On Sufficient Lyapunov Conditions for Fixed-Time Stability of Hybrid Dynamical Systems

Michael Tang, Miroslav Krstic, Jorge Poveda

Abstract— **We study the property of fixed-time stability (FxTS) for hybrid dynamical systems (HDS) that combine continuous-time and discrete-time dynamics, possibly modeled via set-valued maps. For such systems, we provide sufficient Lyapunov characterizations for certifying FxTS of general closed sets. In this work, FxTS of a closed set is defined to require two properties: Lyapunov stability and fixed time convergence. For fixed time convergence, the system's trajectories must converge to the set of interest within a finite hybrid time that is uniformly bounded over all initializations. Our main contributions are two-fold: we first show that if the HDS admits solutions that flow for sufficiently long time, as well as a suitable Lyapunov function with a sufficient "fixed-time" decrease along flows and nonincrease during jumps, then the HDS will exhibit FxTS. Afterwards, we extend this result by showing that, under a mild dwell time condition, FxTS can be established for HDS for which the Lyapunov function is allowed to mildly increase during jumps. While our theoretical results pave the way for the general analysis of a broad class of fixed-time stable hybrid algorithms and systems, we illustrate them using two representative cases: a hybrid system with dwell-time conditions on the jumps, and systems exhibiting arbitrarily fast switching between a finite number of fixed-time stable vector fields.**

I. INTRODUCTION

Many real-world processes can be modeled as hybrid dynamical systems (HDS), i.e., systems that exhibit both continuous-time and discrete time-time dynamics. The HDS framework, formalized in works such as [1] and [2], offers powerful tools to study the stability and robustness properties of a large class of systems, including electrical systems, power grids, cyber-physical systems, biological systems, etc. Various methods have been studied in the literature to assess the stability and asymptotic behavior of HDS, including Lyapunov functions [2], [3], small gain theorems [4], [5], passivity [6], [7], etc. Specialized results have also been derived for particular classes of hybrid systems, such as switched systems [1], piecewise affine systems [8], sampled-data systems [9], [10], etc. Despite the richness of results in the stability analysis of HDS, the majority have primarily focused on studying asymptotic and exponential stability, where the trajectories of the system converge to a point or set of interest as time grows to infinity. On the other hand, results on finite-time stability [11] have been developed for special cases of hybrid systems, such as impulsive dynamical systems [12], [13], switched systems [14], stochastic systems [15], etc. These results have been generalized relatively recently for HDS [16], where the authors derive conditions involving nonsmooth Lyapunov-like functions to certify finite-time stability of closed sets in well-posed HDS.

While finite-time stable systems are enticing due to their ability to achieve exact convergence before a given time, they are often highly sensitive to the initialization of the system. That is, the amount of time needed to converge to the point of interest, i.e., the settling time, may grow unbounded with respect to the initial condition. This limitation has motivated the study of a stronger property called fixed-time stability (FxTS) [17], where convergence occurs by a finite time *that is additionally uniformly bounded over all possible initializations*. Following the introduction of sufficient Lyapunov conditions to verify FxTS in continuous-time systems [18] and estimate their settling time, FxTS has received significant attention due to its ability to address challenges in a wide range of engineering domains, such as neural networks [19], optimization [20], [21], robotics [22], and aerospace systems [23], among others. While FxTS is a highly desirable property in systems with stringent performance requirements, the tools to study FxTS in HDS are quite scarce. The existing methods are only applicable to a restrictive set of systems, such as impulsive systems [24] and delayed switched systems [25]. Moreover, existing results do not consider the possibility of having HDS with set-valued dynamics, a situation that is common in many applications and models of systems that involve hybrid dynamics, and which is also relevant for the study of robustness properties [2].

In this paper, we study FxTS properties for hybrid dynamic inclusions and derive different types of sufficient Lyapunov-based conditions that exploit different properties of the solutions. For these types of systems, FxTS requires fixed-time convergence within a uniform *hybrid* time, which considers the cumulative amount of time a solution spends flowing and the total number of jumps it makes. However, we focus on situations where the fixed-time convergence property is induced mostly via the flows. In this sense, our main contribution is to establish two novel sets of Lyapunov conditions for certifying FxTS of general closed sets in hybrid inclusions, where the Lyapunov function is allowed to be non-smooth. The first result assumes a "fixed-time" decrease condition during flows and a non-increase condition during jumps. The second result allows for increase of the Lyapunov function during jumps provided an additional

This work was supported in part by NSF grants ECCS CAREER 2305756, CMMI 2228791, and AFOSR FA9550-22-1-0211.

M. Tang and J. I. Poveda are with the Electrical and Computer Engineering Department, University of California at San Diego, La Jolla, CA 92037 USA (e-mail: myt001@ucsd.edu; jipoveda@ucsd.edu).

M. Krstic is with the Mechanical and Aerospace Engineering Department, University of California at San Diego, La Jolla, CA 92037 USA (e-mail: mkrstic@ucsd.edu).

dwell time condition holds. The results are then demonstrated via examples of hybrid systems with jumps satisfying dwell times, as well as in the context of switching systems with arbitrarily fast switching signals.

