Max-min Hub Pricing in Payment Channel Networks

Guoliang Xue, Alena Chang, Xuanli Lin, Ruozhou Yu, Dejun Yang

***Abstract*—Payment Channel Networks (PCNs) offer an efficient off-chain alternative to the blockchain for transactions. Router nodes in PCNs facilitate transactions between non-adjacent nodes in exchange for a fee. PCN topology tends to be centralized, with a select number of routers known as *hubs* dominating all payment services. The fee-setting choices of hubs in order to maximize their revenue present fertile grounds for the study of PCN communications and economics. In this paper, we conduct a comprehensive analysis of the Hub Price-Setting (HPS) game. In particular, we define approximate Best Response strategies (ϵ-BR) as well as approximate Nash equilibria (ϵ-NE). We prove that for any $\epsilon > 0$, an ϵ-BR always exists, and can be computed in polynomial time. We also prove that for some $\epsilon > 0$, an ϵ-NE may not exist. We furthermore introduce the notion of conservative estimate and present a max-min approach to the HPS game. Extensive evaluation results demonstrate the power of our proposed approach.**

***Index Terms*—Payment channel network, Lightning network, game theory, approximate Nash equilibrium, max-min approach.**

1. Introduction

The blockchain offers a platform for secure transactions by way of decentralized consensus. However, it suffers from lackluster throughput and high settlement latencies, making for a lack of scalability [12]. Payment Channel Networks (PCNs) offer a medium which circumvents this obstacle [6], [8], [9], [11]. A PCN consists of a network of nodes connected by off-chain payment channels. A payment channel has a capacity which denotes the maximum payment it can route between its incident nodes. Two adjacent nodes may use a payment channel to settle as many transactions as desired. Two non-adjacent nodes may rely on a smart contract [11] to construct a payment path. The payment path's constituent nodes sans the end nodes–known as *routers*–charge a transaction fee to forward a payment [10]. Both of the aforementioned scenarios enable nodes to carry out transactions while evading the blockchain.

An example of a widely used PCN is the Lightning Network (LN) [8]. First implemented in 2017, as of February 2023 it boasts roughly 16,000 online nodes and over 76,000 active channels [2]. With their favorable throughput, lower settlement latencies, and lower transaction fees compared to the blockchain, PCNs will likely endure as an efficient platform for off-chain transactions. A user node in a PCN wishes to make payments with the lowest fees, while a router node in a PCN generates revenue from fees. Proposed fee-setting frameworks for PCNs include [3], which aims to keep channels balanced, and [4], which seeks to maximize revenue.

The routers set their fees to maximize their revenue in a selfish and competitive manner. This goal is particularly elusive because a router must be strategic with its fee-setting. If a router sets its fee too high, it loses its economic advantage over rival routers and on-chain transaction fees, driving away potential client users. If it sets its fee too low, it enjoys a reliable clientele, but may earn low revenue.

In [10], the competition between two routers is studied using a game-theoretic approach. Their *two-hub model* consists of two routers (hubs) providing payment services to a set of nodes. The (sender) nodes intend to send payments to a recipient node via one of the hubs. Some of the senders share channels with only one of the two hubs, while other senders share channels with both hubs. The senders sharing channels with both hubs possess no inherent loyalty towards a single hub. The two hubs hence occupy a market in which they must compete for the attention of the senders they share via fee-setting. A hub chooses its fee so as to maximize its revenue based on the senders' demands.

It has been shown in [10] that pure Nash equilibria as well as best response strategies may or may not exist. In this paper, we study the existence of *approximate* Nash equilibria and *approximate* best response strategies. **We prove that approximate Nash equilibria may not exist, but approximate best response strategies always exist. We further propose a max-min approach to the game and use extensive numerical results to demonstrate the power of the max-min approach**.

The main contributions of this paper are as follows.

- We study the Hub Price-Setting (HPS) game and define approximate best response strategies as well as approximate Nash equilibria.
- We prove that for any $\epsilon > 0$, an ϵ-BR always exists, and can be computed in polynomial time. In contrast, we prove that for some $\epsilon > 0$, an ϵ-NE may not exist.
- We introduce the notion of conservative estimate and present a max-min approach to the HPS game.
- We present extensive evaluation results to demonstrate the power of our proposed approach.

The rest of this paper is organized as follows. In §2, we present the system model. In §3, we present the HPS game setting. In §4, we demonstrate the existence and computation of ϵ-BR. We also prove the non-existence of ϵ-NE. In §5, we present our max-min approach. We present evaluation results in §6, and conclude the paper in §7.

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2. SYSTEM MODEL

In this section, we present the system model. We use i to denote the index of *routers*, and use k to denote the index of *users*. We use $\neg i$ to denote the logical negation of i. In other words, when $i = 1$, $\neg i$ denotes 2; when $i = 2$, $\neg i$ denotes 1.

A. Routers and Users

There are two competing routers, denoted by r_1 and r_2. Router r_i has a *balance* $B_i > 0$ and a *reserved price* $RE_i \geq 0$, $i = 1, 2$. As in [10], we assume that the reserved price of the two routers are the same, and denote this common value by RE, i.e., $RE_1 = RE_2 = RE$. Router r_i can set a *price* $p_i \geq RE$, which is called the *toll fee* of r_i, $i = 1, 2$.

There are K *users*: $u_1, u_2, \ldots, u_K$. User u_k has a *cost upper-bound* $c_k > 0$ and a *demand* $\delta_k > 0$, $k = 1, 2, \ldots, K$. We assume both routers possess knowledge of all users' cost upper-bounds and demands. In practice, the routers can use historic data to estimate these values. We denote the set of users by Ω, i.e., $\Omega = \{u_k | 1 \leq k \leq K\}$. The set Ω is the union of three disjoint subsets: Ω_1, Ω_2, Ω_0.

