mathrm{Cl}^{\times}(V);\ \sigma_a(x) = ax\hat{a}^{-1} \in V \text{ for all } x \in V\}.$$

The following facts about the Lipschitz group $\Gamma(V)$ are well known. For each $a \in \Gamma(V)$, σ_a restricts to a linear transformation of V that preserves the quadratic form $Q(x)$. Moreover, we have an exact sequence

$$1 \longrightarrow \mathbb{R}^{\times} \longrightarrow \Gamma(V) \xrightarrow{\sigma} \mathrm{O}(V) \longrightarrow 1. \tag{10}$$

Definition 7 The *pin group* $\mathrm{Pin}(V)$ and the *spin group* $\mathrm{Spin}(V)$ are defined as

$$\begin{aligned} \mathrm{Pin}(V) &= \{a \in \Gamma(V);\ \bar{a}a = \pm 1\}, \\ \mathrm{Spin}(V) &= \{a \in \Gamma(V);\ \bar{a}a = \pm 1,\ \hat{a} = a\}. \end{aligned} \tag{11}$$

The pin group and the spin group are double covers of the orthogonal group and special orthogonal group respectively. This can be summarized by the exact sequences

$$\begin{aligned} 1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\sigma} \mathrm{O}(V) \longrightarrow 1, \\ 1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\sigma} \mathrm{SO}(V) \longrightarrow 1. \end{aligned}$$

Proposition 8 *The Lipschitz group $\Gamma(V)$ is the subgroup of $Cl^{\times}(V)$ generated by $\mathbb{R}^{\times}$ and the invertible vectors in V. Moreover, every element in $\Gamma(V)$ can be expressed as a product of $\mathbb{R}^{\times}$ and at most $n = \dim V$ invertible vectors.*

Corollary 9 *If $a \in Cl(V)$, then $a \in \Gamma(V)$ if and only if $a\bar{a} \in \mathbb{R}^{\times}$ and $ax\tilde{a} \in V$ for all $x \in V$.*

Proof Suppose first $a \in \Gamma(V)$, then we can write $a = \alpha v_1 v_2 \cdots v_k$ as a product of vectors $v_1, \dots, v_k \in V$ that are not null and $\alpha \in \mathbb{R}^{\times}$. It is clear that $a\bar{a} = \alpha^2 Q(v_1) \cdots Q(v_k) \in \mathbb{R}^{\times}$. For each $x \in V$, since $\bar{a} = a^{-1}(a\bar{a})$, we have

$$ax\tilde{a} = ax\hat{\bar{a}} = ax\hat{a}^{-1}(a\bar{a}) \in V. \tag{12}$$

Conversely, if $a\bar{a} \in \mathbb{R}^{\times}$, then $a \in \mathrm{Cl}^{\times}(V)$. And $ax\tilde{a} \in V$ for all $x \in V$ implies that

$$\hat{a}xa^{-1} = -\frac{\overline{ax\tilde{a}}}{a\bar{a}} \in V \qquad \text{for all } x \in V.$$

Applying the grade involution, we see that $ax\hat{a}^{-1} = \hat{a}xa^{-1} \in V$ for all $x \in V$. Thus, $a \in \Gamma(V)$. □

[1] Sometimes this group is called Clifford-Lipschitz group or Clifford group.

Corollary 10 *If $a \in \Gamma(V)$, then $\bar{a}xa \in V$ for all $x \in V$.*

Proof Write $a = \alpha v_1 v_2 \cdots v_k$ as a product of vectors $v_1, \dots, v_k \in V$ that are not null and $\alpha \in \mathbb{R}^\times$, then we have

$$\bar{a}xa = (-1)^k \alpha^2 v_k \cdots v_1 x v_1 \cdots v_k = (-1)^k \alpha^2 v_k \cdots v_1 x \widetilde{v_k \cdots v_1} \in V$$

for all $x \in V$ because $v_k \cdots v_1 \in \Gamma(V)$. □

2.3 Monogenic Functions

In this subsection we review the definitions of the Dirac operator and monogenic functions in the context of Clifford analysis. Some comprehensive works on Clifford analysis include [3, 10, 12] as well as references therein.

We identify the vector space V with $\mathbb{R}^{p,q}$ and introduce the Dirac operator D on $\mathbb{R}^{p,q}$

$$D = e_1 \frac{\partial}{\partial x_1} + \cdots + e_p \frac{\partial}{\partial x_p} - e_{p+1} \frac{\partial}{\partial x_{p+1}} - \cdots - e_{p+q} \frac{\partial}{\partial x_{p+q}}. \tag{13}$$

This operator can be applied to functions on the left and on the right. The choice of signs in (13) is compatible with [5, 6, 15], and it is discussed further in Appendix.

If f is a twice-differentiable function on $\mathbb{R}^{p,q}$ with values in $\mathrm{Cl}(\mathbb{R}^{p,q})$ or a left $\mathrm{Cl}(\mathbb{R}^{p,q})$-module and g is a twice-differentiable function on $\mathbb{R}^{p,q}$ with values in $\mathrm{Cl}(\mathbb{R}^{p,q})$ or a right $\mathrm{Cl}(\mathbb{R}^{p,q})$-module,

$$DDf = -\Box_{p,q} f, \quad gDD = -\Box_{p,q} g, \quad \text{where} \quad \Box_{p,q} = \sum_{j=1}^{p} \frac{\partial^2}{\partial x_j^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}.$$

Thus, we can think of the Dirac operator D as a factor of the wave operator $\Box_{p,q}$ on $\mathbb{R}^{p,q}$.

Definition 11 Let $U \subseteq \mathbb{R}^{p,q}$ be an open set, and $\mathcal{M}$ a left $\mathrm{Cl}(\mathbb{R}^{p,q})$-module. A differentiable function $f : U \to \mathcal{M}$ is called *left monogenic* if

$$Df = e_1 \frac{\partial f}{\partial x_1} + \cdots + e_p \frac{\partial f}{\partial x_p} - e_{p+1} \frac{\partial f}{\partial x_{p+1}} - \cdots - e_{p+q} \frac{\partial f}{\partial x_{p+q}} = 0$$

at all points in U.

Similarly, let $\mathcal{M}'$ be a right $\mathrm{Cl}(\mathbb{R}^{p,q})$-module, then a differentiable function $g : U \to \mathcal{M}'$ is called *right monogenic* if

$$gD = \frac{\partial g}{\partial x_1} e_1 + \cdots + \frac{\partial g}{\partial x_p} e_p - \frac{\partial g}{\partial x_{p+1}} e_{p+1} - \cdots - \frac{\partial g}{\partial x_{p+q}} e_{p+q} = 0$$

at all points in U.

