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Brief paper

Distributed continuous-time time-varying optimization for networked Lagrangian systems with quadratic cost functions[☆]

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ABSTRACT

In this paper, the distributed time-varying optimization problem is investigated for networked Lagrangian systems with parametric uncertainties. Usually, in the literature, to address some distributed control problems for nonlinear systems, a networked virtual system is constructed, and a tracking algorithm is designed such that the agents' physical states track the virtual states. It is worth pointing out that such an idea requires the exchange of the virtual states and hence necessitates communication among the group. In addition, due to the complexities of the Lagrangian dynamics and the distributed time-varying optimization problem, there exist significant challenges. This paper proposes distributed time-varying optimization algorithms that achieve zero optimum-tracking errors for the networked Lagrangian agents without the communication requirement. The main idea behind the proposed algorithms is to construct a reference system for each agent to generate a reference velocity using absolute and relative physical state measurements with no exchange of virtual states needed, and to design adaptive controllers for Lagrangian systems such that the physical states are able to track the reference velocities and hence the optimal trajectory. The algorithms introduce mutual feedback between the reference systems and the local controllers via physical states/measurements and are amenable to implementation via local onboard sensing in a communication unfriendly environment. Specifically, first, a base algorithm is proposed to solve the distributed time-varying optimization problem for networked Lagrangian systems under switching graphs. Then, based on the base algorithm, a continuous function is introduced to approximate the signum function, forming a continuous distributed optimization algorithm and hence removing the chattering. Such a continuous algorithm is convergent with bounded ultimate optimum-tracking errors, which are proportion to approximation errors. Finally, numerical simulations are provided to illustrate the validity of the proposed algorithms.

1. Introduction

In distributed optimization of networked systems, each member has a local cost function, and the goal is to cooperatively minimize the sum of all the local cost functions. This paper focuses on distributed continuous-time optimization algorithms, and the results on discrete-time ones can be referred to Yang et al. (2019) and the references therein. In the distributed continuous-time optimization problem, the agents are governed by certain dynamics described by differential equations, and the objective is to design control inputs for the agents such that the agents' physical states reach the optimal solution. In the literature, there are some distributed continuous-time optimization algorithms (He et al., 2017; Wang et al., 2015; Zhang et al., 2017; Zou et al., 2021, 2020), and these results assume time-invariant local cost functions for the agents. However, in many practical applications, e.g., the economic dispatch problem (Cherukuri & Cortes, 2016), the local cost functions might be time-varying, which reflects the fact that the optimal point might be changing over time forming an optimal trajectory. Hence, it is meaningful to investigate the distributed time-varying optimization problem.

We are focusing on developing distributed continuous-time time-varying optimization algorithms, which have various applications in practice, e.g., the coordination of a team of robots. For instance, by constructing quadratic objective functions for the agents, the distributed time-varying optimization algorithms

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can be applied to solve the distributed average tracking of multi-agent systems. A few distributed time-varying optimization algorithms are established for single integrators (Cherukuri & Cortes, 2016; Ning et al., 2017; Sun et al., 2023, 2017), double integrators (Rahili & Ren, 2017), and agents with nonlinear dynamics (Huang et al., 2020). In reality, a broad class of robots can be modeled by nonlinear Lagrangian dynamics, e.g., the planar elbow manipulator and autonomous vehicles (Spong et al., 2006). The nonlinear Lagrangian dynamics with parametric uncertainties, which are the focus of this paper, are more complicated than single and double integrators, and are different from and cannot be included as special cases by the model in Huang et al. (2020). The complexity of the dynamics creates more challenges to solve the distributed time-varying optimization problem.

Some results addressing distributed coordination problems (e.g., distributed optimization) for nonlinear agents introduce distributed observers or virtual systems at a higher level, where the agents communicate their observer states (virtual states independent of the agents' physical states/measurements) with neighbors such that the observer states or virtual states reach consensus on the desired optimal point/trajectory. Then control algorithms are designed for the agents to track the virtual states (serving as reference trajectories). However, due to the lack of physical states/feedback (e.g., agent positions) in the observers, the reference trajectories generated by such an approach do not explicitly take into account the physical agents' interaction with the environment and their capability. Also, such an approach cannot be implemented based on local measurements via onboard sensors without communication in a communication unfriendly environment.

In this paper, we propose communication-free distributed time-varying optimization algorithms for networked Lagrangian agents with parametric uncertainties. The main idea of the proposed algorithms is constructing a reference system for each agent, which is driven by the physical states instead of virtual states between neighbors and generates a reference velocity, and then designing adaptive controllers such that the agents' physical states track their reference velocities, and hence the optimal trajectory. The algorithms introduce mutual influence/feedback between reference systems and local controllers via physical states/measurements and are amenable to implementation via local onboard sensing in a communication unfriendly environment. Due to the coupling and mutual influence of the reference systems and the agents' dynamics, there are significant new challenges in the convergence analysis. In particular, the reference systems are rewritten as coupled and perturbed networked second-order systems by taking the tracking errors between agents' velocities and their own reference states as disturbances. Due to the use of the nonlinear functions (e.g., the signum function) in the construction of the reference systems, the coupled and perturbed networked systems have disturbances inside and outside the nonlinear functions, and the general input-to-state stability analysis might not be directly applicable. This requires rigorous analysis on the impact of disturbance on the optimum-tracking performance of the perturbed systems. To this end, this paper carefully examines the perturbed systems, and obtains the input-to-state-like stability from the disturbances to optimum-tracking errors (e.g., Proposition 1). These intermediate results facilitate the convergence analysis of the proposed algorithms for the networked Lagrangian agents. To be exact, we first design a base algorithm for the networked nonlinear Lagrangian systems to achieve exact optimum tracking under switching graphs. Built on the base algorithm, we then propose a continuous variant by replacing the signum function in the reference systems with a smooth nonlinear function to generate continuous control torques for the Lagrangian systems and hence remove the chattering caused by the signum function.

Comparison with Related Works: The works of He et al. (2017), Wang et al. (2015), Zhao et al. (2017), Zou et al. (2021, 2020) and Zhang et al. (2017) focus on solving the distributed time-invariant optimization problem. Due to the complexities of the agents' dynamics and switching graphs, the distributed time-varying optimization algorithms developed for integrator agents (Cherukuri & Cortes, 2016; Ning et al., 2017; Rahili & Ren, 2017; Sun et al., 2023, 2017) cannot be directly applied to address the case considered in the paper. More importantly, the proposed algorithms in this paper rely purely on physical states without the need for exchange of virtual states and can be implemented in communication unfriendly applications. In contrast, the communication of virtual states between neighbors is necessary in Huang et al. (2020), Zhang et al. (2017) and Zou et al. (2021, 2020). The structure of the proposed algorithms is inspired by Wang et al. (2020), where the consensus and leader-following tracking of networked Lagrangian systems are addressed. However, the problem considered in this paper is more complex and challenging, and includes the consensus and leader-following tracking of networked agents as special cases.

Some preliminary results of this paper (e.g., Lemma 1) are presented in Ding et al. (2022). Different from Ding et al. (2022), the current paper introduces two distributed time-varying optimization algorithms under switching graphs, one of which capable of generating continuous control torques for networked Lagrangian agents. In addition, this paper contains more detailed proofs and additional simulation results.

2. Preliminaries

2.1. Notation

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of all real and positive real numbers, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. For a matrix $A \in \mathbb{R}^{p\times p}$, let $\lambda_1(A) \le \cdots \le \lambda_p(A)$ denote its eigenvalues. For a vector $x \in \mathbb{R}^p$, define $\text{sgn}(x) = [\text{sgn}(x_1), \ldots, \text{sgn}(x_p)]^T$ where $\text{sgn}(x_i) = 1$ if $x_i > 0$, $\text{sgn}(x_i) = 0$ if $x_i = 0$, and $\text{sgn}(x_i) = -1$ if $x_i < 0$. Let $\mathbf{0}_m$ and $\mathbf{1}_m$ denote the m dimensional zero and all-ones vector, respectively. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. For a time-varying signal x, let the kth time derivative of x be denoted by $x^{(k)}$, where k is a non-negative integer, and in particular, $x^{(0)} = x$ and $x^{(1)} = \dot{x}$. For a time-varying function $f(q,t)$, its gradient, denoted by $\nabla f(q,t) \in \mathbb{R}^p$ with $q \in \mathbb{R}^p$ and $t \in \mathbb{R}_{\ge 0}$, is the partial derivative of $f(q,t)$ with respect to q, and its Hessian, denoted by $H(q,t) \in \mathbb{R}^{p\times p}$, is the partial derivative of the gradient $\nabla f(q,t)$ with respect to q. Define $\mathcal{L}_\infty^p = \{x : [0,\infty) \to \mathbb{R}^p | \sup_{t\ge 0} \|x(t)\|_\infty < \infty\}$ and $\mathcal{L}_2^p = \{x : [0,\infty) \to \mathbb{R}^p | [\int_0^\infty x^T(t)x(t)\mathrm{d}t]^{1/2} < \infty\}$.

