Short-maturity options on realized variance in local-stochastic volatility models

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Abstract

We derive the short-maturity asymptotics for prices of options on realized variance in local-stochastic volatility models. We consider separately the short-maturity asymptotics for out-of-the-money and in-the-money options cases. The analysis for the out-of-the-money case uses large deviations theory and the solution for the rate function involves solving a two-dimensional variational problem. In the special case when the Brownian noises in the asset price dynamics and the volatility process are uncorrelated, we solve this variational problem explicitly. For the correlated case, we obtain upper and lower bounds for the rate function, as well as an expansion around the at-the-money point. Numerical simulations of the prices of variance options in a local-stochastic volatility model with bounded local volatility are in good agreement with the asymptotic results for sufficiently small maturity. The leading-order asymptotics for at-the-money options on realized variance is dominated by fluctuations of the asset price around the spot value, and is computed in closed form.

1 Introduction

Options on realized variance are derivative contracts whose payoff is linked to the annualized realized variance of the return of some asset, which can be a stock, index, interest rate, exchange rate, or futures on some asset. They are related to variance swaps, which are instruments that pay an amount equal to the realized variance at maturity.

Denoting the price of the asset on a set of discrete sampling time points $\{S_k\}_{k=0}^n$ which are uniformly spaced $0 = t_0 < t_1 < \cdots < t_n = T$, i.e. $t_i - t_{i-1} = \tau$ for every $i = 1, 2, \ldots, n$, where τ is the time step of the sampling period expressed in years (for daily sampling $\tau = \frac{1}{252}$). The annualized realized variance is given by

$$\mathrm{RV}_{T,n} = \frac{1}{n\tau} \sum_{i=1}^{n} \log^2(S_i/S_{i-1}) \,. \tag{1.1}$$

Denote the discrete-time sum

$$P_n(T) = \sum_{i=1}^{n} \log^2(S_i/S_{i-1}) \,. \tag{1.2}$$

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In practice, this sum is approximated, for sufficiently large n, with the quadratic variation of the log-asset price

$$P(T) = [\log S, \log S]_T \,. \tag{1.3}$$

If the asset price is assumed to follow a diffusion of the form $dS_t/S_t = \sigma_t dW_t + (r-q)dt$, where σ_t is an arbitrary stochastic process and W_t is a standard Brownian motion, the quadratic variation of the log-price is

$$[\log S, \log S]_T = \int_0^T \sigma_s^2 ds \,. \tag{1.4}$$

It was noted by Jarrow et al. (2013) [21] that the limit of the expectation $\mathbb{E}[P_n(T)]$ of the discrete-time approximation does not always coincide with the expectation of the continuous time quantity $\mathbb{E}[P(T)]$, since convergence in probability does not imply convergence in L_1 norm. For example, this does not hold for the 3/2 model of stochastic volatility model for certain values of the model parameters.

We will consider in this paper the continuous-time limit, and will use for the underlying of the variance swaps and options with the quadratic variation of $\log S_t$. One must keep in mind that the convergence of the discrete-time result to the continuous time counterpart will have to be checked for each case.

In this paper, we study the short-maturity asymptotics for variance options in Markovian local-stochastic volatility models. The short maturity asymptotics of options on realized variance and options on VIX has been studied by Alós, Garcia-Loríte and Gonzalez [1] and by Lacombe, Muguruza and Stone [26] in a class of multi-factor rough volatility models.

Another case which is implicitly covered in the literature is that of variance options under Markovian stochastic volatility models, which are equivalent to Asian options under local volatility models. The short maturity asymptotics of these options have been studied, for both out-of the-money and at-the-money regimes, in [30]. We will further elaborate on the connection between Asian options in local volatility models and variance options in Section 2.

To the best of our knowledge, the short-maturity asymptotics for variance options in local-stochastic volatility models has never been rigorously studied in the literature. In this paper, we present explicit results for the leading-order asymptotics, considering both cases of out-of-the-money and at-the-money variance options.

Recently, CBOE has started trading futures on realized variance since Monday 23 September 2024. These futures are called `Cboe S&P 500 Variance Futures` (Ticker: VA) and are cash-settled futures contracts based on the realized variance of the S&P 500 index.[4] They will reflect the market view of prices of cash-settled variance swaps. Options on these futures contracts are similar to options on realized variance. Our results can be applied directly to the pricing of these options.

[4] https://www.cboe.com/variance-futures-pipeline-hub/

1.1 Literature review

Pricing options on realized variance has been discussed in the literature under several types of models. We give next a brief literature review.

A class of popular models with practitioners for pricing variance products are forward variance models. Such models describe the dynamics of the instantaneous forward variance. An example of such models are the Bergomi models [6, 4, 5]. The pricing of variance options under the Bergomi models was discussed in [4, 5].

Pricing of variance options in the Heston model requires the distribution of the time integral of a CIR process $\int_0^T V_s ds$ where $dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dZ_t$. The Laplace transform of the density of $\int_0^T V_s ds$ is known in closed form; see e.g. [17]. Therefore standard transform methods can be used to price the variance options in this model; see for example Sepp (2008) [36] and Sepp (2008) [35] which also allow jumps.

Carr, Geman, Madan and Yor (2005) [8] proposed a method for pricing options on realized variance in exponential Lévy models using a Laplace transform method. Carr and Itkin (2009) [10] proposed a new asymptotic method for pricing variance and volatility swaps and options on these swaps which yields a closed-form expression for the fair price of these instruments, if the underlying process is modeled by a Lévy process with stochastic time change. Carr and Lee (2010) [11] obtained robust model-free hedges and price bounds for options on the realized variance of the returns on an underlying price process that is a positive continuous semimartingale. Kallsen, Muhle-Karbe and Voß (2011) [22] studied the pricing of variance options and determined semi-explicit formulas in general affine models allowing for jumps, stochastic volatility, and the leverage effect using the Laplace transform approach. Drimus (2012) [15] discussed the pricing and hedging of options on realized variance in the 3/2 stochastic volatility model using transform-based methods. Drimus (2012) [16] studied the pricing of options on realized variance in a general class of Log-OU stochastic volatility models. Torricelli (2013) [38] studied joint pricing on an asset and its realized variance for various stochastic volatility models.

Although the valuation of variance swaps and options on realized variance is more convenient in continuous time, in actual applications these instruments are defined in discrete time with daily time sampling. Thus, the discrete-time case received a lot of attention in the literature. Sepp (2012) [37] analyzed the effect of discrete sampling on the valuation of options on the realized variance in the Heston model.

The systematic bias of the time discretization and the asymptotics for variance swaps was studied for a few popular stochastic volatility models by Bernard and Cui (2013) [7]. Keller-Ressel and Muhle-Karbe (2013) [23] found that the difference between options on discretely sampled realized variance and the continuous time limit strongly depends on whether or not the stock price process has jumps. They proposed an approximation method based on correcting prices of options on quadratic variation by their asymptotic results, and an exact method using a novel randomization approach and applying Fourier-Laplace techniques. Lian, Chiarella and Kalev (2014) [27] obtained an accurate approximation for

the characteristic function of the discretely sampled realized variance, which yielded semi-analytical pricing formulae for variance options and other derivatives. Zheng and Kwok (2014) [43] used a saddlepoint approximation method to price options on discrete realized variance, and the same authors derived in [41] closed-form pricing formulas for discretely sampled variance swaps. Zheng and Kwok (2014) [42] developed efficient fast Fourier transform algorithms to price and hedge options on discrete realized variance and other products under time-inhomogeneous Lévy processes. Zheng and Kwok (2015) [44] used the partially exact and bounded approximations to derive efficient and accurate analytic approximation formulas for pricing options on discrete realized variance under affine stochastic volatility models with jumps. Zheng, Yuen and Kwok (2016) [45] developed recursive algorithms for pricing pricing variance options and volatility swaps on discrete realized variance under general time-changed Lévy processes. Drimus, Farkas and Gourier (2016) [14] studied the valuation of options on discretely sampled variance by analyzing the discretization effect and obtaining an analytical correction term to be applied to the value of options on continuously sampled variance under general stochastic volatility dynamics. Cui, Kirkby, and Nguyen (2017) [12] developed a transform-based method to price swaps and options related to discretely-sampled realized variance under a general class of stochastic volatility models with jumps. A survey on recent results in the pricing of derivatives on discrete realized variance can be found in Kwok and Zheng (2022) [25].

Moreover, various properties for variance options have been studied in the literature. Carr, Geman, Madan and Yor (2011) [9] analyzed the property of monotonicity in maturity for call options at a fixed strike for realized variance option and options on quadratic variation normalized to unit. Griessler and Keller-Ressel [20] showed that options on variance are typically underpriced if quadratic variation is substituted for the discretely sampled realized variance for a class of models including independently time-changed Lévy models and Sato processes with symmetric jumps.

Finally, our work is related to the short-maturity asymptotics for path-dependent option prices that have been studied in the recent literature. The main tools are the sample-path large deviation principle for small-time diffusion processes that dates back to [39] and the contraction principle from large deviations theory (see e.g. [40, 13]). Most of such works have been focused on the short-maturity asymptotics for Asian options. The first short-maturity asymptotics result for Asian options was obtained in [30] for local volatility models. The out-of-the-money (OTM) case relies on large deviations for small-time diffusion processes, and the rate function is a one-dimensional variational problem that can be solved explicitly [30]. Similar studies have been carried out for Asian options under the CEV model [31]. The short-maturity forward start Asian option has been studied in [28]. By a combination of a Gaussian process approximation and Malliavin calculus, [24] studied both pricing and hedging for short-maturity Asian options for local volatility models.

1.2 Summary of the paper

The outline of the paper is as follows. In Section 2 we introduce the model for the underlying and specify technical assumptions on the model parameters. Section 3 contains the main results. The short-maturity asymptotics for OTM variance options are given in Section 3.1. We show that by using large deviations theory, the leading-order term in the short-maturity asymptotics for OTM variance options can be formulated as a two-dimensional variational problem. In Section 3.2 we discuss the solution of this variational problem. For the particular case when the Brownian noises in the asset price dynamics and the volatility process are uncorrelated, we solve the variational problem for the rate function explicitly by reducing it to a one-dimensional optimization problem. The argument of this optimization problem includes the rate function for short-maturity Asian options under local volatility models obtained in [30]. For the correlated case, we obtain upper and lower bounds for the solution of the variational problem. In Section 3.3, we solve the variational problem for perfectly correlated and anti-correlated asset and volatility. In Section 3.4, we give an explicit result for the rate function of the variance options in an expansion in log-moneyness, which is convenient for use in practical applications. Finally, in Section 3.5, further short-maturity asymptotics results are obtained when the variance options are at-the-money (ATM).

