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Instructors' resource use in multivariable calculus: a case study

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ABSTRACT

This case study examines the relations between the instructional resources, the instructors' pedagogy and the multivariable calculus content. The resources used included digital resources, 3D-printed models and other physical resources. Pedagogy practices included both collaborative and individual in-class learning activities, instructor demonstrations and homework. Content included topics that typically appear in multivariable calculus. The 35 specific topics were grouped into the general areas of introductory/foundational ideas, differential calculus, integral calculus and vector calculus. We also consider the instructor's assessment of the activities they use. Data were obtained from three instructors working at three different types of institutions in the United States. The study finds that instructors use digital resources primarily for demonstrations; 3D-printed materials are more likely to be used in collaborative vs. individual learning opportunities; instructors design more activities for the introductory/foundational and differential calculus topic areas of the course; the integral calculus, and to an even greater degree, the vector calculus portions of the course are lacking in resource-supported activities and instructors sharing materials could be beneficial to others who are just beginning to incorporate the use of resources in their multivariable calculus teaching. The study suggests instructional needs and informs about instructor practices for multivariable calculus topics.

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KEYWORDS
Multivariable calculus; digital resources; TPCK; 3D-printed models

1. Introduction

It is common for a quantity in the physical world to depend on two or more quantities. This makes the calculus of multivariable functions an indispensable tool for modelling physical phenomena and, hence, an essential subject for students of science, technology, engineering, mathematics and other disciplines. However, learning multivariable calculus (MVC) is

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often a challenge for students. In their previous studies of algebra, pre-calculus and single-variable functions in calculus, students are mostly restricted to examining phenomena that can be modelled in a two-dimensional setting. Then, when progressing to MVC, students suddenly find themselves in a situation that requires representing and interpreting ideas graphically in three-dimensional (3D) space. This change in dimensions turns the previous 'close' two-dimensional (2D) representations into 'distant' 3D representations (Parzysz, 1988). This is a new situation for students, as they must now generalise from the 2D context of their previous studies to the 3D context required by two-variable functions. However, this is not a direct generalisation as multivariable functions have their own peculiarities (Jones & Dorko, 2015; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). In this regard, visualisation is a vital and fundamental source of ideas to sustain the conceptual understanding of mathematics (Tall, 1991; Zimmermann, 1991). Visualisation plays a pivotal role in facilitating the interrelation of different representations (Arcavi, 2003; Gutiérrez, 1996; Presmeg, 1997) and is necessary to attain the synergism of representations that corresponds with understanding mathematical objects (Duval, 2006) and particularly two-variable functions (Habre, 2001; Yerushalmy, 1997). However, many students favour algebraic methods and may avoid visualisation (Eisenberg & Dreyfus, 1991; Haciomeroglu et al., 2010), which makes it critical to study how best to leverage resources to help students visualise MVC ideas. While some work has been done to study the use of digital resources in MVC (e.g. Alves, 2014) and the use of other physical resources (e.g. Wangberg & Johnson, 2013), researchers have made calls for more work in this area (Martínez-Planell & Trigueros, 2021).

This study aims to contribute to the gap in the literature regarding the use of resources in the teaching of MVC. By resources, we mean digital technology, digitally generated technology (e.g. 3D-printed models) or other physical artifacts. The case study considers three instructors and how they incorporate resources in their teaching, considering the interplay between their choice of resource, the MVC topics for which they incorporate resources, the pedagogical strategies they use, and how they rate the instructional activities they use in their classrooms. For this case study, we consider data from three MVC instructors who reported using the digital resource CalcPlot3D, 3D-printed models and other physical resources in their MVC instruction.

We use the following research questions to understand the relation between MVC content, resource use, pedagogical practices, and instructor self-assessment of activity quality.

- Which resources used by the participating instructors favour the promotion of student engagement and active learning practices?
- Which topic areas of MVC seem more amenable to resource adoption by the participating instructors?
- Which topic areas of MVC were the focus of student-centred activities implemented by the participating instructors?
- Which topic areas of MVC have activities that the participating instructors assess as good or excellent quality?

2. Literature review

The role of digital and non-digital resources has been a research topic in the mathematics education community for a long time. Adler (2000) considered the type of resources and how to leverage them to improve student learning, emphasising that the effectiveness of resources lies in how they are used in the classroom context. We consider how resources are utilised in the MVC classroom by keeping track of the instructors' pedagogical practices. Lagrange et al. (2003) reported that much research literature underscored the potential of digital resources for teaching and learning mathematics but that they were frequently only slowly integrated into educational institutions. This study aims to investigate how three MVC instructors integrate different types of resources, including digital resources, into their MVC teaching. Hegedus et al. (2017) stressed that digital resources should provide collaborative and active learning opportunities. They observed that digital tools may support the visualisation of mathematical concepts and that this, in turn, can help students obtain a deeper understanding. In our study, we study the integration of technology while keeping track of instructor's pedagogical practices. In so doing, we highlight examples of student-centred learning opportunities, particularly when the learning opportunities are meant to help facilitate student visualisation of MVC concepts and relationships.

The survey article by Martínez-Planell and Trigueros (2021) suggests an increasing number of studies dealing with the teaching and learning of MVC. This includes studies of definitions, geometric understandings, and introductory topics of MVC (Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012, 2019; Trigueros & Martínez-Planell, 2010; Yerushalmy, 1997); the teaching and learning of multivariable differential calculus (Borji et al., 2024a, 2025; Martínez-Planell et al., 2015, 2017; Moreno-Arotzena et al., 2021; Tall, 1992; Weber, 2015) and the integration of multivariable functions (Bašić & Milin Šipuš, 2022; Borji et al., 2024b; Jones & Dorko, 2015; Martínez-Planell & Trigueros, 2020; McGee & Martínez-Planell, 2014). The content for this study is divided into these three categories and vector calculus, about which there is very little research.