2.2. Graph theory

For a multi-agent system consisting of N agents, the interaction topology can be modeled by a switching graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, which maps from $\mathbb{R}_+ \cup \{0\}$ to a finite set of undirected graphs $\mathscr{G} = \{\mathcal{G}_k, k = 1, \ldots, G\}$ with $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$. For each undirected graph $\mathcal{G}_k$, an edge, denoted by $(i,j) \in \mathcal{E}_k$, means that agent i and j can obtain information from each other at time t. The edges (i,j) and (j,i) are equivalent. It is assumed that $(i,i) \notin \mathcal{E}_k$ $\forall k = 1, \ldots, G$. The switching between graphs is modeled by a switching signal $\sigma : \mathbb{R}_+ \cup \{0\} \to \{1, \ldots, G\}$. Denote by $t_0, t_1, \ldots$ with $t_0 = 0$ an infinite sequence of time instances at which σ switches, and any two consecutive switching time instances satisfy the standard dwell-time condition that $t_\kappa - t_{\kappa-1} \ge T_S$ $\forall \kappa = 1, 2, \ldots$ where T_S is some positive constant. The neighbor set of node i at time t is denoted by $\mathcal{N}_i(t) = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}_{\sigma(t)}\}$. By arbitrarily assigning an orientation for every edge in $\mathcal{G}_{\sigma(t)}$ at any time t, let $B(t) = \left[b_{ij}(t)\right] \in \mathbb{R}^{N\times|\mathcal{E}_{\sigma(t)}|}$ denote the incidence matrix associated with graph $\mathcal{G}_{\sigma(t)}$ at time t, where $b_{ij}(t) = -1$ if edge s_j leaves node i, $b_{ij}(t) = 1$ if it enters node i, and $b_{ij}(t) = 0$ otherwise. An undirected path between nodes i_1 and i_k is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, where $i_k \in \mathcal{V}$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

Assumption 1. All the graphs in the set $\mathscr{G}$ are connected.

2.3. Agents' dynamics

In this paper, we consider N Lagrangian systems, and the equations of motion of the ith Lagrangian system can be described by (Spong et al., 2006)

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the generalized position (or configuration), $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p\times p}$ is the Coriolis and centrifugal matrix, $g_i(q_i) \in \mathbb{R}^p$ is the gravitational torque, and $\tau_i \in \mathbb{R}^p$ is the exerted control torque. Three well-known properties associated with the dynamics (1) are listed as follows (Ghapani et al., 2016; Spong et al., 2006).

Property 1. *The inertial matrix $M_i(q_i)$ is symmetric and uniformly positive definite, and there exist positive constants $k_{\bar{C}}$ and $k_{\bar{g}}$ such that $\|C_i(q_i, \dot{q}_i)\|_2 \le k_{\bar{C}} \|\dot{q}_i\|_2$ and $\|g_i(q_i)\|_2 \le k_{\bar{g}}$, $\forall i \in \mathcal{V}$.*

Property 2. *The Coriolis and centrifugal matrix $C_i(q_i, \dot{q}_i)$ can be suitably chosen such that the matrix $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew-symmetric.*

Property 3. *The dynamics (1) depend linearly on an unknown constant parameter vector $\vartheta_i \in \mathbb{R}^m$, that is, for any $x, y \in \mathbb{R}^p$, it holds that $M_i(q_i)x + C_i(q_i, \dot{q}_i)y + g_i(q_i) = Y_i(q_i, \dot{q}_i, y, x)\vartheta_i$, where $Y_i(q_i, \dot{q}_i, y, x)$ is the regressor matrix.*

3. Problem statement

In the distributed time-varying optimization problem, each Lagrangian agent aims to cooperatively track the optimal trajectory determined by the group objective function. Let $q^*(t) \in \mathbb{R}^p$ denote the optimal trajectory, and it is defined as

$$q^*(t) = \arg\min_{\zeta(t)} \left\{ \sum_{i=1}^{N} f_i[\zeta(t), t] \right\}, \tag{2}$$

where $f_i[\zeta(t), t] : \mathbb{R}^p \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is the local cost function associated with agent $i \in \mathcal{V}$. In the rest of the paper, it is assumed that $q^* \in \mathcal{L}_\infty^p$. This assumption is satisfied in most applications in practice. It is assumed that $f_i[\zeta(t), t]$ is known only to agent i. To find $q^*(t)$ defined in (2) is equivalent to find the optimal solution

$$\{q_1^*(t), \ldots, q_N^*(t)\} = \arg\min_{\{q_1(t),\ldots,q_N(t)\}} \left\{ \sum_{i=1}^{N} f_i[q_i(t), t] \right\},$$
$$\text{subject to } q_i(t) = q_j(t) \quad \forall i \neq j,$$

where $q_i^*(t) = q_j^*(t) = q^*(t)$ $\forall i \neq j$. In this paper, the goal is to design the control torques τ_i, $i \in \mathcal{V}$, for the agent (1) such that each agent's position $q_i(t)$ is capable of tracking $q_i^*(t) = q^*(t)$. That is, design τ_i for each agent i such that $\lim_{t\to\infty}[q_i(t) - q^*(t)] = \mathbf{0}_p$ $\forall i \in \mathcal{V}$. We make the following assumptions on the cost functions.

Assumption 2. Each cost function $f_i(q_i, t)$, $i \in \mathcal{V}$, is twice continuously differentiable both in $q_i \in \mathbb{R}^p$ and t, and strongly convex in q_i and uniformly in t. That is, the Hessian matrix $H_i(q_i, t)$ is always positive definite and there exists a positive constant $\underline{m}$ such that $\lambda_j(H_i(q_i, t)) \ge \underline{m}$ $\forall j \in \{1, \ldots, p\}$, $\forall i \in \mathcal{V}$ holds uniformly in t.

Assumption 3. For any $i \in \mathcal{V}$, the gradient of the cost function $f_i(q_i, t)$ can be written as $\nabla f_i(q_i, t) = H(t)q_i + g_i(t)$, where $H(t)$ is a matrix-valued function, and $g_i(t)$ is a time-varying function. In addition, there exist positive constants $\bar{H}$ and $\bar{g}$ such that $\sup_{t\in[0,\infty)} \left\|H^{(l)}(t)\right\|_2 \le \bar{H}$ and $\sup_{t\in[0,\infty)} \left\|g_i^{(l)}(t)\right\|_2 \le \bar{g}$ $\forall l = 0, 1, 2$, $\forall i \in \mathcal{V}$,

In Assumption 2, the uniform strong convexity of the objective functions guarantees that the optimal trajectory q^* is unique for all $t \ge 0$, and it also ensures that $H_i(q_i, t)$ $\forall i \in \mathcal{V}$ is invertible for all t. Assumptions 2 and 3 are some similar/equivalent assumptions that are used in prior related works (Ding et al., 2024; Huang et al., 2020; Ning et al., 2017; Rahili & Ren, 2017; Simonetto et al., 2016). Assumptions 2 and 3 can be satisfied in many situations in practice, such as energy minimization (Ghapani et al., 2017), motion coordination (Sun et al., 2023), and distributed average tracking (Chen et al., 2015; Rahili & Ren, 2017).

Example 1 (*Distributed Average Tracking*)**.** In a networked system, each agent has a local reference signal $r_i(t) \in \mathbb{R}^p$, and the objective is to design control inputs such that the agents' physical states track the average of the group reference signals, i.e., $\lim_{t\to\infty}[q_i(t) - \frac{1}{N}\sum_{j=1}^{N} r_i(t)] = \mathbf{0}_p$ $\forall i \in \mathcal{V}$. Distributed average tracking algorithms have found applications in region following formation control (Chen & Ren, 2017), coordinated path planning (Švestka & Overmars, 1998), and distributed optimization problems (Shi et al., 2023). If the cost function are constructed as $f_i(q_i, t) = \|q_i(t) - r_i(t)\|_2^2$, the distributed time-varying optimization algorithms can be applied to address the distributed average tracking of networked agents. Note that Assumption 2 holds trivially from the above construction of $f_i(q_i, t)$. Also, the boundedness assumptions of r_i, $\dot{r}_i$ and $\ddot{r}_i$ are commonly placed when dealing with the distributed average tracking of networked agents (Ghapani et al., 2017), and such boundedness assumptions imply that Assumption 3 holds.