Section 4 discusses the application of the asymptotic results to pricing variance options in the local-stochastic volatility model. Numerical tests of the asymptotic results are presented. For the uncorrelated case $\rho = 0$ we compare the expansion in log-moneyness with an exact solution of the variational problem. The expansion is also tested by comparing with general bounds, and by comparing with Monte Carlo simulations in a model with bounded local volatility.

Some background of large deviations theory is presented in Appendix A. The technical proofs of all the results in the main paper are provided in Appendix B. Finally, some additional technical results are provided in Appendix C.

2 Model Setup

In this paper, we are interested in studying variance options under a local-stochastic volatility model. Suppose that under the risk-neutral probability measure $\mathbb{Q}$ the underlying asset S_t and the variance process V_t follow a local-stochastic volatility model of the form:

$$\frac{dS_t}{S_t} = (r-q)dt + \eta(S_t)\sqrt{V_t}\rho dZ_t + \eta(S_t)\sqrt{V_t}\sqrt{1-\rho^2}dW_t, \tag{2.1}$$

$$\frac{dV_t}{V_t} = \mu(V_t)dt + \sigma(V_t)dZ_t, \tag{2.2}$$

where r is the risk-free rate, q is the dividend yield, and W_t, Z_t are two independent standard Brownian motions, where the functions $\eta(\cdot), \sigma(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and $\mu(\cdot) : \mathbb{R}^+ \to \mathbb{R}$

are assumed to be time-homogeneous for simplicity.

The payoff of variance options is linked to the realized variance of the asset price. In continuous time this is related to the quadratic variation of the log-price, which is expressed in the model (2.1), (2.2) as

$$[\log S]_T = \int_0^T V_s \eta^2(S_s) ds \,. \tag{2.3}$$

The fair strike of a variance swap with maturity T is defined as

$$F_V(T) = \mathbb{E}\left[\int_0^T V_s \eta^2(S_s) ds\right]. \tag{2.4}$$

The call and put variance options prices are given by

$$\begin{aligned} C(T) &= e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s \eta^2(S_s) ds - K\right)^+\right], \\ P(T) &= e^{-rT}\mathbb{E}\left[\left(K - \frac{1}{T}\int_0^T V_s \eta^2(S_s) ds\right)^+\right], \end{aligned} \tag{2.5}$$

where $K > 0$ is the strike price and $T > 0$ is the maturity. A variance call option is out-of-the-money (OTM) if $K > F_V(T)$, in-the-money (ITM) if $K < F_V(T)$ and at-the-money (ATM) if $K = F_V(T)$. A variance put option is OTM if $K < F_V(T)$, ITM if $K > F_V(T)$ and ATM if $K = F_V(T)$.

We first assume that $\eta(\cdot), \mu(\cdot)$ and $\sigma(\cdot)$ are uniformly bounded for simplicity.

Assumption 1. *We assume that $\eta(\cdot), \mu(\cdot)$ and $\sigma(\cdot)$ are uniformly bounded:*

$$\sup_{x\in\mathbb{R}^+} \eta(x) \le M_\eta, \qquad \sup_{x\in\mathbb{R}^+} |\mu(x)| \le M_\mu, \qquad \sup_{x\in\mathbb{R}^+} \sigma(x) \le M_\sigma. \tag{2.6}$$

In the remainder of the paper, we often further assume that $\eta(\cdot)$ is decreasing, which will be stated explicitly where this is assumed, such that our model satisfies the leverage effect in finance. More precisely, when $\eta(\cdot)$ is not a constant function, we assume that $\eta(\cdot)$ is strictly decreasing so that its inverse function $\eta^{-1}(\cdot)$ exists. We also provide the following assumptions about the Lipschitz continuity.

Assumption 2. *We assume that η is ℓ_η-Lipschitz and σ is ℓ_σ-Lipschitz.*

In addition, we impose the following assumption on $\eta(\cdot)$ and $\sigma(\cdot)$ that appear in the diffusion terms of (2.1)-(2.2) that is needed for the small-time large deviations estimates for (2.1)-(2.2).

Assumption 3. *We assume that $\inf_{x\in\mathbb{R}^+} \sigma(x) > 0$ and $\inf_{x\in\mathbb{R}^+} \eta(x) > 0$. Moreover, there exist some constants $M, \alpha > 0$ such that for any $x, y \in \mathbb{R}^+$, $|\sigma(e^x) - \sigma(e^y)| \le M|x-y|^\alpha$ and $|\eta(e^x) - \eta(e^y)| \le M|x-y|^\alpha$.*

To satisfy the leverage effect that is commonly observed in finance, it is often assumed in the literature that $\eta(\cdot)$ is monotonically decreasing. That is, the larger value of the asset price S_t, the smaller value of the volatility function $\eta(S_t)$.

When $\eta(\cdot) \equiv 1$, the local-stochastic volatility model (2.1)-(2.2) reduces to the stochastic volatility model:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sqrt{V_t}\left(\sqrt{1-\rho^2}dW_t + \rho dZ_t\right), \\ \frac{dV_t}{V_t} &= \mu(V_t)dt + \sigma(V_t)dZ_t, \end{aligned} \tag{2.7}$$

where r is the risk-free rate, q is the dividend yield, and W_t and Z_t are two independent Brownian motions, and for variance options, the call and put options prices in (2.5) reduce to:

$$C(T) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s ds - K\right)^+\right], \qquad P(T) = e^{-rT}\mathbb{E}\left[\left(K - \frac{1}{T}\int_0^T V_s ds\right)^+\right], \tag{2.8}$$

where $K > 0$ is the strike price. This shows that the variance option prices $C(T), P(T)$ in the stochastic volatility model (2.7) are identical to those of Asian options in the local volatility model with local volatility function $\sigma(V)$. The short-maturity asymptotics for Asian options under local volatility models have been studied in [30, 31, 32, 24].

Certain models used in financial practice, such as the Heston model, do not satisfy the Assumptions 1 and 3. However, our arguments can be extended to these models as well, under alternative assumptions; see Remark 5.

In the rest of the paper, we focus on the variance options under the local-stochastic volatility model (2.1)-(2.2) and study the call option price $C(T)$ and put option price $P(T)$ given in (2.5). Note the short-maturity limit of the fair strike of the variance swap (2.4) is

$$F_V(0) = \lim_{T\to 0} F_V(T) = \eta^2(S_0)V_0\,. \tag{2.9}$$

In this paper, we are interested in the short-maturity asymptotics ($T \to 0$) for $C(T)$ and $P(T)$. We distinguish two cases: the out-of-the-money (OTM) case and the at-the-money (ATM) case. In the short maturity limit, by (2.9), the moneyness is measured with respect to $F_V(0) = \eta^2(S_0)V_0$. More explicitly, the OTM call option corresponds to $V_0\eta^2(S_0) < K$, and the OTM put option corresponds to $V_0\eta^2(S_0) > K$. The ATM case for both call and put options corresponds to $V_0\eta^2(S_0) = K$. We omit the discussions for in-the-money (ITM) case here, which can be analyzed via put-call parity.

3 Main Results

3.1 OTM case

In the short-maturity regime, i.e. as $T \to 0$, we have $\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds$ converges a.s. to $V_0\eta^2(S_0)$. Therefore, the OTM case for variance call corresponds to $V_0\eta^2(S_0) < K$ and the OTM case for variance put options corresponds to $V_0\eta^2(S_0) > K$. We are interested in studying the OTM short-maturity asymptotics for variance call and put options. We have the following main result.

Theorem 4. *Suppose that Assumptions 1 and 3 hold, and that the correlation parameter is in the range $-1 < \rho < +1$.*

(i) For OTM variance call options, i.e. $V_0\eta^2(S_0) < K$, we have

$$\lim_{T\to 0} T\log C(T) = -\mathcal{I}_\rho(S_0, V_0, K), \tag{3.1}$$

where

$$\begin{aligned}
&\mathcal{I}_\rho(S_0, V_0, K)\\
&= \inf_{\substack{g(0)=\log S_0, h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt = K}} \left\{ \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}} - \frac{\rho h'(t)}{\sigma(e^{h(t)})} \right)^2 dt \right.\\
&\qquad \left. + \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})} \right)^2 dt \right\}.
\end{aligned} \tag{3.2}$$

(ii) For OTM variance put options, i.e. $V_0\eta^2(S_0) > K$, we have

$$\lim_{T\to 0} T\log P(T) = -\mathcal{I}_\rho(S_0, V_0, K), \tag{3.3}$$

where $\mathcal{I}_\rho(S_0, V_0, K)$ is defined in (3.2).

In Theorem 4, $\mathcal{I}_\rho(S_0, V_0, K)$ is the rate function from large deviations theory. It is written as the solution to a variational problem optimizing over two functions g, h. In the ATM limit, i.e. $V_0\eta^2(S_0) = K$, by letting $h'(t) \equiv 0$ and $g'(t) \equiv 0$ in (3.2), one gets $\mathcal{I}_\rho(S_0, V_0, K) = 0$, which corresponds to the law of large numbers limit.

The edge cases $\rho = \pm 1$ corresponding to perfectly correlated/anticorrelated asset and volatility processes, require a separate treatment due to the singular coefficient in the first term of (3.2), and will be covered in Proposition 14.