We often regard $\mathrm{Cl}(\mathbb{R}^{p,q})$ itself as a $\mathrm{Cl}(\mathbb{R}^{p,q})$-module and speak of left or right monogenic functions with values in $\mathrm{Cl}(\mathbb{R}^{p,q})$. Modules over $\mathrm{Cl}(\mathbb{R}^{p,q})$ are well understood since the seminal paper by Atiyah, Bott and Shapiro [1]; they treated modules over complex and real Clifford algebras. The result is saying that such modules are completely reducible and that there is a classification of the irreducible ones. More detailed exposition of this topic can be found in, for example, [5, 6]. Hence, without loss of generality one can consider functions with values in one of these irreducible $\mathrm{Cl}(\mathbb{R}^{p,q})$-modules.

For the remainder of this subsection we discuss some properties of the Dirac operator D.

Lemma 12 *The Dirac operator D is independent of the choice of orthogonal basis of $V \simeq \mathbb{R}^{p,q}$ satisfying (3).*

Proof Let $\{e'_1, \dots, e'_n\}$ be another basis of V satisfying (3), and let D' be the Dirac operator associated to this basis. There exists a matrix $T \in \mathrm{O}(p,q)$ such that

$$\begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix} = T \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\partial}{\partial x'_1} \\ \vdots \\ \frac{\partial}{\partial x'_n} \end{pmatrix} = T \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

Then

$$\begin{aligned} D' &= \begin{pmatrix} e'_1 & \cdots & e'_n \end{pmatrix} \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x'_1} \\ \vdots \\ \frac{\partial}{\partial x'_n} \end{pmatrix} \\ &= \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} T^{\mathsf{T}} \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix} T \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \\ &= D. \end{aligned}$$

□

Furthermore, we can define the Dirac operator in a basis independent fashion. Let V^* denote the dual space of V, and treat the non-degenerate symmetric bilinear form B on V as an element of $V^* \otimes V^*$. The form B induces isomorphisms $V \simeq V^*$ and $V \otimes V \simeq V^* \otimes V^*$. Denote by B^* the element in $V \otimes V$ corresponding to B. Also, let V' denote the space of all translation-invariant first order differential operators on V. Naturally, V and V' are isomorphic via

$$V \ni v \longleftrightarrow \partial_v \in V',$$

where ∂_v denotes the directional derivative in the direction of vector v. This isomorphism induces a map $d : V \otimes V \simeq V \otimes V' \to \mathcal{D}(V)$, where $\mathcal{D}(V)$ denotes the space of all V-valued translation-invariant first order differential operators on V:

$$d(u \otimes v) = u\partial_v.$$

Proposition 13 *The image $d(B^*) \in \mathcal{D}(V)$ is precisely the Dirac operator D defined by (13).*

Proof Let $\{e_1, \dots, e_n\}$ be any basis of V, and let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^*, then

$$B^* = \sum_{i,j} B^*(e_i^*, e_j^*) e_i \otimes e_j \quad \in V \otimes V \quad \text{and} \quad d(B^*) = \sum_{i,j} B^*(e_i^*, e_j^*) e_i \frac{\partial}{\partial x_j} \in \mathcal{D}(V).$$

The matrix with entries $B^*(e_i^*, e_j^*)$ is the inverse of the matrix with entries $B(e_i, e_j)$. If $\{e_1, \dots, e_n\}$ is an orthogonal basis of V satisfying (3), it is clear that $d(B^*)$ equals the Dirac operator D defined by (13) relative to the basis $\{e_1, \dots, e_n\}$. □

Corollary 14 *For any basis $\{e_1, \dots, e_n\}$ of V, let B^{-1} be the inverse of the matrix with entries $B(e_i, e_j)$, then*

$$D = \big(e_1 \cdots e_n\big)\, \mathrm{B}^{-1} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = \sum_{\lambda,\mu=1}^{n} e_\lambda [\mathrm{B}^{-1}]_{\lambda\mu} \frac{\partial}{\partial x_\mu}. \tag{14}$$

Expression (14) essentially serves as the definition of the Dirac operator in [6]; this definition is standard in physics literature.

3 Vahlen Matrices

We introduce Vahlen matrices, which are certain 2×2 matrices with entries in $\mathrm{Cl}(V)$, and explain the relation between Vahlen matrices and conformal transformations on V. The results of this section are well known, especially in the positive definite case [2, 8, 16, 22] (and many other works). The last two references are particularly relevant, since they deal with the indefinite signature case.

3.1 Algebra Isomorphism $\mathrm{Cl}(V \oplus \mathbb{R}^{1,1}) \simeq \mathrm{Mat}(2, \mathrm{Cl}(V))$

We denote by $\mathrm{Mat}(2, \mathrm{Cl}(V))$ the algebra of 2×2 matrices with entries in $\mathrm{Cl}(V)$. The key ingredient is the $(1, 1)$ periodicity of Clifford algebras (see, for example, [8, 10, 16]):

Proposition 15 *The Clifford algebra* $Cl(V \oplus \mathbb{R}^{1,1})$ *is isomorphic to the matrix algebra* $\mathrm{Mat}(2, Cl(V))$.

Proof First, we construct a Clifford map $j : V \oplus \mathbb{R}^{1,1} \to \mathrm{Mat}(2, \mathrm{Cl}(V))$ by defining

$$j(v) = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, \quad j(e_-) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j(e_+) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad v \in V.$$

We need to check that $j(x)j(x) = -\hat{Q}(x)$ for all $x \in V \oplus \mathbb{R}^{1,1}$. Writing $x = v + x_-e_- + x_+e_+$,

$$j(x) = j(v + x_-e_- + x_+e_+) = \begin{pmatrix} v & x_- - x_+ \\ x_- + x_+ & -v \end{pmatrix},$$

then

$$\begin{aligned} j(x)j(x) &= \begin{pmatrix} v & x_- - x_+ \\ x_- + x_+ & -v \end{pmatrix}^2 \\ &= \begin{pmatrix} v^2 + (x_-)^2 - (x_+)^2 & 0 \\ 0 & v^2 + (x_-)^2 - (x_+)^2 \end{pmatrix} = -\hat{Q}(x), \end{aligned}$$

since $v^2 = -Q(v)$ by (9). This proves that $j : V \oplus \mathbb{R}^{1,1} \to \mathrm{Mat}(2, \mathrm{Cl}(V))$ is a Clifford map. By the universal property of Clifford algebras, this map extends to an algebra homomorphism $\hat{j} : \mathrm{Cl}(V \oplus \mathbb{R}^{1,1}) \to \mathrm{Mat}(2, \mathrm{Cl}(V))$.