4. Distributed time-varying optimization of networked Lagrangian agents

4.1. The base algorithm

For each agent $i \in \mathcal{V}$, construct a reference system as

$$\dot{v}_i = -\alpha\dot{q}_i - \gamma \sum_{j\in\mathcal{N}_i(t)} \text{sgn}[\alpha(q_i - q_j) + \dot{q}_i - \dot{q}_j] + \varphi_i, \tag{3}$$

where α and γ are some positive constants to be determined, and φ_i is defined by

$$\varphi_i = -\alpha F_i(q_i, t) - \dot{F}_i(q_i, t), \tag{4}$$

$$F_i(q_i, t) = H_i^{-1}(q_i, t)\Big[\frac{\partial}{\partial t}\nabla f_i(q_i, t) + \beta\nabla f_i(q_i, t)\Big], \tag{5}$$

with β being a positive constant to be determined. Note that Assumptions 2 and 3 guarantee the existence of φ_i, $i \in \mathcal{V}$. Define

$$s_i = \dot{q}_i - v_i. \tag{6}$$

The adaptive controller for the Lagrangian system (1) is given by

$$\tau_i = -K_i s_i + Y_i(q_i, \dot{q}_i, v_i, \dot{v}_i)\hat{\vartheta}_i, \tag{7}$$

$$\dot{\hat{\vartheta}}_i = -\Gamma_i Y_i^T(q_i, \dot{q}_i, v_i, \dot{v}_i)s_i, \tag{8}$$

where K_i and Γ_i are symmetric positive definite matrices, and $\hat{\vartheta}_i$ is the estimate of ϑ_i. In the algorithm, the reference system (3) generates a desired reference velocity v_i for each agent i, and the adaptive controller (7)–(8) is used to drive each agent's velocity $\dot{q}_i$ to track its local v_i. Then, by the cascaded structure of the proposed algorithm, one might expect q_i to track the optimal trajectory. In the reference system (3), the term $-\alpha\dot{q}_i+\varphi_i$ is introduced to minimize the cost functions and the signum function term is for coordination purpose. Note also that $\dot{F}_i(q_i,t)$ could be estimated from $F_i(q_i,t)$ using a filter.

Remark 1. It is worth emphasizing that the algorithm (3)–(8) does not rely on exchange of virtual variables between neighbors. Especially, the reference system (3) is driven by agents' physical state information, i.e., q_i, $\dot{q}_i$, q_i-q_j and $\dot{q}_i-\dot{q}_j$. Such design excludes the usage of communication channels, and can be implemented by onboard sensors. This feature distinguishes this algorithm from existing results on distributed optimization of networked Lagrangian systems, e.g., Zhang et al. (2017) and Zou et al. (2021, 2020), where inter-agent communication is required. In addition, the algorithm (7)–(8) with $\dot{v}_i$ defined in (3) addresses the distributed time-varying optimization problem with zero optimum-tracking error, while the results in Zou et al. (2021, 2020) are limited to distributed time-invariant optimization, and the work of Zhang et al. (2017) only addresses a special case of time-varying cost functions with nonzero bounded optimum-tracking errors. It is also worth pointing out that such design results in the fact that the reference systems and agents' dynamics are highly coupled. The convergence analysis of such coupled systems is quite complex and challenging. However, in the literature, when distributed control problems are addressed for nonlinear systems, networked virtual systems are constructed completely independent of the agents' dynamics, which makes the analysis much easier and more straightforward compared with our convergence analysis later.

4.2. Convergence analysis

Before moving on to the convergence analysis, an essential lemma is presented.

Lemma 1. *Let $\gamma\in\mathbb{R}_+$, $z=[z_1^T,\ldots,z_N^T]^T$, and $s=[s_1^T,\ldots,s_N^T]^T$, where $z_i\in\mathbb{R}^p$ and $s_i\in\mathbb{R}^p$ $\forall i\in\mathcal{V}$. It holds that $-\gamma z^T[B(t)\otimes I_p]\mathrm{sgn}([B^T(t)\otimes I_p](z+s))\le-\gamma\|[B^T(t)\otimes I_p]z\|_1+2\gamma\|[B^T(t)\otimes I_p]s\|_1$.*

Proof. Define $\mathcal{P}=\{1,\ldots,p\}$, and let $z_{i,k}$ and $s_{i,k}$ denote the kth entry in vector z_i and s_i. It holds that $-\gamma z^T[B(t)\otimes I_p]\mathrm{sgn}([B^T(t)\otimes I_p](z+s))=-\gamma\sum_{(i,j)\in\mathcal{E}(t)}(z_i-z_j)^T\mathrm{sgn}(z_i-z_j+s_i-s_j)=-\gamma\sum_{k\in\mathcal{P}}\sum_{(i,j)\in\mathcal{E}(t)}\Lambda_{i,j}^k$, where $\Lambda_{i,j}^k=(z_{i,k}-z_{j,k})\mathrm{sgn}(z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k})$. For any $k\in\mathcal{P}$, define $\mathcal{E}_0^k(t)=\{(i,j)\in\mathcal{E}(t)|z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k}=0\}$. Note that $\Lambda_{i,j}^k=0$ if $(i,j)\in\mathcal{E}_0^k(t)$. It then holds that $-\gamma z^T[B(t)\otimes I_p]\mathrm{sgn}\big([B^T(t)\otimes I_p](z+s)\big)=-\gamma\sum_{k\in\mathcal{P}}\sum_{(i,j)\in\mathcal{E}(t)\setminus\mathcal{E}_0^k(t)}\Lambda_{i,j}^k$. For any $(i,j)\in\mathcal{E}(t)\setminus\mathcal{E}_0^k(t)$, it holds that $-\gamma\Lambda_{i,j}^k=-\gamma|z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k}|+\gamma\frac{(s_{i,k}-s_{j,k})(z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k})}{|z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k}|}\le-\gamma|z_{i,k}-z_{j,k}+s_{i,k}-s_{j,k}|+\gamma|s_{i,k}-s_{j,k}|\le-\gamma\big||z_{i,k}-z_{j,k}|-|s_{i,k}-s_{j,k}|\big|+\gamma|s_{i,k}-s_{j,k}|$. For any $k\in\mathcal{P}$, define $\mathcal{E}_+^k(t)=\{(i,j)\in\mathcal{E}(t)||z_{i,k}-z_{j,k}|\ge|s_{i,k}-s_{j,k}|\}$ and $\mathcal{E}_-^k(t)=\{(i,j)\in\mathcal{E}(t)||z_{i,k}-z_{j,k}|<|s_{i,k}-s_{j,k}|\}$. Then, it follows that $-\gamma\sum_{(i,j)\in\mathcal{E}(t)\setminus\mathcal{E}_0^k(t)}\Lambda_{i,j}^k\le-\gamma\sum_{(i,j)\in\mathcal{E}_+^k(t)\setminus\mathcal{E}_0^k(t)}(|z_{i,k}-z_{j,k}|-2|s_{i,k}-s_{j,k}|)+\gamma\sum_{(i,j)\in\mathcal{E}_-^k(t)}|z_{i,k}-z_{j,k}|\le-\gamma\sum_{(i,j)\in\mathcal{E}}|z_{i,k}-z_{j,k}|+2\gamma\sum_{(i,j)\in\mathcal{E}_+^k(t)\setminus\mathcal{E}_0^k(t)}|s_{i,k}-s_{j,k}|+\gamma\sum_{(i,j)\in\mathcal{E}_0^k(t)}|z_{i,k}-z_{j,k}|+2\gamma\sum_{(i,j)\in\mathcal{E}_-^k(t)}|s_{i,k}-s_{j,k}|\le-\gamma\sum_{(i,j)\in\mathcal{E}(t)}|z_{i,k}-z_{j,k}|+2\gamma\sum_{(i,j)\in\mathcal{E}(t)}|s_{i,k}-s_{j,k}|$. Hence, $-\gamma z^T[B(t)\otimes I_p]\mathrm{sgn}([B^T(t)\otimes I_p](z+s))\le-\gamma\sum_{k\in\mathcal{P}}\sum_{(i,j)\in\mathcal{E}(t)}(|z_{i,k}-z_{j,k}|-2|s_{i,k}-s_{j,k}|)=-\gamma\sum_{(i,j)\in\mathcal{E}(t)}\|z_i-z_j\|_1+2\gamma\sum_{(i,j)\in\mathcal{E}(t)}\|s_i-s_j\|_1=-\gamma\|[B^T(t)\otimes I_p]z\|_1+2\gamma\|[B^T(t)\otimes I_p]s\|_1$. ■

Using the definition of s_i in (6), the reference system (3) can be rewritten as

$$\dot{q}_i=v_i+s_i, \tag{9}$$

$$\dot{v}_i=-\alpha v_i-\alpha s_i+\varphi_i-\gamma\sum_{j\in\mathcal{N}_i(t)}\mathrm{sgn}[\alpha(q_i-q_j)+v_i-v_j+s_i-s_j]. \tag{10}$$

By the system reformulation, the reference system (3) (i.e., (9)–(10)) can be viewed as a group of networked perturbed double-integrators with disturbances s_i, $i\in\mathcal{V}$. The following proposition shows that the system (9)–(10) is input-to-state-like stable from the disturbances (i.e., s_i, $i\in\mathcal{V}$) to the optimum-tracking errors (i.e., $q_i(t)-q^*(t)$, $i\in\mathcal{V}$). That is, optimum-tracking errors are bounded and convergent to zero if the disturbances are bounded in a certain sense and convergent to zero.