Despite a growing body of research on MVC, there is not much research on the use of resources in MVC (Trigueros et al., 2023). There are a few studies that report improved learning with the use of physical manipulatives, including tangible surfaces and 3D-printed models (e.g. McGee et al., 2012, 2015; Paul, 2018; Sherer et al., 2013; Wangberg, 2020; Wangberg & Johnson, 2013). A few articles consider the use of digital technology like GeoGebra, Maple, and Mathematica as an aid in visualisation, to foment student discussion, and to help bridge the gap between single and multivariable calculus (e.g. Alves, 2012; Ingar, 2014). Some of the studies using digital technologies focus on specific topics or areas of MVC; for example, Trigueros et al. (2023) considered basic ideas of two-variable functions, Rojas Flórez et al. (2019) studied directional derivative, Ingar and Silva (2019) investigated extrema of two-variable functions, Alves (2014) also considered critical points of a two-variable function focusing on motivating the second derivative test by using digital resources to compare the graphs of second-order Taylor polynomials, the graph of the function, and its level curves in neighbourhoods of the critical points; Henriques (2006) looked at multivariable integration, particularly for drawing complex regions and finding limits of integration, and VanDieren et al. (2020) considered the vector cross product. Some research examines the effects of using digital resources for an entire MVC course (e.g. Both Carvalho & Pereira, 2011; Gemechu et al., 2018; Habre, 2001). Some more recent articles consider virtual or augmented reality (e.g. Cheong et al., 2023; Jones et al., 2022; Kang et al., 2020; Karabina et al., 2023). Yet, overall, there is much to learn about resources, especially digital resources, and how they can support the teaching and learning of MVC. Further, all these studies primarily focus on student learning. Only one published study of which we know, considers the use of resources for teaching MVC from the point of view of the instructors' needs (Moore-Russo et al., 2024). In this study, the authors found that the instructors' beliefs in the importance of visualisation led them to incorporate resources and that ease-of-use, affordability, and accessibility played important roles in resource adoption choices. However, this research did not delve into particular MVC topics, the student-centredness of designed activities, and the interrelation between content, resources and pedagogy. The present research extends the work of that previous study, including data about the relationship between content, pedagogy, and resources for teaching MVC.

3. Theoretical framework

The Technological Pedagogical Content Knowledge (TPCK) model (Koehler & Mishra, 2009; Mishra & Koehler, 2006) serves as a theoretical framework for this study. This framework of instructor knowledge builds on the Pedagogical Content Knowledge model initially introduced by Shulman (1986) while also considering the role of technology in instructional decisions and actions. Koehler and Mishra (2009) situate the flexible knowledge needed for teaching as a 'complex interaction among three bodies of knowledge: content, pedagogy, and technology' (p. 60). We adopt Mishra and Koehler's (2006) terms 'pedagogy' and 'content' to mean respectively the 'processes and practices or methods of teaching and learning and how [they] encompass educational purposes, values, and aims (p. 1026)' and 'the actual subject matter that is to be learned or taught (p. 1025)' respectively. We also adopt their TPCK stance on technologies as being both more traditional items (e.g. pencils and chalkboards) and what they refer to as newer technologies, including

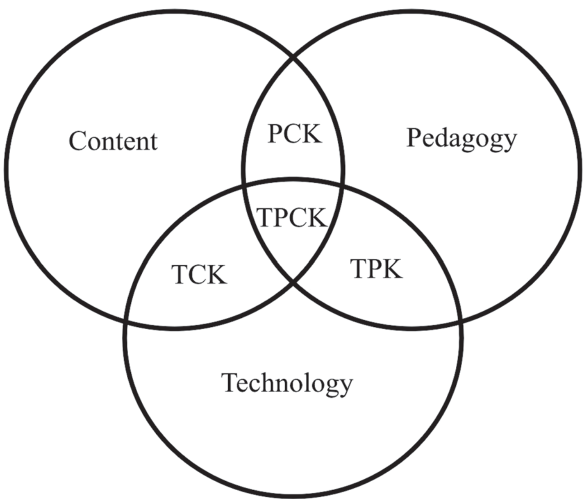

Figure 1. TPCK: Technological pedagogical content knowledge.

digital resources that include computers, tablets and mobile devices, as well as the digital applications they use.

Mishra and Koehler (2006) do not treat technology, pedagogy, and content as mutually exclusive domains. Instead, there is an interplay between the three, as noted in Figure 1, by the overlapping intersections in the TPCK Venn Diagram. In fact, certain resources are more likely to be employed for instructional tasks due to the affordances and capabilities of the resources (Koehler & Mishra, 2008), the fit between the characteristics of the resources and the characteristics of the task at hand (Goodhue & Thompson, 1995) and the instructor's perceptions of how useful and easy-to-use the resources are (Davis, 1989).

4. Methodology

We used a convenience sample of three MVC professors who employ the CalcPlot3D digital platform, digitally generated 3D-printed models and other physical artifacts to aid their MVC instruction. The professors taught at three different institutions in three different states in the USA. Professor A has taught the course over 25 times, professor H over 15 times and professor T over 15 times in their respective academic careers. Professor A teaches at a public two-year community college with over 8000 students in the Northeast, Professor H teaches at a small selective undergraduate four-year college with about 4000 students in the Midwest and Professor T teaches at a large public comprehensive research university with over 35,000 students in the East. The three participated for a second year in a project involving CalcPlot3D and 3D-printed models. There is bias due to participant selection since we specifically chose participants who were using a digital resource, CalcPlot3D, but also leveraging additional resources, such as 3D surfaces, to improve their MVC instruction. The data came directly from the professors through surveys and informal interviews. Follow-up questions posed to each professor were used to clarify their responses when needed, and all three were asked to review the final report of the results and this paper to ensure their thoughts and actions were represented accurately.

The instructors' knowledge was not explicitly studied. Rather, it was assumed that due to their experience in teaching multivariable calculus, all three had a rather robust content knowledge. Similarly, all three instructors have led educational workshops on the use of technology and 3D-printed materials. So, not only were the instructors able to adapt different types of resources to model various multivariable calculus topics taught; they have taught others how to do so. This speaks to their technological knowledge. As part of the same workshops, a section on student engagement and visualisation has been emphasised, which provides evidence of their pedagogical knowledge. This study has a basic assumption that the synergy of the three types of knowledge and their overlaps in TPCK comes into play in the activities the instructors incorporate in their own classrooms.

The primary data were instructors' reports of the instructional activities they use. There were 138 activities reported by the three instructors. These activities were given in response to a list of 35 common MVC topics, grouped into general content areas (introductory/foundational information, differential calculus, integral calculus and vector calculus) during analysis. The list of topics was essentially taken from the table of contents of a popular calculus textbook (Stewart, 2006). The professors were given this list and asked to respond to questions about the types of resources they used in teaching each topic, describe the activities in which the resources were used and rate the quality of each activity. An activity was the unit of analysis. The professors were asked to self-analyse their instruction using the following predetermined codes under each italicised coding category: resource type (digital resources, 3D-printed models, other), pedagogical practice (instructor demonstration, individual in-class student activity, collaborative in-class student activity, student homework) and instructor rating. The instructors using one of three ratings:

- *Excellent* (Fully met all instructional goals) – The instructors were excited about the activity as it was and felt that any additional time spent modifying it would lead to no (or very minimal) improvements.
- *Pretty Good* (Met instructional goals) – The instructors felt that it could be improved and that with some modification the improvements would be noticeable.
- *Needs Work* (Did not fully meet instructional goals) – The instructors felt that if the activity was modified, the improvements could be substantial.
- *No Longer Used* (Did not meet instructional goals) – Instructors felt either (1) it was not a constructive use of time in the course and a complete overhaul of the activity was needed or (2) external factors (e.g. physical classroom layout, number of students, access to materials) made it difficult to implement the activity.