Users in Ω_1 are *locked-in* with r_1, *i.e.*, they can be served by r_1, but not by r_2. Users in Ω_2 are locked-in with r_2, *i.e.*, they can be served by r_2, but not by r_1. Users in Ω_0 are *flexible*, *i.e.*, they can be served by both r_1 and r_2. We use the term *locked-in user* to denote a user in $\Omega_1 \cup \Omega_2$, and use the term *flexible user* to denote a user in Ω_0.

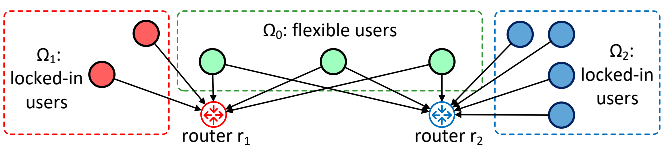


Fig. 1. System model of the HPS game. Links from users to routers are uplinks. Down-links (from router to recipients) are not shown.

Fig. 1 illustrates the system model. Users in Ω_1 are locked-in with r_1. Users in Ω_2 are locked-in with r_2. Users in Ω_0 are flexible. Since this paper concentrates on the competition between r_1 and r_2, down-links are not shown in the figure.

Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_K$. We denote $\min\{RE, c_k | 1 \leq k \leq K\}$ by c_0, and denote $\max\{c_k | 1 \leq k \leq K\}$ by $c_{\max}$. Hence we have $c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_K = c_{\max}$. In the rest of this paper, we use $\mathcal{R}$ to denote the set of real numbers. We use $\mathcal{P}$ to denote the interval $[RE, c_K]$. Unless specified otherwise, p_i denotes the price of r_i, $i = 1, 2$.

B. User Preference to Routers

Before proceeding, we define the following notations.

$$S_1(x) = \{u_k \in \Omega_1 | c_k \geq x\},\ d_1(x) = \sum_{u_k \in S_1(x)} \delta_k,\ x \in \mathcal{R}, \quad (1)$$

$$S_2(x) = \{u_k \in \Omega_2 | c_k \geq x\},\ d_2(x) = \sum_{u_k \in S_2(x)} \delta_k,\ x \in \mathcal{R}, \quad (2)$$

$$S_0(x) = \{u_k \in \Omega_0 | c_k \geq x\},\ d_0(x) = \sum_{u_k \in S_0(x)} \delta_k,\ x \in \mathcal{R}. \quad (3)$$

$S_1(p_1)$ is the set of users in Ω_1 that can use router r_1, and $d_1(p_1)$ is aggregated demand of the users in $S_1(p_1)$. $S_2(p_2)$ is the set of users in Ω_2 that can use router r_2, and $d_2(p_2)$ is aggregated demand of the users in $S_2(p_2)$. $S_0(p_1)$ is the set of users in Ω_0 that can use router r_1, and $d_0(p_1)$ is aggregated demand of the users in $S_0(p_1)$. $S_0(p_2)$ is the set of users in Ω_0 that can use router r_2, and $d_0(p_2)$ is aggregated demand of the users in $S_0(p_2)$. We have the following lemma whose proof is straightforward, and omitted.

Lemma 1: For $i = 0, 1, 2$, $d_i(x)$ is monotonically non-increasing, and $S_i(x)$ is order reversing. In other words, $x < y$ implies $d_i(x) \geq d_i(y)$ and $S_i(x) \supseteq S_i(y)$. Furthermore, the values of both $d_i(x)$ and $S_i(x)$ remain constant when x varies in the interval $(c_{k-1}, c_k]$, $k = 1, 2, \ldots, K$. □

Given a user u_k and a router r_i, either u_k cannot use r_i (e.g. $u_k \in \Omega_1$ and $i = 2$, or $c_k < p_i$) or u_k can use r_i. When u_k can use both r_1 and r_2, it *prefers* the router with a lower price. In the following, we present a detailed analysis of user preferences.

a. If $u_k \in \Omega_1$, u_k cannot use r_2.
b. If $u_k \in \Omega_2$, u_k cannot use r_1.
c. If $u_k \in \Omega$ and $p_1 > c_k$, u_k will not use r_1.
d. If $u_k \in \Omega$ and $p_2 > c_k$, u_k will not use r_2.
e. If $u_k \in \Omega_1$ and $p_1 \leq c_k$, u_k is willing to use r_1.
f. If $u_k \in \Omega_2$ and $p_2 \leq c_k$, u_k is willing to use r_2.
g. If $u_k \in \Omega_0$ and $p_1 = p_2 \leq c_k$, c_k is willing to use either r_1 or r_2, with no preference.
h. If $u_k \in \Omega_0$ and $p_1 \leq c_k < p_2$, c_k is willing to use r_1, but not r_2.
i. If $u_k \in \Omega_0$ and $p_1 < p_2 \leq c_k$, c_k is willing to use either r_1 or r_2, but prefers r_1 to r_2.
j. If $u_k \in \Omega_0$ and $p_2 \leq c_k < p_1$, c_k is willing to use r_2, but not r_1.
k. If $u_k \in \Omega_0$ and $p_2 < p_1 \leq c_k$, c_k is willing to use either r_1 or r_2, but prefers r_2 to r_1.

C. Assignment of User Demand to Routers

Now we are ready to describe the demand-router association. Since $p_1 \geq RE$ and $p_2 \geq RE$, user u_k will not be served by either r_1 or r_2 if $c_k < RE$ (refer to properties c and d in section 2-B). Without loss of generality, we assume that $c_k \geq RE$ for $k = 1, 2, \ldots, K$.