II. PRELIMINARIES

A. Notation

The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{K}_\infty$, denoted $\alpha \in \mathcal{K}_\infty$, if $\alpha(0) = 0$, it is strictly increasing, and it is unbounded. Moreover, if $\mathcal{A}$ is closed, and $x \in \mathbb{R}^n$, we use $|x|_\mathcal{A}$ to denote its distance to $\mathcal{A}$, i.e $|x|_\mathcal{A} = \inf_{y\in\mathcal{A}} |x - y|$, where $|\cdot|$ is the Euclidean norm. We introduce the function $[\cdot]_+ : \mathbb{R} \to \mathbb{R}_{\geq 0}$, defined as $[\cdot]_+ := \max(\cdot, 0)$. Given a continuous and locally Lipschitz function V, the generalized gradient (in the sense of Clarke [26]) at a point z, denoted $\partial V(z)$, is the convex hull of all limits of sequences $\nabla V(z_i)$, where $z_i \to z$ and V is differentiable at each z_i.

B. Hybrid Dynamical Systems

A hybrid dynamical system $\mathcal{H}$ with data (C, F, D, G) is modeled by the dynamics

$$\dot{x} \in F(x), \quad x \in C \tag{1a}$$

$$x^+ \in G(x), \quad x \in D, \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping that defines the continuous-time dynamics, which are allowed to evolve on the flow set $C \subset \mathbb{R}^n$. Similarly, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping that defines the discrete-time dynamics, which are allowed to evolve whenever the state x is in the jump set $D \subset \mathbb{R}^n$. In (1b), x^+ represents the value of the state x after a "jump", i.e., a discrete-time update. Throughout this paper, we assume (1) satisfies the *hybrid basic conditions* [2, Assumption 6.5]:

- C and D are closed, where $C \subset \mathrm{dom} F$ and $D \subset \mathrm{dom} G$
- F is outer semicontinuous, locally bounded on C, and convex valued
- G is outer semicontinous and locally bounded on D

Solutions to (1) are given by functions $\phi(t, j)$, which are parameterized by a continuous time index $t \geq 0$ and a discrete-time index $j \in \{0, 1, 2, ...\}$. The domain of a solution ϕ, denoted $\mathrm{dom}\, \phi$, is as defined in [2, Def 2.3]. For the reader's convenience, we reproduce the definition here.

Definition 1: A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if $E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_J$. E is a hybrid time domain if it is the union of a non-decreasing sequence of compact hybrid time domains, namely E is the union of compact hybrid time domains E_j where $E_0 \subset E_1 \subset ... \subset E_j...$, etc.

A solution ϕ to (1) is defined as follows, taken from [2, Def 2.6]

Definition 2: $\phi : \mathrm{dom}\, \phi \to \mathbb{R}^n$ is a solution to (1) if $\phi(0, 0) \in C \cup D$ and

1) For all $j \in \mathbb{N}$ such that $I^j = \{t : (t, j) \in \mathrm{dom}\, \phi\}$ has nonempty interior:
   - $\phi(t, j) \in C \quad \forall t \in \mathrm{int}\ I^j$
   - $\frac{d}{dt}\phi(t', j) \in F(\phi(t', j))$ for almost all $t' \in I^j$
2) For all $(t, j) \in \mathrm{dom}\, \phi$ such that $(t, j + 1) \in \mathrm{dom}\, \phi$:
   - $\phi(t, j) \in D$
   - $\phi(t, j + 1) \in G(\phi(t, j))$

A solution ϕ is called maximal if it cannot be extended to another solution, and it is complete if its domain is unbounded. The set of all maximal solutions to $\mathcal{H}$ is denoted $\mathcal{S}_\mathcal{H}$, and the set of all maximal solutions to $\mathcal{H}$ with initial conditions belonging to set E is denoted $\mathcal{S}_\mathcal{H}(E)$. For more details on the definitions and notations, we refer the readers to [2].

Given a complete solution ϕ to $\mathcal{H}$, we denote its "jump count" by $\mathcal{J}(\phi) := \sup_{(t,j)\in\mathrm{dom}\,\phi} j$. Given a solution ϕ, if $\mathcal{J}(\phi) \geq 1$ and $\mathrm{dom}\, \phi \cap ([0, \infty) \times \{0, ..., \mathcal{J}(\phi) - 1\}) = \cup_{i=0}^{\mathcal{J}(\phi)-1}([t_i, t_{i+1}], i)$, we let $\tau_i(\phi) := t_{i+1} - t_i$ for $i \leq \mathcal{J}(\phi) - 1$. This essentially represents the amount of time the solution spends flowing during its i-th instance of flow.

In this paper, we are interested in studying the following property.