Based on the activity descriptions provided by the instructors as well as their descriptions of the activities, the resource team identified the activities that were reported as both collaborative and individual in-class activities as being *student-centred*. By student-centred activities, we mean that instructional intentions were for students to be actively engaged in doing mathematical tasks during that particular portion of class time. The *teacher-centred activities* were ones that did not require active student participation or engagement. For this study, only instructor demonstrations, where teachers used the resources (e.g. as examples in their lectures) to help students understand or visualise the topic at hand, were coded as teacher-centred.

5. Results

5.1. General results

Table 1 displays the counts for several coding categories. However, based on the research questions, this section will primarily discuss the more qualitative aspects of the data. By 'activity', we mean an instructor-planned intervention using digital resources, 3D-printed materials, or other physical resources.

5.2. Technology-pedagogy interface

We start by considering the first research question: Which resources used by the participating instructors favour the promotion of student engagement and active learning practices? See Figures 2 and 3. Of the 138 activities, there were 100 for which instructors reported using digital resources. Of these, only eight activities were rated as either collaborative in-class student activity or individual in-class student activity, which may be

Table 1. Activity counts by pedagogical practice and instructor rating for each content area and resource type.

	Resource type		
Content area	Digital resource	3D-printed model	Other physical resource
Introductory/foundational	**39 activities** pedagogical practice 5 ind act, 37 demos, 10 hwk[a] instructor rating 9 excellent, 30 good	**3 activities** pedagogical practice 3 collab instructor rating 2 excellent, 1 good	**11 activities** pedagogical practice 3 collab, 4 demos instructor rating 7 good, 4 not used
Differential Calculus	**19 activities** pedagogical practice 2 collab, 1 ind, 18 demos, 4 hwk instructor rating 9 excellent, 9 good, 1 needs work	**10 activities** pedagogical practice 10 collab instructor rating 7 excellent, 2 good, 1 needs work	**4 activities** pedagogical practice 1 collab, 3 hwk instructor rating 1 excellent, 3 good
Integral Calculus	**21 activities** pedagogical practice 21 demos, 1 hwk instructor rating 3 excellent, 14 good, 4 needs work	**6 activities** pedagogical practice 6 collab instructor rating 3 good, 3 not used	**0 activities**
Vector Calculus	**21 activities** pedagogical practice 21 demos, 4 hwk instructor rating 3 excellent, 15 good 3 needs work	**4 activities** pedagogical practice 3 collab, 1 demo instructor rating 2 excellent, 2 not used	**0 activities**

[a]Instructor H allows students to use CalcPlot3D for all homework, but she did not detail how this resource is leveraged. For this reason, we did not count her homework assignments.

considered student-centred (8%). Some multipart activities were rated as both collaborative and instructor demonstrations. With this in mind, 97 of the 100 activities leveraging digital resources involved instructor demonstrations. Of the 23 activities using 3D-printed surfaces, 22 were collaborative in-class student activities (96%). Finally, of the 15 activities requiring other physical resources, four were collaborative in-class student activities (36%) and seven were instructor demonstrations or individual homework (64%). This shows that most CalcPlot3D activities were instructor demonstrations, and most 3D-printed surface activities were designed for collaborative learning. Of the 138 activities, 100 (72%) involved digital technology (CalcPlot3D), 23 (17%) used 3D-printed models and 15 (11%) used other resources. This speaks to the instructors' TPK. All three seem to be capable of leveraging resources, be they digital or non-digital, in their teaching, but there was a clear preference among the participating instructors for using digital technology even though these resources had a much lower percentage of student-centred activities than the 3D-printed and other resources.

5.3. Technology-content interface

We now consider the second research question, inquiring into which topic areas of MVC seem more amenable to resource adoption by the participating instructors. This deals with TCK, by considering the specific activities and the more general content areas that were

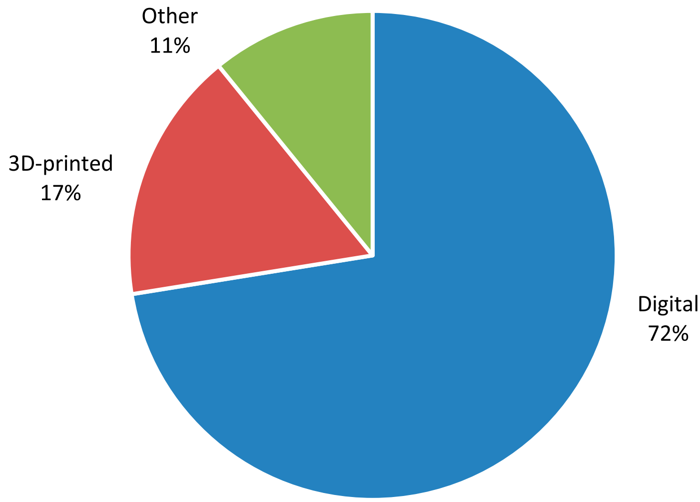

Figure 2. Pie chart showing the distribution of activities by type of resource.

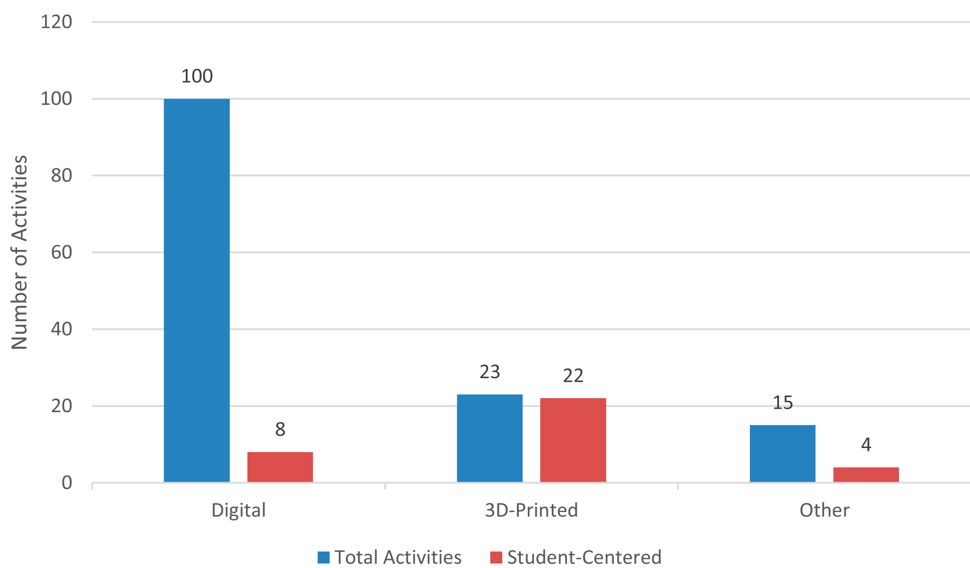

Figure 3. Bar chart comparison of total number of activities vs. student-centred activities by type of resource.

