AN EXTENSION OF GOW'S THEOREM

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In memory of our good friend and esteemed colleague Gary Seitz

ABSTRACT. We extend Gow's theorem to finite groups G whose generalized Fitting subgroup is $\mathbf{Z}(G)S$ for a quasisimple Lie-type group S of simply connected type in characteristic p, and whose center $\mathbf{Z}(G)$ has p'-order.

A result of Rod Gow [Gow, Theorem 2] asserts that the product $a^G b^G$ of any two regular semisimple classes in a finite simple group of Lie type G contains every nontrivial semisimple element $x \in G$. This result has been used in many applications. It has also been extended to any quasisimple Lie-type group of simply connected type: the product $a^G b^G$ of any two regular semisimple classes in G contains every non-central semisimple element $x \in G$, see [GT, Lemma 5.1].

In this paper we will further extend Gow's theorem. Let p be a prime and let $\underline{\boldsymbol{G}}$ be a simple, simply connected algebraic group defined over $\overline{\mathbb{F}}_p$. Let $F : \underline{\boldsymbol{G}} \to \underline{\boldsymbol{G}}$ be a Steinberg endomorphism, so that

$$S := \underline{\boldsymbol{G}}^F$$

be quasisimple. (In particular, we do not view $\mathrm{PSL}_2(9)$ as $\mathrm{Sp}_4(2)'$, $\mathrm{SU}_3(3)$ as $G_2(2)'$, or $\mathrm{SL}_2(8)$ as ${}^2G_2(3)'$.)

We will consider finite groups G with

$$F^*(G) = \mathbf{Z}(G)S \text{ and } p \nmid |\mathbf{Z}(G)|, \tag{1}$$

(so $\mathbf{C}_G(S) = \mathbf{C}_G(F^*(G)) = \mathbf{Z}(G)\mathbf{Z}(S)$ is a p'-group), and aim to show that the product $a^G b^G$ of two particular conjugacy classes in G will cover all elements $g \in G$ of a certain kind. Before going on we state a special case of a consequence of our main result which is less technical. Versions of this result have already been used in [AGS, GTT].

Corollary 1. *Let G be as above. Assume that $a \in G$ has order prime to p, $|\mathbf{C}_S(a)|$ has order prime to p and that $s \in S \smallsetminus \mathbf{Z}(S)$ is semisimple. Then $s^t = [a, u]$ for some $t, u \in S$.*

In fact, one need not assume that g has order prime to p. See our main result Theorem 6.

Let a and b be elements of G, and set $G_1 := \langle S, a, b\rangle$. Then $S \lhd G_1$, and hence $S \lhd F^*(G_1)$. It follows that $F(G_1)$, as well as any quasisimple subnormal subgroup $T \neq S$ of G_1, centralizes S. Hence by (1), $F^*(G_1) = \mathbf{Z}(G_1)S$. Furthermore,

$$\mathbf{Z}(G_1) \leq \mathbf{C}_G(S) = \mathbf{Z}(G)\mathbf{Z}(S)$$

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is also a p'-subgroup. Hence, for our purposes, we may assume

$$G = \langle S, a, b \rangle. \tag{2}$$

Let St denote the Steinberg character of S, and for $x \in G$, write x_p for the p-part of x. By [F1, Corollary D], St extends to a rational-valued character St_G of G (called the *basic p-Steinberg character* of G). Moreover, by [F2, Theorem C], there is a Sylow p-subgroup P and a p-subgroup D of G, of order

$$|D| = p^d = |G/S|_p,$$

such that $P = Q \rtimes D$ for a Sylow p-subgroup Q of S and the following statement holds. For any element $x \in G$, $\mathsf{St}_G(x) \neq 0$ if and only if $x_p \in D$ (up to conjugation), in which case

$$\mathsf{St}_G(x) = \pm|\mathbf{C}_S(x)|_p.$$

In view of these results, the proper generalization to G of regular semisimple classes in S will be that $a, b \in G$ satisfy

$$a_p \in D \text{ up to conjugation in } G, \text{ and } p \nmid |\mathbf{C}_S(a)|, \tag{3}$$

and

$$b_p \in D \text{ up to conjugation in } G, \text{ and } p \nmid |\mathbf{C}_S(b)|, \tag{4}$$

Certainly, $g \in G$ can belong to $a^G b^G$ only when it does so in the solvable group G/S, so we will assume

$$gS \in (aS)^{G/S}(bS)^{G/S}, \text{ and } g_p \in D \text{ up to conjugation in } G. \tag{5}$$

For instance, if G/S is abelian, then the first condition in (5) is equivalent to $g \in abS$.