Definition 3: Consider the HDS $\mathcal{H}$ given by (1), and let $\mathcal{A} \subset C \cup D$ be a closed set. The set $\mathcal{A}$ is said to be:

1) *uniformly globally stable* for $\mathcal{H}$ if there exists $\alpha \in \mathcal{K}_\infty$ such that each $\phi \in \mathcal{S}_\mathcal{H}(C \cup D)$ satisfies $|\phi(t, j)|_\mathcal{A} \leq \alpha(|\phi(0, 0)|_\mathcal{A})$ for each $(t, j) \in \mathrm{dom}\, \phi$.
2) *uniformly globally fixed-time attractive (UGFxTA)* if there exists a function $\mathcal{T} : \mathbb{R}^n \to [0, \infty)$, called the hybrid settling time function, such that for each $\phi \in \mathcal{S}_\mathcal{H}$,
   - $\sup_{(t,j)\in\mathrm{dom}\ \phi} t + j \geq \mathcal{T}(\phi(0, 0))$
   - $|\phi(t, j)|_\mathcal{A} = 0$ for each $(t, j) \in \mathrm{dom}\, \phi$ with $t+j \geq \mathcal{T}(\phi(0, 0))$

   where $\sup_{x\in C\cup D} \mathcal{T}(x) < \infty$.
3) *uniformly globally fixed time stable (UGFxTS)* if it is uniformly globally stable and UGFxTA.

III. MAIN RESULTS

In this section, we present sufficient Lyapunov conditions to certify UGFxTS for system (1). To establish these conditions, given a set $U \subset \mathbb{R}^n$ with $C \subset U$ and a continuous function $V : U \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz on a neighborhood of C, we introduce the following auxiliary functions:

$$u_C(x) = \begin{cases} \max\limits_{v\in F(x)} \max\limits_{\zeta\in\partial V(x)} v^\top \zeta & \text{if } x \in C \\ -\infty & \text{else,} \end{cases} \tag{2a}$$

where F and C come from (1a), and

$$u_D(x) = \begin{cases} \max\limits_{\zeta\in G(x)} V(\zeta) - V(x) & \text{if } x \in D \\ -\infty & \text{else,} \end{cases} \tag{2b}$$

where G and D come from (1b). It follows that for each solution ϕ to (1) and each t at which $\frac{d}{dt}V(\phi(t, j))$ exists, the following estimate holds:

$$\frac{d}{dt}V(\phi(t, j)) \leq u_C(\phi(t, j)). \tag{3}$$

Similarly, for each solution ϕ to (1) and each j for which $(t_{j+1}, j), (t_{j+1}, j+1) \in \operatorname{dom}\phi$, we have

$$V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \leq u_D(\phi(t_{j+1}, j)). \quad (4)$$

With these functions in mind, we can now state our main results, which provide sufficient Lyapunov conditions to establish UGFxTS of a closed set $\mathcal{A}$ for (1).

A. Nonincrease During Jumps

First, we consider the case where the Lyapunov function V exhibits sufficient "fixed-time" decrease during flows and is nonincreasing during jumps. In this case, UGFxTS can be guaranteed provided solutions spend a sufficient amount of time flowing, and that there exists a uniform bound on the number of jumps each solution undergoes by that time.

Theorem 1: Consider the HDS $\mathcal{H}$ given by (1) and a closed set $\mathcal{A} \subset C \cup D$. Suppose there exists a continuous function $V : C \cup D \to \mathbb{R}_{\geq 0}$, that is locally Lipschitz on a neighborhood of C, and $c_1, c_2 > 0, p_1 \in (0,1), p_2 > 1$, $J^* \in \mathbb{N}$ such that the following holds:

(a) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that V satisfies

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad (5)$$

for each $x \in C \cup D \cup G(D)$.

(b) The functions u_C, u_D defined in (2) satisfy

$$u_C(x) \leq -c_1 V^{p_1}(x) - c_2 V^{p_2}(x) \qquad x \in C \quad (6a)$$
$$u_D(x) \leq 0 \qquad x \in D \quad (6b)$$

(c) For every $\xi \in (C \cup D) \setminus \mathcal{A}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies

$$\sup_{(t,j)\in\operatorname{dom}\phi} t \geq T^* := \frac{1}{c_1(1-p_1)} + \frac{1}{c_2(p_2-1)}, \quad (7)$$

and

$$(T^*, j) \in \operatorname{dom}\phi \implies j \leq J^*.$$

Then, the set $\mathcal{A}$ is UGFxTS for $\mathcal{H}$ where the settling time function $\mathcal{T}$ satisfies $\mathcal{T}(\xi) \leq T^* + J^*$ for all $\xi \in C \cup D$.