Routers r_1 and r_2 are competing against each other. Router r_i's primary goal is to earn as much as possible. Router r_i's secondary goal is reduce its opponent's earning as much as possible [7], [10]. For technical rigour, we present detailed case analysis of the competition in the following.

Case 1: $p_1 = p_2$. Since $p_1 = p_2$, we have $S_0(p_1) = S_0(p_2)$ and $d_0(p_1) = d_0(p_2)$. Given the selfish and competitive nature, router r_i will compete for the demand of users in $S_0(p_i) = S_0(p_{\neg i})$ first, serving the locked-in users in $S_i(p_i)$ only if there is left over balance.

Case 1a: $p_1 = p_2$ and $B_1 < \frac{d_0(p_1)}{2}$. In this case, r_1 will serve B_1 demand from the users in $S_0(p_1)$, and zero demand from the users in $S_1(p_1)$. Router r_2 will serve $\min\{B_2, d_2(p_2) +$

$d_0(p_2)-B_1\}$ demand from users in $S_2(p_2)\cup S_0(p_2)$, including $\min\{B_2, d_0(p_2)-B_1\}$ demand from the users in $S_0(p_2)$.

Case 1b: $p_1 = p_2$ and $B_2 < \frac{d_0(p_2)}{2}$. In this case, r_2 will serve B_2 demand from the users in $S_0(p_2)$, and zero demand from the users in $S_2(p_2)$. Router r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)-B_2\}$ demand from users in $S_1(p_1)\cup S_0(p_1)$, including $\min\{B_1, d_0(p_1)-B_2\}$ demand from the users in $S_0(p_1)$.

Case 1c: $p_1 = p_2$ and $\min\{B_1, B_2\} \geq \frac{d_0(p_1)}{2}$. In this case, router r_1 will serve $\min\{B_1, d_1(p_1) + \frac{d_0(p_1)}{2}\}$ demand from users in $S_1(p_1)\cup S_0(p_1)$, including $\frac{d_0(p_1)}{2}$ demand from users in $S_0(p_1)$. Router r_2 will serve $\min\{B_2, d_2(p_2) + \frac{d_0(p_2)}{2}\}$ demand from users in $S_2(p_2)\cup S_0(p_2)$, including $\frac{d_0(p_2)}{2}$ demand from users in $S_0(p_2)$.

Summary of Case 1: When $p_1 = p_2$, router r_1 will serve $\min\{B_1, d_1(p_1) + \max\{d_0(p_1)-B_2, \frac{d_0(p_1)}{2}\}\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, router r_2 will serve $\min\{B_2, d_2(p_2)+ \max\{d_0(p_2)-B_1, \frac{d_0(p_2)}{2}\}\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$.

Case 2: $p_1 < p_2$. According to Lemma 1, $S_0(p_2) \subseteq S_0(p_1)$ and $d_0(p_2) \leq d_0(p_1)$. Since $p_1 < p_2$, each user $u_k \in S_0(p_2)$ prefers r_1 to r_2 because r_1 charges less. Due to the competition nature [7], r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, including $\min\{B_1, d_0(p_2)\}$ demand from users in $S_0(p_2)$.

Case 2a: $p_1 < p_2$ and $B_1 \geq d_0(p_2)$. In this case, router r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, including $d_0(p_2) = \min\{B_1, d_0(p_2)\}$ demand from users in $S_0(p_2)$. Router r_2 will serve $\min\{B_2, d_2(p_2)\}$ demand from users in $S_2(p_2)$, and zero demand from users in $S_0(p_2)$.

Case 2b: $p_1 < p_2$ and $B_1 < d_0(p_2)$. In this case, router r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, including $B_1 = \min\{B_1, d_0(p_2)\}$ demand from users in $S_0(p_2)$. Router r_2 will serve $\min\{B_2, d_2(p_2) + d_0(p_2)-B_1\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$, including $\min\{B_2, d_0(p_2)-B_1\}$ demand from users in $S_0(p_2)$.

Summary of Case 2: When $p_1 < p_2$, router r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, including $\min\{B_1, d_0(p_2)\}$ demand from users in $S_0(p_2)$; Router r_2 will serve $\min\{B_2, d_2(p_2) + \max\{0, d_0(p_2)-B_1\}\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$.

Case 3: $p_1 > p_2$. Since $p_1 > p_2$, each user $u_k \in S_0(p_1)$ prefers r_2 to r_1 because r_2 charges less. According to Lemma 1, $S_0(p_1) \subseteq S_0(p_2)$ and $d_0(p_1) \leq d_0(p_2)$. Due to the competition nature [7], r_2 will serve $\min\{B_2, d_2(p_2) + d_0(p_2)\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$, including $\min\{B_2, d_0(p_1)\}$ demand from users in $S_0(p_1)$.

Case 3a: $p_1 > p_2$ and $B_2 \geq d_0(p_1)$. In this case, router r_2 will serve $\min\{B_2, d_2(p_2) + d_0(p_2)\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$, including $d_0(p_1) = \min\{B_2, d_0(p_1)\}$ demand from users in $S_0(p_1)$. Router r_1 will serve $\min\{B_1, d_1(p_1)\}$ demand from users in $S_1(p_1)$, and zero demand from users in $S_0(p_1)$.

Case 3b: $p_1 > p_2$ and $B_2 < d_0(p_1)$. In this case, router r_2 will serve $\min\{B_2, d_2(p_2) + d_0(p_2)\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$, including $B_2 = \min\{B_2, d_0(p_1)\}$ demand from users in $S_0(p_1)$. Router r_1 will serve $\min\{B_1, d_1(p_1) + d_0(p_1)-B_2\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$, including $\min\{B_1, d_0(p_1)-B_2\}$ demand from users in $S_0(p_1)$.