more amenable to resource adoption. Of the 138 activities, 53 were used for introductory/foundational ideas (38%), 33 for differential calculus (24%), 27 for integral calculus (20%) and 25 for vector calculus (18%). See Figure 4. This shows that activities tend to be concentrated in the initial part of the MVC course (introductory ideas and differential calculus) rather than for topics that occur in the later portion of the course (integral and vector calculus).

As for resource type, the 100 activities that used digital resources were spread across all four content areas. Most activities (39%) were employed for introductory/foundational

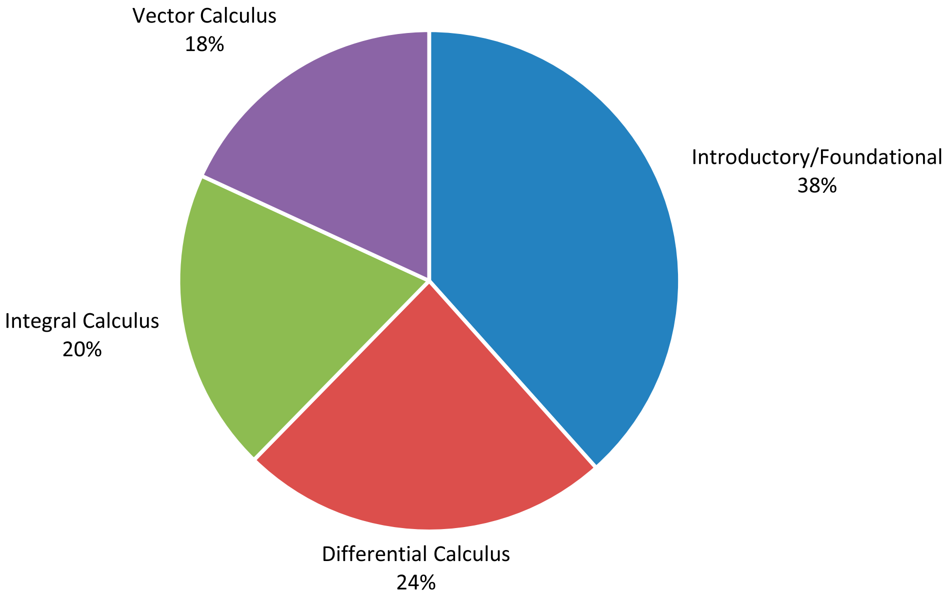

Figure 4. Pie chart showing the distribution of activities by content area.

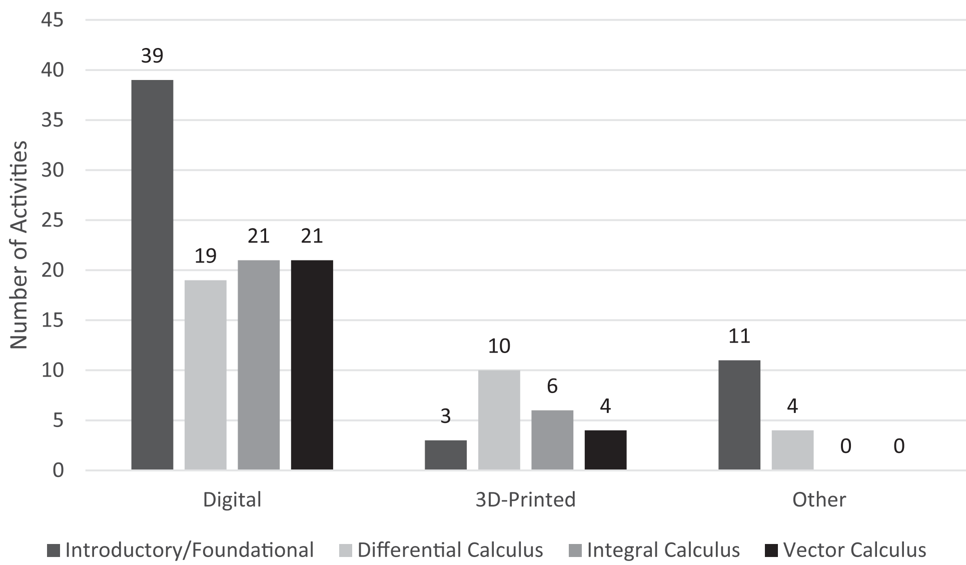

Figure 5. Bar chart showing the number of activities in each content area by type of resource.

ideas with a nearly even split for differential (19%), integral (21%) and vector (21%) calculus, covering every individual topic area except limits and continuity and the chain rule. See Figure 5.

There were 13 topics in common for which the three instructors used digital resources. We consider two examples to give an idea of the nature and potential of the activities using

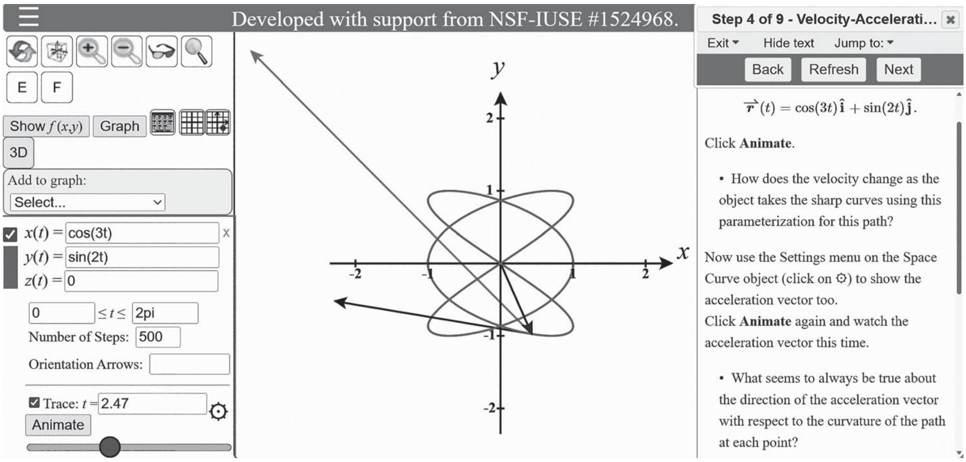

Figure 6. Screenshot of CalcPlot3D displaying an interactive helix with movable point and the resulting position, velocity, and acceleration vectors.