Proof: The proof follows similar ideas as in [18, Lemma 1] and [16, Theorem 3.5] by exploiting the "fixed-time" decrease condition (6a) and the non-increase condition (6b). Pick $\phi \in \mathcal{S}_{\mathcal{H}}(C \cup D)$ and arbitrary $(t,j) \in \operatorname{dom}\,\phi$. Let $0 \leq t_0 \leq t_1 \leq \ldots \leq t_{j+1} = t$ satisfy

$$\operatorname{dom}\,\phi \cap ([0,t] \times \{0,1,...,j\}) = \bigcup_{i=0}^{j} ([t_i, t_{i+1}] \times \{i\}).$$

We can assume that $V(\phi(s,i)) > 0$, since otherwise we would have $\phi(s,i) \in \mathcal{A}$. First, consider the case where $V(\xi) > 1$. Then, from (3), we have $\frac{d}{ds}V(\phi(s,i)) \leq -c_2 V^{p_2}(\phi(s,i))$ for almost all s, which yields $V^{-p_2}(\phi(s,i))dV(\phi(s,i)) \leq -c_2 ds$. For simplicity, we denote $V_{p,q} = V(\phi(t_p, q))$. Integrating over $[t_i, t_{i+1}]$ yields $\frac{1}{1-p_2}(V_{i+1,i}^{1-p_2} - V_{i,i}^{1-p_2}) \leq -c_2(t_{i+1} - t_i)$. Similarly, for $i \in \{1,...,j\}$, we have $\phi(t_i, i-1) \in D$ and $V_{i,i} - V_{i,i-1} \leq 0$. But since $1 - p_2 < 0$, we also have $\frac{1}{1-p_2}\left(V_{i,i}^{1-p_2} - V_{i,i-1}^{1-p_2}\right) \leq 0$. We then obtain $\frac{1}{1-p_2}(V_{j+1,j}^{1-p_2} - V_{0,0}^{1-p_2}) \leq -\sum_{i=0}^{j} c_2(t_{i+1} - t_i) = -c_2 t$. This can be rearranged into $V(\phi(t,j)) \leq \left(\left(c_2(p_2-1)t + V^{1-p_2}(\xi)\right)^{\frac{1}{p_2-1}}\right)^{-1}$, which holds for each $(t,j) \in \operatorname{dom}\,\phi$. It is clear that $V(\phi(t,j)) \leq 1$ for $t \geq \frac{1}{c_2(p_2-1)}$. Now pick $(t^*, j^*) \in \operatorname{dom}\phi$ such that $V(\phi(t^*, j^*)) \leq 1$. Applying the same techniques as above, we arrive at the following for $t \geq t^*$

$$\frac{1}{1-p_1}(V^{1-p_1}(\phi(t,j)) - V^{1-p_1}(\phi(t^*, j^*))) \leq -c_1 t + c_1 t^*,$$

which can be rearranged into $V(\phi(t,j)) \leq (1 - (1-p_1)c_1(t-t^*))^{\frac{1}{1-p_1}}$. As we recall, this holds for $V(\phi(t,j)) > 0$. We can apply (5) to obtain

$$|\phi(t,j)|_{\mathcal{A}} \leq \alpha_1^{-1}\left(\left[1 - (1-p_1)c_1(t-t^*)^{\frac{1}{1-p_1}}\right]_+\right),$$

where the right hand side vanishes for $t \geq t^* + \frac{1}{(1-p_1)c_1}$. Hence, UGFxTS is proven with settling time upper bounded by $T^* + J^*$. ■

Remark 1: In item (c) of Theorem 1, the existence of J^* is required to guarantee that the fixed time convergence occurs in hybrid time, which accounts for jumps. Note that this conditions allows for solutions that jump after reaching $\mathcal{A}$, since the uniform bound on jumps only needs to hold before $t = T^*$.

Remark 2: If we are only concerned with achieving global fixed-time converegence in t time (not hybrid time), then we do not require such J^* to exist in item (c). This can be mostly attributed to the fact that the computed T^* does not depend on J^*. This observation is particularly useful when establishing UGFxTS for switching systems with globally fixed-time stable modes under arbitrary switching. We address this idea further in Section IV. □

Condition (6a) is highly reminiscent of the Lyapunov inequality introduced in [18] to establish global fixed-time stability for continuous-time autonomous systems. In Theorem 1, we essentially show that the same bound implies UGFxTS for hybrid dynamical systems as long as the system spends enough time flowing. Moreover, compared to the finite-time counterpart [16], our computed T^* holds uniformly over all initial conditions $\xi \in C \cup D \setminus \mathcal{A}$.

Even though assuming non-increase of V during jumps is relatively standard when studying resets and switching systems, it is still of interest to consider settings that permit a mild increase of the Lyapunov function during jumps. This is particularly useful for studying the robustness of condition (6b).

B. Allowed Increase During Jumps

We now consider the case where V is allowed to increase during jumps. Specifically, we allow for increases of the form

$$u_D(x) \leq \gamma V(x), \quad x \in D \quad (8)$$

where $\gamma \geq 0$. Increases of this form are relatively commonplace throughout the literature, e.g [2, Prop 3.29], [27], [28].

We can now present the second main result of the paper, which makes use of the constants

$$\delta_i := (1+\gamma)^{1-p_i},$$

for $i = 1, 2$. The following result can be established by inductively estimating the system's trajectories at various part of the solution's domain. Due to space limitations, the proof is omitted.