Proposition 1. *Consider a group of N agents, and their interaction is described by the graph $\mathcal{G}_{\sigma(t)}$. Each agent's dynamics are given by (9)–(10). Suppose that Assumptions 1–3 hold. Let α, β, and γ be chosen such that*

$$\alpha\in\mathbb{R}_+,\ \beta>\max\{\kappa_1,\kappa_2\},\ \text{and } \gamma>\pi, \tag{11}$$

where

$$\kappa_1=\frac{\bar{H}}{\underline{m}}\left(2+\frac{3}{2\alpha}+\frac{3\bar{H}}{2\underline{m}}\right)+\frac{\alpha}{2}, \tag{12}$$

$$\kappa_2=\frac{\bar{H}}{\underline{m}}\left(2+\frac{1}{2\alpha}+\frac{\bar{H}}{2\underline{m}}\right)+\frac{3\alpha}{2}, \tag{13}$$

$$\pi=\frac{\bar{g}}{\underline{m}}(\beta+1)\left(\alpha+1+\frac{\bar{H}}{\underline{m}}\right)(N-1), \tag{14}$$

and $\bar{g}$, $\bar{H}$, $\underline{m}$ are given in Assumptions 2 and 3. Then, the following two statements hold.

(I) If $s_i\in\mathcal{L}_\infty^p$ $\forall i\in\mathcal{V}$, it holds that $q_i-q^*\in\mathcal{L}_\infty^p$ $\forall i\in\mathcal{V}$.
(II) If $\lim_{t\to\infty}s_i(t)=\mathbf{0}_p$ $\forall i\in\mathcal{V}$, it holds that $\lim_{t\to\infty}[q_i(t)-q^*(t)]=\mathbf{0}_p$ $\forall i\in\mathcal{V}$.

Proof. First, the following statements are proved, which then are used to prove Statements (I) and (II):

(i) $q_i-\frac{1}{N}\sum_{j=1}^N q_j\in\mathcal{L}_\infty^p$ and $v_i-\frac{1}{N}\sum_{j=1}^N v_j\in\mathcal{L}_\infty^p$ $\forall i\in\mathcal{V}$ if $s_i\in\mathcal{L}_\infty^p$ $\forall i\in\mathcal{V}$, and $\lim_{t\to\infty}[q_i(t)-\frac{1}{N}\sum_{j=1}^N q_j(t)]=\mathbf{0}_p$ and $\lim_{t\to\infty}[v_i(t)-\frac{1}{N}\sum_{j=1}^N v_j(t)]=\mathbf{0}_p$ $\forall i\in\mathcal{V}$ if $\lim_{t\to\infty}s_i=\mathbf{0}_p$ $\forall i\in\mathcal{V}$;
(ii) $\sum_{j=1}^N\nabla f_j(q_j,t)\in\mathcal{L}_\infty^p$ and $\sum_{j=1}^N[v_j+F_j(q_j,t)]\in\mathcal{L}_\infty^p$ if $s_i\in\mathcal{L}_\infty^p$ $\forall i\in\mathcal{V}$, and $\lim_{t\to\infty}\sum_{j=1}^N\nabla f_j(q_j,t)=\mathbf{0}_p$ if $\lim_{t\to\infty}s_i(t)=\mathbf{0}_p$ $\forall i\in\mathcal{V}$.

Proof of Statement (i). From Assumption 3, it holds that $\varphi_i=-\alpha\beta q_i-\beta\dot{q}_i+D_1(t)q_i+D_2(t)\dot{q}_i+\widetilde{g}_i(t)$, where $D_1(t)=-\alpha H^{-1}(t)\dot{H}(t)-H^{-1}(t)\ddot{H}(t)+[H^{-1}(t)\dot{H}(t)]^2$, $D_2(t)=-H^{-1}(t)\dot{H}(t)$, and $\widetilde{g}_i(t)=-H^{-1}(t)\{\beta\dot{g}_i(t)+\ddot{g}_i(t)+[\alpha I_p-\dot{H}(t)H^{-1}(t)][\beta g_i(t)+\dot{g}_i(t)]\}$. Let $q=[q_1^T,\ldots,q_N^T]^T$, $v=[v_1^T,\ldots,v_N^T]^T$, $s=[s_1^T,\ldots,s_N^T]^T$, and $\varphi=[\varphi_1^T,\ldots,\varphi_N^T]^T$. Note that the right hand side of the closed-loop dynamics is discontinuous because of the signum function, and the solution should be understood in terms of differential inclusion by using non-smooth analysis (Filippov, 1988). Define $x=(M\otimes I_p)q$, $y=(M\otimes I_p)v$, and $\xi=[x^T,y^T]^T$, where

$M = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. Then, it holds that $\dot{\xi} \overset{a.e.}{\in} \mathcal{K}[f](\xi)$, where *a.e.* stands for "almost everywhere" and $f = [f_x^T, f_y^T]^T$ with

$$f_x(\xi) = y + (M \otimes I_p)s, \tag{15}$$

$$f_y(\xi) = -\alpha y - \alpha s + (M \otimes I_p)\varphi - \gamma[B(t) \otimes I_p]\mathrm{sgn}([B^T(t) \otimes I_p](\alpha x + y + s)). \tag{16}$$

Define the function $V(\xi) = \xi^T(P \otimes I_{Np})\xi$, where

$$P = \frac{1}{2}\begin{bmatrix} 2\alpha\beta + \alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}. \tag{17}$$

Note that the matrix P is positive definite if $\alpha\beta > 0$. It also holds that $\lambda_1(P)\|\xi\|_2^2 \le V(\xi) \le \lambda_2(P)\|\xi\|_2^2$. In the following, it is shown that $V(\xi)$ is a common ISS-Lyapunov triple for switched system for ξ (Mancilla-Aguilar & Garcıa, 2001) by letting $\mathcal{G}_{\sigma(t)} = \mathcal{G}_k$.

The set-valued Lie-derivative is given by Shevitz and Paden (1994) $\dot{\tilde{V}} \subseteq \mathcal{K}[U_1] + \mathcal{K}[U_2] + \mathcal{K}[U_3]$, where $U_1 = -\alpha^2\beta x^Tx - \beta y^Ty + \alpha x^T(I_N \otimes D_1)x + y^T[\alpha I_{Np} + I_N \otimes D_2]y + x^T[\alpha(I_N \otimes D_2) + (I_N \otimes D_1) + I_{Np}]y$, $U_2 = x^T(\alpha\beta I_{Np} + I_N \otimes D_2)s + y^T(I_N \otimes D_2 - \beta I_{Np})s$, $U_3 = (\alpha x+y)^T\widetilde{g}(t) - \gamma(\alpha x+y)^T[B(t)\otimes I_p]\mathrm{sgn}([B^T(t)\otimes I_p](\alpha x+y+s))$ with $\widetilde{g} = [\widetilde{g}_1^T, \ldots, \widetilde{g}_N^T]^T$, and the fact that $M^2 = M$ has been used.

Consider U_1. It follows from Assumptions 2 and 3 that $\|I_N \otimes D_1\|_2 = \|D_1\|_2 \le \frac{\bar{H}}{\underline{m}}(\alpha + 1 + \frac{\bar{H}}{\underline{m}})$, and similarly, it holds that $\|I_N \otimes D_2\|_2 \le \frac{\bar{H}}{\underline{m}}$. Note that $\alpha x^T(I_N \otimes D_1)x \le \alpha\|x\|_2\|I_N \otimes D_1\|_2\|x\|_2 \le \alpha\frac{\bar{H}}{\underline{m}}(\alpha + 1 + \frac{\bar{H}}{\underline{m}})\|x\|_2^2$, $y^T(I_N \otimes D_2)y \le \frac{\bar{H}}{\underline{m}}\|y\|_2^2$, and $x^T[\alpha(I_N \otimes D_1) + (I_N \otimes D_1) + I_{Np}]y \le [\frac{\bar{H}}{\underline{m}}(2\alpha + 1 + \frac{\bar{H}}{\underline{m}}) + \alpha^2](\frac{\alpha}{2}\|x\|_2^2 + \frac{1}{2\alpha}\|y\|_2^2)$. Then, it holds that $U_1 \le -\alpha^2[\beta - \frac{\bar{H}}{\underline{m}}(2 + \frac{3}{2\alpha} + \frac{3\bar{H}}{2\alpha\underline{m}}) - \frac{\alpha}{2}]\|x\|_2^2 - [\beta - \frac{\bar{H}}{\underline{m}}(2 + \frac{1}{2\alpha} + \frac{\bar{H}}{2\alpha\underline{m}}) - \frac{3\alpha}{2}]\|y\|_2^2 \le -\alpha^2(\beta - \kappa_1)\|x\|_2^2 - (\beta - \kappa_2)\|y\|_2^2$, where κ_1 and κ_2 are given in (12) and (13), respectively.