Proposition 2. *The following statements hold.*

(i) *If $p > 2$ and $Q \in \mathrm{Syl}_p(S)$, then $\mathbf{C}_G(Q) = \mathbf{Z}(G)\mathbf{Z}(S)\mathbf{Z}(Q)$.*
(ii) *If $g \in G \smallsetminus \mathbf{Z}(G)\mathbf{Z}(S)$ and $g_p \in D$, then p divides $[S : \mathbf{C}_S(g)]$.*

Proof. Since $Z := \mathbf{Z}(G)\mathbf{Z}(S)$ is a p'-group centralizing Q and normal in G, we may work in $\bar{G} := G/Z$ and identify Q with $\bar{Q} := QZ/Z$. Note that $Z \cap S = \mathbf{Z}(S)$. Moreover, as S is perfect, we have $\mathbf{C}_G(S/\mathbf{Z}(S)) = \mathbf{C}_G(S) = Z$. It follows that

$$\bar{S} := S/\mathbf{Z}(S) \lhd \bar{G} \leq \mathrm{Aut}(\bar{S}),$$

i.e. $\bar{G}$ is almost simple.

Consider any element $x \in \bar{C} := \mathbf{C}_{\bar{G}}(\bar{Q})$. Then $H := \langle \bar{S}, x \rangle \leq \bar{G}$ is also almost simple, whence $\mathbf{O}_{p'}(H) = 1$, and $R := \langle \bar{Q}, x_p \rangle$ is a Sylow p-subgroup of H centralized by x. It follows that $R = \mathbf{O}_p(\mathbf{N}_H(R))$. By [GLS, Corollary 3.1.4], $\mathbf{O}_p(\mathbf{N}_H(R)) = F^*(\mathbf{N}_H(R))$, whence x belongs to $\mathbf{C}_{\mathbf{N}_H(R)}(R) \leq R$ and so $x = x_p$ is a p-element. Thus $\bar{C}$ is a p-group.

Similarly, $\bar{Q} = \mathbf{O}_p(\mathbf{N}_{\bar{S}}(\bar{Q})) = F^*(\mathbf{N}_{\bar{S}}(\bar{Q}))$, and so

$$\bar{C} \cap \bar{S} = \mathbf{C}_{\bar{S}}(\bar{Q}) = \mathbf{C}_{\mathbf{N}_{\bar{S}}(\bar{Q})}(\bar{Q}) \leq \bar{Q}.$$

It follows that $\bar{C} \cap \bar{S} = \mathbf{Z}(\bar{Q})$.

(i) Now we assume $p > 2$ and show that $\bar{C} \leq \bar{S}$, which implies that $\mathbf{C}_G(Q) = \mathbf{Z}(Q)Z$. Assume the contrary: $\bar{C} \not\leq \bar{S}$. Since $\bar{C}$ is a p-group, we can find a p-element $x \in \bar{C} \smallsetminus \bar{S}$; in particular, $[x, \bar{Q}] = 1$. Now $R := \langle x, \bar{Q} \rangle$ is Sylow p-subgroup of $H := \langle \bar{S}, x \rangle$ and $x \in \mathbf{Z}(R)$. As $\bar{S} \lhd H \leq \bar{G} \leq \mathrm{Aut}(\bar{S})$, we still have $\mathbf{O}_{p'}(H) = 1$. Now, if $p > 2$ then $\mathbf{Z}(R) \leq F^*(H) = \bar{S}$ by [GGLN, Corollary 1.2], and hence $x \in \bar{S}$, contrary to the choice of x.

(ii) Assume the contrary that $p \nmid [S : \mathbf{C}_S(g)]$. Conjugating g suitably, we may assume that $g \in \mathbf{C}_G(Q)$ with $Q \in \mathrm{Syl}_p(S)$ as before.