Remark 5. *Several models popular in financial practice, for example the Heston-type models with $\sigma(V) = \sigma V^{-1/2}$ for some $\sigma > 0$ in Eqn. (2.2), do not satisfy Assumptions 1 and 3. Although Theorem 4 is obtained under the Assumptions 1 and 3, one can see that these*

results hold also for this case as long as $\mathbb{Q}(\{(\log S_{Tt}, \log V_{Tt}), 0 \leq t \leq 1\} \in \cdot)$ *satisfies a sample-path large deviation principle (which is true for the Heston-type and CEV-type SDEs without Assumptions 1 and 3 (see e.g. [2, 34])). One can also easily check that p-th moments of* V_t *are bounded on* $[0, T]$ *for any* $p > 1$ *(see e.g. [29]) as in the proof of Theorem 4. On the other hand, we do not pursue Heston-type models in our paper because they are disfavored by market data on options on realized variance. More precisely, they produce a down-sloping implied volatility smile for variance options, which is different from the observed trend (up-sloping smile); see e.g. [15].*

In the next section, we will analyze the variational problem (3.2). We will show that in the special case when $\rho = 0$, we can solve the variational problem (3.2) in closed form, and for general ρ, we will derive upper and lower bounds on the rate function. We give also explicit solutions in the limit $\rho = \pm 1$ of perfectly correlated and anti-correlated models. In the limit of a stochastic volatility model $\eta(\cdot) \equiv 1$, the rate function reduces to the rate function for Asian options in the local volatility model which was computed previously in [30]. We confirm that the result obtained here reduces to the known result in the limit $\eta(\cdot) \equiv 1$. We also derive an expansion of the rate function around the ATM point which can be used for practical applications.

3.2 Variational problem

3.2.1 Zero correlation

In this section, we will show that for the special case $\rho = 0$, we can solve the variational problem (3.2) in closed form. When $\rho = 0$, the underlying asset S_t follows a stochastic volatility model of the form:

$$\frac{dS_t}{S_t} = (r - q)dt + \eta(S_t)\sqrt{V_t}dW_t, \tag{3.4}$$

$$\frac{dV_t}{V_t} = \mu(V_t)dt + \sigma(V_t)dZ_t, \tag{3.5}$$

where W_t and Z_t are two independent standard Brownian motions. When $\rho = 0$, we are able to solve the variational problem (3.2) in closed form. The solution takes a simpler form for monotonic local volatility $\eta(x)$, but this assumption is not essential and can be relaxed. The general case of non-monotonic $\eta(x)$ can be treated by separating the range of x into domains of monotonicity. For definiteness, we give next the result for a monotonic non-increasing $\eta(x)$.

Proposition 6. *Suppose the assumptions in Theorem 4 hold. Assume that the local volatility function* $\eta(x)$ *is non-increasing. When* $\rho = 0$*, the variational problem (3.2) has the following solution.*

(i) For OTM variance call options, i.e. $V_0\eta^2(S_0) < K$, we have

$$\mathcal{I}_0(S_0, V_0, K) = \inf_z \left\{ \frac{1}{K^2} \left(\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(G_c(z)) - \eta^2(x)\right)}} \right)^2 \left(\eta^2(G_c(z))z - K\right) + \mathcal{J}(V_0, z) \right\}, \quad (3.6)$$

where $G_c(z)$ satisfies the equation

$$\frac{\int_{S_0}^{G_c(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(G_c(z)) - \eta^2(x)}}}{\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(G_c(z)) - \eta^2(x)}}} = \frac{z}{K}, \quad (3.7)$$

and

$$\mathcal{J}(V_0, z) := \inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt = z} \frac{1}{2} \int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})} \right)^2 dt. \quad (3.8)$$

(ii) For OTM variance put options, i.e. $V_0\eta^2(S_0) > K$, we have

$$\mathcal{I}_0(S_0, V_0, K) = \inf_z \left\{ \frac{1}{K^2} \left(\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x) - \eta^2(G_p(z))\right)}} \right)^2 \left(K - \eta^2(G_p(z))z\right) + \mathcal{J}(V_0, z) \right\}, \quad (3.9)$$

where $G_p(z)$ satisfies the equation

$$\frac{\int_{S_0}^{G_p(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(x) - \eta^2(G_p(z))}}}{\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(x) - \eta^2(G_p(z))}}} = \frac{z}{K}, \quad (3.10)$$

and $\mathcal{J}(V_0, z)$ is defined in (3.8).

Remark 7. *Note that when the options are ATM as $T \to 0$, i.e. when $V_0\eta^2(S_0) = K$, the rate function $\mathcal{I}_0(S_0, V_0, K) = 0$, which is consistent with the law of large numbers. Notice that when $V_0\eta^2(S_0) = K$, if we take $z = V_0$, then $\mathcal{J}(V_0, V_0) = 0$ since we can take $h(t) \equiv h(0) = \log V_0$ in (3.8), and moreover, with $z = V_0$, one can check that $G_c(z) = G_p(z) = S_0$ since when $z = V_0$, we have $\frac{z}{K} = \frac{V_0}{K} = \frac{1}{\eta^2(S_0)}$, and one can check that*

$$\lim_{G_c(z)\to S_0} \frac{\int_{S_0}^{G_c(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(G_c(z)) - \eta^2(x)}}}{\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(G_c(z)) - \eta^2(x)}}} = \lim_{G_p(z)\to S_0} \frac{\int_{S_0}^{G_p(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(x) - \eta^2(G_p(z))}}}{\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(x) - \eta^2(G_p(z))}}} = \frac{1}{\eta^2(S_0)}. \quad (3.11)$$

Hence, we conclude that when $V_0\eta^2(S_0) = K$, the optimal $z = V_0$ in (3.6) and (3.9) and $G_c(z) = G_p(z) = \frac{1}{\eta^2(S_0)}$.

We illustrate the application of Proposition 6 in Section 4.1 on a particular local volatility function $\eta(x)$ given by the Tanh model. For this case the equations (3.7) and (3.10) have unique solutions for $G_c(z), G_p(z)$. In general they may have multiple solutions, and the infimum in (3.6) and (3.9) must be taken also over all solutions of these equations.

Remark 8. *The optimization problem*

$$\mathcal{J}(V_0, z) = \inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt = z} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \tag{3.12}$$

has already been solved in [30]. In particular, Proposition 8 in [30] showed that

$$\mathcal{J}(V_0, z) = \begin{cases} \frac{1}{2}F^{(-)}(f_1)G^{(-)}(f_1), & \text{for } z > V_0, \\ \frac{1}{2}F^{(+)}(h_1)G^{(+)}(h_1), & \text{for } z < V_0, \end{cases} \tag{3.13}$$

where f_1 is the solution of the equation $e^{f_1} - z/V_0 = G^{(-)}(f_1)/F^{(-)}(f_1)$ and h_1 is the solution of the equation $z/V_0 - e^{-h_1} = G^{(+)}(h_1)/F^{(+)}(h_1)$. The functions $F^{(\pm)}(x), G^{(\pm)}(x)$ are defined as

$$G^{(-)}(x) := \int_0^x \frac{\sqrt{e^x - e^y}}{\sigma(V_0 e^y)}dy, \quad F^{(-)}(x) := \int_0^x \frac{dy}{\sigma(V_0 e^y)\sqrt{e^x - e^y}}dy \tag{3.14}$$

$$G^{(+)}(x) := \int_0^x \frac{\sqrt{e^{-y} - e^{-x}}}{\sigma(V_0 e^{-y})}dy, \quad F^{(+)}(x) := \int_0^x \frac{dy}{\sigma(V_0 e^{-y})\sqrt{e^{-y} - e^{-x}}}dy. \tag{3.15}$$

In particular, when $\sigma(\cdot) \equiv \sigma_0$, the solution simplifies and is given by (Proposition 12 in [30])

$$\mathcal{J}(V_0, z) = \begin{cases} \frac{1}{\sigma_0^2}\left(\frac{1}{2}\beta^2 - \beta\tanh\left(\frac{\beta}{2}\right)\right), & \text{for } z > V_0, \\ \frac{2}{\sigma_0^2}\xi(\tan\xi - \xi), & \text{for } z < V_0, \end{cases} \tag{3.16}$$

where β is the solution of the equation $\frac{1}{\beta}\sinh\beta = \frac{z}{V_0}$ for $z \geq V_0$ and ξ is the solution in the interval $[0, \frac{\pi}{2}]$ of the equation $\frac{1}{2\xi}\sin(2\xi) = \frac{z}{V_0}$ for $z \leq V_0$.

Remark 9. *Note that for call options in (3.6), if we let z such that $G_c(z) = S_0$, then $\mathcal{I}_0(S_0, V_0, K) \leq \mathcal{J}(V_0, z)$. Moreover, when $G_c(z) \to S_0$, it follows from (3.7) that the left hand side of (3.7) converges to $\frac{1}{\eta^2(S_0)}$ such that $z \to \frac{K}{\eta^2(S_0)}$. Thus, when $G_c(z) = S_0$, we have $z = \frac{K}{\eta^2(S_0)}$. Hence, we obtain the upper bound $\mathcal{I}_0(S_0, V_0, K) \leq \mathcal{J}\left(V_0, \frac{K}{\eta^2(S_0)}\right)$. It is similar to check that a similar upper bound holds for the put options in (3.9).*

Proposition 6 solves the variational problem (3.2) when $\rho = 0$ and obtains a simplified expression for $\mathcal{I}_0(S_0, V_0, K)$. As a corollary from the proof of Proposition 6, we are able to obtain the optimal g and h that solve the variational problem (3.2) for the uncorrelated case $\rho = 0$.