Summary of Case 3: When $p_1 > p_2$, router r_2 will serve $\min\{B_2, d_2(p_2) + d_0(p_2)\}$ demand from users in $S_0(p_2)\cup S_2(p_2)$, including $\min\{B_2, d_0(p_1)\}$ demand from users in $S_0(p_1)$; Router r_1 will serve $\min\{B_2, d_1(p_1) + \max\{0, d_0(p_1)-B_2\}\}$ demand from users in $S_0(p_1)\cup S_1(p_1)$.

3. The HPS Game Setting

As in [10], we model the hub price-setting problem using a static game with complete information [5]. We denote this game by HPS. The two *player*s in HPS are r_1 and r_2. The strategy of r_i is router r_i's price p_i, for $i = 1, 2$. The strategy space of r_i is $\mathcal{P} = [RE, c_{\max}]$, $i = 1, 2$. Let p_1 be a strategy of r_1, and p_2 be a strategy of r_2, we call (p_1, p_2) a *strategy profile*.

A. Demand Association

Let (p_1, p_2) be a strategy profile. It follows from Section 2-C that the total demand $D_1(p_1, p_2)$ served by r_1 is

$$\begin{cases} \min\{B_1, d_1(p_1) + d_0(p_1)\}, & \text{if } p_1 < p_2 \quad (4a)\\ \min\{B_1, d_1(p_1) + \max\{d_0(p_1) - B_2, \frac{d_0(p_1)}{2}\}\}, & \text{if } p_1 = p_2 \quad (4b)\\ \min\{B_1, d_1(p_1) + \max\{0, d_0(p_1) - B_2\}\}, & \text{if } p_1 > p_2 \quad (4c)\end{cases}$$

The total demand $D_2(p_1, p_2)$ served by r_2 is

$$\begin{cases} \min\{B_2, d_2(p_2) + \max\{0, d_0(p_2) - B_1\}\}, & \text{if } p_1 < p_2 \quad (5a)\\ \min\{B_2, d_2(p_2) + \max\{d_0(p_2) - B_1, \frac{d_0(p_2)}{2}\}\}, & \text{if } p_1 = p_2 \quad (5b)\\ \min\{B_2, d_2(p_2) + d_0(p_2)\}, & \text{if } p_1 > p_2 \quad (5c)\end{cases}$$

B. Utility Functions

The utility of r_1 corresponding to the strategy profile (p_1, p_2) is $U_1(p_1, p_2) = p_1 \times D_1(p_1, p_2)$, i.e.,

$$\begin{cases} p_1\times\min\{B_1, d_1(p_1)+d_0(p_1)\}, & \text{if } p_1 < p_2 \quad (6a)\\ p_1\times\min\{B_1, d_1(p_1)+\max\{d_0(p_1)-B_2, \frac{d_0(p_1)}{2}\}\}, & \text{if } p_1 = p_2 \quad (6b)\\ p_1\times\min\{B_1, d_1(p_1)+\max\{0, d_0(p_1)-B_2\}\}, & \text{if } p_1 > p_2 \quad (6c)\end{cases}$$

The utility of r_2 corresponding to the strategy profile (p_1, p_2) is $U_2(p_1, p_2) = p_2 \times D_1(p_1, p_2)$, i.e.,

$$\begin{cases} p_2\times\min\{B_2, d_2(p_2)+\max\{0, d_0(p_2) - B_1\}\}, & \text{if } p_1 < p_2 \quad (7a)\\ p_2\times\min\{B_2, d_2(p_2)+\max\{d_0(p_2)-B_1, \frac{d_0(p_2)}{2}\}\}, & \text{if } p_1 = p_2 \quad (7b)\\ p_2\times\min\{B_2, d_2(p_2)+d_0(p_2)\}, & \text{if } p_1 > p_2 \quad (7c)\end{cases}$$

C. Best Response Strategy

Let $p_2 \in \mathcal{P}$ be given. A price $p_1^{\text{BR}}(p_2) \in \mathcal{P}$ is said to be a *best response strategy* (BR) of r_1 corresponding to the strategy p_2 of r_2, if

$$U_1(p_1^{\text{BR}}(p_2), p_2) \geq U_1(q_1, p_2), \forall q_1 \in \mathcal{P}. \quad (8)$$

Let $p_1 \in \mathcal{P}$ be given. A price $p_2^{\text{BR}}(p_1) \geq RE$ is said to be a *best response strategy* of r_2 corresponding to the strategy p_1 of r_1, if

$$U_2(p_1, p_2^{\text{BR}}(p_1)) \geq U_2(p_1, q_2), \forall q_2 \in \mathcal{P}. \quad (9)$$

D. Nash Equilibrium

A strategy profile (p_1, p_2) is called a *Nash Equilibrium* (NE) if

$$U_1(p_1, p_2) \geq U_1(q_1, p_2), \quad \forall q_1 \in \mathcal{P} \tag{10}$$

$$U_2(p_1, p_2) \geq U_2(p_1, q_2), \quad \forall q_2 \in \mathcal{P} \tag{11}$$

In other words, (p_1, p_2) is an NE if p_1 is a BR of r_1 corresponding to the strategy p_2 of r_2, and p_2 is a BR of r_2 corresponding to the strategy p_1 of r_1.

4. Analysis of the HPS Game

It has been shown in [10] that best response strategies and pure NEs may or may not exist in the HPS game. We define the concept of ϵ-BR and ϵ-NE. We show that an ϵ-BR always exists for $\epsilon > 0$, yet an ϵ-NE may not always exist.