Theorem 2: Consider the HDS system $\mathcal{H}$ given by (1) and a closed set $\mathcal{A} \subset C \cup D$, where there exists a continuous function $V : C \cup D \to \mathbb{R}_{\geq 0}$, that is locally Lipschitz on a neighborhood of C, and $c_1, c_2 > 0$, $p_1 \in (0,1)$, $p_2 > 1$, $\gamma \geq 0$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (5), (6a) and (8) hold. Moreover, suppose that every solution $\phi \in \mathcal{S}_{\mathcal{H}}((C \cup D) \setminus \mathcal{A})$ is complete, and each complete $\phi \in \mathcal{S}_{\mathcal{H}}((C \cup D) \setminus \mathcal{A})$ with $\mathcal{J}(\phi) \geq 1$ has $\tau_\phi^* \geq \tau_D$ for some τ_D satisfying

$$\tau_D > \max\left(\frac{\delta_1 - 1}{c_1(1-p_1)}, \frac{1-\delta_2}{c_2(p_2-1)}\right), \tag{9}$$

where $\tau_\phi^* = \inf\{\tau_i(\phi) : i \leq \mathcal{J}(\phi) - 1, \phi(t_{i+1}, i) \notin \mathcal{A}\}$. Then, the set $\mathcal{A}$ is UGFxTS for $\mathcal{H}$.

Remark 3: We define the minimum dwell time, τ_ϕ^*, of a solution, ϕ, in the manner presented in Theorem 2 since it allows us to generalize our results to include complete solutions that jump once they enter $\mathcal{A}$. Note that we no longer require a condition of the form (7) since the dwell time condition (9) implies that all complete solutions will flow "sufficiently enough". However, the required amount of time spent flowing for fixed time convergence is a somewhat complicated expression. To address this, we assume completeness for all maximal solutions in our systems of interest. This is a reasonable assumption, and can be obtained by applying standard results, such as [2, Prop 6.10]. □

Remark 4: It is useful to note that one obvious dwell time condition to ensure UGFxTS is $\tau_D \geq T^*$, which is an immediate consequence of Theorem 1. Hence, the merit in Theorem 2 is that condition (9) allows for $\tau_D < T^*$, which can be guaranteed when $\delta_1 < 2$. We do not need to be concerned about the other term, since it is always the case that $1 - \delta_2 < 1$. □

Remark 5: Even though Theorem 2 requires a dwell time condition to compensate for increases during jumps, the results can be generalized to account for chattering and flows that are shorter than the required dwell time. Systems that permit such solutions can be reduced to the ones we consider by over-approximating consecutive instances of chattering and short flows with a modified jump map. This is indeed possible since we assume decrease of V during flows. To simplify our presentation, we leave this generalization for future work. □

IV. EXAMPLES AND APPLICATIONS

A. FxTS System with Periodic Jumps

We illustrate our results via two applications: systems with resets, and switching systems. Consider the plant

$$\dot{z} = f(z), \quad z \in \mathbb{R}^n, \tag{10}$$

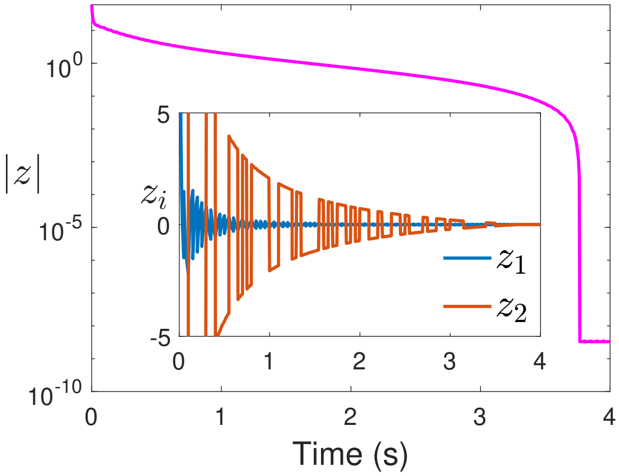

Fig. 1. The trajectories of the system (12) where a state component is periodically chosen to switch sign, i.e $\Lambda \in \left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$

where $f(\cdot)$ is a continuous, non-Lipschitz vector field which we assume has globally FxTS equilibrium at the origin $z = 0$. Moreover, we assume that there exists a smooth Lyapunov function $\tilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ to certify FxTS of the origin. In other words, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|z|) \leq \tilde{V}(z) \leq \alpha_2(|z|), \tag{11}$$