To show that $\hat{j}$ is an isomorphism, note that for $a \in \mathrm{Cl}(V) \subset \mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$, we have

$$\hat{j}(a) = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix},$$

and, for an arbitrary element $a + be_- + ce_+ + de_-e_+ \in \mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$, where $a, b, c, d \in \mathrm{Cl}(V)$, we have

$$\begin{aligned} &\hat{j}(a + be_- + ce_+ + de_-e_+) \\ &\quad = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & \hat{b} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & \hat{c} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & \hat{d} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\quad = \begin{pmatrix} a + d & b - c \\ \hat{b} + \hat{c} & \hat{a} - \hat{d} \end{pmatrix}. \end{aligned} \tag{15}$$

It is clear that $\hat{j}$ is a bijection and hence an algebra isomorphism. □

From now on we use isomorphism $\hat{j}$ to identify $\mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$ with $\mathrm{Mat}(2, \mathrm{Cl}(V))$. The following result can be deduced from Eq. (15) describing this isomorphism.

Lemma 16 *The three involutions of $Cl(V \oplus \mathbb{R}^{1,1})$ in terms of* $\mathrm{Mat}(2, Cl(V))$ *are given by*

$$\text{the grade involution:} \quad \widehat{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \hat{a} & -\hat{b} \\ -\hat{c} & \hat{d} \end{pmatrix},$$

$$\text{the reversal:} \quad \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix},$$

$$\text{the Clifford conjugation:} \quad \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}.$$

Definition 17 For a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}(2, \mathrm{Cl}(V))$, define its *pseudo-determinant* as $\Delta(A) = a\tilde{d} - b\tilde{c}$.

Observe that, for all $x \in V \oplus \mathbb{R}^{1,1}$,

$$\hat{Q}(x) = \Delta(j(x)). \tag{16}$$

3.2 Vahlen Matrices

We introduce another key ingredient of this paper—Vahlen matrices.

Definition 18 A matrix $A \in \mathrm{Mat}(2, \mathrm{Cl}(V))$ is called a *Vahlen matrix* if it is in the Lipschitz group $\Gamma(V \oplus \mathbb{R}^{1,1})$.

Introduce a basis $\{n_0, n_\infty\}$ for $\mathbb{R}^{1,1} \subset \mathrm{Mat}(2, \mathrm{Cl}(V))$:

$$n_0 = \frac{e_- + e_+}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad n_\infty = \frac{e_- - e_+}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Maks [16] has a very useful characterization of Vahlen matrices. We supply a proof of his criteria.

Proposition 19 ([16]) *If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}(2, Cl(V))$, then $A \in \Gamma(V \oplus \mathbb{R}^{1,1})$ if and only if*

1. $a\bar{a},\ b\bar{b},\ c\bar{c},\ d\bar{d} \in \mathbb{R}$,
2. $a\bar{c},\ b\bar{d} \in V$,
3. $av\bar{b} - bv\bar{a},\ cv\bar{d} - dv\bar{c} \in \mathbb{R}$ *for all* $v \in V$,
4. $av\bar{d} - bv\bar{c} \in V$ *for all* $v \in V$,
5. $a\tilde{b} = b\tilde{a},\ c\tilde{d} = d\tilde{c}$, *and*
6. *the pseudo-determinant* $\Delta(A) = a\tilde{d} - b\tilde{c}$ *is a nonzero real number.*

Proof By Corollary 9 and linearity, it is sufficient to show that this list of conditions is equivalent to $A\bar{A} \in \mathbb{R}^\times$ and $Ax\tilde{A} \in V \oplus \mathbb{R}^{1,1}$ for all $x \in V \cup \{n_0, n_\infty\}$. Direct calculation shows

$$
\begin{aligned}
An_0\tilde{A} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} b\bar{d} & b\bar{b} \\ d\bar{d} & d\bar{b} \end{pmatrix}, \\
An_\infty\tilde{A} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} a\bar{c} & a\bar{a} \\ c\bar{c} & c\bar{a} \end{pmatrix}, \\
Av\tilde{A} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} av\bar{d} - bv\bar{c} & av\bar{b} - bv\bar{a} \\ cv\bar{d} - dv\bar{c} & cv\bar{b} - dv\bar{a} \end{pmatrix}.
\end{aligned}
$$

These expressions being in $V \oplus \mathbb{R}^{1,1}$ for all $v \in V$ is equivalent to the first four criteria. From the calculation

$$
\begin{aligned}
A\bar{A} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \\
&= \begin{pmatrix} a\tilde{d} - b\tilde{c} & -a\tilde{b} + b\tilde{a} \\ c\tilde{d} - d\tilde{c} & -\tilde{c}b + d\tilde{a} \end{pmatrix},
\end{aligned}
\tag{17}
$$

we see that $A\bar{A} \in \mathbb{R}^\times$ is equivalent to the last two criteria. □

Recall the definition of the pin group (11). Computation (17) implies the following description of $\mathrm{Pin}(V \oplus \mathbb{R}^{1,1})$ and a property of the pseudo-determinants.

Corollary 20 *The pin group* $\mathrm{Pin}(V \oplus \mathbb{R}^{1,1})$*, as a subset of* $\mathrm{Mat}(2, Cl(V))$*, consists of Vahlen matrices A with the pseudo-determinant* $\Delta(A) = a\tilde{d} - b\tilde{c} = \pm 1$.

Corollary 21 *For Vahlen matrices* $A, B \in \mathrm{Mat}(2, Cl(V))$*, we have* $\Delta(AB) = \Delta(A)\Delta(B)$.

Proof Since A and B are Vahlen matrices,

$$
\Delta(A)\Delta(B) = A\bar{A}B\bar{B} = A(B\bar{B})\bar{A} = AB\overline{AB} = \Delta(AB).
$$

□

At this point, we would like to mention that Cnops [8] has a more refined set of criteria for a matrix $A \in \mathrm{Mat}(2, \mathrm{Cl}(V))$ to be a Vahlen matrix. His criteria reduce to Ahlfors' criteria given in [2] when V has positive definite signature. (However, Cnops uses the sign convention $v^2 = Q(v)$ in his definition of a Clifford algebra.)