Consider U_2. Note that $\alpha\beta x^Ts \le \alpha\beta\|x\|_1\|s\|_\infty$ and $x^T(I_N \otimes D_2)s \le \frac{\bar{H}}{\underline{m}}\|x\|_2\|s\|_2 \le \sqrt{Np}\frac{\bar{H}}{\underline{m}}\|x\|_1\|s\|_\infty$. It follows that $U_2 \le (\alpha\beta + \sqrt{Np}\frac{\bar{H}}{\underline{m}})\|x\|_1\|s\|_\infty + (\beta + \sqrt{Np}\frac{\bar{H}}{\underline{m}})\|y\|_1\|s\|_\infty \le \kappa_3(\|x\|_1 + \|y\|_1)\|s\|_\infty \le \kappa_3\sqrt{2Np}\|\xi\|_2\|s\|_\infty$, where $\kappa_3 = \max\{\alpha\beta + \sqrt{Np}\frac{\bar{H}}{\underline{m}}, \beta + \sqrt{Np}\frac{\bar{H}}{\underline{m}}\}$.

Consider U_3. Note that $\|\alpha x + y\|_1 \le \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\|\alpha(x_i - x_j) + y_i - y_j\|_1 \le \max_{i\in\mathcal{V}}\{\sum_{j=1,j\neq i}^N\|\alpha(x_i - x_j) + y_i - y_j\|_1\} \le (N-1)\|[B^T(t) \otimes I_p](\alpha x + y)\|_1$, where $x_i = q_i - \frac{1}{N}\sum_{j=1}^N q_j$ and $y_i = v_i - \frac{1}{N}\sum_{j=1}^N v_j$. It follows from Assumption 3 that $\|\widetilde{g}\|_\infty \le \max_{i\in\mathcal{V}}\{\|\widetilde{g}_i\|_\infty\} \le \|\widetilde{g}_i\|_2 \le \frac{\bar{g}}{\underline{m}}(\beta + 1)(\alpha + 1 + \frac{\bar{H}}{\underline{m}})$. Note that $(\alpha x + y)^T\widetilde{g} \le \|\alpha x + y\|_1\|\widetilde{g}\|_\infty \le \frac{\bar{g}}{\underline{m}}(\beta + 1)(\alpha + 1 + \frac{\bar{H}}{\underline{m}})(N-1)\left\|[B^T(t) \otimes I_p](\alpha x + y)\right\|_1 = \pi\|[B^T(t) \otimes I_p](\alpha x + y)\|_1$, where π is given in (14). From Lemma 1, it follows that $-\gamma(\alpha x + y)^T[B(t) \otimes I_p]\mathrm{sgn}\big([B^T(t) \otimes I_p](\alpha x + y + s)\big) \le -\gamma\|[B^T(t) \otimes I_p](\alpha x + y)\|_1 + 2\gamma\|[B^T(t)\otimes I_p]s\|_1 \le -\gamma\|[B^T(t)\otimes I_p](\alpha x+y)\|_1 + \gamma N^2(N-1)p\|s\|_\infty$. It holds that $U_3 \le -(\gamma - \pi)\|[B^T(t)\otimes I_p](\alpha x+y)\|_1 + \kappa_4\|s\|_\infty$, where $\kappa_4 = \gamma N^2(N-1)p$.

Since the signum function is measurable and locally essentially bounded, then the Filippov solutions always exist and are absolutely continuous (Cortes, 2008). Hence, x and y are continuous. Note that, for any $\tilde{U}_1 \in \mathcal{K}[U_1]$, $\tilde{U}_1 \le -\alpha^2(\beta - \kappa_1)\|x\|_2^2 - (\beta - \kappa_2)\|y\|_2^2$ holds. Similar facts hold for any $\tilde{U}_2 \in \mathcal{K}[U_2]$ and $\tilde{U}_3 \in \mathcal{K}[U_3]$. By Shevitz and Paden (1994), it holds that $\dot{V} \in \dot{\tilde{V}}$. Hence, it holds that $\dot{V} \le -\alpha^2(\beta - \kappa_1)\|x\|_2^2 - (\beta - \kappa_2)\|y\|_2^2 + \kappa_3\sqrt{2Np}\|\xi\|_2\|s\|_\infty + \kappa_4\|s\|_\infty - (\gamma - \pi)\left\|[B^T(t) \otimes I_p](\alpha x + y)\right\|_1 \le -c_m\|\xi\|_2^2 + c_M\|\xi\|_2\|s\|_\infty + \kappa_4\|s\|_\infty$, where $c_m = \min\{\alpha^2(\beta - \kappa_1), \beta - \kappa_2\}$ and $c_M = \kappa_3\sqrt{2Np}$. Note that $c_M\|\xi\|_2\|s\|_\infty \le \eta c_m\|\xi\|_2^2 + \frac{c_M^2}{4\eta c_m}\|s\|_\infty^2 = \eta c_m\|\xi\|_2^2 + \kappa_5\|s\|_\infty^2$, where $\eta \in (0,1)$ and $\kappa_5 = \frac{c_M^2}{4\eta c_m}$. It then follows that $\dot{V} \le -(1-\eta)c_m\|\xi\|_2^2 + \kappa_5\|s\|_\infty^2 + \kappa_4\|s\|_\infty \le -\kappa_6 V + 2\rho(\|s\|_\infty)$, where $\kappa_6 = \frac{(1-\eta)c_m}{\lambda_2(P)}$ and $\rho(r) = \max\{\kappa_5 r^2, \kappa_4 r\}$ is a class $\mathcal{K}$ function (Khalil, 2002, p. 144). Such inequalities hold for any $\mathcal{G}_k$, $k = 1, \ldots, G$, and hence, $V(\xi)$ is indeed a common ISS-Lyapunov triple. For any $t \in [t_0, \infty)$ with $t_0 \ge 0$, it holds that $V[\xi(t)] \le e^{-\kappa_6(t-t_0)}V[\xi(t_0)] + \frac{2}{\kappa_6}\sup_{\tau\in[t_0,t]}\rho(\|s(\tau)\|_\infty)$. If $s_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$, then $\sup_{\tau\in[0,\infty)}\rho(\|s(\tau)\|_\infty) < \infty$, then $V \in \mathcal{L}_\infty^1$, which implies that $x \in \mathcal{L}_\infty^{Np}$ and $y \in \mathcal{L}_\infty^{Np}$.

Note that $\sup_{\tau\in[t_0,t]}\rho(\|s(\tau)\|_\infty) = \rho(\sup_{\tau\in[t_0,t]}\|s(\tau)\|_\infty)$. Given any $\varpi > 0$, there is $\varpi_1 > 0$ such that $\rho(\varpi_1) \le \frac{\varpi\kappa_6}{4}$. If $\lim_{t\to\infty} s_i = \mathbf{0}_p$ $\forall i \in \mathcal{V}$, then there exists $T_1 > 0$ such that $\sup_{\tau\in[T_1,\infty)}\|s(\tau)\|_\infty \le \varpi_1$. Let $t_0 \ge T_1$. For $t \ge t_0$, it follows that $V[\xi(t)] \le e^{-\kappa_6(t-t_0)}V[\xi(t_0)] + \frac{\varpi}{2}$. There exists $T_2 \ge t_0$ such that $e^{-\kappa_6(t-t_0)}V[\xi(t_0)] \le \frac{\varpi}{2}$ $\forall t \ge T_2$. Then, it holds that $V[\xi(t)] \le \varpi$ $\forall t \ge \max\{T_1, T_2\}$. This shows that $\lim_{t\to\infty}\xi = \mathbf{0}_{2Np}$, i.e., $\lim_{t\to\infty} x = \lim_{t\to\infty} y = \mathbf{0}_{Np}$. By the definitions of x and y, the proof of Statement (i) is completed.