and $\tilde{V}$ satisfies the following

$$\nabla \tilde{V}(z)^\top f(z) \leq -c_1 V^{p_1}(z) - c_2 V^{p_2}(z),$$

for some $c_1, c_2 > 0, p_1 \in (0,1), p_2 > 1$ and all $z \in \mathbb{R}^n$. We will also make the key assumption that $\tilde{V}$ satisfies $\tilde{V}(\Lambda z) - \tilde{V}(z) \leq 0$ for all $z \in \mathbb{R}^n$ and $\Lambda \in \mathbb{R}^{n \times n}$ with $\overline{\lambda}(\Lambda^\top \Lambda) \leq 1$, where $\overline{\lambda}(\cdot)$ denotes the maximum eigenvalue. This is a relatively easy condition to obtain, as it holds for all positive definite quadratic Lyapunov functions and all Lyapunov functions of the form $\tilde{V}(z) = \alpha(|z|)$, where $\alpha \in \mathcal{K}_\infty$. Suppose that every τ_D seconds, the state is passed through an arbitrary linear contraction. If we define the state $x = [z, \tau]^\top$, this system can be formally expressed in the framework of (1) using:

$$F(x) = \begin{pmatrix} f(z) \\ 1 \end{pmatrix}, \quad C = \mathbb{R}^n \times [0, \tau_D] \tag{12a}$$

$$G(x) = \left\{ \begin{pmatrix} \Lambda z \\ 0 \end{pmatrix} : \overline{\lambda}(\Lambda^\top \Lambda) \leq 1 \right\}, \quad D = \mathbb{R}^n \times \{\tau_D\}. \tag{12b}$$

where $\Lambda \in \mathbb{R}^{n \times n}$. It is trivial to verify that F, C, and D satisfy their respective properties of the hybrid basic conditions. It is also straightforward to establish local boundedness and outer semicontinuity of G on D.

We can then use the Lyapunov function $V(x) = \tilde{V}(z)$ to establish that the set $\mathcal{A} = \{0\} \times [0, \tau_D]$ for this system is UGFxTS. For all $x \in C$, we have $u_C(x) \leq -c_1 V^{p_1} - c_2 V^{p_2}$. Similarly, for all $x \in D$, we have $u_D(x) \leq 0$. Thus, by

Theorem 1, we conclude that the set $\mathcal{A}$ for (12) is UGFxTS for every $\tau_D > 0$ with hybrid settling time

$$\mathcal{T}^* = \left(\frac{1}{c_1(1-p_1)} + \frac{1}{c_2(p_2-1)}\right)\left(1 + \frac{1}{\tau_D}\right).$$

If we are not concerned with the jumps (i.e, we are only interested in the t settling time), then the settling time of (10) can be computed to be T^*, where T^* is given in item (c) of Theorem 1.

This example can model many interesting settings, such as when a random set of states are periodically chosen to switch sign, which emerges in cyber-security applications. In this case, Λ would be any diagonal matrix with 1's and -1's on the diagonal. We simulate this exact scenario with the plant

$$\dot{z} = -\frac{Qz}{|Qz|^{\frac{7}{9}}} - \frac{Qz}{|Qz|^{-\frac{4}{5}}}, \quad Q = \begin{bmatrix} 3 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \succ 0, \tag{13}$$

where every $\tau_D = 0.05$ seconds, either z_1 or z_2 switches sign. It can be verified that the plant (13) is FxTS with Lyapunov function $\tilde{V}(z) = |z|^2$. The trajectories of this system with periodic sign switches are shown in Figure 1.

B. *Switching Systems with FxTS Modes*

To demonstrate another relevant application of our results, we study FxTS for switching systems where each mode is FxTS. In particular, we consider systems of the form:

$$\dot{x} = f_\sigma(x), \tag{14}$$

where $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous, but non-Lipschitz, vector field, parameterized by a switching signal $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ that is piecewise constant and takes values in the set of modes $\Sigma = \{1, 2, ..., N\}$. Moreover, for each $\sigma \in \Sigma$, the closed set $\mathcal{A}$ for the system (14) is UGFxTS, which is certified via a common Lyapunov function V. More formally, we make the following assumption:

Assumption 1: There exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying the following

1) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}),$$

for each $x \in \mathbb{R}^n$.

2) For each $\sigma \in \Sigma$, there exists $a_\sigma, b_\sigma > 0$, $p_\sigma \in (0, 1)$, $q_\sigma > 1$ such that

$$\nabla V(x)^\top f_\sigma(x) \leq -a_\sigma V^{p_\sigma} - b_\sigma V^{q_\sigma}.$$

Under Assumption 1, we can directly apply our results to show that the overall system (14) is UGFxTS under arbitrary switching. To establish this property, we observe that it suffices to establish UFxTS of the differential inclusion formed from the closed convex hull of the vector fields f_σ, which follows from [2, Ch. 2] and [29]. We can model this differential inclusion in the framework of (1), with:

$$F(x) = \overline{\text{co}} \bigcup_{\sigma \in \Sigma} f_\sigma(x), \quad C = \mathbb{R}^n \tag{15a}$$