3.3 Conformal Space

In order to relate Vahlen matrices to conformal transformations, we observe that the embedding (6) can be rewritten as

$$
(j \circ \hat{\imath})(v) = \begin{pmatrix} v & -v^2 \\ 1 & -v \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -v \end{pmatrix}. \tag{18}
$$

This particular presentation encourages us to reinterpret the twisted adjoint action σ of the Lipschitz group as an action on a spinor-like object with two components. We describe a construction of the conformal space due to Maks [16] that makes this idea precise.

Definition 22 The *pre-conformal space* W_{pre} is the set of products $\{Ae\}$, where

$$e = \frac{1 + e_- e_+}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $A \in \text{Mat}(2, \text{Cl}(V))$ ranges over all Vahlen matrices.

The group of Vahlen matrices $\Gamma(V \oplus \mathbb{R}^{1,1})$ acts on W_{pre} by multiplication on the left.

We see that any element in the pre-conformal space W_{pre} must have a matrix realization

$$\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \tag{19}$$

with entries $x, y \in \text{Cl}(V)$. And since these entries form the first column of a Vahlen matrix, by Proposition 19, they satisfy $x\bar{x},\ y\bar{y} \in \mathbb{R}$ and $x\bar{y} \in V$. Simplifying notations, we drop the right column and write (x, y) or $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for an element of W_{pre} represented by (19).

In order to map the pre-conformal space W_{pre} into the null cone $\mathcal{N}$ of $V \oplus \mathbb{R}^{1,1}$ (recall equation (5)) in a way that is compatible with (18), we introduce a map $\gamma : W_{\text{pre}} \to \text{Mat}(2, \text{Cl}(V))$

$$\gamma(Ae) = Aen_\infty \widetilde{Ae} = An_\infty \tilde{A}. \tag{20}$$

Equivalently, we can use Lemma 16 and express (20) in matrix form

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{y} & \bar{x} \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & y\bar{x} \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & -x\bar{y} \end{pmatrix}. \tag{21}$$

From this expression and Eq. (16), we see that γ takes values in the null cone $\mathcal{N}$ of $V \oplus \mathbb{R}^{1,1}$.

For convenience we state the following result (see, for example, [14]).

Lemma 23 (Witt's Extension Theorem) *Let $\mathcal{V}$ be a finite-dimensional real vector space together with a non-degenerate symmetric bilinear form. If $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ is an isometric isomorphism of two subspaces $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$, then φ extends to an isometric isomorphism $\hat{\varphi} : \mathcal{V} \to \mathcal{V}$.*

Recall that $\varpi : V \oplus \mathbb{R}^{1,1} \setminus \{0\} \twoheadrightarrow P(V \oplus \mathbb{R}^{1,1})$ is the quotient map.

Lemma 24 *The map $\varpi \circ \gamma : W_{\text{pre}} \to N(V)$, which by abuse of notation we also denote by γ, is surjective.*

Proof By the Witt's extension theorem, for each null vector $x \in \mathcal{N}$, there exists an orthogonal transformation in $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$ that takes n_∞ to x. Since the twisted adjoint action σ is surjective onto $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$, there exists a Vahlen matrix A_x such that $\sigma_{A_x}(n_\infty) = x$. By Eqs. (12) and (20), we have

$$\gamma(A_x e) = A_x n_\infty \widetilde{A_x} \sim A_x n_\infty \widehat{A_x}^{-1} = \sigma_{A_x}(n_\infty) = x.$$

Therefore, it becomes an equality $\gamma(A_x e) = x$ in $N(V)$. This proves $\gamma : W_{\mathrm{pre}} \to N(V)$ is surjective. □

Lemma 25 *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a Vahlen matrix. If* $\gamma(Ae)$ *is proportional to* n_∞*, then* $c = 0$ *and* $a, d \in \Gamma(V)$.

Proof By Eq. (12), we have

$$n_\infty \sim \gamma(Ae) = A n_\infty \tilde{A} \sim A n_\infty \hat{A}^{-1}.$$

Write $A n_\infty \hat{A}^{-1} = \lambda n_\infty$ for some $\lambda \in \mathbb{R}^\times$, then

$$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = A n_\infty = \lambda n_\infty \hat{A} = \lambda \begin{pmatrix} -\hat{c} & \hat{d} \\ 0 & 0 \end{pmatrix}.$$

We conclude that $c = 0$ and $a = \lambda \hat{d}$. From Proposition 19 we see that $\Delta(A) = \lambda^{-1} a\bar{a} \in \mathbb{R}^\times$ and $\lambda^{-1} a v \tilde{a} \in V$ for all $v \in V$. Corollary 9 implies a and $d = \lambda^{-1}\hat{a} \in \Gamma(V)$. □

Lemma 26 *Let* $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ *be two Vahlen matrices. Then* $\gamma(A_1 e)$ *and* $\gamma(A_2 e)$ *are proportional if and only if there exists an* $r \in \Gamma(V)$ *such that* $(a_1, c_1) = (a_2 r, c_2 r)$.

Proof Suppose first $(a_1, c_1) = (a_2 r, c_2 r)$ for some $r \in \Gamma(V)$, then

$$\gamma(A_1 e) = \begin{pmatrix} a_1\bar{c}_1 & a_1\bar{a}_1 \\ c_1\bar{c}_1 & -a_1\bar{c}_1 \end{pmatrix} = r\bar{r} \begin{pmatrix} a_2\bar{c}_2 & a_2\bar{a}_2 \\ c_2\bar{c}_2 & -a_2\bar{c}_2 \end{pmatrix} = r\bar{r}\gamma(A_2 e).$$

Since $r\bar{r} \in \mathbb{R}^\times$, we conclude that $\gamma(A_1 e)$ and $\gamma(A_2 e)$ are proportional.

Conversely, suppose $\gamma(A_1 e) = \lambda\gamma(A_2 e)$ for some $\lambda \in \mathbb{R}^\times$. Then

$$A_1 n_\infty \widetilde{A_1} = \gamma(A_1 e) = \lambda\gamma(A_2 e) = \lambda A_2 n_\infty \widetilde{A_2},$$

which implies

$$\gamma(A_2^{-1} A_1 e) = A_2^{-1} A_1 n_\infty \widetilde{A_2^{-1} A_1} = \lambda n_\infty.$$

Writing $A_2^{-1}A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, Lemma 25 implies that $c = 0$ and $a \in \Gamma(V)$, and thus

$$A_2^{-1}A_1 e = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = ea.$$

In other words, $A_1 e = A_2 ea$, or in matrix form

$$\begin{pmatrix} a_1 & 0 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 a & 0 \\ c_2 a & 0 \end{pmatrix}.$$

This finishes the proof. □

This lemma inspires the following definition of the conformal space.