Suppose first that $p > 2$. Then $g \in \mathbf{Z}(G)\mathbf{Z}(S)\mathbf{Z}(Q)$ by (i), and so $g_p \in S$. But g_p is conjugate to an element in D by assumption and $D \cap S = 1$, so $g_p = 1$. It follows that $g \in \mathbf{Z}(G)\mathbf{Z}(S)$, a contradiction.

Thus we have $p = 2$. Then $\mathsf{St}_G(1) = |Q| = |\mathbf{C}_S(g)|_p = \pm\mathsf{St}_G(g)$. On the other hand, St is trivial at $\mathbf{Z}(S)$, so the generalized center of St_G contains $Z = \mathbf{Z}(G)\mathbf{Z}(S)$ and hence equals G as G/Z is almost simple with socle $\bar{S}$. As the generalized center of St_G contains g, we conclude that $g \in Z$, again a contradiction. □

Fix any element $g \in G$ satisfying (5). Then $S\mathbf{C}_G(g) \leq G$, so

$$\mathbb{Z} \ni [G : S\mathbf{C}_G(g)] = \frac{|G| \cdot |\mathbf{C}_S(g)|}{|S| \cdot |\mathbf{C}_G(g)|} = \frac{[G : \mathbf{C}_G(g)]}{[S : \mathbf{C}_S(g)]}. \tag{6}$$

Write

$$\frac{[G : \mathbf{C}_G(g)]_p}{[S : \mathbf{C}_S(g)]_p} = p^e. \tag{7}$$

Lemma 3. *Let X be a finite group, which is abelian-by-cyclic, that is, X has a normal abelian subgroup $A \lhd X$ such that X/A is cyclic. Suppose $x, y, z \in X$ are such that*

$$X = \langle x, y \rangle \text{ and } z \equiv xy \pmod{[X, X]}.$$

Then

$$\sum_{\alpha \in \mathrm{Irr}(X)} \frac{\alpha(x)\alpha(y)\overline{\alpha(z)}}{\alpha(1)} = |X/[X, X]|.$$

Proof. The condition $z \equiv xy \pmod{[X, X]}$ implies that

$$\sum_{\alpha \in \mathrm{Irr}(X),\ \alpha(1)=1} \frac{\alpha(x)\alpha(y)\overline{\alpha(z)}}{\alpha(1)} = |X/[X, X]|.$$

Hence it suffices to show that the contribution of any non-linear $\alpha \in \mathrm{Irr}(X)$ to the sum in the statement is 0. Consider any irreducible constituent λ of $\alpha|_A$. Suppose λ is not X-invariant. As $X = \langle x, y \rangle$, we may assume that λ is not x-invariant, in which case $\alpha(x) = 0$ by Clifford's theorem and the contribution is 0 as claimed.

Suppose now that λ is X-invariant. Then for any $a \in A$ and $t \in X$, as $\lambda(1) = 1$ we have

$$\lambda(tat^{-1}a^{-1}) = \lambda(tat^{-1})/\lambda(a) = 1,$$

whence $[t, a] \in \mathrm{Ker}(\lambda)$ and $\mathrm{Ker}(\lambda) \lhd X$. It follows that $A/\mathrm{Ker}(\lambda) \leq \mathbf{Z}(X/\mathrm{Ker}(\lambda))$. But X/A is cyclic, so $X/\mathrm{Ker}(\lambda)$ is abelian. Now λ is the unique irreducible constituent of α_A, so $\mathrm{Ker}(\lambda) \leq \mathrm{Ker}(\alpha)$, and hence α, viewed as an irreducible character of $X/\mathrm{Ker}(\lambda)$, must be linear, contrary to the assumption $\alpha(1) > 1$. □

Proposition 4. *Under the assumptions* (1)–(5)*, assume in addition that G/S is abelian-by-cyclic. Then*

$$\Sigma_1 := \sum_{\chi \in \mathrm{Irr}(G|\mathsf{St})} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \cdot |g^G|$$

is a rational integer whose p-part is at most p^{d+e}.