$$G(x) = 0, \quad D = \emptyset, \tag{15b}$$

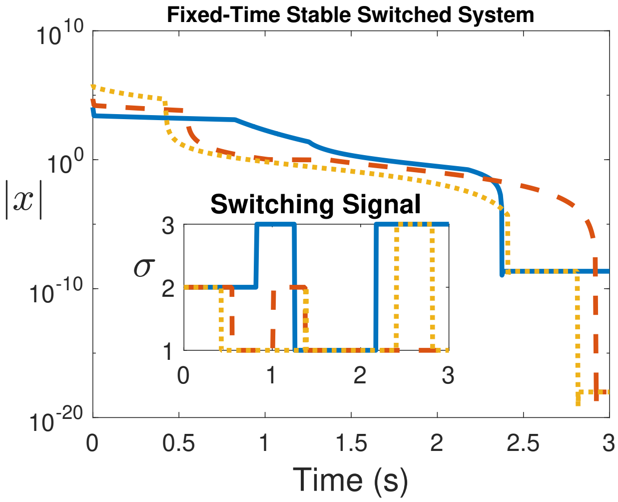

Fig. 2. Switched system with FxTS modes and the corresponding switching signals

where $\overline{\text{co}}$ denotes the closure of the convex hull. We would now like to show that the set $\mathcal{A}$ is UGFxTS for (15). We can consider the Lyapunov function V from Assumption 1. Next, let $k := \min_\sigma \min\{a_\sigma, b_\sigma\}$ and pick $p \in [\max_\sigma p_\sigma, 1)$, $q \in (1, \min_\sigma q_\sigma]$. We can observe that for each $x \in \mathbb{R}^n$ and $f(x) \in F(x)$, we have $\nabla V(x)^\top f(x) \leq -\frac{k}{2}V^p - \frac{k}{2}V^q$. Hence, for all $x \in C$, it follows that $u_C(x) \leq -\frac{k}{2}V^p - \frac{k}{2}V^q$. For $x \in D$, we have $u_D(x) = 0$. Moreover, since the solutions of (15) never jump, we have $J^* = 0$. Thus, by Theorem 1, we establish that the set $\mathcal{A}$ for system (15) is UGFxTS with hybrid settling time

$$\mathcal{T}^* = T^* = \left(\frac{2}{k(1-p)} + \frac{2}{k(q-1)}\right).$$

This also establishes UGFxTS of $\mathcal{A}$ for system (14) under arbitrary switching.

We present some numerical studies of system (15) with $z \in \mathbb{R}^2$ and FxTS modes of the form:

$$f_\sigma(z) = -\frac{Q_\sigma z}{|Q_\sigma z|^{\eta_{\sigma,1}}} - \frac{Q_\sigma z}{|Q_\sigma z|^{\eta_{\sigma,2}}}, \tag{16}$$

where Q_σ are given by

$$Q_1 = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 4 & 1 \\ 1 & 7 \end{bmatrix},$$

and $\eta_{1,1} = \frac{1}{3}, \eta_{1,2} = -\frac{2}{3}, \eta_{2,1} = \frac{1}{5}, \eta_{2,2} = -\frac{3}{7}, \eta_{3,1} = \frac{3}{4}, \eta_{3,2} = -\frac{1}{7}$. Assumption 1 is satisfied with the common Lyapunov function $\tilde{V}(z) = |z|^2$, which implies that the origin of (16) is UGFxTS for each fixed switching signal with dwell time $\tau_D > 0$. Moreover, we can apply our previous observations to further conclude that (16) converges to the origin in fixed time under arbitrary switching. The settling time in t can be computed to be upper bounded by 729.65, which is quite conservative when compared with the trajectories plotted in Figure 2. However, it is important to note that the importance of our result lies in the fact that we can guarantee a uniform settling time exists. Improving the computation of the settling time bound may be of interest for future work.

We would like to mention that the two given examples could be generalized into one by simply replacing the state z in (15b) with Λz. This would be referred to as a *switching system with resets* [30], [31]. However, to simplify our presentation and emphasize the different key assumptions of our main results, we consider separate examples.

V. CONCLUSION

We have introduced the notion of fixed-time stability of closed sets for hybrid dynamical systems and derived nonsmooth Lyapunov-based methods to establish such properties. One result, which assumes a "fixed-time" decrease along flows and non-increase during jumps, requires sufficient cumulative flow over all solutions. The second result offsets potential increases during jumps by requiring all solutions to satisfy a dwell time condition. It is of future interest to investigate ways to relax the dwell time requirement, possibly by imposing a more lenient *persistence of flows* condition or leveraging the notions of *average dwell time* and *average activation time*, which are common in the study of asymptotic stability properties of hybrid systems. The results of this paper open the door to the systematic study of novel Lyapunov-based fixed-time stable hybrid algorithms, including those encountered in the context of event-triggered control, reset control, switching control, and synergistic hybrid control, to name just a few, thus establishing an important link between the vast literature on fixed-time stabilization [18], [32] and hybrid control systems [2], [33].

REFERENCES

[1] D. Liberzon, *Switching in systems and control*. Springer, 2003, vol. 190.

[2] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[3] M. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[4] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Transactions on Automatic control*, vol. 59, no. 6, pp. 1395–1410, 2014.