Definition 27 The *conformal space* W of V is the pre-conformal space W_{pre} modulo the relation that (x_1, y_1) and (x_2, y_2) are equivalent if and only if there exists an $r \in \Gamma(V)$ such that $(x_1, y_1) = (x_2 r, y_2 r)$.

Theorem 28 *The map $\gamma : W \to N(V)$ is well defined and is a bijection. Moreover, γ intertwines the actions of $\Gamma(V \oplus \mathbb{R}^{1,1})$, that is, $\gamma(AX) = \sigma_A(\gamma(X))$ for all Vahlen matrices A and all $X \in W$.*

Proof By Lemma 26, γ is well defined and injective, and, by Lemma 24, γ is surjective.

For an arbitrary element in W represented by $X \in W_{\text{pre}}$, by Eq. (12), we have

$$\gamma(AX) = AXn_\infty \tilde{X}\tilde{A} \sim AXn_\infty \tilde{X}\hat{A}^{-1} = \sigma_A(Xn_\infty \tilde{X}) = \sigma_A(\gamma(X)).$$

This becomes an equality in $N(V)$ and proves that the two actions of Vahlen matrices commute with γ. □

3.4 Geometry of the Conformal Space *W*

Recall that Eq. (18) was our inspiration for the conformal space, so we wish to identify $v \in V$ with $(v, 1) \in W$, but we first need to show that $(v, 1) \in W$. Indeed, $A = \left(\begin{smallmatrix} v & 1 \\ 1 & 0 \end{smallmatrix}\right)$ satisfies the conditions of Proposition 19, thereby is a Vahlen matrix. Geometrically, A is the composition of the inversion $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ followed by the translation $\left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right)$, and so $\gamma(Ae) = An_\infty \tilde{A}$ can be obtained from n_∞ by applying the inversion and mapping n_∞ into n_0, then translating the result by v.

Maks [16] categorizes the points (x, y) in W into three classes.

- $y\bar{y} \neq 0$. Then we have $y^{-1} = \bar{y}/(y\bar{y})$ and

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & -x\bar{y} \end{pmatrix} \sim \begin{pmatrix} xy^{-1} & x\bar{x}(y\bar{y})^{-1} \\ 1 & -xy^{-1} \end{pmatrix} = \gamma\begin{pmatrix} xy^{-1} \\ 1 \end{pmatrix}.$$

  Thus, (x, y) can be identified with $(xy^{-1}, 1)$ in W and in turn with xy^{-1} in V.

- $y\bar{y} = 0$ and $x\bar{x} \neq 0$. In this case, we have $x^{-1} = \bar{x}/(x\bar{x})$ and

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y\bar{x} & x\bar{x} \\ y\bar{y} & y\bar{x} \end{pmatrix} \sim \begin{pmatrix} -yx^{-1} & 1 \\ y\bar{y}(x\bar{x})^{-1} & yx^{-1} \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ yx^{-1} \end{pmatrix}.$$

  Thus, (x, y) can be identified with $(1, yx^{-1})$ in W. The Vahlen matrix that represents the inversion is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and we recognize $(1, yx^{-1})$ as the inversion of $(yx^{-1}, 1)$. In other words, (x, y) represents the inversion of a null vector $yx^{-1} \in V$ and does not belong to V.
- $x\bar{x} = y\bar{y} = 0$ and $x\bar{y} \neq 0$. This class is empty in the Euclidean cases $p = 0$ or $q = 0$. Otherwise, we can think of (x, y) as the limiting point of the line generated by the non-zero null vector $x\bar{y} \in V$.

In terms of this identification of W with $N(V)$ and embedding $V \hookrightarrow W$, the four types of conformal transformations from Proposition 3 can be represented by the following Vahlen matrices:

1. parallel translations: $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, where $b \in V$;
2. orthogonal linear transformations: $\left(\begin{smallmatrix} a & 0 \\ 0 & \hat{a} \end{smallmatrix}\right)$, where $a \in \Gamma(V)$;
3. dilations: $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$, where $\lambda > 0$;
4. the inversion: $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By Proposition 3 and Eq. (10), every Vahlen matrix can be written as a finite product of matrices of these four types.

Let $U \subset V$ be a non-empty connected open subset. When $p+q > 2$, by Theorem 4, every conformal transformation $U \to \mathbb{R}^{p,q}$ can be described by a Vahlen matrix acting through

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = (ax + b)(cx + d)^{-1} \tag{22}$$

where $cx + d$ is invertible for all $x \in U$. Whenever we write a Vahlen matrix acting on x in this fashion, we always assume that $cx + d$ is invertible.

Before we go into the next section, we need to show that the product $(x\tilde{c}+\tilde{d})(cx+d)$ is a real number.

Lemma 29 *If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix, then $\tilde{a}a,\ \tilde{b}b,\ \tilde{c}c,\ \tilde{d}d \in \mathbb{R}$.*

Proof By Corollary 10, the product

$$\bar{A}n_\infty A = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tilde{d}c & \tilde{d}d \\ -\tilde{c}c & -\tilde{c}d \end{pmatrix}$$

lies in $V \oplus \mathbb{R}^{1,1}$, which implies $\tilde{c}c,\ \tilde{d}d \in \mathbb{R}$. Similarly, $\bar{A}n_0 A \in V \oplus \mathbb{R}^{1,1}$ leads to $\tilde{a}a,\ \tilde{b}b \in \mathbb{R}$. □

Corollary 30 *If $x \in V$, the product $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}$.*

Proof Since $cx + d$ is an entry of a Vahlen matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ax+b & a \\ cx+d & c \end{pmatrix},$$

by Lemma 29, $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}$. □

Consequently, when $cx + d$ is invertible, we have $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}^\times$. It is worth mentioning that Maks [16] claims an even stronger result that if an entry in a Vahlen matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible, then that entry belongs to $\Gamma(V)$.