A. Supremum and Maximum of the Utility Function

We define the supremum and maximum of the utility function of a router as follows:

$$U_i^{\sup}(p_{\neg i}) = \sup_{p_i \in \mathcal{P}} U_i(p_i, p_{\neg i}), \ i = 1, 2. \tag{12}$$

$$U_i^{\max}(p_{\neg i}) = \max_{p_i \in \mathcal{P}} U_i(p_i, p_{\neg i}), \ i = 1, 2. \tag{13}$$

Theorem 1: For $i \in \{1, 2\}$ and any strategy $p_{\neg i} \in \mathcal{P}$, $U_i^{\sup}(p_{\neg i})$ can be computed in $O(K^2)$ time. □

Proof. Let n be such that $c_{n-1} \leq p_{\neg i} < c_n$. Then $U_i(x, p_{\neg i})$ is a linear function in each of the open intervals (c_{k-1}, c_k) for $k = 1, 2, \ldots, n-1$, $(c_{n-1}, p_{\neg i})$, $(p_{\neg i}, c_n)$, and (c_{k-1}, c_k) for $k = n+1, 2, \ldots, K$. Let

$$X_1 = U_i(p_{\neg i}, p_{\neg i}), \tag{14}$$

$$X_2 = \max\{U_i(c_k, p_{\neg i}) | k = 0, 1, 2, \ldots, K\}, \tag{15}$$

$$Y_1 = \max\{\lim_{x \to c_{n-1}{}^+} U_i(x, p_{\neg i}), \lim_{x \to p_{\neg i}{}^-} U_i(x, p_{\neg i})\}, \tag{16}$$

$$Y_2 = \max\{\lim_{x \to p_{\neg i}{}^+} U_i(x, p_{\neg i}), \lim_{x \to c_n{}^-} U_i(x, p_{\neg i})\}, \tag{17}$$

$$Z_k = \max\{\lim_{x \to c_{k-1}{}^+} U_i(x, p_{\neg i}), \lim_{x \to c_k{}^-} U_i(x, p_{\neg i})\}, \quad 1 \leq k \leq K. \tag{18}$$

Then $U_i^{\sup}(p_{\neg i}) = \max\{X_1, X_2, Y_1, Y_2, \max_{1 \leq k \leq K, k \neq n} Z_k\}$. □

The above proof also shows an algorithm to compute $U_i^{\sup}(p_{\neg i})$ using $O(K)$ utility evaluations. Since each utility evaluation can be accomplished in $O(K)$ time, we have designed an $O(K^2)$ time algorithm for computing $U_i^{\sup}(p_{\neg i})$.

B. Non-existence of BR and NE

Lemma 2 and Lemma 3 were proved in [10]. We present them here to make the current paper self-contained.

Lemma 2: $U_i^{\max}(p_{\neg i})$ does not always exist. Hence BR does not always exist in HPS. □

Proof. We prove this by an example. In this example, we have $B_1 = 8.0$, $B_2 = 8.0$, $RE = 1.0$; $K = 3$; $c_1 = 2.0, \delta_1 = 4.0$, $c_2 = 6.0, \delta_2 = 1.5$, $c_2 = 6.0, \delta_3 = 1.5$; $\Omega_0 = \{u_1\}$, $\Omega_1 = \{u_2\}$, $\Omega_2 = \{u_3\}$.

When r_2 plays $p_2 = 2.0$, r_1 does not have a best response strategy. Since $\lim_{p \to 2.0^-} U_1(p, 2.0) = \lim_{p \to 2.0^-} p \times (4.0 + 1.5) = 11.0$, r_1 can earn a utility arbitrarily close to $U_1^{\sup}(2.0) = 11.0$, by setting $p_1 < 2.0$ but very close to 2.0. However, it will never reach a utility of 11.0. When p_1 is set to 2.0, the utility of r_1 drops to $2.0 \times (2.0 + 1.5) = 7.0$. The largest utility r_1 can earn by setting $p_1 > 2.0$ is 9.0, achieved by setting $p_1 = 6.0$. □

Lemma 3: An NE does not always exist in HPS. □

This can be viewed as a corollary of Theorem 3, since an NE is a special case of an approximate NE.

C. Approximate BR and Approximate NE

Lemma 2 shows that a BR does not always exist. We introduce the concept of approximate best response strategy [1] in the following.

Definition 1: Let $\epsilon \geq 0$ be given real number. Let $i \in \{1, 2\}$ and a strategy $p_{\neg i} \in \mathcal{P}$ be given. A strategy $p_i^{\epsilon}(p_{\neg i}) \in \mathcal{P}$ is called an ϵ-approximate best response strategy of r_i (with respect to $p_{\neg i}$), denoted as ϵ-BR, if

$$U_i(p_i^{\epsilon}(p_{\neg i}), p_{\neg i}) \geq U_i^{\sup}(p_{\neg i}) - \epsilon. \tag{19}$$

We use ϵ-BR to denote an ϵ-approximate best response strategy, where i and $p_{\neg i}$ can be implied from the context.□

Theorem 2: Let $\epsilon > 0$ be any positive real number. Let $i \in \{1, 2\}$ and a strategy $p_{\neg i} \in \mathcal{P}$ be given. An ϵ-BR always exists, and can be computed in polynomial time. □

Proof. This follows from the definition of $U_i^{\sup}(p_{\neg i})$. □

Note that ϵ is a **positive** real number is Theorem 2. This is in sharp contrast with Lemma 2 which corresponds to the case where ϵ is 0.