Remark 10. *Denote g_0, h_0 the optimizers of the variational problem for the uncorrelated case $\rho = 0$. They are given explicitly as follows.*

(i) For OTM variance call options, i.e. $V_0\eta^2(S_0) < K$, then the optimal g_0 is given by

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\left(\eta^2(G_c(z_c)) - \eta^2(x)\right)}} = \frac{\int_{S_0}^{G_c(z_c)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(G_c(z_c)) - \eta^2(x))}}}{K} \int_0^t e^{h_0(s)}ds, \quad (3.17)$$

where the optimal $h_0(t) = \log V_0 + f_0(t; z_c)$ with

$$\begin{cases} \int_0^{f_0(t;z)} \frac{dy}{\sigma(V_0e^y)\sqrt{e^{\alpha_-} - e^y}} = F^{(-)}(\alpha_-(z))t & \text{when } z > V_0, \\ \int_0^{f_0(t;z)} \frac{dy}{\sigma(V_0e^y)\sqrt{e^y - e^{-\alpha_+}}} = -F^{(+)}(\alpha_+(z))t & \text{when } z < V_0, \end{cases} \quad (3.18)$$

for any $0 \le t \le 1$, where $\alpha_+ = \alpha_+(z)$ is the solution of the equation

$$\frac{z}{V_0} - e^{-\alpha_+} = \frac{G^{(+)}(\alpha_+)}{F^{(+)}(\alpha_+)}, \quad (3.19)$$

with

$$G^{(+)}(\alpha_+) = \int_0^{\alpha_+} \frac{\sqrt{e^{-y} - e^{-\alpha_+}}}{\sigma(V_0e^{-y})}dy, \quad F^{(+)}(\alpha_+) = \int_0^{\alpha_+} \frac{1}{\sigma(V_0e^{-y})} \frac{1}{\sqrt{e^{-y} - e^{-\alpha_+}}}dy\,, \quad (3.20)$$

and $\alpha_- = \alpha_-(z)$ is the solution of the equation

$$e^{\alpha_-} - \frac{z}{V_0} = \frac{G^{(-)}(\alpha_-)}{F^{(-)}(\alpha_-)}, \quad (3.21)$$

with

$$G^{(-)}(\alpha_-) = \int_0^{\alpha_-} \frac{\sqrt{e^{\alpha_-} - e^y}}{\sigma(V_0e^y)}dy, \quad F^{(-)}(\alpha_-) = \int_0^{\alpha_-} \frac{1}{\sigma(V_0e^y)} \frac{1}{\sqrt{e^{\alpha_-} - e^y}}dy\,. \quad (3.22)$$

(ii) For OTM variance put options, i.e. $V_0\eta^2(S_0) > K$, then the optimal g_0 is given by

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\left(\eta^2(x) - \eta^2(G_p(z_p))\right)}} = \frac{\int_{S_0}^{G_p(z_p)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(x) - \eta^2(G_p(z_p)))}}}{K} \int_0^t e^{h_0(s)}ds, \quad (3.23)$$

where the optimal $h_0(t) = \log V_0 + f_0(t; z_p)$, with $f_0(t; z)$ being defined in (i).

3.2.2 Non-zero correlation

In this section, we discuss the general $\rho \neq 0$ case. We first recall the variational problem for the rate function, see (3.2)

$$\mathcal{I}_\rho(S_0, V_0, K) = \inf_{\substack{g(0)=\log S_0, h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}} \Lambda_\rho[g,h], \tag{3.24}$$

where

$$\Lambda_\rho[g,h] := \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}} - \frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2 dt + \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt\,. \tag{3.25}$$

The optimal g and h for the variational problem (3.24) can be determined implicitly via the Euler-Lagrange equations. First, let us define:

$$\begin{aligned}
\Lambda_\rho[g,h;\lambda] :=& \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}} - \frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2 dt + \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&+ \lambda\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt \\
=& \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&- \frac{\rho}{1-\rho^2}\int_0^1 \frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\frac{h'(t)}{\sigma(e^{h(t)})}dt + \lambda\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt,
\end{aligned} \tag{3.26}$$

where λ is the Lagrange multiplier. The optimal g and h for the variational problem (3.24) satisfy the Euler-Lagrange equations for the functional Λ_ρ. They are given by $\frac{\delta\Lambda_\rho}{\delta g} = \frac{d}{dt}\left(\frac{\delta\Lambda_\rho}{\delta g'}\right)$ and $\frac{\delta\Lambda_\rho}{\delta h} = \frac{d}{dt}\left(\frac{\delta\Lambda_\rho}{\delta h'}\right)$ which leads to

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{1-\rho^2}\frac{g'(t)}{\eta^2(e^{g(t)})e^{h(t)}} - \frac{\rho}{1-\rho^2}\frac{h'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}\sigma(e^{h(t)})}\right) \\
&= -\frac{1}{1-\rho^2}\frac{(g'(t))^2\eta'(e^{g(t)})e^{g(t)}}{\eta^3(e^{g(t)})e^{h(t)}} + \frac{\rho}{1-\rho^2}\frac{g'(t)h'(t)\eta'(e^{g(t)})e^{g(t)}}{\eta^2(e^{g(t)})\sqrt{e^{h(t)}}\sigma(e^{h(t)})} \\
&\qquad + 2\lambda e^{h(t)}\eta(e^{g(t)})\eta'(e^{g(t)})e^{g(t)},
\end{aligned} \tag{3.27}$$

and

$$\frac{d}{dt}\left(\frac{1}{1-\rho^2}\frac{h'(t)}{\sigma^2(e^{h(t)})}-\frac{\rho}{1-\rho^2}\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}\sigma(e^{h(t)})}\right)$$
$$=-\frac{1}{2(1-\rho^2)}\frac{(g'(t))^2}{\eta^2(e^{g(t)})e^{h(t)}}-\frac{1}{1-\rho^2}\frac{(h'(t))^2\sigma'(e^{h(t)})e^{h(t)}}{\sigma^3(e^{h(t)})}$$
$$+\frac{\rho}{1-\rho^2}\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\frac{h'(t)\sigma'(e^{h(t)})e^{h(t)}}{\sigma^2(e^{h(t)})}+\frac{\rho}{2(1-\rho^2)}\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\frac{h'(t)}{\sigma(e^{h(t)})}$$
$$+\lambda e^{h(t)}\eta^2(e^{g(t)}), \tag{3.28}$$

with the constraints $g(0)=\log S_0$, $h(0)=\log V_0$ and $\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K$. The transversality condition gives $g'(1)=h'(1)=0$.

Even though it seems impossible to solve the Euler-Lagrange equations (3.27)-(3.28) (and hence the variational problem for $\mathcal{I}_\rho(S_0,V_0,K)$) in closed form, we can obtain the following lower and upper bounds for $\mathcal{I}_\rho(S_0,V_0,K)$.

Proposition 11. *For any $\rho\in(-1,1)$, we have*

$$\frac{1}{1+|\rho|}\mathcal{I}_0(S_0,V_0,K)\le\mathcal{I}_\rho(S_0,V_0,K)\le\frac{1}{1-|\rho|}\mathcal{I}_0(S_0,V_0,K), \tag{3.29}$$

where $\mathcal{I}_0(S_0,V_0,K)$ is computed in closed form in Proposition 6.

The bounds in Proposition 11 can be used to obtain bounds for the asymptotics of the OTM variance options. We also notice that Proposition 11 works well when $|\rho|$ is small. Indeed, when $|\rho|$ is small, we expect that the optimal g,h for the variational problem (3.24) should be close to g_0,h_0 which are the optimal solutions for the variational problem (3.24) when $\rho=0$ that can be solved analytically; see Corollary 10. This helps us establish the following upper bound for $\mathcal{I}_\rho(S_0,V_0,K)$ which we expect to work well when $|\rho|$ is small.

Proposition 12. *For any $\rho\in(-1,1)$, we have*

$$\mathcal{I}_\rho(S_0,V_0,K)\le\Lambda_\rho[g_0,h_0], \tag{3.30}$$

where $\Lambda_\rho[\cdot,\cdot]$ is defined in (3.25) and g_0,h_0 are the optimal solutions for the variational problem (3.24) when $\rho=0$.

Next, we provide another upper bound for $\mathcal{I}_0(S_0,V_0,K)$, which is an extension to the upper bound in Remark 9 when $\rho=0$.

Proposition 13. *For any $\rho\in(-1,1)$, we have*

$$\mathcal{I}_\rho(S_0,V_0,K)\le\frac{1}{1-\rho^2}\mathcal{J}\left(V_0,\frac{K}{\eta^2(S_0)}\right), \tag{3.31}$$

where $\mathcal{J}(\cdot,\cdot)$ is defined in (3.8).

Although Proposition 11 works well when $|\rho|$ is small, when $|\rho|\to1$, the upper bound in Proposition 11 becomes trivial. Next, we will analyze the $|\rho|\to1$ case in detail.

3.3 Perfectly correlated and anti-correlated cases

In this section, we consider the case of perfect correlation $\rho = +1$ and perfectly anti-correlated asset price and volatility $\rho = -1$. We show that for these cases the variational problem for $\mathcal{I}_\rho(S_0, V_0, K)$ given in Theorem 4 can be further analyzed. We have the following result.

Proposition 14. *Denote $\mathcal{I}_{\pm 1}(S_0, V_0, K)$ the rate function for the perfectly correlated/anti-correlated case $\rho = \pm 1$. Then we have*

$$\mathcal{I}_{\pm 1}(S_0, V_0, K) = \inf_{\substack{h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(\mathcal{F}_\pm(e^{h(t)}))dt=K}} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt, \tag{3.32}$$

where $\mathcal{F}_\pm(\cdot)$ is defined as:

$$\int_{S_0}^{\mathcal{F}_\pm(x)} \frac{dy}{y\eta(y)} = \int_{V_0}^{x} \frac{\pm dy}{\sqrt{y}\sigma(y)}, \qquad \text{for any } x > 0. \tag{3.33}$$

Remark 15. *For the special case $\eta(\cdot) \equiv \eta_0$, the optimization problem*

$$\mathcal{J}(V_0, K/\eta_0^2) = \inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt=K/\eta_0^2} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \tag{3.34}$$

has already been solved in [30]. The solution is given in Proposition 8 in [30] and reads

$$\mathcal{J}(V_0, K/\eta_0^2) = \begin{cases} \frac{1}{2}F^{(-)}(f_1)G^{(-)}(f_1), & \text{for } K/\eta_0^2 > V_0, \\ \frac{1}{2}F^{(+)}(h_1)G^{(+)}(h_1), & \text{for } K/\eta_0^2 < V_0, \end{cases} \tag{3.35}$$

where f_1 is the solution of the equation $e^{f_1} - K/(\eta_0^2 V_0) = G^{(-)}(f_1)/F^{(-)}(f_1)$ and h_1 is the solution of the equation $K/(\eta_0^2 V_0) - e^{-h_1} = G^{(+)}(h_1)/F^{(+)}(h_1)$, where $\mathcal{F}^{(\mp)}(\cdot)$ and $\mathcal{G}^{(\mp)}(\cdot)$ are defined in (3.14)-(3.15).