4 Conformal Invariance of Monogenic Functions

Monogenic function might not stay monogenic under translations by conformal transformations. That is, starting with a monogenic function f and a Vahlen matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the composition function $f(Ax)$ need not be monogenic, where Ax is defined by (22). On the other hand, it is well known that in the positive definite case (see [4, 18]), the function

$$J_A(x) f(Ax) = \frac{(cx+d)^{-1}}{|(x\tilde{c} + \tilde{d})(cx+d)|^{n/2-1}} f(Ax) \tag{23}$$

is monogenic. We extend this result to the case of the quadratic form Q on the underlying vector space $V \simeq \mathbb{R}^{p,q}$ having arbitrary signature.

Lemma 31 *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a Vahlen matrix, then*

$$\begin{aligned} Ax - Ay &= \Delta(A)(y\tilde{c} + \tilde{d})^{-1}(x-y)(cx+d)^{-1} \\ &= \Delta(A)(x\tilde{c} + \tilde{d})^{-1}(x-y)(cy+d)^{-1}, \end{aligned} \tag{24}$$

for all $x, y \in V$ *such that* $cx + d$ *and* $cy + d$ *are invertible.*

Proof A proof by direct calculation is possible (see [2, 19]). Here, we give a proof by induction based on the fact that every orthogonal transformation in $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$ is a composition of translations, orthogonal transformations in $\mathrm{O}(V)$, dilations, and the inversion (Proposition 3). It is straightforward to show that the formula holds for the Vahlen matrices listed in Subsection 3.4 that produce translations, orthogonal transformations, dilations, and the inversion on V. Then we show that if the formula is true for Vahlen matrices $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, then the formula also holds for $A_{21} = A_2 A_1 = \begin{pmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{pmatrix}$. That is, we need to show

$$\begin{aligned} &\Delta(A_{21})(y\tilde{c}_{21} + \tilde{d}_{21})^{-1}(x-y)(c_{21}x + d_{21})^{-1} \\ &\quad = \Delta(A_2)\Delta(A_1)((A_1 y)\tilde{c}_2 + \tilde{d}_2)^{-1}(y\tilde{c}_1 + \tilde{d}_1)^{-1}(x-y)(c_1 x + d_1)^{-1}(c_2(A_1 x) + d_2)^{-1}. \end{aligned}$$

By Corollary 21, it is sufficient to show

$$(y\tilde{c}_{21}+\tilde{d}_{21})^{-1}=((A_1y)\tilde{c}_2+\tilde{d}_2)^{-1}(y\tilde{c}_1+\tilde{d}_1)^{-1}$$
$$(c_{21}x+d_{21})^{-1}=(c_1x+d_1)^{-1}(c_2(A_1x)+d_2)^{-1}.$$

These two identities are related to each other through the reversal operation, so it is sufficient to show

$$c_{21}x+d_{21}=(c_2a_1+d_2c_1)x+(c_2b_1+d_2d_1)=(c_2(A_1x)+d_2)(c_1x+d_1). \quad (25)$$

This identity follows from $(A_1x)(c_1x+d_1)=a_1x+b_1$.

The second identity in (24) follows from the first by applying the reversal. □

Corollary 32 *The partial derivatives of the conformal transformation produced by a Vahlen matrix $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ can be expressed as*

$$\begin{aligned}\frac{\partial(Ax)}{\partial x_\mu}&=\Delta(A)(x\tilde{c}+\tilde{d})^{-1}e_\mu(cx+d)^{-1}\\ &=\frac{\Delta(A)}{(x\tilde{c}+\tilde{d})(cx+d)}(cx+d)e_\mu(cx+d)^{-1}.\end{aligned} \quad (26)$$

Proof Using Eq. (24), we obtain

$$\frac{\partial(Ax)}{\partial x_\mu}=\lim_{h\to 0}\frac{A(x+he_\mu)-Ax}{h}=\Delta(A)(x\tilde{c}+\tilde{d})^{-1}e_\mu(cx+d)^{-1}.$$

□

This expression for the partial derivatives allows us to verify directly that the mapping $x\mapsto Ax$ is a conformal transformation (compare with (4)).

Lemma 33 *The pull-back*

$$A^*B=\Omega_A^2B,$$

where the conformal factor is

$$\Omega_A(x)=\frac{\Delta(A)}{(x\tilde{c}+\tilde{d})(cx+d)}.$$

Proof Write $Ax=\sum_{\kappa=1}^n(Ax)_\kappa e_\kappa$, then, by (26),

$$(cx+d)e_\mu(cx+d)^{-1}=\frac{(x\tilde{c}+\tilde{d})(cx+d)}{\Delta(A)}\sum_{\kappa=1}^n\frac{\partial(Ax)_\kappa}{\partial x_\mu}e_\kappa \quad (27)$$

By (9), we have

$$-2B(e_\mu, e_\nu) = (cx+d)(e_\mu e_\nu + e_\nu e_\mu)(cx+d)^{-1}.$$

Inserting $(cx+d)^{-1}(cx+d)$ between e_μ and e_ν and applying Eq. (27), we obtain

$$\begin{aligned} -2B(e_\mu, e_\nu) &= \left[\frac{(x\tilde{c}+\tilde{d})(cx+d)}{\Delta(A)}\right]^2 \sum_{\kappa,\lambda=1}^{n} (e_\kappa e_\lambda + e_\lambda e_\kappa) \frac{\partial (Ax)_\kappa}{\partial x_\mu} \frac{\partial (Ax)_\lambda}{\partial x_\nu} \\ &= -2\Omega_A(x)^{-2} \sum_{\kappa,\lambda=1}^{n} B(e_\kappa, e_\lambda) \frac{\partial (Ax)_\kappa}{\partial x_\mu} \frac{\partial (Ax)_\lambda}{\partial x_\nu}, \end{aligned} \tag{28}$$

and the result follows. □

Corollary 34 *Write* B *for the matrix of the bilinear form with entries $B(e_i, e_j)$ and* B^{-1} *for its inverse. We have:*

$$\sum_{\lambda,\mu=1}^{n} [\mathrm{B}^{-1}]_{\lambda\mu} e_\lambda (cx+d)^{-1} \frac{\partial (Ax)_\kappa}{\partial x_\mu} = \Omega_A(x)(cx+d)^{-1} \sum_{\nu=1}^{n} [\mathrm{B}^{-1}]_{\nu\kappa} e_\nu. \tag{29}$$