Definition 2: Let $\epsilon \geq 0$ be given. A strategy profile (p_1, p_2) is called an ϵ-approximate NE, denoted as ϵ-NE, if p_1 is an ϵ-BR corresponding to p_2 and p_2 is an ϵ-BR corresponding to p_1. □

Theorem 3: For a given $\epsilon > 0$, an ϵ-NE may not exist in the HPS game. □

Proof. We use an example to show the non-existence of 0.5-NE. Such an example shows the non-existence of ϵ-NE for $\epsilon = 0.5$. The non-existence of ϵ-NE for other values of ϵ can be proved similarly. We choose to use $\epsilon = 0.5$ for ease of understanding.

The instance of the two-router game is given by $B_1 = 8.0$, $B_2 = 8.0$, $RE = 1.0$; $K = 3$; $c_1 = 2.0, \delta_1 = 4.0$, $c_2 = 6.0, \delta_2 = 1.5$, $c_2 = 6.0, \delta_3 = 1.5$; $\Omega_0 = \{u_1\}$, $\Omega_1 = \{u_2\}$, $\Omega_2 = \{u_3\}$. Assume that $(\bar{p}_1, \bar{p}_2)$ be a 0.5-NE. We will derive a contradiction. We first establish some facts.

(a1) *The maximum possible utility for r_1 is* 11.0.
This is achieved when r_1 plays strategy $p_1 = 2.0$ and r_2 plays strategy $p_2 > 2.0$. r_1 is serving all demand in $\Omega_0 \cup \Omega_1$. If $p_1 > 2.0$, r_1 will not serve any demand in Ω_0.

(a2) *When r_1 plays strategy $p_1 = 6.0$, r_1 has a utility of* 9.0, *regardless of what strategy r_2 plays.*

(a3) $\bar{p}_1 \in [\frac{17}{11}, 2] \cup [\frac{17}{3}, 6]$.
Since $(\bar{p}_1, \bar{p}_2)$ is a 0.5-NE, $U_1(\bar{p}_1, \bar{p}_2) \geq 9 - 0.5 = 8.5$.
When $p_1 < \frac{17}{11}$, $U_1(p_1, p_2) < \frac{17}{11} \times (4 + 1.5) < 8.5, \forall p_2 \in \mathcal{P}$. Hence $\bar{p}_1 \geq \frac{17}{11}$.
When $p_1 > 6$, $U_1(p_1, p_2) = 0, \forall p_2 \in \mathcal{P}$. Hence $\bar{p}_1 \leq 6$.

When $p_1 > 2$, $U_1(p_1, p_2) = p_1 \times 1.5, \forall p_2 \in \mathcal{P}$. Hence $\bar{p}_1 > 2$ and $U_1(\bar{p}_1, \bar{p}_2) \geq 8.5$ implies $\bar{p}_1 \geq \frac{8.5}{1.5} = \frac{17}{3}$.

Similarly, we have

(b1) *The maximum possible utility for* r_2 *is* 11.0.

(b2) *When* r_2 *plays strategy* $p_2 = 6.0$, r_2 *has a utility of* 9.0, *regardless of what strategy* r_1 *plays.*

(b3) $\bar{p}_2 \in [\frac{17}{11}, 2] \cup [\frac{17}{3}, 6]$.

Next, we have the following.

(a4) *If* $\bar{p}_1 = \frac{17}{11}$, *we must have* $\bar{p}_2 \in [\frac{17}{3}, 6]$.
When $p_2 < \frac{17}{11}$, $U_2(\frac{17}{11}, p_2) = p_2 \times (4+1.5) < 8.5$. Hence $\bar{p}_2 \geq \frac{17}{11}$.
Since $U_2(\frac{17}{11}, \frac{17}{11}) = \frac{17}{11} \times (2+1.5) < 8.5$, $\bar{p}_2 \neq \frac{17}{11}$.
When $p_2 \in (\frac{17}{11}, 2]$, $U_2(\frac{17}{11}, p_2) = p_2 \times 1.5 < 8.5$. Hence $\bar{p}_2 \notin (\frac{17}{11}, 2]$. It follows from (b3) that $\bar{p}_2 \in [\frac{17}{3}, 6]$.

(a5) *If* $\bar{p}_1 \in (\frac{17}{11}, 2]$, *we must have* $\bar{p}_2 \in [\frac{17}{11}, \bar{p}_1) \cup [\frac{17}{3}, 6]$.
When $p_2 = \bar{p}_1$, we have $U_2(\bar{p}_1, p_2) = p_2 \times (2+1.5) < 8.5$. Hence $\bar{p}_2 \neq \bar{p}_1$.
When $p_2 < \frac{17}{11}$, $U_2(\bar{p}_1, p_2) = p_2 \times (4+1.5) < 8.5$. Hence $\bar{p}_2 \geq \frac{17}{11}$.
When $p_2 > \bar{p}_1$, we have $U_2(\bar{p}_1, p_2) = p_2 \times 1.5$. Since $U_2(\bar{p}_1, \bar{p}_2) \geq 8.5$, $\bar{p}_2 \geq \bar{p}_1$ implies $\bar{p}_2 \in [\frac{17}{3}, 6]$.

(a6) *If* $\bar{p}_1 \in [\frac{17}{3}, 6]$, *we must have* $\bar{p}_2 \in [\frac{21}{11}, 2]$.
When $\bar{p}_1 \in [\frac{17}{3}, 6]$, the maximum possible utility of r_2 is 11.0, by playing strategy $p_2 = 2.0$. Since $U_2(\bar{p}_1, \bar{p}_2) \geq 10.5$, we must have $\bar{p}_2 \leq 2$. Since $U_2(\bar{p}_1, \bar{p}_2) \geq 11.0 - 0.5 = 10.5$, we must have $\bar{p}_2 \geq \frac{10.5}{5.5} = \frac{21}{11}$.

Similarly, we can have the following.

(b4) *If* $\bar{p}_2 = \frac{17}{11}$, *we must have* $\bar{p}_1 \in [\frac{17}{3}, 6]$.