Proof of Statement (ii). Define $\chi = \sum_{j=1}^N \nabla f_j(q_j, t)$ and $\psi = \sum_{j=1}^N[v_j + F_j(q_j, t)]$. One has $\dot{\chi} = -\chi + H(t)\psi + \sum_{j=1}^N H(t)s_j$ and $\dot{\psi} = -\alpha\psi - \alpha\sum_{j=1}^N s_j$. Note that $\dot{\psi} = -\alpha\psi$ is a stable linear time-invariant (LTI) system. Then, from the properties of input-to-state stability, it holds that $\psi \in \mathcal{L}_\infty^p$ if $s_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$ and $\lim_{t\to\infty}\psi = \mathbf{0}_p$ if $\lim_{t\to\infty} s_i = \mathbf{0}_p$ $\forall i \in \mathcal{V}$. Note also that $\dot{\chi} = -\chi$ is an exponentially stable LTI system. Then, it follows from Assumption 3 that $\chi \in \mathcal{L}_\infty^p$ if $s_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$ and $\lim_{t\to\infty}\chi = \mathbf{0}_p$ if $\lim_{t\to\infty} s_i = \mathbf{0}$ $\forall i \in \mathcal{V}$.

Proof of Statement (I). Since $\sum_{j=1}^N f_j(q, t)$ is strongly convex in q, it follows that $N\underline{m}\|\bar{q} - q^*\|_2^2 \le [\sum_{j=1}^N \nabla f_j(\bar{q}, t) - \sum_{j=1}^N \nabla f_j(q^*, t)]^T(\bar{q} - q^*) = [\sum_{j=1}^N \nabla f_j(\bar{q}, t) - \chi]^T(\bar{q} - q^*) + [\chi - \sum_{j=1}^N \nabla f_j(q^*, t)]^T(\bar{q} - q^*)$, where $\bar{q} = \frac{1}{N}\sum_{j=1}^N q_j$ Then, it holds that $N\underline{m}\|\bar{q} - q^*\|_2 \le \left\|\sum_{j=1}^N \nabla f_j(\bar{q}, t) - \chi\right\|_2 + \left\|\chi - \sum_{j=1}^N \nabla f_j(q^*, t)\right\|_2$. From Assumptions 2 and 3 that $\|\sum_{j=1}^N \nabla f_j(\bar{q}, t) - \chi\|_2 \le \sum_{j=1}^N\|\nabla f_j(\bar{q}, t) - \nabla f_j(q_j, t)\|_2 \le \sum_{j=1}^N \bar{H}\left\|q_j - \bar{q}\right\|_2$. Since $\sum_{j=1}^N f_j(q^*, t) = \mathbf{0}_p$, then $N\underline{m}\|\bar{q} - q^*\|_2 \le \left\|\chi\right\|_2 + \sum_{j=1}^N \bar{H}\left\|q_j - \bar{q}\right\|_2$. By Statements (i) and (ii), it holds that $q_i - \bar{q} \in \mathcal{L}_\infty^p$ and $\chi \in \mathcal{L}_\infty^p$ if $s_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$. Then it follows that $\|\bar{q} - q^*\|_2 < \infty$, and hence $q_i - q^* \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.

Proof of Statement (II). Since $\lim_{t\to\infty} s_i = \mathbf{0}_p$ $\forall i \in \mathcal{V}$, it follows from Statement (ii) that $\lim_{t\to\infty}\chi = \mathbf{0}_p$. Then, from the analysis in the proof of Statement (I), it holds that $\lim_{t\to\infty} N\underline{m}\|\bar{q}(t) - q^*(t)\|_2 \le \lim_{t\to\infty}\sum_{j=1}^N \bar{H}\|q_j(t) - \bar{q}(t)\|_2 + \lim_{t\to\infty}\left\|\chi\right\|_2 = 0$, which implies that $\lim_{t\to\infty}\bar{q}(t) = q^*(t)$. By Statement (i), one has $\lim_{t\to\infty} q_i(t) = \bar{q}(t)$, which implies Statement (II). ■

With Proposition 1 at hand, the convergence of the distributed optimization algorithm (7)–(8) can be established by the following theorem.

Theorem 1. *Suppose that Assumptions 1–3 hold, and let α, β, and γ be chosen as in (11). Using the controller (7)–(8) with v_i and $\dot{v}_i$ given by the reference system (3) for the networked Lagrangian system (1) solves the distributed time-varying optimization problem, that is,* $\lim_{t\to\infty}[q_i(t) - q^*(t)] = \mathbf{0}_p$.

Proof. Substituting (7) into (1) and using Property 3 and (8) yield that

$$M_i(q_i)\dot{s}_i + C_i(q_i, \dot{q}_i)s_i = -K_is_i + Y_i(q_i, \dot{q}_i, v_i, \dot{v}_i)\widetilde{\vartheta}_i, \tag{18}$$

$$\dot{\widetilde{\vartheta}}_i = -\Gamma_iY_i^T(q_i, \dot{q}_i, v_i, \dot{v}_i)s_i, \tag{19}$$

where $\widetilde{\vartheta}_i = \hat{\vartheta}_i - \vartheta_i$. For any $i \in \mathcal{V}$, define a Lyapunov function candidate $W_i = \frac{1}{2}s_i^TM_i(q_i)s_i + \frac{1}{2}\widetilde{\vartheta}_i^T\Gamma_i^{-1}\widetilde{\vartheta}_i$. The derivative of W_i along the solution of (18)–(19) is given as $\dot{W}_i = s_i^T M_i(q_i)\dot{s}_i + \frac{1}{2}s_i^T \dot{M}_i(q_i)s_i + \widetilde{\vartheta}_i^T \Gamma_i^{-1}\dot{\widetilde{\vartheta}}_i = -s_i^T K_i s_i - s_i^T C_i(q_i, \dot{q}_i)s_i + \frac{1}{2}s_i^T \dot{M}_i(q_i)s_i = -s_i^T K_i s_i \leq -\lambda_1(K_i)\|s_i\|_2^2 \leq 0$, where (18)–(19) and Property 2 have been used to obtain the second and last equalities, respectively. It then holds that $W_i \in \mathcal{L}_\infty^1\ \forall i \in \mathcal{V}$, which implies that $s_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$, and that $\widetilde{\vartheta}_i \in \mathcal{L}_\infty^m$ and hence $\hat{\vartheta}_i \in \mathcal{L}_\infty^m\ \forall i \in \mathcal{V}$. Integrating over $[0, t]$ on both sides yields that $\lambda_1(K_i)\int_0^t \|s_i(\tau)\|_2^2\, d\tau \leq W_i(0) - W_i(t) < \infty\ \forall t \geq 0$, which implies that $s_i \in \mathcal{L}_2^p\ \forall i \in \mathcal{V}$. Hence, it holds that $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ and $\hat{\vartheta}_i \in \mathcal{L}_\infty^m\ \forall i \in \mathcal{V}$.

Using (6), we can rewrite (3) as the system (9)–(10). Since $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p\ \forall i \in \mathcal{V}$, it follows from Statement (I) of Proposition 1 that $q_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. It then follows from (5) and Assumption 3 that $F_i(q_i, t) \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. Recall from Statement (ii) that $\sum_{j=1}^N [v_j + F_j(q_j, t)] \in \mathcal{L}_\infty^p$, then $\sum_{j=1}^N v_j \in \mathcal{L}_\infty^p$. Recall from Statement (i) that $v_i - \frac{1}{N}\sum_{j=1}^N v_j \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. It then holds that $v_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. From (9), it then holds that $\dot{q}_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. By Assumption 3, it holds that $\varphi_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. From (10), it holds that $\dot{v}_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. Then, by using (18) and Property 1, it holds that $\dot{s}_i \in \mathcal{L}_\infty^p\ \forall i \in \mathcal{V}$. It can thus be shown that $s_i\ \forall i \in \mathcal{V}$ are uniformly continuous. Since $s_i \in \mathcal{L}_2^p\ \forall i \in \mathcal{V}$, then using Barbalat's lemma (Khalil, 2002, p. 323) yields that $\lim_{t\to\infty} s_i(t) = \mathbf{0}_p\ \forall i \in \mathcal{V}$. The proof is completed by following Statement (II). ■

4.3. Distributed time-varying optimization algorithm removing chattering

Note that the control torques may involve the chattering issue in practice, since $\dot{v}_i$ in (3) introduces the signum function in the controller (7). This subsection focuses on the distributed time-varying optimization algorithm that generates continuous control inputs, and hence removes the chattering issue while application in practice. Inspired by Burton and Zinober (1986), we introduce a differentiable function $h(\cdot)$ to approximate and replace the signum function, which results in continuous control torques for the Lagrangian agents and removes the effects of chattering. The function $h(\cdot)$ is given by

$$h(r) = \frac{r}{\|r\|_2 + \varepsilon}, \tag{20}$$

where $r \in \mathbb{R}^p$ and ε is a positive constant. After replacing the signum function in (3) with the function (20), the reference system for agent $i \in \mathcal{V}$ becomes

$$\dot{v}_i = -\alpha\dot{q}_i - \gamma \sum_{j\in\mathcal{N}_i(t)} h[\alpha(q_i - q_j) + \dot{q}_i - \dot{q}_j] + \varphi_i, \tag{21}$$

where φ_i is defined in (4). As in Section 4.2, a similar preliminary lemma is presented.