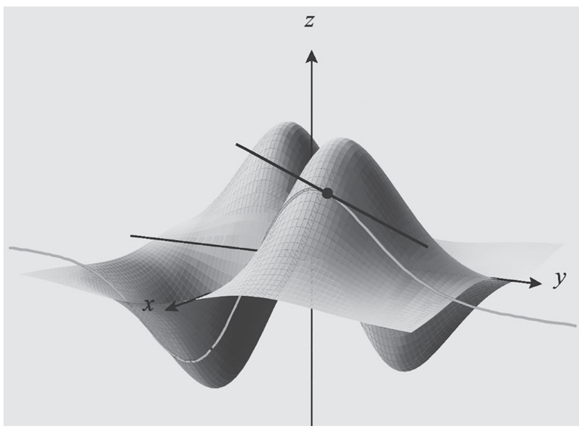

Figure 7. CalcPlot3D surface with a moveable point along a trace and resulting tangent line in the *y* direction.

digital resources in these common content topics. Referring to the topics of motion in space (velocity and acceleration), one instructor said,

> I do a lot with CalcPlot3D here to help students gain intuition for how a vector-valued function can represent motion in the plane or in space (Figure 6). The Velocity/Acceleration Exploration uses CalcPlot3D to help students investigate a whole series of carefully selected examples of space curves (mostly plane curves) to improve their intuition for this topic and help them solidify certain insights about the relationship between the velocity vector and the curve, between the acceleration vector and the curve and between the acceleration vector and the velocity vector at any given point. Also, to know the relationship between the velocity and acceleration vectors when the motion has a constant speed.

Another instructor commented, 'CalcPlot3D to show the traces and partial derivatives as slopes in class to demonstrate what partial derivatives show us geometrically' (Figure 7).

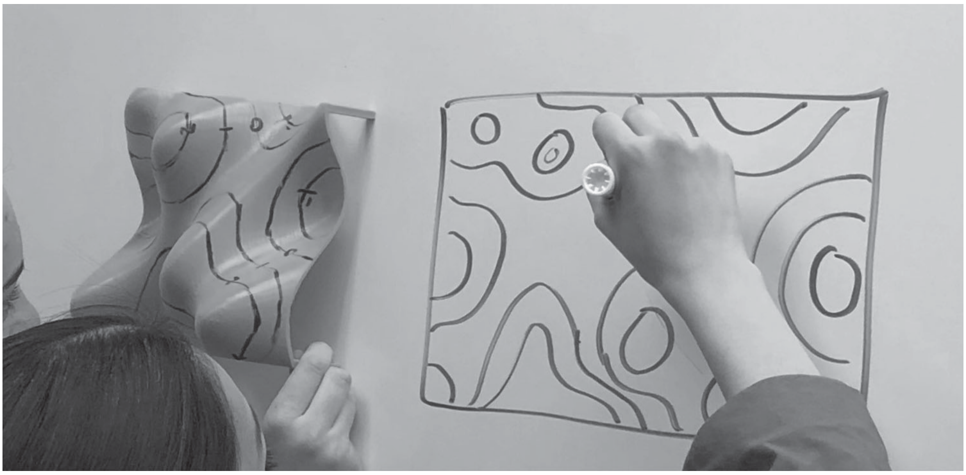

Figure 8. Students using a 3D-printed surface to draw contour plots.

Regarding the activities using 3D-printed models, 10 of the 23 activities (43%) used with 3D-printed models were for differential calculus (Figure 5). This suggests that instructors might find differential calculus more natural or amenable to designing activities based on 3D-printed models. Also, there are two topics for which the three instructors used 3D-printed models (functions of several variables, partial derivatives) and five topics for which two of the three instructors used 3D-printed models (maxima and minima; Lagrange multipliers; triple integrals in rectangular, cylindrical and spherical coordinates). There were six other topics for which a single instructor used 3D-printed models. This suggests that the use of 3D-printed models and their implementation might merit further exploration.

We consider two examples to give an idea of the nature and potential of the 3D-printed activities in these common content topics. An example of a topic for which the three instructors used 3D-printed surfaces was described by one as a contour plot activity where students work in groups of two to three.

> They are given a mountainous surface, a whiteboard marker, and a ruler. Following a scaffolded worksheet, students 'walk' a mathematical bug along the surface at four different elevations. Then, they draw a 2D representation of their curves on the board (Figure 8). Then, [the instructor does] a live demo on how to use CalcPlot3D to create a contour plot. Then, students use CalcPlot3D to create their own contour plot.

This activity could help address difficulties in geometric understanding of two-variable functions observed in the literature (Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010).

Another example concerns a collaborative activity for Lagrange multipliers completed by small groups of students. The instructor shared,

> [Students] are given a clear (thermoformed) mountainous surface and a whiteboard marker. Through this activity we motivate where the Lagrange Multiplier equations come from and the geometric significance of the Lagrange points. Students line up the surface above its contour plot (Figure 9). Then they trace a circular 'hiking' path along the surface (by looking at the constraint curve that is included on the contour plot). On the surface, students mark the