In the particular limit when $\sigma(\cdot) \equiv \sigma_0$, this simplifies further and can be expressed in closed form through $J_{\mathrm{BS}}(k)$, the rate function for Asian options in the Black-Scholes model, given by (see Proposition 12 in [30])

$$\mathcal{J}(V_0, K/\eta_0^2) = \frac{1}{\sigma_0^2} J_{\mathrm{BS}}(K/(\eta_0^2 V_0)) := \begin{cases} \frac{1}{\sigma_0^2}\left(\frac{1}{2}\beta^2 - \beta\tanh\left(\frac{\beta}{2}\right)\right), & \text{for } K > \eta_0^2 V_0, \\ \frac{2}{\sigma_0^2}\xi(\tan\xi - \xi), & \text{for } K < \eta_0^2 V_0, \end{cases} \tag{3.36}$$

where β is the solution of the equation $\frac{1}{\beta}\sinh\beta = \frac{K}{\eta_0^2 V_0}$ for $K \geq \eta_0^2 V_0$ and ξ is the solution in the interval $[0, \frac{\pi}{2}]$ of the equation $\frac{1}{2\xi}\sin(2\xi) = \frac{K}{\eta_0^2 V_0}$ for $K \leq \eta_0^2 V_0$.

Remark 16. *More generally, the variational problem (3.32) can be solved analytically. By a change of variable it can be reformulated as the variational problem for the asymptotics of Asian options in local volatility models which was solved in [30]. To see this, let us first define* $\mathcal{G}_\pm(x) := x\eta^2(\mathcal{F}_\pm(x))$ *for any* $x > 0$ *and assume that the inverse function* $\mathcal{G}_\pm^{-1}(\cdot)$ *exists. Next, let* $g(t) = \log \mathcal{G}_\pm(e^{h(t)})$ *in (3.32), such that* $h(t) = \log \mathcal{G}_\pm^{-1}(e^{g(t)})$*. Then, we can rewrite (3.32) as*

$$\mathcal{I}_{\pm 1}(S_0, V_0, K) = \inf_{\substack{g(0)=\log \mathcal{G}_\pm(V_0) \\ \int_0^1 e^{g(t)}dt = K}} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\hat{\sigma}(e^{g(t)})}\right)^2 dt\,, \tag{3.37}$$

where $\hat{\sigma}(S) := \frac{1}{S}\mathcal{G}_\pm^{-1}(S)\mathcal{G}_\pm'(\mathcal{G}_\pm^{-1}(S))\sigma(\mathcal{G}_\pm^{-1}(S))$ *for any* $S > 0$*. Then, (3.37) is exactly the rate function for Asian option for local volatility models with the local volatility* $\hat{\sigma}(\cdot)$*, the spot asset price* $\mathcal{G}_\pm(V_0)$ *and the strike price* K*; see [30]. The variational problem (3.37) has been solved analytically in [30].*

Note that Proposition 14 concerns the case $\rho = \pm 1$. When ρ is close to ± 1, the factor $\frac{1}{2(1-\rho^2)}$ in (3.25) is large, and we expect that the choice of g, h to make the first term in (3.25) zero becomes a good choice since otherwise the first term in (3.25) would become large when ρ is close to ± 1. Using this intuition, we obtain the following result, that provides an upper bound for $\mathcal{I}_\rho(S_0, V_0, K)$ in (3.24) and we expect that this provides a good approximation when ρ is close to ± 1.

Proposition 17. *For any* $\rho \in (-1, 1)$*,*

$$\mathcal{I}_\rho(S_0, V_0, K) \le \inf_{\substack{h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(\mathcal{F}_\rho(e^{h(t)}))dt = K}} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt, \tag{3.38}$$

where $\mathcal{F}_\rho(\cdot)$ *is defined as:*

$$\int_{S_0}^{\mathcal{F}_\rho(x)} \frac{dy}{y\eta(y)} = \int_{V_0}^{x} \frac{\rho dy}{\sqrt{y}\sigma(y)}, \qquad \text{for any } x > 0. \tag{3.39}$$

Remark 18. *For the special case* $\eta(\cdot) \equiv \eta_0$*, the upper bound in Proposition 17 can be solved in closed form as discussed in Remark 15. More generally, the upper bound in Proposition 17 can be reformulated and solved as discussed in Remark 16.*

The previous simplification of the variational problem still requires solving a variational problem of a single function or can be reduced further to a finite-dimensional optimization problem.

3.4 Expansion of the rate function around the ATM point

In this section we consider the expansion of the rate function around the ATM point. Our approach is inspired by the recent work [33] that conducts an expansion of the rate function around the ATM point for Asian options under the local-stochastic volatility model (2.1)-(2.2) to solve a two-dimensional variational problem. We mention also the expansion used in [3, 19] to solve a variational problem for the rate function for a stochastic volatility model with rough volatility. The expansion approach has the advantage of providing explicit formulas that are easier to use in practice than a direct numerical solution of the variational problem for the rate function.

Recall that the ATM case corresponds to strike $K = V_0\eta^2(S_0)$. Thus, we are seeking the solution of the variational problem in an expansion around the ATM point using the log-moneyness $x := \log\left(\frac{K}{V_0\eta^2(S_0)}\right)$ as a small parameter.

The result is formulated in terms of the coefficients in the expansion of the local volatility function around S_0

$$\eta(S) = \eta_0 + \eta_1 \log\frac{S}{S_0} + \eta_2 \log^2\frac{S}{S_0} + O\left(\log^3(S/S_0)\right), \tag{3.40}$$

and analogous for the expansion of the volatility-of-volatility function around V_0

$$\sigma(V) = \sigma_0 + \sigma_1 \log\frac{V}{V_0} + \sigma_2 \log^2\frac{V}{V_0} + O\left(\log^3(V/V_0)\right). \tag{3.41}$$

More explicitly, $\eta_0 = \eta(S_0), \eta_1 = S_0\eta'(S_0)$ and $\sigma_0 = \sigma(V_0), \sigma_1 = V_0\sigma'(V_0)$. We have the following expansion of the rate function in powers of $x := \log\left(\frac{K}{V_0\eta_0^2}\right)$.

Proposition 19. *Suppose that $\sigma(\cdot), \eta(\cdot)$ are twice continuously differentiable such that the expansions (3.41) and (3.40) are valid. Then we have the following expansion of the rate function in powers of the log-moneyness $x := \log\left(\frac{K}{V_0\eta^2(S_0)}\right)$:*

$$\begin{aligned}\mathcal{I}_\rho(S_0, V_0, V_0\eta^2(S_0)e^x) &= \frac{3}{2\left(\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2 V_0\right)}x^2 \\ &- \frac{3}{10(\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2V_0)^3}\left(\sigma_0^4 + 2\sigma_0^3\left(3\sigma_1 + 7\eta_1\rho\sqrt{V_0}\right) + \beta_2\sigma_0^2 + \beta_1\sigma_0 + \beta_0\right)x^3 \\ &+ O(x^4),\end{aligned} \tag{3.42}$$

as $x \to 0$, with

$$\beta_0 = 16\eta_1^2V_0^2\left(\eta_1^2 + 6\eta_0\eta_2\right), \tag{3.43}$$

$$\beta_1 = 8\eta_1\rho V_0\left(3\eta_1\rho\sigma_1 + 7\eta_1^2\sqrt{V_0} + 12\eta_0\eta_2\sqrt{V_0}\right), \tag{3.44}$$

$$\beta_2 = 4\left(6\eta_1\rho\sigma_1\sqrt{V_0} + 6\eta_0\eta_2\rho^2V_0 + \eta_1^2\left(5 + 7\rho^2\right)V_0\right). \tag{3.45}$$

The optimal g, h that solve the variational problem (3.24) admit the expansion

$$g(t) = g_0(t) + xg_1(t) + O(x^2), \qquad h(t) = h_0(t) + xh_1(t) + O(x^2), \tag{3.46}$$

as $x \to 0$, where $g_0(t) \equiv \log S_0$, $h_0(t) \equiv \log V_0$ and

$$g_1(t) = \frac{3}{2}\frac{(2\eta_1\sqrt{V_0} + \rho\sigma_0)\sqrt{V_0}}{\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2 V_0}(2t - t^2), \tag{3.47}$$

$$h_1(t) = \frac{3}{2}\frac{\sigma_0(\sigma_0 + 2\eta_1\rho\sqrt{V_0})}{\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2 V_0}(2t - t^2). \tag{3.48}$$

The functions $g_2(t), h_2(t)$ are given in Appendix C.

In the limit $\eta(x) \equiv 1$ the rate function for variance options reduces to the rate function for Asian options in a local volatility model with volatility $\sigma(v)$ [30]. The expansion of this rate function in powers of log-strike was computed for the latter case in Corollary 16 of [30]. We can check that the two results agree indeed. By taking $\eta_0 = 1, \eta_2 = \eta_3 = 0$ in (3.42), we get

$$\mathcal{I}_\rho(S_0, V_0, V_0\eta^2(S_0)e^x)\Big|_{\eta(x)\equiv 1} = \frac{3}{2\sigma_0^2}x^2 - \frac{3}{10\sigma_0^3}(\sigma_0 + 6\sigma_1)\,x^3 + O(x^4), \tag{3.49}$$

which agrees indeed with the first two terms in equation (40) of [30].

3.5 ATM case

We give in this section the leading short maturity asymptotics for ATM variance options. Recall that in the short maturity limit $T \to 0$ the ATM case corresponds to variance options with strike $K = V_0\eta^2(S_0)$. Unlike the OTM case, the T scaling for the ATM variance options will be seen to be different. In probabilistic language, the ATM regime corresponds to fluctuations associated with the central limit theorem regime, which is in contrast to the large deviations regime that dominates the short maturity asymptotics for the OTM case. We have the following result.