Lemma 2. *Let $\gamma \in \mathbb{R}_+$, and $z_i, s_i \in \mathbb{R}^p$, $i \in \mathcal{V}$. It holds that $-\gamma\sum_{(i,j)\in\mathcal{E}(t)}(z_i - z_j)^T h(z_i - z_j + s_i - s_j) \leq -\gamma\sum_{(i,j)\in\mathcal{E}(t)}(\|z_i - z_j\|_2 - 2\|s_i - s_j\|_2 - \varepsilon)$.*

Proof. From the definition of $h(\cdot)$ in (20), it follows that $-\gamma\sum_{(i,j)\in\mathcal{E}(t)}(z_i - z_j)^T h(z_i - z_j + s_i - s_j) = -\gamma\sum_{(i,j)\in\mathcal{E}(t)}\|z_i - z_j + s_i - s_j\|_2 + \gamma\sum_{(i,j)\in\mathcal{E}(t)}\varepsilon + \gamma\sum_{(i,j)\in\mathcal{E}(t)}\frac{(s_i - s_j)^T(z_i - z_j + s_i - s_j) - \varepsilon^2}{\|z_i - z_j + s_i - s_j\|_2 + \varepsilon}$. It follows that $(s_i - s_j)^T(z_i - z_j + s_i - s_j) \leq \|s_i - s_j\|_2\|z_i - z_j + s_i - s_j\|_2$. Then, $-\gamma\sum_{(i,j)\in\mathcal{E}(t)}(z_i - z_j)^T h(z_i - z_j + s_i - s_j) \leq -\gamma\sum_{(i,j)\in\mathcal{E}(t)}\|z_i - z_j + s_i - s_j\|_2 + \gamma\sum_{(i,j)\in\mathcal{E}(t)}\|s_i - s_j\|_2 - \gamma\sum_{(i,j)\in\mathcal{E}(t)}\frac{\varepsilon\|s_i - s_j\|_2 + \varepsilon^2}{\|z_i - z_j + s_i - s_j\|_2 + \varepsilon} + \gamma\sum_{(i,j)\in\mathcal{E}(t)}\varepsilon$. Define $\mathcal{E}_+(t) = \{(i,j) \in \mathcal{E}(t)|\ \|z_i - z_j\|_2 \geq \|s_i - s_j\|_2\}$ and $\mathcal{E}_-(t) = \{(i,j) \in \mathcal{E}(t)|\ \|z_i - z_j\|_2 < \|s_i - s_j\|_2\}$. Since $\frac{\varepsilon\|s_i - s_j\|_2 + \varepsilon^2}{\|z_i - z_j + s_i - s_j\|_2 + \varepsilon} \geq 0\ \forall (i,j) \in \mathcal{E}(t)$, it then holds that $-\gamma\sum_{(i,j)\in\mathcal{E}(t)}(z_i - z_j)^T h(z_i - z_j + s_i - s_j) \leq -\gamma\sum_{(i,j)\in\mathcal{E}(t)}|\ \|z_i - z_j\|_2 - \|s_i - s_j\|_2\ | + \gamma\sum_{(i,j)\in\mathcal{E}(t)}(\varepsilon + \beta\|s_i - s_j\|_2) \leq -\gamma\sum_{(i,j)\in\mathcal{E}_+(t)}\|z_i - z_j\|_2 + \gamma\sum_{(i,j)\in\mathcal{E}_+(t)}\|s_i - s_j\|_2 - \gamma\sum_{(i,j)\in\mathcal{E}_-(t)}\|s_i - s_j\|_2 + \gamma\sum_{(i,j)\in\mathcal{E}_-(t)}\|z_i - z_j\|_2 + \gamma\sum_{(i,j)\in\mathcal{E}(t)}(\varepsilon + \|s_i - s_j\|_2)$. The statement follows from the definition of $\mathcal{E}_-(t)$. ■

Theorem 2. *Suppose that Assumptions 1–3 hold, and let α and β be chosen as in (11), and γ be chosen such that $\gamma > \pi\sqrt{p}$, where π is given in (14). Using the controller (7)–(8) with $\dot{v}_i$ defined in (21) for the networked Lagrangian system (1) solves the distributed time-varying optimization problem with bounded optimum-tracking errors, and the ultimate optimum-tracking errors satisfy $\lim_{t\to\infty}\|q_i(t) - q^*(t)\|_2 \leq (\frac{\bar{H}}{\underline{m}} + 1)\left[\frac{\varepsilon p N(N-1)\lambda_2(P)}{(1-\eta)c_m\lambda_1(P)}\right]^{1/2}\ \forall i \in \mathcal{V}$, where $c_m = \min\{\alpha^2(\beta - \kappa_1), \beta - \kappa_2\}$, $\eta \in (0, 1)$, and κ_1, κ_2, and P are given in (12), (13), and (17), respectively.*

Proof. The same notational symbols are used as in the proofs of Proposition 1 and Theorem 1. Using Lemma 2 and following the analysis in the proof of Proposition 1 imply that Statements (i) and (ii) still hold. In addition, it holds that $\dot{V} \leq -\kappa_6 V + 2\rho(\|s\|_\infty) + \bar{\varepsilon}$, where $\bar{\varepsilon} = \frac{1}{2}\varepsilon\gamma N(N-1)$ and $\rho(\cdot)$ is given in Proposition 1. This implies that $V[\xi(t)] \leq e^{-\kappa_6(t-t_0)}V[\xi(t_0)] + \frac{2}{\kappa_6}\rho(\sup_{\tau\in[t_0,t]}\|s(\tau)\|_\infty) + \frac{\bar{\varepsilon}}{\kappa_6}$ for $t \geq t_0$ and $t_0 \geq 0$. Given any $\varpi > 0$, there exists T such that $V[\xi(t)] \leq \varpi + \frac{\bar{\varepsilon}}{\kappa_6}\ \forall t \geq T$. Then, it follows that $\lim_{t\to\infty} V[\xi(t)] = \frac{\bar{\varepsilon}}{\kappa_6}$, which implies that ξ converges to the set $\{\xi|\ \|\xi\|_2 \leq [\frac{\bar{\varepsilon}}{\kappa_6\lambda_1(P)}]^{1/2}\}$. That is, $\lim_{t\to\infty}\|q_i(t) - \bar{q}(t)\|_\infty \leq [\frac{\bar{\varepsilon}}{\kappa_6\lambda_1(P)}]^{1/2}$, where $\bar{q} = \frac{1}{N}\sum_{j=1}^N q_j$. Recall that $\lim_{t\to\infty}\sum_{j=1}^N \nabla f_i(q_i, t) = \mathbf{0}_p$ (since $\lim_{t\to\infty} s_i(t) = \mathbf{0}_p\ \forall i \in \mathcal{V}$), and by Proof of Statement (II), one has $\lim_{t\to\infty}\|\bar{q}(t) - q^*(t)\|_2 \leq \frac{\bar{H}}{\underline{m}}[\frac{\bar{\varepsilon}p}{\kappa_6\lambda_1(P)}]^{1/2}$. It follows that $\lim_{t\to\infty}\|q_i(t) - q^*(t)\|_2 \leq \lim_{t\to\infty}\|q_i(t) - \bar{q}(t)\|_2 + \lim_{t\to\infty}\|\bar{q}(t) - q^*(t)\|_2 \leq (\frac{\bar{H}}{\underline{m}} + 1)[\frac{\bar{\varepsilon}p}{\kappa_6\lambda_1(P)}]^{1/2}$, which completes the proof. ■

From Theorem 2, the ultimate optimum-tracking errors are proportional to the value of ε, which controls the approximation errors of $h(\cdot)$. The optimum-tracking errors can be made arbitrary small by selecting sufficiently small ε. Moreover, one can set $\varepsilon = \epsilon_1 e^{-\epsilon_2 t}$ or some positive functions that are convergent to zero, and then $h(\cdot)$ becomes a time-varying approximation function, and it can be shown that $\lim_{t\to\infty}[q_i(t) - q^*(t)] = \mathbf{0}_p\ \forall i \in \mathcal{V}$. There are other continuous functions that can be used to approximate the signum function, such as $\mathrm{sat}(\frac{r}{\varepsilon})$ and $\tanh(\frac{r}{\varepsilon})$, where $\varepsilon \in \mathbb{R}_+$. The convergence can be proved by following a similar line of analysis in Lemma 2, Proposition 1, and Theorem 2, which is omitted. The chattering in control systems is caused by the discontinuity in the control design. By the approximation idea, the resulting control torque (7) is continuous, and hence, the chattering is removed. See the comparison of Figs. 3 and 5 in Section 5 for illustration. In addition, the function $h(r)$ is approaching to the discontinuous signum function when ε is approaching to zero, which in term indicates that the effect of removing the chattering is decreased. Therefore, the value of ε can be chosen to trade off the requirement of ensuring certain optimum-tracking performance with that of ensuring continuous control torques.