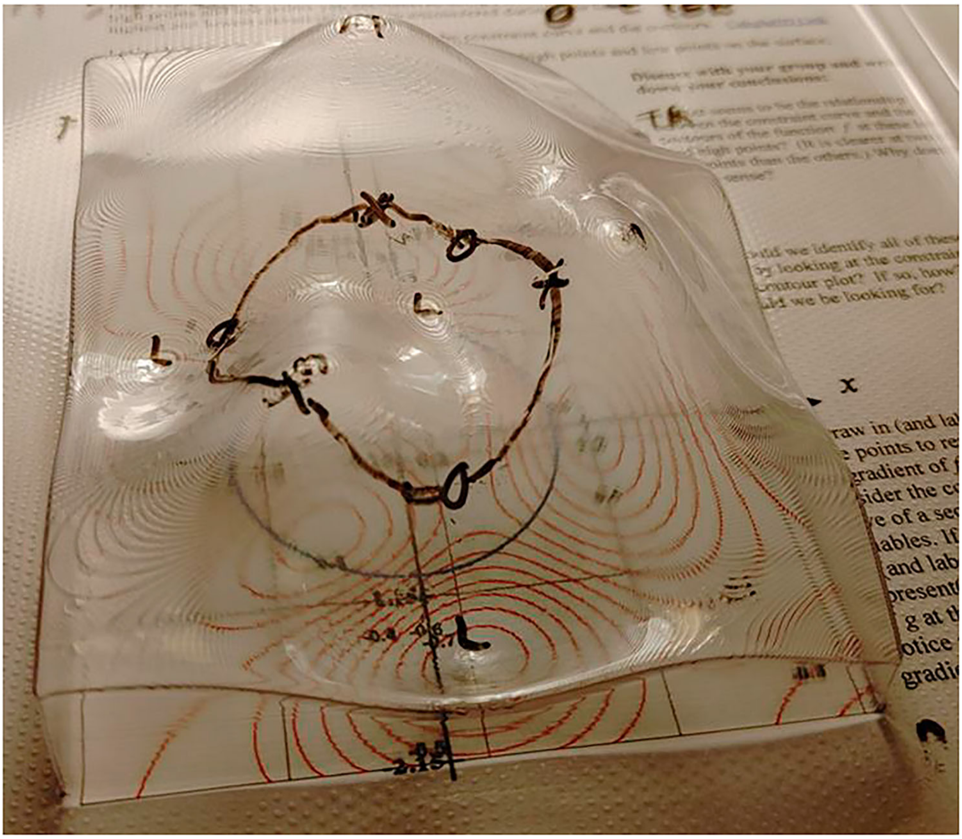

Figure 9. Thermoformed surface and contour plot for Lagrange multiplier activity.

> relative max and min [points] along their hike as well as the highest and lowest elevations they encounter on their hike.

CalcPlot3D allows a user to save a URL. This instruction uses one of the pre-saved URLs that depicts 'the surface, the constraint curve on the surface, and the contours of the surface in three dimensions [to help students] observe that the contours and constraint curve are tangent at these local high/low points'. The students then are asked to mark these points on the 2D contour plot to observe the same tangent relationship. For one of the points, students must draw in the gradient of the surface and the gradient of the constraint so they can see that the two are parallel. The instructor then shows another pre-saved CalcPlot3D URL to depict the contour plot and the constraint in 2D with a manipulable slider to show the parallel relationship between the gradients at all the Lagrange points.

Eleven of the 15 activities used resources other than digital or 3D-printed materials addressed introductory/foundational topics (Figure 5). Some of these other resources included K'nex rods, a homemade desktop coordinate system, large vector props made from wooden dowels, extendable vectors, a hill outside class and static images and contour plots on paper. For the most part, the use of these resources predated the adoption of digital and digitally based resources and attested to instructors' initial TPCK as well as for their motivations for later adopting digital and digitally generated resources. To show an example using other physical resources, one instructor uses a hill on campus and has

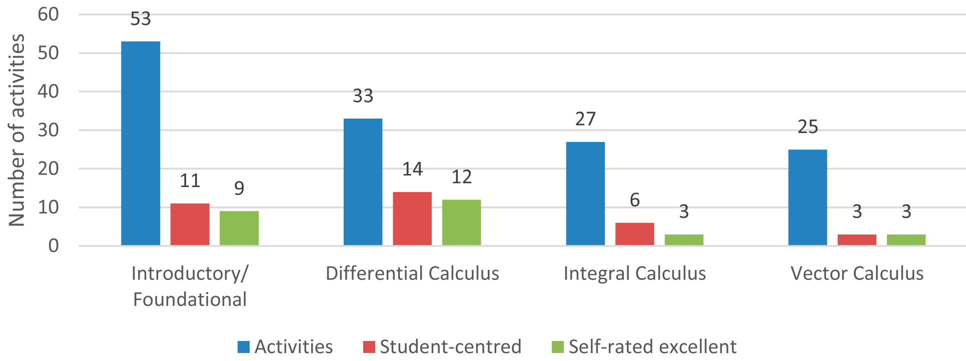

Figure 10. Bar chart comparing the total number of activities to those that were student-centred and self-rated as excellent by content area.

the students go on a class field trip to preview upcoming topics. The students stand at different locations on the hill and the instructor leads a walk around and discussion to have the students physically explore the concepts behind partial derivatives including the ideas of gradients, directional derivatives and extrema. Such an activity can potentially help students address observations in learning differential calculus of two-variable functions, as discussed by Martínez-Planell et al. (2015, 2017).

5.4. Pedagogy-content interface

For the third research question, we consider which MVC topic areas were the focus of student-centred activities implemented by the participating instructors. That is, we consider their pedagogical practices to determine which class activities (by content area) appear to allow for more student-centredness (i.e. individual and collaborative in-class activities). Eleven of the 53 activities for introductory/foundational ideas (21%), 14 of the 33 differential calculus activities (42%), 6 of the 27 integral calculus activities (22%) and 3 of the 25 vector calculus activities (12%) were student-centred. See Figure 10. All three instructors reported valuing student engagement, but their TPCK allowed them to design more student-centred activities for differential calculus than other MVC content areas, especially compared to vector calculus.

There were only two topics (functions of several variables, partial derivatives) for which all three instructors designed either individual or collaborative in-class student-centred activities and five topics (maxima and minima, Lagrange multipliers and triple integrals in rectangular, cylindrical and spherical coordinates) for which two of the three instructors designed student-centred activities. These topics suggest commonalities in the instructors' PCK. At the same time, there were 11 topics for which a single instructor designed student-centred activities. This underscores that there may be the potential of sharing activities.

We now consider the activities that involve instructor demonstrations, which are less student-centred than the other in-class activities just reported and which also stem from the instructors' TPCK. Demonstrations include 41 of the 53 activities for introductory/foundational ideas (77%), 18 of the 33 for differential calculus (55%), 21 of the 27 for integral calculus (78%) and 22 of the 25 for vector calculus (88%). So, the activities

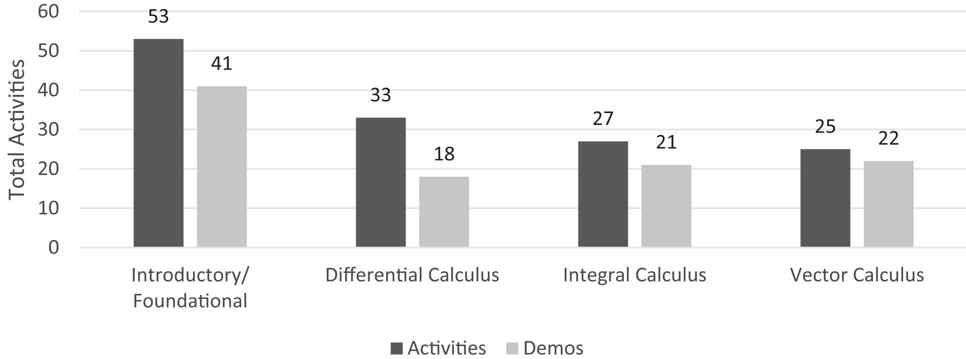

Figure 11. Bar chart comparing the total number of activities to those that only involved instructor demonstrations by content area.

designed for introductory/foundational ideas, integral calculus and vector calculus are mainly instructor demonstrations (Figure 11).