(b5) *If* $\bar{p}_2 \in (\frac{17}{11}, 2]$, *we must have* $\bar{p}_1 \in [\frac{17}{11}, \bar{p}_2) \cup [\frac{17}{3}, 6]$.

(b6) *If* $\bar{p}_2 \in [\frac{17}{3}, 6]$, *we must have* $\bar{p}_1 \in [\frac{21}{11}, 2]$.

Now we are ready to establish a contradiction.

Case 1: $\bar{p}_1 = \frac{17}{11}$. We must have $\bar{p}_2 \in [\frac{17}{3}, 6]$ by (a4). By (b6), we must have $\bar{p}_1 \in [\frac{21}{11}, 2]$. This contradicts $\bar{p}_1 = \frac{17}{11}$.

Case 2: $\bar{p}_1 \in (\frac{17}{11}, 2]$. We must have $\bar{p}_2 \in [\frac{17}{11}, \bar{p}_1) \cup [\frac{17}{3}, 6]$ by (a5).

Case 2a: $\bar{p}_2 \in [\frac{17}{3}, 6]$. We derive $\bar{p}_1 \in [\frac{21}{11}, 2]$ using (b6). Since $U_2(\bar{p}_1, \frac{20}{11}) = 10$, we conclude that $\bar{p}_2 \notin [\frac{17}{3}, 6]$ because $U_2(\bar{p}_1, p_2) = p_2 \times 1.5 \leq 9, \forall p_2 \in [\frac{17}{3}, 6]$. This is a contradiction.

Case 2b: $\bar{p}_2 \in [\frac{17}{11}, \bar{p}_1)$. By (b5), we must have $\bar{p}_1 \in [\frac{17}{11}, \bar{p}_2) \cup [\frac{17}{3}, 6]$. Since $\bar{p}_1 \in (\frac{17}{11}, 2]$, we have $\bar{p}_1 \in [\frac{17}{11}, \bar{p}_2)$. Now we have $\bar{p}_2 < \bar{p}_1 < \bar{p}_2$, a contradiction.

Case 3: $\bar{p}_1 \in [\frac{17}{3}, 6]$. By (a6), we have $\bar{p}_2 \in [\frac{21}{11}, 2]$. Since $U_1(\frac{20}{11}, \bar{p}_2) = \frac{20}{11} \times (4+1.5) = 10$, we conclude that $\bar{p}_1 \notin [\frac{17}{3}, 6]$ because $U_1(\bar{p}_1, p_2) = p_1 \times 1.5 \leq 9, \forall p_1 \in [\frac{17}{3}, 6]$. This is a contradiction.

In summary, we have proved that $(\bar{p}_1, \bar{p}_2)$ is not a 0.5-NE. This contradiction completes the proof. □

5. The Max-min Approach to HPS

Theorem 2 shows that for any given $\epsilon > 0$ and a strategy $p_{\neg i}$ of $r_{\neg i}$, we can compute a ϵ-BR of r_i in polynomial time. This gives us hope that an ϵ-BR dynamic process [1] can be used to compute an ϵ-NE. However, Theorem 3 casts doubts in the aforementioned ϵ-BR dynamics, as an ϵ-NE may not always exist. In this section, we present an Max-min approach to tackle the HPS game.

A. Conservative Estimate for the Utility of r_i

For $p_1 \in \mathcal{P}$, we define the *conservative estimate* for r_1 by

$$U_1^{\min}(p_1) = \min\{U_1(p_1, p_2) | p_2 \in \mathcal{P}\}. \quad (20)$$

For $p_2 \in \mathcal{P}$, we define the *conservative estimate* for r_2 by

$$U_2^{\min}(p_2) = \min\{U_2(p_1, p_2) | p_1 \in \mathcal{P}\}. \quad (21)$$

Theorem 4: $U_1^{\min}(p_1) = U_1(p_1, RE)$. $U_2^{\min}(p_2) = U_2(RE, p_2)$. In other words, the minimum utility for router r_i playing strategy p_i is achieved when router $r_{\neg i}$ is playing strategy $p_{\neg i} = RE$. Furthermore, the conservative estimates $U_1^{\min}(p_1)$ and $U_2^{\min}(p_2)$ can be computed in $O(K)$ time. □

Proof. When $r_{\neg i}$ plays strategy $p_{\neg i}$, router serves the minimum user demand. This proves the theorem. □

B. The Max-min Strategy of r_i

The *optimal conservative strategy* for router r_i is

$$p_i^{\text{opt}} = \arg\max_{p_i \in \mathcal{P}}(U_i^{min}(p_i)), \quad i = 1, 2. \quad (22)$$

Theorem 5: Both p_1^{opt} and p_2^{opt} can be computed in polynomial time. Furthermore, we have

$$U_1(p_1^{\text{opt}}, p_2^{\text{opt}}) \geq U_1^{\min}(p_1^{\text{opt}}), \quad (23)$$

$$U_2(p_1^{\text{opt}}, p_2^{\text{opt}}) \geq U_2^{\min}(p_2^{\text{opt}}). \quad (24)$$

In other words, when both routers play their optimal conservative strategies, router r_i's actual utility is at least as good as $U_i^{\min}(p_i^{\text{opt}})$. □

Proof. Inequalities (23) and (24) are direct consequences of the definition. We prove that p_1^{opt} can be computed in $O(K^2)$ time. The time complexity for computing p_2^{opt} can be proved similarly.