Proof Let ∂A denote the matrix of partial derivatives with $\frac{\partial (Ax)_i}{\partial x_j}$ in the (ij)-entry. Then Eq. (28) can be rewritten as

$$(\partial A)^{\mathsf{T}} \mathrm{B} (\partial A) = \Omega_A^2 \mathrm{B},$$

which in turn can be rewritten as

$$(\partial A) \mathrm{B}^{-1} (\partial A)^{\mathsf{T}} = \Omega_A^2 \mathrm{B}^{-1}$$

or, equivalently,

$$\Omega_A^2(x)[\mathrm{B}^{-1}]_{\nu\kappa} = \sum_{\lambda,\mu=1}^{n} [\mathrm{B}^{-1}]_{\lambda\mu} \frac{\partial (Ax)_\nu}{\partial x_\lambda} \frac{\partial (Ax)_\kappa}{\partial x_\mu}.$$

Hence, using (26),

$$\begin{aligned} \sum_{\nu=1}^{n} [\mathrm{B}^{-1}]_{\nu\kappa} e_\nu &= \Omega_A^{-2}(x) \sum_{\lambda,\mu,\nu=1}^{n} [\mathrm{B}^{-1}]_{\lambda\mu} \frac{\partial (Ax)_\nu}{\partial x_\lambda} \frac{\partial (Ax)_\kappa}{\partial x_\mu} e_\nu \\ &= \Omega_A^{-2}(x) \sum_{\lambda,\mu=1}^{n} [\mathrm{B}^{-1}]_{\lambda\mu} \frac{\partial (Ax)}{\partial x_\lambda} \frac{\partial (Ax)_\kappa}{\partial x_\mu} \\ &= \Omega_A^{-1}(x) \sum_{\lambda,\mu=1}^{n} [\mathrm{B}^{-1}]_{\lambda\mu} (cx+d) e_\lambda (cx+d)^{-1} \frac{\partial (Ax)_\kappa}{\partial x_\mu}, \end{aligned}$$

and the result follows. □

To each Vahlen matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we associate a function

$$J_A(x) = \frac{(cx+d)^{-1}}{|(x\tilde{c}+\tilde{d})(cx+d)|^{n/2-1}}$$

defined for all $x \in V$ such that $cx+d$ is invertible. Note that, by Corollary 30, the expression $(x\tilde{c}+\tilde{d})(cx+d)$ in the denominator is always real.

Lemma 35 *Given Vahlen matrices A_1 and A_2, we have*

$$J_{A_2A_1}(x) = J_{A_1}(x)J_{A_2}(A_1x)$$

wherever both sides are defined.

Proof Let $j_A(x) = cx+d$, then J_A can be expressed as

$$J_A(x) = \frac{(j_A(x))^{-1}}{|\widetilde{j_A(x)}j_A(x)|^{n/2-1}},$$

and Eq. (25) can be restated as

$$j_{A_2A_1}(x) = j_{A_2}(A_1x)j_{A_1}(x).$$

Therefore,

$$\begin{aligned} J_{A_2A_1}(x) &= \frac{(j_{A_2A_1}(x))^{-1}}{|\widetilde{j_{A_2A_1}(x)}j_{A_2A_1}(x)|^{n/2-1}} \\ &= \frac{(j_{A_1}(x))^{-1}}{|\widetilde{j_{A_1}(x)}j_{A_1}(x)|^{n/2-1}} \frac{(j_{A_2}(A_1x))^{-1}}{|\widetilde{j_{A_2}(A_1x)}j_{A_2}(A_1x)|^{n/2-1}} \\ &= J_{A_1}(x)J_{A_2}(A_1x). \end{aligned}$$

□

Theorem 36 *Let $U \subseteq V$ be an open set, $\mathcal{M}$ a left $Cl(V)$-module, and $f : U \to \mathcal{M}$ a differentiable function. For each Vahlen matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,*

$$D\big(J_A(x)f(Ax)\big) = \Omega_A(x)J_A(x)(Df)(Ax) \tag{30}$$

for all $x \in A^{-1}(U) = \{v \in V;\ Av \in U\}$ such that $cx+d$ is invertible. In particular, if f is left monogenic, so is $J_A(x)f(Ax)$.

Similarly, if $\mathcal{M}'$ a right $Cl(V)$-module, and $g : U \to \mathcal{M}'$ a differentiable function,

$$\big(g(Ax)\widetilde{J_A}\big)D = \Omega_A(x)(gD)(x)\widetilde{J_A}(x) \tag{31}$$

for all $x \in A^{-1}(U) = \{v \in V;\ Av \in U\}$ *such that* $cx + d$ *is invertible, where*

$$\widetilde{J_A}(x) = \frac{(x\tilde{c} + \tilde{d})^{-1}}{|(x\tilde{c} + \tilde{d})(cx + d)|^{n/2-1}}.$$

In particular, if g *is right monogenic, so is* $g(Ax)\widetilde{J_A}(x)$.

Proof Choose a basis $\{e_1, \dots, e_n\}$ of V. By Eqs. (14) and (29),

$$\begin{aligned} D\big(J_A(x) f(Ax)\big) &= (DJ_A)(x) f(Ax) + \sum_{\kappa,\lambda,\mu=1}^{n} e_\lambda [\mathrm{B}^{-1}]_{\lambda\mu} J_A(x) \frac{\partial (Ax)_\kappa}{\partial x_\mu} \frac{\partial f}{\partial x_\kappa} \\ &= (DJ_A)(x) f(Ax) + \Omega_A(x) J_A(x) \sum_{\kappa,\nu=1}^{n} e_\nu [\mathrm{B}^{-1}]_{\nu\kappa} \frac{\partial f}{\partial x_\kappa} \\ &= (DJ_A)(x) f(Ax) + \Omega_A(x) J_A(x) (Df)(Ax). \end{aligned} \tag{32}$$

Thus, it remains to prove that $J_A(x)$ is left monogenic wherever it is defined.

If A is one of the Vahlen matrix listed in Subsection 3.4 that produces a translation, orthogonal linear transformation, dilation on V, then J_A is just a constant function, hence monogenic. If $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ represents the inversion on V, then

$$J_A(x) = \operatorname{sgn}(x^2) \frac{x}{|x^2|^{n/2}}. \tag{33}$$

By direct calculation,

$$\begin{gathered} Dx = -n, \qquad D(x^2) = -2x, \qquad D|x^2| = -2\operatorname{sgn}(x^2) x, \\ D\left(\operatorname{sgn}(x^2) \frac{x}{|x^2|^{n/2}}\right) = -\operatorname{sgn}(x^2) \frac{n}{|x^2|^{n/2}} + \frac{nx^2}{|x^2|^{n/2+1}} = 0, \end{gathered}$$

and $J_A(x)$ is monogenic too.