There were 13 topics where the three instructors used classroom demonstrations (vectors; equations of lines and planes; cylinders and quadric surfaces; vector functions and space curves; motion in space-velocity and acceleration; functions of several variables; partial derivatives; tangent planes and linear approximations; directional derivatives and the gradient vector; maxima and minima; Lagrange multipliers; vector fields; fundamental theorem of line integrals). There were also 14 other topics in which two of the three instructors used classroom demonstrations. So, there is an ample collection of available demonstrations, which is the most common pedagogical practice used with digital technology implementation. This speaks of the instructors' PCK since it tells us that a demonstration by the instructor is the most utilised pedagogical practice for introducing a wide variety of topic areas throughout the MVC course.

The fourth research question considers which topic areas of MVC have activities the participating instructors assess as being of good or excellent quality. This may also be related to the PCK interface since we assume that the instructors gave these activities high ratings due to the perceived pedagogical value and the activities themselves are designed to address MVC content. Not many activities (only 20%) were self-rated as excellent. Of these, nine dealt with introductory/foundational topics (33%), 12 with differential calculus (44%), 3 with integral calculus (11%) and 3 with vector calculus (11%). So, we see that a higher proportion of differential calculus activities are rated as excellent, and there is a lack of highly rated activities in integral and vector calculus.

In examining instructors' PCK, there was only one topic for which all three instructors self-rated their activity as excellent (partial derivatives) and five topics for which two of the three instructors self-rated their activity as excellent (motion in space-velocity and acceleration; functions of several variables; directional derivatives and the gradient vector; maxima and minima; Lagrange multipliers). At the same time, there were 11 other topics for which a single instructor self-rated an activity as excellent.

The participating instructors designed activities that they self-rated as good or excellent in 32 of the 35 common MVC topics in the course that were listed for data gathering. To provide an idea of the types of activities that were self-rated as excellent and their potential to support student learning, we include in Table 2 a representative list of some of these activities. This list underscores the possible benefits of instructors sharing their high-quality teaching materials. The list includes only 14 of the 35 course topics. Ten of the 14 activities on the list are instructor demonstrations. This suggests that activities of high quality are needed in more topics and that more of them should involve active student engagement and communication. One can also see the relative lack of highly rated activities for integral calculus, particularly triple integrals in different coordinate systems, as well as for multiple topics in vector calculus. Table 2 includes a citation to related mathematics education literature when the activity proposed by an instructor in this study addresses observations from the corresponding research.

We now consider if any patterns were noted specific to particular instructors in the sample. First, pedagogical practices, as determined by student-centredness (collaborative and individual in-class activities), vary by content area and instructor. For example, for the introductory/foundational information, only one of the 27 activities of instructor H (4%) was student-centred. In contrast, in differential calculus, five of the 13 activities (38%) reported by the same instructor were student-centred. So, the pedagogical practice seems to depend on the content area. Similarly, in the topics of integral and vector calculus, instructor T designed six of his 13 activities (46%) to be student-centred, collaborative activities. In comparison, none of the nine activities of Instructor A in these areas were student-centred. So, pedagogical practices also seem to depend on the instructor. The choice of resource type also seems to be instructor dependent. For example, Instructor H used CalcPlot3D 23 times when discussing introductory/foundational ideas, while Instructor T used it only four times. Similarly, when teaching smaller classes, Instructor T used 3D-printed models in 12 activities, while Instructor A used them for three activities.

5.5. Discussion and summary

Moore-Russo et al. (2024) found that concern for students' learning and the conviction that geometrical understanding is crucial in multivariable calculus motivated their sample of instructors to consider various resources in their teaching. They found that the interviewed instructors were sensitive to students' needs and that, over time, their teaching shifted from being instructor-centred (demonstrations presented in class) to being more student-centred (activities to engage students, often in collaborative group work) as called forth by Hegedus et al. (2017). The present study examines instructors' TPCK by relating the resources instructors use, the pedagogy employed and the content at hand.

The first research question relates to instructors' TPK and inquires into what resource use by the participating instructors led to student engagement in active learning practices. We found that most CalcPlot3D activities were instructor demonstrations (97%), and most 3D-printed surface activities (96%) were designed for collaborative learning. As Adler (2000) observed, how instructors use resources is crucial. In particular, instructors' beliefs in the efficacy of lecturing and demonstrations, together with perceived time constraints, could play a role in slowing the movement toward more collaborative and active learning opportunities.

To answer the second research question, we observed that the area of introductory/foundational ideas is more amenable to resource adoption in the sense that it was more frequently the target of activities, mainly for demonstrations using digital technology. This

Table 2. Samples of activities that were self-rated as excellent.