Proof. As mentioned above, St extends to St_G. Hence, by Gallagher's theorem [Is, (6.17)], any $\chi \in \mathrm{Irr}(G|\mathsf{St})$ is of the form

$$\chi = \mathsf{St}_G \alpha$$

with $\alpha \in \mathrm{Irr}(G/S)$. Using (2), (5) and Lemma 3, we see that

$$\sum_{\alpha \in \mathrm{Irr}(G/S)} \frac{\alpha(a)\alpha(b)\overline{\alpha(g)}}{\alpha(1)}$$

is a rational integer whose p-part is at most p^d.

On the other hand, by (3) and (4) we see that

$$\frac{\mathsf{St}_G(a)\mathsf{St}_G(b)\overline{\mathsf{St}_G(g)}}{\mathsf{St}_G(1)} \cdot |g^G| = \pm \frac{|\mathbf{C}_S(g)|_p \cdot |G|_p \cdot |G|_{p'}}{|S|_p \cdot |\mathbf{C}_G(g)|_p \cdot |\mathbf{C}_G(g)|_{p'}} = \pm \frac{[G : \mathbf{C}_G(g)]_p}{[S : \mathbf{C}_S(g)]_p} \cdot [G : \mathbf{C}_G(g)]_{p'}$$

is p^e times a p'-integer. Hence the statement follows. □

Recall that St is the only p-defect zero character of S. By the main result of [Hu], all the remaining characters of S belong to p-blocks of maximal defect. The next result deals with these characters.

Proposition 5. *Under the assumptions* (1)–(5)*, assume in addition that G/S has a cyclic Sylow p-subgroup and a normal p-complement, and that $g \notin \mathbf{Z}(G)\mathbf{Z}(S)$. Then*

$$\Sigma_2 := \sum_{\chi \in \mathrm{Irr}(G) \smallsetminus \mathrm{Irr}(G|\mathsf{St})} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \cdot |g^G|$$

is p^{d+e+1} times an algebraic integer.

Proof. By the hypothesis we can write $G/S = (H/S) \rtimes D$ for some normal subgroup $H \geq S$ of G. Note that any $\chi \in \mathrm{Irr}(G) \smallsetminus \mathrm{Irr}(G|\mathsf{St})$ lies above some $\theta \in \mathrm{Irr}(H)$ which does not lie above St. Suppose θ is not G-invariant. As $G = \langle H, a, b\rangle$, we may assume that θ is not a-invariant, in which case $\chi(a) = 0$ by Clifford's theorem, and the contribution of χ to Σ_2 is 0.

Hence we need to count the total contribution to Σ_2 of the characters $\chi \in \mathrm{Irr}(G|\theta)$, where $\theta \notin \mathrm{Irr}(H|\mathsf{St})$ is G-invariant. Since G/H is cyclic, any such θ extends to a character χ_1 of G, and we may write

$$\mathrm{Irr}(G|\theta) = \{\chi_1\mu \mid \mu \in \mathrm{Irr}(G/H)\}.$$

By the assumption $\theta \notin \mathrm{Irr}(H|\mathsf{St})$, every irreducible constituent of $\theta|_S$ belongs to an S-block B_S of maximal p-defect.

Conjugating g suitably, we may assume that $g_p \in D$. Note that $g_{p'} \in H$, so $g = g_p g_{p'}$ belongs to

$$K := \langle H, g_p\rangle \lhd G.$$

Set

$$\chi_2 := (\chi_1)|_K \in \mathrm{Irr}(K|\theta).$$

Now the p-block B of H that contains θ covers B_S, and $p \nmid |H/S|$, so B has maximal defect, see e.g. [N, Theorem 9.26]. But $K/H \hookrightarrow D$ is a p-group, so by [N, Corollary 9.6] there is a unique p-block B_2 of K that covers B. In particular,

$$\mathrm{Irr}(K|\theta) \subseteq \mathrm{Irr}(B_2).$$

Moreover, B is K-invariant as θ is K-invariant, whence B_2 is of maximal defect by [N, Theorem 9.17]. It follows that B_2 contains a character χ_0 of height zero, and so of p'-degree.