Theorem 20. *Suppose that Assumptions 1, 2 hold. Further assume that $\eta(\cdot)$ is twice differentiable with $\sup_{x\in\mathbb{R}^+}|(\eta^2)''(x)| < \infty$ and there exists some $C' \in (0,\infty)$ such that $\max_{0\le t\le T}\mathbb{E}[(S_t)^4] \le C'$ for any sufficiently small $T > 0$.*

For ATM variance call and put options, i.e. $K = V_0\eta^2(S_0)$, we have

$$\begin{aligned}\lim_{T\to 0}\frac{C(T)}{\sqrt{T}} &= \lim_{T\to 0}\frac{P(T)}{\sqrt{T}} \\ &= \frac{1}{\sqrt{6\pi}}(\eta^2(S_0)V_0)\sqrt{4V_0(S_0\eta'(S_0))^2 + \sigma^2(V_0) + 4\rho S_0\eta'(S_0)\sqrt{V_0}\sigma(V_0)}.\end{aligned} \tag{3.50}$$

Note that the small-maturity asymptotics of the ATM option prices is of the order $O(\sqrt{T})$, in contrast with that of the OTM options which are exponentially suppressed in the order of $O(e^{-1/T})$. This is similar to the behavior obtained for Asian options in [30].

4 Applications and Numerical Tests

We present in this section the application of our asymptotic results for the pricing of variance options. Following the same approach as that used for Asian options in [30], the prices of these options are represented in Black-Scholes form as

$$\begin{aligned} C(K,T) &= e^{-rT}[F_V(T)N(d_1) - KN(d_2)]\,, \\ P(K,T) &= e^{-rT}[-F_V(T)N(-d_1) + KN(-d_2)]\,, \end{aligned} \tag{4.1}$$

where $F_V(T)$ is the varswap fair strike defined in (2.4), $N(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable with mean 0 and variance 1 and $d_{1,2} = \frac{1}{\Sigma_V\sqrt{T}}(\log(F_V(T)/K) \pm \frac{1}{2}\Sigma_V^2 T)$, with $\Sigma_V(K,T)$ being the *implied volatility of a variance option*. This volatility is defined such that the price of the variance option is equal to the Black-Scholes price given above in (4.1).

The short-maturity limit in Theorem 4 implies a prediction for the implied volatility of the variance options

$$\lim_{T\to 0} \Sigma_V^2(K,T) := \Sigma_V^2(K) = \frac{\log^2(K/F_V(T))}{2\mathcal{I}_\rho(S_0,V_0,K)}\,. \tag{4.2}$$

Using the expansion of the rate function around the ATM point in Proposition 19 yields an expansion of the asymptotic implied volatility $\Sigma_V(K)$ in powers of log-strike:

$$\Sigma_V(K) = \Sigma_{V,\mathrm{ATM}} + s_V x + O(x^2)\,, \tag{4.3}$$

where

$$\Sigma_{V,\mathrm{ATM}} = \frac{1}{\sqrt{3}}\sqrt{\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2 V_0}\,, \tag{4.4}$$

and

$$s_V = \frac{\sigma_0^4 + 2\sigma_0^3\left(3\sigma_1 + 7\eta_1\rho\sqrt{V_0}\right) + \beta_2\sigma_0^2 + \beta_1\sigma_0 + \beta_0}{10\sqrt{3}(\sigma_0^2 + 4\rho\sigma_0\sqrt{V_0}\eta_1 + 4\eta_1^2 V_0)^{3/2}}\,, \tag{4.5}$$

where $\beta_{0,1,2}$ are given above in (3.43) - (3.45).

We will use the approximation (4.3) for the numerical tests of the asymptotic expansion in the next section.

For the numerical tests, we will assume the local-stochastic volatility model with log-normal volatility

$$dS_t/S_t = \eta(S_t)\sqrt{V_t}dB_t\,, \qquad dV_t/V_t = \sigma dZ_t\,, \tag{4.6}$$

where (B_t, Z_t) are two standard Brownian motions correlated with correlation ρ, and the local volatility function is given by

$$\eta(S) = f_0 + f_1\tanh\left(\log(S/S_0) - x_0\right)\,. \tag{4.7}$$

This is the so-called Tanh model which was used in Forde and Jacquier (2011) [18] to test the predictions of their asymptotic results for the uncorrelated local-stochastic volatility model. This model was also used in [29] for numerical tests of the short maturity asymptotics for VIX options.

The local volatility function (4.7) is expanded in powers of the log-asset $\log(S/S_0)$ as

$$\eta(S) = \eta_0 + \eta_1 \log \frac{S}{S_0} + \eta_2 \log^2 \frac{S}{S_0} + \cdots , \tag{4.8}$$

with

$$\eta_0 = f_0 - f_1 \tanh x_0 , \quad \eta_1 = \frac{f_1}{\cosh^2 x_0} , \quad \eta_2 = \frac{f_1}{\cosh^2 x_0} \tanh x_0 . \tag{4.9}$$

4.1 The uncorrelated case

In this section we present a test for the solution of the variational problem (3.2) for the rate function which does not rely on an expansion in log-moneyness. For the uncorrelated case $\rho = 0$ an exact solution for the variational problem was given in Proposition 6. We present a test of the accuracy of the rate function obtained by expanding in log-moneyness in Proposition 17 by comparing with the exact solution following from Proposition 6.

For this test, we take the constant vol-of-vol $\sigma(v) \equiv \sigma$ and $\mu(v) \equiv 0$. The rate function $\mathcal{J}(V_0, K/\eta_0^2)$ corresponding to $\sigma(v) \equiv \sigma$ is the same as the rate function for Asian options in the Black-Scholes model and is known exactly from Proposition 12 in [30]. The result of Proposition 6 can be reformulated in a simpler form as follows.

i) For the OTM variance call options, we have

$$\mathcal{I}_0(S_0, V_0, K) = \inf_y \left(\frac{1}{K} F_c(y) + \frac{1}{\sigma^2} J_{\rm BS}(yK/V_0) \right) , \tag{4.10}$$

where $J_{\rm BS}(k)$ is defined in (3.36) which is the rate function for Asian options in the Black-Scholes model and is given in closed form in Proposition 12 of [30], and

$$F_c(y) := \frac{1}{2} \left(I_1^c(g_c(y)) \right)^2 \left[\left(\eta(g_c(y)) \right)^2 y - 1 \right] , \tag{4.11}$$

where $g_c(y)$ is the solution of the equation $\frac{I_2^c(g_c)}{I_1^c(g_c)} = y$ and

$$I_1^c(g_c) := \int_{S_0}^{g_c} \frac{\eta(x) dx}{x \sqrt{\eta^2(g_c) - \eta^2(x)}} , \quad I_2^c(g_c) := \int_{S_0}^{g_c} \frac{dx}{x \eta(x) \sqrt{\eta^2(g_c) - \eta^2(x)}} . \tag{4.12}$$

ii) For the OTM variance put options, we have

$$\mathcal{I}_0(S_0, V_0, K) = \inf_y \left(\frac{1}{K} F_p(y) + \frac{1}{\sigma^2} J_{\rm BS}(yK/V_0) \right) , \tag{4.13}$$

with

$$F_p(y) := \frac{1}{2}\left(I_1^c(g_p(y))\right)^2\left[1-\left(\eta(g_p(y))\right)^2 y\right], \tag{4.14}$$

where $g_p(y)$ is the solution of the equation

$$\frac{I_2^p(g_p)}{I_1^p(g_p)} = y,$$

where

$$I_1^p(g_p) := \int_{S_0}^{g_p} \frac{\eta(x)dx}{x\sqrt{\eta^2(x)-\eta^2(g_p)}}, \quad I_2^p(g_p) := \int_{S_0}^{g_p} \frac{dx}{x\eta(x)\sqrt{\eta^2(x)-\eta^2(g_p)}}. \tag{4.15}$$

This formulation follows from Proposition 6 by the change of variable $y = z/K$.

The local volatility function is chosen as the Tanh model (4.7) with model parameters

$$V_0 = 1.0, \quad \sigma = 2.0, \quad f_0 = 1, \quad f_1 = -0.5, \quad x_0 = 0\,. \tag{4.16}$$

The choice of V_0 is chosen such that the numerical value of the parameter is sufficiently large $V_0T \sim O(1)$ for maturities $T \sim 1$ year. This ensures that the testing is distinctive from that of pricing Asian options in the local volatility model. If this parameter is small $V_0T \ll 1$ then the model predictions are effectively the same as those for Asian options in the local volatility model with local volatility $\eta(x)$. This case was covered by [30].

The left plot in Figure 1 compares the exact rate function $\mathcal{I}_0(S_0, V_0, K)$ (solid black) with the approximation of Proposition 17 by expansion to $O(x^3)$ (dashed blue). The exact result for the rate function $\mathcal{I}_0(S_0, V_0, K)$ is used to compute the asymptotic implied volatility of the variance options $\Sigma_V(K)$ which is shown in the right plot of Figure 1 as a function of log-moneyness $x = \log(K/(\eta_0^2 V_0))$ (solid black curve). This is compared with the linear approximation by an expansion in log-moneyness to $O(x^3)$ from Proposition 17, see (4.23) (dotted black).

From this plot we conclude that the approximation by expansion in x is reasonably good for sufficiently small log-moneyness, for options close to the ATM point. The approximation error increases away from this point.

4.2 Bounds

We discuss further in this section the two bounds on the rate function $\mathcal{I}_\rho(S_0, V_0, K)$ proved in Section 3. Proposition 11 gives upper and lower bounds for the correlated case in terms of the rate function for $\rho = 0$. Proposition 13 gives an upper bound in terms of the rate function $\mathcal{J}$ for Asian options in the local volatility model. We present in this section several tests for the tightness of these bounds.