Topic	Instructor's description of activity rated as excellent
Dot product	'Students … complete an independent interactive exploration activity using CalcPlot3D in which they explore various aspects of the dot product of two vectors and how its value depends on both the length of the given vectors and the angle between them' (see Barniol & Zavala, 2014a, 2014b).
Cross product	'I first teach students how to calculate the cross product of a pair of vectors in class on the board. Then, we use CalcPlot3D to observe the geometric relationship between the cross product vector and the two vectors forming it. We also take some time to discuss other properties of the cross product and how the formula $\|u \times v\| = \|u\|\|v\| \sin\theta$ can be used to determine the max and min values of the magnitude of the cross product of two vectors of a given length. Students then complete an interactive guided exploration in which they explore the right-hand rule, max and min values of the cross product for two vectors of specific lengths and other relevant properties of the cross-product' (see Moore-Russo et al., 2017; VanDieren et al., 2017, 2020).
Equations of lines and planes	'CalcPlot3D script that visually illustrates the process of finding the line of intersection of two planes' (see Moore-Russo et al., 2013).
Cylinders and quadric surfaces	'CalcPlot3D script that illustrates using traces to graph a paraboloid' (Trigueros & Martínez-Planell, 2010).
Partial derivatives	'After covering partial derivatives as a skill and then using CalcPlot3D to discuss the geometric meaning of both the first and second partials in each direction, I have students get into groups of four students again and give each group one of the same three mountainous surfaces. The activity sheet then directs them to first verbally explain what they expect to see on a surface with various combinations of positive, negative, or zero partials and second partials. Then, they are asked to locate a point on the surface that corresponds to each of these and write the corresponding numbers next to these points on the surface' (see Wangberg, 2020; Wangberg & Johnson, 2013).
Directional derivatives	'I use CalcPlot3D to show how directional derivatives can be thought of as slope along the graph of a function, similar to partial derivatives' (see Martínez-Planell et al., 2017).
Gradient vector	'Pre-saved CalcPlot3D URL, which shows the … feature of adding the gradient vector at a point. We visualise the gradient in a 2D contour plot to discuss the direction and length of the gradient vector, then we also look at it in 3D' (see Moreno-Arotzena et al., 2021).
Maximum and minimum values	'Students work in groups of two or three. They are given a mountainous surface and a whiteboard marker. They are asked to mark all of the saddle points, then select one saddle point and 'walk' a mathematical bug around so that it stays at the same elevation. Through this exercise, we find out that saddle points are the only place where contour lines intersect' (see Alves, 2012; Ingar, 2014).
Lagrange multipliers	'Visual verification of answers absolute extrema on a closed set. Pre-saved CalcPlot3D URLS which show the surface, the contour plot on the surface, and the constraint curve (2D version and a 3D version)' (see Mkhatshwai, 2021; Xhonneux, 2011).
Double integrals over rectangles	'Pre-saved CalcPlot3D URL that shows Riemann prisms to estimate the volume of the solid below a surface. We show an increasing number of prisms and discuss the different sample points that one could use (midpoint, upper-right, etc.)' (see Borji et al., 2024b; Martínez-Planell & Trigueros, 2020).
Double integrals in polar coordinates	'CalcPlot3D script that shows the volume inside a sphere and above a cone as a difference of the volume of 2 solids'.
Vector fields	'I use CalcPlot3D in class to show vector fields in R^3 so that we can view them from several angles. For homework, I have links to pre-plotted 3D vector fields so that they can match these with the formulas and identify the direction of the curl vectors for each' (see Bollen et al., 2017).
Line integrals	'Pre-saved CalcPlot3D URL that shows a 2D vector field and curve, so that we can discuss the concept of work visually' (see Jones, 2020).
Surface integrals	'I use CalcPlot3D to illustrate how the grid curves of a surface parametrisation dice a surface into parallelogram-shaped bits'.

might relate to instructors' TCK. This content area is also more likely to leverage resources other than digital or 3D-printed for in-class activities. While this might reflect instructors' concern for student learning, it might also show their initial belief that demonstrations and instructor explanations are as effective as collaborative and active learning opportunities. In the case of 3D-printed models, we found that it seems to be more natural for the instructors in our study to use them for differential calculus, changing distant 2D representations to close 3D representations, as Parzysz (1988) described.

For the third research question, which mainly focused on PCK, we observed that all areas allow for student-centred activities using resources. There were more student-centred activities for differential calculus followed by student-centred activities for introductory/foundational ideas. The areas of integral calculus, and particularly vector calculus, are comparatively lacking in the design of activities that allow for active student engagement and communication. Activities for 3D-printed models, although fewer in number, were much more likely to involve active or collaborative student engagement than those involving digital resources. We also found that differential calculus has more activities that instructors gave the highest ratings followed by the activities designed for introductory/foundational topic areas. The topic areas in integral and vector calculus do not show comparable development of activities that leverage resources for visualisation.

The content addressed, pedagogy used and resources employed vary by instructor. The general patterns that were noted: digital technology is preferred to other resource use across all instructors in the sample; the use of in-class instructor demonstrations vastly outnumbers the use of student-centred activities; there are fewer activities for 3D-printed models, although these were the activities that tended to be more student-centred; and the area of vector calculus is vastly overlooked in terms of resource-leveraged activity development, in comparison with the other general areas of MVC. These patterns suggest that MVC instructors' TPCK may not be uniform across MVC topic areas or across resource types.

There were six topics for which only one instructor used 3D-printed models, 11 topics for which only one instructor designed student-centred activities and 11 topics for which an activity was self-rated as excellent by only one instructor. This suggests that sharing materials might help instructors incorporate 3D-printed models into other topics, make the MVC course more student-centred and disseminate activities rated as excellent. Consistent with Moore-Russo et al. (2024), who argued that instructors could benefit from support on how to teach MVC to engage students by leveraging instructional resources, our findings suggest that establishing a means to facilitate instructors sharing student-centred activities, activities they self-rate as excellent and activities that employ 3D-printed models, might be valuable for other instructors and could improve MVC teaching. This study makes a case for establishing a means to do so. The sharing of materials might help speed up the adoption of resources, as Lagrange et al. (2003) called forth, particularly for teaching MVC.

In the areas of integral calculus and vector calculus, there are fewer activities, very few of which were student-centred or rated as excellent. Correspondingly, most of the relatively few integral and vector calculus activities are instructor demonstrations. This perceived inadequacy or imbalance in these two areas of MVC compared with the areas of introductory/foundational ideas and differential calculus could result from several factors. This might be a consequence of the content itself, with certain specific topics more readily lending themselves to the incorporation of resources; a consequence of these areas (integral and vector calculus) being taught towards the end of the course; or a consequence of a lack of digital capabilities or other tools to properly treat these two areas. It is also possible that the instructors studied may have learned to use resources for the introductory/foundational and differential calculus areas following the general sequence of the MVC as they revise their own instructional practices. So, they have yet to refine all aspects of their MVC instruction but are intending on creating new activities that leverage resources across the entire MVC, but these plans have not yet come to fruition. More research is needed to understand this issue better. Documenting that these two areas (integral and vector calculus) may need more consideration and effort in terms of activity creation, especially high-quality activities that are student-centred, is a contribution of our study.

The MVC course had a number of activities to address introductory/foundational ideas. However, not many of them were student-centred or rated excellent; classroom demonstrations constituted a large percentage of these activities. While this suggests the possibility of re-examining existing teaching practices to increase student engagement with the content, this might be more of a function of time constraint issues exerting pressure on instructional choices than pedagogical beliefs playing a role in the promotion of student engagement. Further study is needed here.

In terms of content, this study suggests that more activities that use resources to improve the teaching of integral and vector calculus are needed. Regarding pedagogy, there is a preference for instructor demonstrations and what might be considered an insufficient number of highly rated, student-centred activities. For resources, we observe that more work is needed to explore the pedagogical potential of 3D-printed models. Overall, this study contributes to a better understanding of resource use and its relationship with pedagogy and content in teaching MVC. It also contributes to a better understanding of MVC instructors' TPCK.

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Data availability statement

The data that support the findings of this study are available from the corresponding author, [RMP], upon reasonable request in the https://tinyurl.com/MVCInstructorData.

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