By Eq. (32), $J_A(x) f(Ax)$ is left monogenic whenever f and J_A are left monogenic. Thus, by Lemma 35, $J_{A_2 A_1}$ is left monogenic whenever J_{A_1} and J_{A_2} are left monogenic. Since each Vahlen matrix can be written as a finite product of Vahlen matrices realizing translations, orthogonal transformations, dilations and the inversion on V, this proves that J_A is left monogenic for all Vahlen matrices A. This finishes the proof of (30).

The proof of (31) is similar. □

5 Appendix: A Note on the Definition of the Dirac Operator

We introduced the Dirac operator with plus and minus signs in (13), and we have shown that this is a natural construction from the point of view of basis independence. Since this phenomenon does not happen in the positive definite case, it is perhaps worthwhile to show that the classical formula Eq. (23) fails if the Dirac operator were defined differently, for example, having all positive signs:

$$D^* = e_1 \frac{\partial}{\partial x_1} + \cdots + e_p \frac{\partial}{\partial x_p} + e_{p+1} \frac{\partial}{\partial x_{p+1}} + \cdots + e_{p+q} \frac{\partial}{\partial x_{p+q}}.$$

Note that this differential operator depends on the choice of basis—a different choice of orthogonal basis of $V \simeq \mathbb{R}^{p,q}$ satisfying (3) typically leads to a different operator. This can be seen by trying to adapt the proof of Lemma 12 for D^*.

As a first example, consider $\mathbb{R}^{2,1}$ with orthogonal generators of $\mathrm{Cl}(\mathbb{R}^{2,1})$ satisfying $e_1^2 = -1$, $e_2^2 = -1$ and $e_3^2 = 1$. The function $f(x) = x_1 e_1 - x_2 e_2$ is in the kernel of both D^* and D, but if we consider a Vahlen matrix

$$A = \begin{pmatrix} \cosh\alpha + e_2 e_3 \sinh\alpha & 0 \\ 0 & \cosh\alpha + e_2 e_3 \sinh\alpha \end{pmatrix}$$

producing a hyperbolic rotation in the e_2e_3-plane inside $\mathbb{R}^{2,1}$, we find that $J_A(x) f(Ax)$ is not in the kernel of D^*. Indeed, by direct computation,

$$J_A(x) = \cosh\alpha - e_2 e_3 \sinh\alpha,$$
$$Ae_1 = e_1, \quad Ae_2 = e_2 \cosh(2\alpha) + e_3 \sinh(2\alpha), \quad Ae_3 = e_2 \sinh(2\alpha) + e_3 \cosh(2\alpha).$$

Therefore, we have

$$\begin{aligned} J_A(x) f(Ax) &= (\cosh\alpha - e_2 e_3 \sinh\alpha)\big(x_1 e_1 - (x_2 \cosh(2\alpha) + x_3 \sinh(2\alpha)) e_2\big) \\ &= x_1 e_1 (\cosh\alpha - e_2 e_3 \sinh\alpha) \\ &\quad - (x_2 \cosh(2\alpha) + x_3 \sinh(2\alpha))(e_2 \cosh\alpha - e_3 \sinh\alpha). \end{aligned}$$

Then D^* acting on the first term yields

$$D^*\big[x_1 e_1 (\cosh\alpha - e_2 e_3 \sinh\alpha)\big] = -(\cosh\alpha - e_2 e_3 \sinh\alpha),$$

whereas the second term results in

$$D^*\big[-(x_2 \cosh(2\alpha) + x_3 \sinh(2\alpha))(e_2 \cosh\alpha - e_3 \sinh\alpha)\big] = \cosh(3\alpha) + e_2 e_3 \sinh(3\alpha).$$

They clearly do not cancel each other. To contrast, if we use the true Dirac operator,

$$\begin{aligned} &D\big[x_1 e_1 (\cosh\alpha - e_2 e_3 \sinh\alpha)\big] = -(\cosh\alpha - e_2 e_3 \sinh\alpha), \\ &D\big[-(x_2 \cosh(2\alpha) + x_3 \sinh(2\alpha))(e_2 \cosh\alpha - e_3 \sinh\alpha)\big] = \cosh\alpha - e_2 e_3 \sinh\alpha, \end{aligned}$$

as desired.

In general, the true Dirac operator D plays well with orthogonal linear transformations, while the ostensible Dirac operator D^* does not. More precisely, restricting to the

Vahlen matrices producing orthogonal linear transformations on V, that is, matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & \hat{a} \end{smallmatrix}\right)$, where $a \in \Gamma(V)$, Eq. (23) becomes

$$J_A(x) f(Ax) = \frac{\hat{a}^{-1}}{|\tilde{a}a|^{n/2-1}} f(ax\hat{a}^{-1}) \sim \hat{a}^{-1} f(ax\hat{a}^{-1}).$$

It was established in Theorem 36 that

$$Df = 0 \qquad \text{implies} \qquad D\big(\hat{a}^{-1} f(ax\hat{a}^{-1})\big) = 0.$$

On the other hand, as the above example shows,

$$\text{when} \quad D^* f = 0, \quad \text{typically} \quad D^*\big(\hat{a}^{-1} f(ax\hat{a}^{-1})\big) \neq 0.$$

The ostensible Dirac operator D^* also does not behave well with respect to the inversion. The constant function $f(x) = 1$ is annihilated by D^* (and D). Applying to this function the inversion realized by $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ results in $J_A(x)$, and $J_A(x)$ is not in the kernel of D^*. For concreteness, consider $\mathbb{R}^{1,1}$ with generators of $\mathrm{Cl}(\mathbb{R}^{1,1})$ satisfying $e_1^2 = -1$, $e_2^2 = 1$. By Eq. (33), we have

$$J_A(x) = \operatorname{sgn}(x^2) \frac{x}{|x^2|} = \frac{x_1 e_1 + x_2 e_2}{(x_2)^2 - (x_1)^2}.$$

Then $D^*[x_1 e_1 + x_2 e_2] = 0$, and

$$D^* J_A(x) = 2 \frac{(x_1 e_1 - x_2 e_2)(x_1 e_1 + x_2 e_2)}{((x_2)^2 - (x_1)^2)^2} = 2 \frac{-(x_1)^2 - (x_2)^2 + 2 x_1 x_2 e_1 e_2}{((x_2)^2 - (x_1)^2)^2},$$

which is clearly non-zero.

Author contributions M.L. proposed the problem for research. Both authors worked on solving the problem and the manuscript. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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