Bound 1. Proposition 11 gives lower and upper bounds on $I_\rho(K)$ in terms of $I_0(K)$. They can be translated into bounds on the implied volatility $\Sigma_V(K)$:

$$\sqrt{1-|\rho|}\Sigma_V(K;\rho=0) \le \Sigma_V(K) \le \sqrt{1+|\rho|}\Sigma_V(K;\rho=0)\,. \tag{4.17}$$

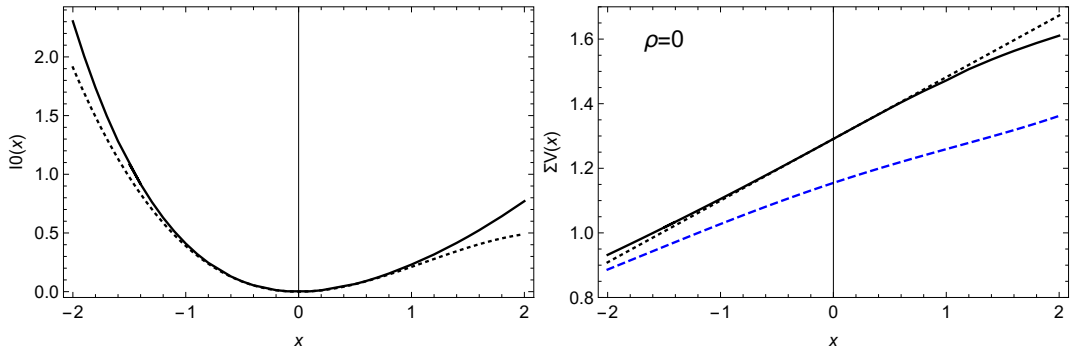

Figure 1: Left: the exact rate function $\mathcal{I}_0(S_0, V_0, K)$ vs log-moneyness $x = \log(K/(\eta_0^2 V_0)$ from Proposition 6 (solid black), comparing with the expansion to $O(x^3)$ from Proposition 17 (dotted black). Right: Asymptotic implied volatility $\Sigma_V(K)$ vs x, for the uncorrelated case $\rho = 0$. The exact result from Proposition 6 is shown as the solid black curve, and the linear approximation obtained by expanding in log-moneyness to $O(x)$ is the dotted black line. The dashed blue curve is the lower bound (4.18). Model parameters are as in (4.16).

Figure 2 shows these bounds for $\rho = \pm 0.5$ (left) and $\rho = \pm 0.7$ (right) as the solid blue and red curves, for the parameters given in (4.16). The bounds depend only on $|\rho|$ so they are identical for positive and negative correlation. The dotted black lines show the linear approximations for $\Sigma_V(K)$ following from Proposition 19. The bounds are well satisfied for a range of strikes around the ATM point, although the upper bound is violated by the linear approximation at the upper range of the strikes considered.

Bound 2. Proposition 13 gives an upper bound on $I_\rho(K)$ in terms of $\mathcal{J}(V_0, K/\eta_0^2)$, the rate function for Asian options in the local volatility model with volatility function $\sigma(v)$. This translates into a lower bound on the asymptotic implied volatility of variance options $\Sigma_V(K)$.

For constant vol-of-vol $\sigma(v) \equiv \sigma$, this bound reads

$$\Sigma_V(K) \geq \sigma\sqrt{1-\rho^2}\,\Sigma_{\mathrm{LN}}^{(\mathrm{BS})}\left(K/(\eta_0^2 V_0)\right)\,, \tag{4.18}$$

where $\Sigma_{\mathrm{LN}}^{(\mathrm{BS})}(k) = \frac{|\log k|}{\sqrt{2J_{\mathrm{BS}}(k)}}$ is the asymptotic implied log-normal volatility for Asian options in the Black-Scholes model, expressed in terms of $J_{\mathrm{BS}}(k)$ introduced in (3.36). For k sufficiently close to 1 (near-ATM Asian options), $\Sigma_{\mathrm{LN}}^{(\mathrm{BS})}(k)$ is well approximated by the series expansion (see equation (55) in [30])

$$\Sigma_{LN}^{(\mathrm{BS})}(k) = \frac{1}{\sqrt{3}}\left(1 + \frac{1}{10}\log k - \frac{23}{2100}\log^2 k + O(\log^3 k)\right). \tag{4.19}$$

The dashed blue curve in Figure 1 (right panel) shows the lower bound (4.18) for $\rho = 0$, comparing with the exact asymptotic implied volatility $\Sigma_V(K)$ (solid black curve).

The model parameters are as in (4.16). The bound is indeed satisfied in the range of log-moneyness covered by the plot.

The lower bound (4.18) is shown also in Figure 2 as the dashed blue curve, for several choices of the correlation $\rho = \pm 0.5$ (left) and $\rho = \pm 0.7$ (right). The bound is independent of the sign of ρ; thus we show on the same plot both cases with either sign for ρ. This bound is violated by the linear approximation for $\rho = -0.7$ for positive log-moneyness larger than about 0.5. The bound is satisfied for all strikes by the linear approximation for $\rho = -0.5$. At the ATM point $K_{\text{ATM}} = \eta_0^2 V_0$ one can see that the bound (4.18) is automatically satisfied by the asymptotic prediction for the ATM volatility (4.4). Indeed, this bound reads

$$\Sigma_V(K_{\text{ATM}}) = \frac{1}{\sqrt{3}}\sigma\sqrt{\sigma^2 + 4\eta_1\rho\sigma\sqrt{V_0} + 4\eta_1^2 V_0} \geq \frac{1}{\sqrt{3}}\sigma(1-\rho^2)\,, \tag{4.20}$$

which is equivalent to $\rho^2\sigma^2 + 4\eta_1\rho\sigma\sqrt{V_0} + 4\eta_1^2 V_0 = (\rho\sigma + 2\eta_1\sqrt{V_0})^2 \geq 0$, which holds indeed. By continuity, this implies that the lower bound (4.18) is always satisfied in a non-vanishing region around the ATM point.

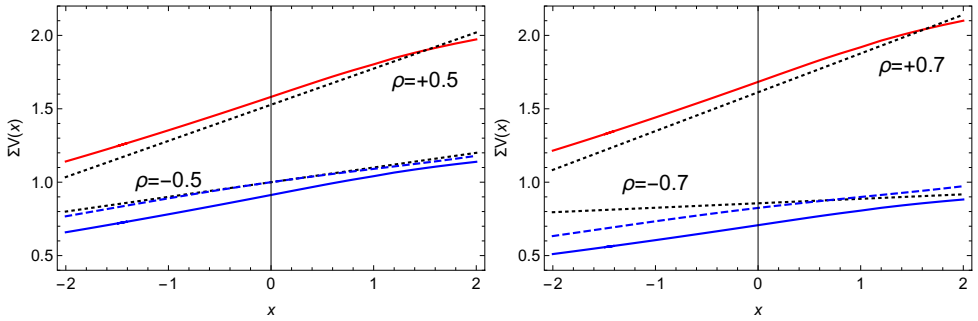

Figure 2: Lower and upper bounds on the implied volatility $\Sigma_V(K)$ from Proposition 11 (bound 1) (solid blue and red curves), and lower bound from Proposition 13 (bound 2) (dashed blue). The dotted black lines show the linear approximation for $\Sigma_V(K)$. Left: $\rho = \pm 0.5$, right: $\rho = \pm 0.7$.

4.3 Monte Carlo tests

The result of Proposition 19 can be reformulated as a prediction for the ATM implied volatility $\Sigma_{V,\text{ATM}}$ and ATM skew s_V of the variance options. They read explicitly

$$\Sigma_{V,\text{ATM}} = \frac{1}{\sqrt{3}}\sqrt{\sigma^2 + 4\eta_1\sigma\rho\sqrt{V_0} + 4\eta_1^2 V_0}\,, \tag{4.21}$$

and

$$s_V = \frac{1}{10\sqrt{3}\left(\sigma^2 + 4\eta_1\sigma\rho\sqrt{V_0} + 4\eta_1^2 V_0\right)^{3/2}} \tag{4.22}$$
$$\cdot \Big(\sigma^4 + 14\sigma^3\eta_1\rho\sqrt{V_0} + 4\sigma^2 V_0\left(6\eta_0\eta_2\rho^2 + \eta_1^2\left(5 + 7\rho^2\right)\right)$$
$$+ 8\sigma\eta_1\rho V_0^{3/2}\left(7\eta_1^2 + 12\eta_0\eta_2\right) + 16\eta_1^2 V_0^2\left(\eta_1^2 + 6\eta_0\eta_2\right)\Big).$$

The result (4.21) coincides with the prediction of Theorem 20 for the short-maturity asymptotics of ATM variance options.

Using these predictions, we construct the following linear approximation for the asymptotic implied volatility of the variance options using the ATM level and skew given in (4.21) and (4.22):

$$\Sigma_V^{\text{lin}}(K) := \Sigma_{V,\text{ATM}} + s_V x\,. \tag{4.23}$$

This is expected to give a good approximation for $\Sigma_V(K)$ for strikes sufficiently close to the ATM point. We will test this approximation for the implied volatility by comparing it with the numerical pricing of the variance options by Monte Carlo simulation.

We will use for this numerical test the Tanh local volatility model (4.7) with model parameters $f_0 = 1, f_1 = -0.1, x_0 = 0$. For the V_t process, we assume $\sigma = 2.0, V_0 = 0.1$. The spot asset price is $S_0 = 1$. The timeline is discretized by $N_T = 2000$ steps, and we use $N_{\text{MC}} = 10^5$ MC paths for the simulation.

Table 1: The ATM volatility level, skew and convexity for the short-maturity asymptotics of the variance options in the Tanh model. The last column shows the forward $F_V(T)$ for $T = 1/12$ computed by MC simulation.

ρ	$\sigma_{V,\text{ATM}}$	s_V	$F_V(T)$
-0.7	1.1806	0.1257	0.1004 ± 0.0001
0	1.1553	0.1553	1.0006 ± 0.0001
$+0.7$	1.1294	0.1053	0.0997 ± 0.0001

The MC simulation results for the implied volatility of variance options are shown in Figure 3 for several choices of ρ for variance options with maturity $T = 1/12$ (orange dots). The bands show the MC errors, defined as $\frac{1}{\sqrt{N_{\text{MC}}}}\text{sd}(\text{pay})$ in terms of the standard deviation of the option payoff for the MC sample.

The asymptotic prediction for the ATM implied vol from the linear approximation (4.23) is shown as the blue dots in Figure 3. The numerical values of the ATM implied vol and the ATM skew from (4.21) and (4.22) for each test case are shown in Table 1. The asymptotic prediction agrees well with the MC simulation for a range of strikes around the ATM point, although it slightly overestimates the simulation result at the ATM point for $\rho = \pm 0.7$.