As χ_0 and χ_2 belong to the same block, we know that the two algebraic integers

$$\omega_{\chi_i}(g) = \frac{\chi_i(g)}{\chi_i(1)} \cdot |g^K|$$

for $i \in \{0, 2\}$ are congruent modulo p. By Proposition 2, $|g^S|$ is divisible by p, so $|g^K|$ is divisible by p as well, see the computation in (6). But $p \nmid \chi_0(1)$, so $p | \omega_{\chi_0}(g)$. It follows that

$$p \text{ divides } \omega_{\chi_2}(g) = \frac{\chi_2(g)}{\chi_2(1)} \cdot |g^K| = \frac{\chi_1(g)}{\chi_1(1)} \cdot |g^K|. \tag{8}$$

Next, (6) applied to $S \lhd H$ with $p \nmid |H/S|$ shows that

$$|g^S|_p = |g^H|_p.$$

On the other hand, g_p centralizes g, and $K = \langle H, g_p \rangle$, so $H\mathbf{C}_K(g) = K$, showing that $g^K = g^H$. Hence $|g^S|_p = |g^K|_p$, and (7) becomes

$$p^e = \frac{|g^G|_p}{|g^K|_p}.$$

Together with (8), we now obtain

$$p^{e+1} \text{ divides } \omega_{\chi_1}(g) = \frac{\chi_1(g)}{\chi_1(1)} \cdot |g^G|.$$

Now, (5) implies that $g \equiv ab \pmod{G/H}$, and so

$$\sum_{\mu \in \mathrm{Irr}(G/H)} \frac{\mu(a)\mu(b)\overline{\mu(g)}}{\mu(1)} = |G/H| = p^d.$$

It follows that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \cdot |g^G| = \omega_{\chi_1}(g) \sum_{\mu \in \mathrm{Irr}(G/H)} \frac{\mu(a)\mu(b)\overline{\mu(g)}}{\mu(1)} = p^d \omega_{\chi_1}(g)$$

is p^{d+e+1} times an algebraic integer. □

Theorem 6. *Under the assumptions* (1)–(5)*, assume in addition that all the following conditions hold.*

(a) *G/S is abelian-by-cyclic.*
(b) *G/S has cyclic Sylow p-subgroups and a normal p-complement.*
(c) *$g \notin \mathbf{Z}(G)\mathbf{Z}(S)$.*

Then $g \in a^G b^G$.

Proof. By Propositions 4 and 5,

$$|g^G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} = \Sigma_1 + \Sigma_2$$

is $p^s(u + p^t v)$, where $u \in \mathbb{Z} \smallsetminus p\mathbb{Z}$, v is an algebraic integer, $0 \le s \le d + e$, and $t \ge 1$. Now if $u + p^t v = 0$, then $v = -u/p^t$ is rational and an algebraic integer, so $v \in \mathbb{Z}$ and $u \in p\mathbb{Z}$, a contradiction. Thus

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \neq 0,$$

and so $g \in a^G b^G$ by Frobenius' character formula. $\Box$

Proof of Corollary 1. To prove the result, we may replace G by $\langle S, a, b\rangle$ with $b := a^{-1}$. Then (3)–(4) hold with $g := s$. Now G/S is cyclic, so by $s \in a^G b^G$. But $a^G = a^S$ and $b^G = b^S$ since $G = \langle S, a\rangle = \langle S, b\rangle$, so the statement follows. $\Box$

In what follows, q is always a power of the prime p. We will use the structure of $\mathrm{Aut}(\bar{S})$ as described in [GLS, Theorem 2.5.12], in particular the notations $\mathrm{Inndiag}(\bar{S})$ and $\mathrm{Outdiag}(\bar{S})$.

Theorem 7. *Under the assumptions* (1)–(5)*, assume in addition that all the following conditions hold for g, $\bar{S} = S/\mathbf{Z}(S)$, and $\bar{G} = G/\mathbf{Z}(G)\mathbf{Z}(S)$.*

(a) $g \notin \mathbf{Z}(G)\mathbf{Z}(S)$.
(b) *If $\bar{S} = \mathrm{PSL}_n(q)$ with $n \geq 3$, or $\bar{S} = P\Omega^+_{2n}(q)$ with $n \geq 4$, or $S = E_6(q)$, then the quotient $\bar{G}/\big(\bar{G} \cap \mathrm{Inndiag}(\bar{S})\big)$ is cyclic.*

Then $g \in a^G b^G$.