We note that p_2^{opt} is either RE or greater than RE. $U_1^{\min}(RE)$ can be evaluated in $O(K)$ time. For $p_1 > RE$, $U_1^{\min}(p_1)$ is computed using Eq. (6c). For $k = 1, 2, \ldots, K$, both $d_1(x)$ and $d_0(x)$ are constants when $x \in (c_{k-1}, c_k]$. Therefore for each $k = 1, 2, \ldots, K$, the maximum of $U_1^{\min}(p_1)$ when $c_{k-1} < p_1 \leq c_k$ is achieved with $p_1 = c_k$. Since we are evaluating $O(K)$ utility values, the time complexity is $O(K^2)$. This proves the theorem. □

6. Evaluation

We use randomly generated test cases to evaluate the performance of the proposed max-min approach to HPS. In this section, we present our evaluation setting, the evaluation results, and our observations.

A. Evaluation Setting

We study the performance of our proposed max-min approach in 10 scenarios, where the number of users varies from 100 to 1000 (i.e. $K = 100, 200, \ldots, 1000$). Each scenario has the following parameters: K, $RE > 0$, $B_1 > 0$, $B_2 > 0$, $C_{\min}$ and $C_{\max}$ such that $C_{\max} > C_{\min} \geq RE$, $\Delta_{\min}$ and $\Delta_{\max}$ such that $\Delta_{\max} > \Delta_{\min} > 0$.

A test case for a given scenario is generated by setting RE, B_1, B_2 to the specified parameters, and randomly generating K users where c_k is a random number in the interval $[C_{\min}, C_{\max}]$, δ_k is a random number in the interval $[\Delta_{\min}, \Delta_{\max}]$ and user u_k is randomly assigned to one of the three sets: $\Omega_1, \Omega_2, \Omega_0$.

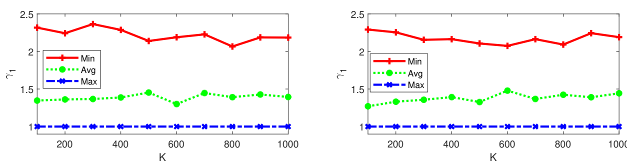
For each scenario, we generate 100 test cases, compute the values p_i^{opt}, $U_i^{\min}(p_i^{\text{opt}})$, $U_i(p_1^{\text{opt}}, p_2^{\text{opt}})$, $i = 1, 2$. We compute the maximum, mean, and minimum of the ratios

$$\gamma_1 = U_1(p_1^{\text{opt}}, p_2^{\text{opt}})/U_1^{\min}(p_1^{\text{opt}}), \quad (25)$$

$$\gamma_2 = U_2(p_1^{\text{opt}}, p_2^{\text{opt}})/U_2^{\min}(p_2^{\text{opt}}). \quad (26)$$

B. Results and Observations

We present the evaluation results using the generated test cases.



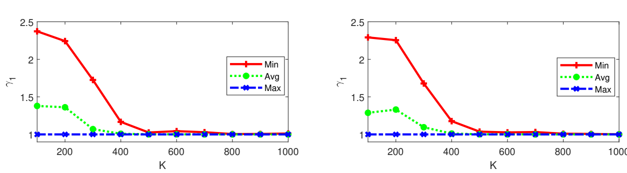
(a) Max, mean, min values of γ_1 (b) Max, mean, min values of γ_2

Fig. 2. Stats of γ_1 and γ_2 in different K sizes (scaling router capacity).

Fig. 2 shows the result where $B_1 = B_2 = K$ (scaling router capacity). We observe that γ_i is always greater than or equal to 1.0. This is consistent with equations (23) and (24) in Theorem 5, which says that the actual utility of router r_i is always greater than or equal to its conservative estimate $U_i^{\min}(p_i^{\text{opt}})$.

We observe that the minimum of γ_i, regardless of K, is exactly 1.0. We attribute this observation to the fact that if γ_1 is 1.0 in one of the 100 test cases, then the minimum of γ_1 will be 1.0. We observe that the mean of γ_i is around 1.38, and maximum is above the 2.0 mark. This shows that $U_i^{\min}(p_i^{\text{opt}})$ is a very conservative estimate. Our results show that the max-min strategy achieves on average about 38% more utility when compared to the conservative estimate.

The max-min approach works well for both r_1 and r_2. As Fig. 2 indicates, the value of γ_1 is comparable to that of γ_2. Other than slight turbulence caused by the randomness of the data, the overall results are similar in both graphs.



(a) Max, mean, min values of γ_1 (b) Max, mean, min values of γ_2

Fig. 3. Stats of γ_1 and γ_2 in different K sizes (fixed router capacity).

Fig. 3 shows the result where $B_1 = B_2 = 200$ (fixed router capacity). For $K = 100$ and 200, we observe a similar pattern as in the scaling router capacity scenario. However the ratios decrease towards 1.0 as K increases, eventually stabilize at 1.0.

This can be explained by the saturation of both routers with increased user demand at larger K values. As K increases, the demand of locked-in user set Ω_1 and Ω_2 also increases, eventually causing both routers to run at their capacities. This results in decreases in γ_i as both routers do not have to compete as hard for customer.

7. Conclusions

In this paper, we have studied the hub fee-setting behavior in payment channel networks. Since pure NE and best response strategies do not always exist, we define approximate best response strategies as well as approximate Nash equilibria. On the negative side, we prove that for some $\epsilon > 0$, an ϵ-NE may not exist. On the positive side, we prove that for any $\epsilon > 0$, an ϵ-BR always exists, and can be computed in polynomial time. Furthermore, we introduce the notion of conservative estimate and present a novel max-min approach to HPS. Our evaluation results demonstrate the power of the max-min approach.

An interesting future research direction is to study the performance of the proposed max-min approach on real payment channel network topologies, such as the ones used in [10], [12]. It is also interesting to investigate the case where the routers only know the distributions of the user demands and cost upper-bounds.

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