[5] D. Nesic and D. Liberzon, "A small-gain approach to stability analysis of hybrid systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 5409–5414.

[6] M. Zefran, F. Bullo, and M. Stein, "A notion of passivity for hybrid systems," in *Proceedings of the 40th IEEE conference on decision and control (Cat. No. 01CH37228)*, vol. 1. IEEE, 2001, pp. 768–773.

[7] A. Bemporad, G. Bianchini, and F. Brogi, "Passivity analysis and passification of discrete-time hybrid systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1004–1009, 2008.

[8] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An lmi approach," in *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)*, vol. 1. IEEE, 2000, pp. 504–509.

[9] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel, "On the robust stability and stabilization of sampled-data systems: A hybrid system approach," in *Proceedings of the 45th IEEE Conference on Decision and Control*. IEEE, 2006, pp. 4873–4878.

[10] N. Sivashankar and P. P. Khargonekar, "Robust stability and performance analysis of sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 38, no. 1, pp. 58–69, 1993.

[11] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[12] S. G. Nersesov and W. M. Haddad, "Finite-time stabilization of nonlinear impulsive dynamical systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 3, pp. 832–845, 2008.

[13] J. Lee and W. M. Haddad, "On finite-time stability and stabilization of nonlinear hybrid dynamical systems," *AIMS Math*, vol. 6, no. 6, pp. 5535–5562, 2021.

[14] G. Chen, Y. Yang, and J. Li, "Finite time stability of a class of hybrid dynamical systems," *IET control theory & applications*, vol. 6, no. 1, pp. 8–13, 2012.

[15] Y. Yang, J. Li, and G. Chen, "Finite-time stability and stabilization of nonlinear stochastic hybrid systems," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 338–345, 2009.

[16] Y. Li and R. G. Sanfelice, "Finite time stability of sets for hybrid dynamical systems," *Automatica*, vol. 100, pp. 200–211, 2019.

[17] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer design, and output feedback," *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[18] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic and Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[19] C. Chen, L. Li, H. Peng, Y. Yang, L. Mi, and H. Zhao, "A new fixed-time stability theorem and its application to the fixed-time synchronization of neural networks," *Neural networks*, vol. 123, pp. 412–419, 2020.

[20] J. I. Poveda, M. Krstić, and T. Başar, "Fixed-time nash equilibrium seeking in time-varying networks," *IEEE Transactions on Automatic Control*, vol. 68, no. 4, pp. 1954–1969, 2022.

[21] J. I. Poveda and M. Krstić, "Nonsmooth extremum seeking control with user-prescribed fixed-time convergence," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 6156–6163, 2021.

[22] Y. Su, C. Zheng, and P. Mercorelli, "Robust approximate fixed-time tracking control for uncertain robot manipulators," *Mechanical Systems and Signal Processing*, vol. 135, p. 106379, 2020.

[23] Q. Chen, S. Xie, M. Sun, and X. He, "Adaptive nonsingular fixed-time attitude stabilization of uncertain spacecraft," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 54, no. 6, pp. 2937–2950, 2018.

[24] Q. Wang and A. Abdurahman, "Fixed-time stability analysis of general impulsive systems and application to synchronization of complex networks with hybrid impulses," *Neurocomputing*, vol. 601, p. 128218, 2024.

[25] G. Chen and Y. Yang, "Fixed-time stability of time-varying hybrid systems with time-delay," *Circuits, Systems, and Signal Processing*, vol. 43, no. 5, pp. 2758–2781, 2024.

[26] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

[27] J. B. Biemond, W. M. H. Heemels, R. G. Sanfelice, and N. van de Wouw, "Distance function design and lyapunov techniques for the stability of hybrid trajectories," *Automatica*, vol. 73, pp. 38–46, 2016.

[28] J. Liu and A. R. Teel, "Lyapunov-based sufficient conditions for stability of hybrid systems with memory," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 1057–1062, 2016.

[29] R. Kamalapurkar, J. A. Rosenfeld, A. Parikh, A. R. Teel, and W. E. Dixon, "Invariance-like results for nonautonomous switched systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 614–627, 2019.

[30] S. Liu, A. Tanwani, and D. Liberzon, "Iss and integral-iss of switched systems with nonlinear supply functions," *Mathematics of Control, Signals, and Systems*, vol. 34, no. 2, pp. 297–327, 2022.

[31] D. E. Ochoa, N. Espitia, and J. I. Poveda, "Prescribed-time stability in switching systems with resets: A hybrid dynamical systems approach," *Systems & Control Letters*, vol. 193, p. 105910, 2024.

[32] A. Polyakov and M. Krstic, "Fixed-time stabilization with a prescribed constant settling time by static feedback for delay-free and input delay systems," *International Journal of Robust and Nonlinear Control*, vol. 34, no. 10, pp. 6393–6415, 2024.

[33] R. G. Sanfelice, *Hybrid feedback control*. Princeton University Press, 2021.