Proof. Recall that $G/S \cong \bar{G}/\bar{S}$ is a subgroup of $O := \mathrm{Out}(\bar{S})$. By Theorem 6, we need to show that $A = G/S$ satisfies both of the conditions (a) and (b) listed therein. Note that both (a) and (b) in Theorem 6 follow from the condition

(9) $\quad A$ admits a normal abelian p'-subgroup B with A/B being cyclic.

In turn, (9) is a consequence of the condition

(10) $\quad O := \mathrm{Out}(\bar{S})$ admits a normal abelian p'-subgroup J with O/J being cyclic.

(Indeed, taking $B := A \cap J$ we have $A/B \hookrightarrow O/J$.)

Set $J := \mathrm{Outdiag}(\bar{S}) := \mathrm{Inndiag}(\bar{S})/\bar{S}$. Now, if $\bar{S}$ is a *twisted* group, i.e. the parameter d for $\bar{S} \cong {}^d\Sigma(q)$ in [GLS, Theorem 2.5.12] is greater than one, then (10) holds (for this choice of J). It remains to consider the untwisted groups, i.e. the ones with $d = 1$.

If $\bar{S} = \mathrm{PSL}_2(q)$ then (10) holds. Suppose that $\bar{S} = \mathrm{PSL}_n(q)$ with $n \geq 3$, or $\bar{S} = P\Omega^+_{2n}(q)$ with $n \geq 4$, or $S = E_6(q)$. Then taking $B := \big(\bar{G} \cap \mathrm{Inndiag}(\bar{S})\big)/\bar{S}$, we see that A/B is cyclic by assumption (b) in Theorem 7, hence (9) holds.

In the remaining cases, $\bar{S}$ is of type B_n, C_n, G_2, F_4, E_7, or E_8, hence O/J is cyclic, and so (10) holds. $\Box$

Next we deduce another consequence of Theorem 6. For the definition of the *reduced Clifford group* $\Gamma^+(\mathbb{F}_q^n) = \mathrm{CSpin}^\epsilon_n(q)$, see e.g. [TZ, §6]; in particular, it contains $\mathrm{Spin}^\epsilon_n(q)$ as a normal subgroup with factor C_{q-1}.

Theorem 8. *Let q be a prime power, and let (G, S) be any of the following pairs of groups:*

(a) $G = \mathrm{GL}_n(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2), (2, 3)$, *and* $S = \mathrm{SL}_n(q)$.
(b) $G = \mathrm{GU}_n(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2), (2, 3), (3, 2)$, *and* $S = \mathrm{SU}_n(q)$.
(c) $G = \mathrm{CSp}_{2n}(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2)$, *and* $S = \mathrm{Sp}_{2n}(q)$.
(d) $G = \mathrm{CSpin}^\epsilon_n(q)$ *with* $n \geq 5$, $2 \nmid q$, *and* $\epsilon = \pm$, *and* $S = \mathrm{Spin}^\epsilon_n(q)$.
(e) $G = \mathrm{GO}^\epsilon_n(q)$ *or* $\mathrm{SO}^\epsilon_n(q)$ *with* $n \geq 5$, $2 \nmid q$, *and* $\epsilon = \pm$, *and* $S = \Omega^\epsilon_n(q)$.

Suppose that $a, b \in G$ are such that $p \nmid |\mathbf{C}_S(a)|$ and $p \nmid |\mathbf{C}_S(b)|$. If $g \in G$ is any non-central p'-element such that $g \in abS$, then $g \in a^G b^G$.

Proof. For all of the above pairs, we have that S is a quasisimple group of Lie type of simply connected type, $S \lhd G$, $F^*(G) = \mathbf{Z}(G)S$, and $\mathbf{Z}(S) \leq \mathbf{Z}(G)$. Furthermore, G/S is abelian of p'-order, and (3), (4), and (5) are all fulfilled. Hence the statement follows from Theorem 6. $\square$

Note that Theorem 6 also applies to $\mathrm{GO}^{\epsilon}_{2n}(q)$ with $2|q$ and $n \geq 3$. But we do not include them in Theorem 8 since the subgroup D is now of order 2 and so conditions (3)–(5) are more complicated than those formulated in Theorem 8.

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