Remark 2. The method of approximating the signum function using (20) has been applied in Rahili and Ren (2017) to remove the chattering. However, this work considers the distributed time-varying optimization problem for networked Lagrangian systems, and the proposed algorithms can be implemented by using on-board sensors taking physical state measurements. The Lagrangian dynamics are more complex compared with single- and double-integrator agents considered in Rahili and Ren (2017). Moreover, the complexities of the problem of

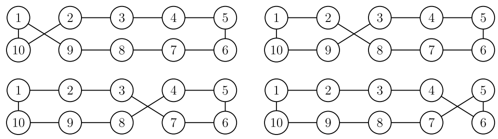

Fig. 1. Graph 1 (top left); Graph 2 (top right); Graph 3 (bottom left); Graph 4 (bottom right).

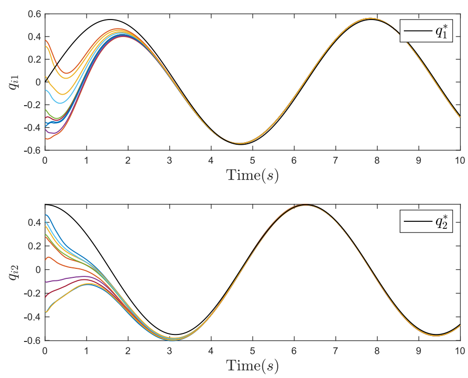

Fig. 2. The position trajectories of Lagrangian agents (1) using the distributed algorithms in Section 4.1.

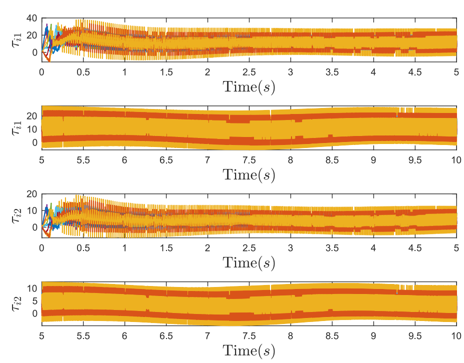

Fig. 3. The control torques of Lagrangian agents (1) using the distributed algorithms in Section 4.1.

interest and agents' dynamics pose challenges in the convergence analysis. For instance, as an intermediate step in the convergence analysis, two statements are established for the networked system (9)–(10), which can be seen as networked second-order systems perturbed by disturbances s_i, $i \in \mathcal{V}$. Hence, they are different from the disturbance-free double-integrator model considered in Rahili and Ren (2017), and there are significant technical challenges.

Remark 3. The structure of the proposed distributed algorithms for networked Lagrangian agents are partially inspired by Wang et al. (2020), where the consensus and/or leader-following of networked Lagrangian systems are investigated. In this paper, the distributed time-varying optimization problem is addressed, which are more complex and challenging and include the consensus and leader-following as special cases. Moreover, while dealing with the distributed time-varying optimization for networked Lagrangian agents, the analysis is quite different from the work of Wang et al. (2020). The nonlinear functions, such as the signum function and the one defined in (20), are used to constructing $\dot{v}_i$, which forms a perturbed closed-loop networked double-integrator systems with s_i as disturbance in the model and inside the nonlinear functions (see (9)–(10) for an example). This paper provide rigorous analysis on the performance of the perturbed systems under bounded and convergent disturbances. In addition, when considering distributed time-varying optimization problem, additional analysis steps are required, see the optimum-tracking steps in the proof of Proposition 1 and Theorem 1 for instance.

Remark 4. As shown in Theorems 1 and 2, the lower bounds of the design parameters (e.g., γ, α, and β) depend on some global information, such as the bounds on the cost functions and the graph. It is worth mentioning that these design parameters are constants, and can be determined off-line. Once they are chosen, one can embed them into each agent and implement the proposed algorithms by using only local information, which implies that the proposed algorithm can be implemented in a distributed way. In addition, one can be conservative and select large enough values for these parameters. Also, one can use some existing algorithms (Giannini et al., 2016; Varagnolo et al., 2013) to estimate the bounds about the cost functions and the graph, and then choose appropriate values for the parameters based on the estimated bounds.

5. Illustrative examples

In this section, we provide examples to illustrate the results in this paper. We consider a group of ten planar manipulators with two revolute joints (Spong et al., 2006, pp. 259–262) ($\mathcal{V} = \{1, \dots, 10\}$). The interaction among these ten agents is characterized as the graph in Fig. 1. The interaction graph starts from Graph 1. Then after every 0.25 s, it switches to the next graph and the process repeats. Each agent $i \in \mathcal{V}$ has a local cost function $f_i(\zeta, t) = [\zeta_1 - 0.1i\sin(t)]^2 + [\zeta_2 - 0.1i\cos(t)]^2$ with $\zeta = [\zeta_1, \zeta_2]^T$, and denote by $q^*(t) = [q_1^*(t), q_2^*(t)]^T$ the optimal trajectory that minimizes $\sum_{i=1}^{10} f_i(\zeta, t)$. In the following algorithm validations, the initial values, $q_i(0) = [q_{i1}(0), q_{i2}(0)]^T$, $v_i(0)$, and $\vartheta_i(0)$, are chosen as follows: for any $i \in \mathcal{V}$ and any $j \in \{1, 2\}$, $q_{ij}(0)$ and $\dot{q}_{ij}(0)$ are generated randomly from the ranges $[-0.5, 0.5]$, and $v_i(0) = \dot{q}_i(0) + 0.1\mathbf{1}_2$, and $\vartheta_i(0) = \mathbf{0}_5$. We first validate the distributed time-varying optimization algorithm (7)–(8) with $\dot{v}_i$ defined in (3). In this simulation, we select $\Gamma_i = 0.09I_5$ and $K_i = 14I_2$ for any $i \in \mathcal{V}$, $\alpha = 1.5$, and $\gamma = 8$. The position trajectories and control torques are presented in Figs. 2 and 3, respectively. From Fig. 2, it shows that all the agents track the optimal trajectory. It can be seen from Fig. 3 that there exists chattering. To remove the chattering, we apply and validate the distributed algorithm in Section 4.3, and set $\gamma = 13$ and $\varepsilon = 0.4$. The position trajectories and control torques are presented in Figs. 4 and 5, respectively. From Fig. 4, it shows that all the agents track the optimal trajectory with bounded errors. Unlike the discontinuous and frequently switching control torques in Fig. 3, it can be seen from Fig. 5 that the control torques are continuous except for the graph switching time instants.

6. Conclusion

This paper investigated the distributed time-varying optimization of networked Lagrangian systems with parametric uncertainties. The proposed algorithms can be implemented by using only on-board sensors and drive the agents to track the optimal trajectory. First, a base algorithm has been designed to achieve zero optimum-tracking error. Built on the base algorithm,

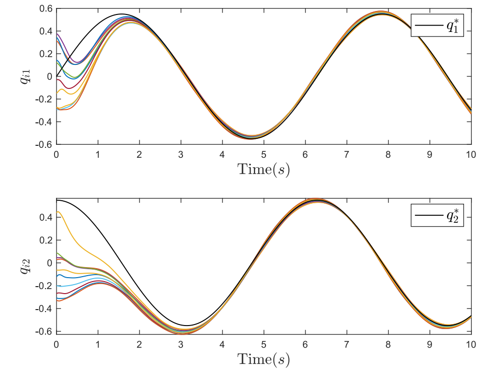

Fig. 4. The position trajectories of Lagrangian agents (1) using the distributed algorithms in Section 4.3.

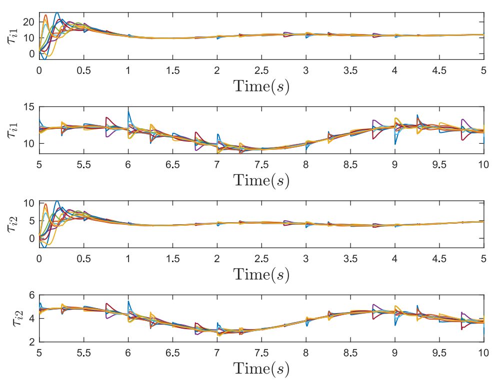

Fig. 5. The control torques of Lagrangian agents (1) using the distributed algorithms in Section 4.3.

a continuous variant has been developed, which is capable of generating continuous control torques for the networked Lagrangian systems and hence reducing the chattering. In the end, numerical examples have been provided to illustrate the obtained results.

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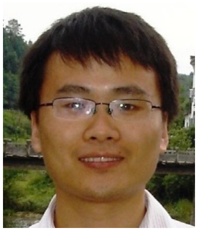

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