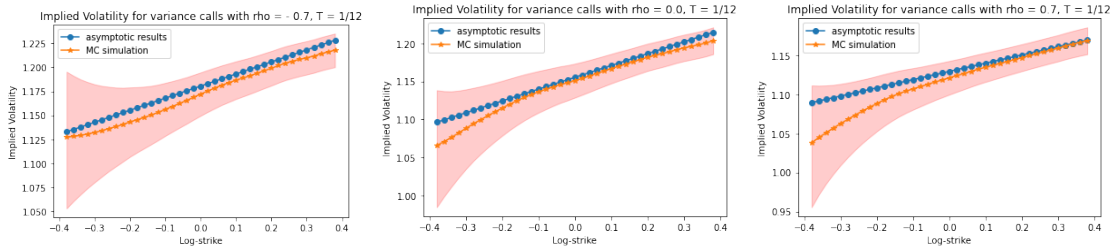

Figure 3: The implied volatility of options on variance with maturity $T = 1/12$ (1 month) in the Tanh LSV model, for three values of the correlation $\rho \in \{-0.7, 0, 0.7\}$. The orange points denote the MC simulation and the blue dots show the asymptotic prediction (4.23).

In order to investigate the size of the subleading corrections of $O(T)$, we show in Figure 4 the same results for variance options with maturity $T = 1/252$ (1 business day), and the same model parameters as in Figure 3. These plots we compare the MC simulation with the asymptotic prediction in a more narrow range of log-moneyness $x \in [-0.1, +0.1]$. For this case the difference between the asymptotic prediction and the simulation results is much smaller. We conclude that the difference between the asymptotic result and the MC simulation in Figure 3 can be attributed to subleading $O(T)$ corrections to the asymptotic result.

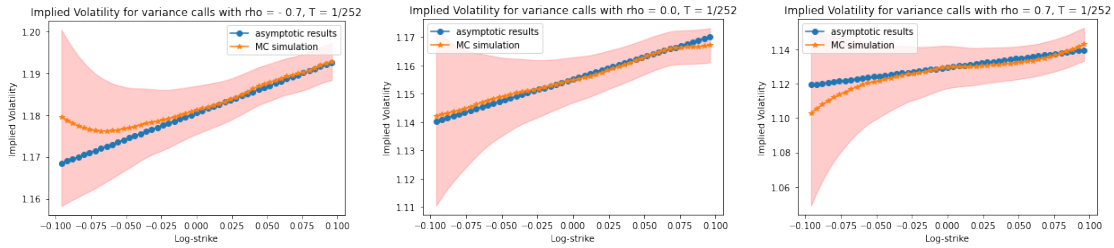

Figure 4: Same as Figure 3 but the maturity of the variance options is $T = 1/252$ (1 day).

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References

[1] E. Alòs, D. García-Lorite, and A. M. Gonzalez. On smile properties of volatility derivatives and exotic products: understanding the VIX skew. *SIAM Journal on Financial Mathematics*, 13(1):32–69, 2022.

[2] P. Baldi and L. Caramellino. General Freidlin-Wentzell large deviations and positive diffusions. *Statistics and Probability Letters*, 81:1218–1229, 2011.

[3] C. Bayer, P.K. Friz, A. Gulisashvili, B. Horvath, and B. Stemper. Short-time near the money skew in rough fractional stochastic volatility models. *Quantitative Finance*, 17:463–480, 2017.

[4] L. Bergomi. Smile dynamics. Société Générale White Paper, 2004.

[5] L. Bergomi. Smile dynamics II. Société Générale White Paper, 2005.

[6] L. Bergomi. *Stochastic Volatility Modeling*. Routledge, 2016.

[7] C. Bernard and Z. Cui. Prices and asymptotics for discrete variance swaps. *Applied Mathematical Finance*, 21:140–173, 2014.

[8] P. Carr, H. Geman, D. Madan, and M. Yor. Pricing options on realized variance. *Finance and Stochastics*, 4:453–475, 2005.

[9] P. Carr, H. Geman, D.B. Madan, and M. Yor. Options on realized variance and convex orders. *Quantitative Finance*, 11(11):1685–1694, 2011.

[10] P. Carr and A. Itkin. Pricing swaps and options on quadratic variation under stochastic time change models - discrete observations case. *Review of Derivatives Research*, 13:141–176, 2009.

[11] P. Carr and R. Lee. Hedging variance options on continuous semimartingales. *Finance and Stochastics*, 14:179–207, 2010.

[12] Z. Cui, J. Kirkby, and D. Nguyen. A general framework for discretely sampled realized volatility derivatives in stochastic volatility models with jumps. *European Journal of Operational Research*, 262(1):381–400, 2017.

[13] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer Verlag, 2nd edition, 1998.

[14] G. Drimus, W. Farkas, and E. Gourier. Valuation of options on discretely sampled variance: a general analytic approximation. *Journal of Computational Finance*, 20(2):39–66, 2016.

[15] G.G. Drimus. Options on realized variance by transform methods: a non-affine stochastic volatility model. *Quantitative Finance*, 12:1679–1694, 2012.

[16] G.G. Drimus. Options on realized variance in Log-OU models. *Applied Mathematical Finance*, 19(5):477–494, 2012.

[17] D. Dufresne. The integrated square root process. Research Paper nr. 90, University of Melbourne, 2001.

[18] M. Forde and A. Jacquier. Small-time asymptotics for an uncorrelated local-stochastic volatility model. *Applied Mathematical Finance*, 2:517–535, 2011.

[19] P.K. Friz, P. Gassiat, and P. Pigato. Short-dated smile under rough volatility: Asymptotics and numerics. *Quantitative Finance*, 22:463–480, 2022.

[20] C. Griessler and M. Keller-Ressel. Convex order of discrete, continuous, and predictable quadratic variation and applications to options on variance. *SIAM Journal on Financial Mathematics*, 5(1):1–19, 2014.

[21] R. Jarrow, Y. Kchia, M. Larsson, and P. Protter. Discretely sampled variance and volatility swaps versus their continuous approximations. *Finance and Stochastics*, 17:305–324, 2013.

[22] J. Kallsen, J. Muhle-Karbe, and M. Voß. Pricing options on variance in affine stochastic volatility models. *Mathematical Finance*, 21(4):627–641, 2011.

[23] M. Keller-Ressel and J. Muhle-Karbe. Asymptotic and exact pricing of options on variance. *Finance and Stochastics*, 17:107–133, 2013.

[24] J. Kim, H. Park, and J. Park. Pricing and hedging short-maturity Asian options in local volatility models. *arXiv:1911.12944*, 2019.

[25] Y.K. Kwok and W. Zheng. *Pricing Models of Volatility Products and Exotic Variance Derivatives*. Chapman and Hall/CRC, New York, 2022.

[26] C. Lacombe, A. Muguruza, and H. Stone. Asymptotics for volatility derivatives in multi-factor rough volatility models. *Mathematics and Financial Economics*, 15:545–577, 2021.

[27] G. Lian, C. Chiarella, and P.S. Kalev. Volatility swaps and voltility options on discretely sampled realized variance. *Journal of Economic Dynamics and Control*, 47:239–262, 2014.

[28] D. Pirjol, J. Wang, and L. Zhu. Short maturity forward start Asian options in local volatility models. *Applied Mathematical Finance*, 26:187–221, 2019.

[29] D. Pirjol, X. Wang, and L. Zhu. Short-maturity asymptotics for VIX and European options in local-stochastic volatility models. *arXiv:2407.16813*, 2024.

[30] D. Pirjol and L. Zhu. Short maturity Asian options in local volatility models. *SIAM Journal on Financial Mathematics*, 7(1):947–992, 2016.

[31] D. Pirjol and L. Zhu. Short maturity Asian options for the CEV model. *Probability in Engineering and Information Sciences*, 33:258–290, 2019.

[32] D. Pirjol and L. Zhu. What is the volatility of an asian option? *Risk*, May:74–79, 2021.

[33] D. Pirjol and L. Zhu. Asian options for local-stochastic volatility models in the short-maturity regime. *To appear in SIAM Journal on Financial Mathematics*, 2025.

[34] S. Robertson. Sample path Large Deviations and optimal importance sampling for stochastic volatility models. *Stochastic Processes and their Applications*, 120:66–83, 2010.

[35] A. Sepp. Pricing options on realized variance in the Heston model with jumps in returns and volatility. *Journal of Computational Finance*, 11(4):33–70, 2008.

[36] A. Sepp. VIX option pricing in a jump-diffusion model. *Risk*, May:84–89, 2008.

[37] A. Sepp. Pricing options on realized variance in the Heston model with jumps in returns and volatility - part II: An approximate distribution of discrete variance. *Journal of Computational Finance*, 16(2):3–32, 2012.

[38] L. Torricelli. Pricing joint claims on an asset and its realized variance in stochastic volatility models. *International Journal of Theoretical and Applied Finance*, 16(01):1350005, 2013.

[39] S.R.S. Varadhan. Diffusion processes in a small time interval. *Communications on Pure and Applied Mathematics*, 20:659–685, 1967.

[40] S.R.S. Varadhan. *Large Deviations and Applications*. SIAM, Philadelphia, 1984.

[41] W. Zheng and Y.K. Kwok. Closed form pricing formulas for discretely sampled generalized variance swaps. *Mathematical Finance*, 24(4):855–881, 2014.

[42] W. Zheng and Y.K. Kwok. Fourier transform algorithms for pricing and hedging discretely sampled exotic variance products and volatility derivatives under additive processes. *Journal of Computational Finance*, 18(2):3–30, 2014.

[43] W. Zheng and Y.K. Kwok. Saddlepoint approximation methods for pricing derivatives on discrete realized variance. *Applied Mathematical Finance*, 21(1):1–31, 2014.

[44] W. Zheng and Y.K. Kwok. Pricing options on discrete realized variance with partially exact and bounded approximations. *Quantitative Finance*, 15(12):2011–2019, 2015.

[45] W. Zheng, C.H. Yuen, and Y.K. Kwok. Recursive algorithms for pricing discrete variance options and volatility swaps under time-changed Lévy processes. *International Journal of Theoretical and Applied Finance*, 19(02):1650011, 2016.

A Background on Large Deviations Theory

We give in this Appendix a few basic concepts of large deviations theory from probability theory which are in the proofs. We refer to Dembo and Zeitouni [13] and Varadhan [40] for more details on large deviations theory and its applications.

Definition 21 (Large Deviation Principle)**.** *A sequence $(P_\epsilon)_{\epsilon\in\mathbb{R}^+}$ of probability measures on a topological space X satisfies the large deviation principle with rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A, we have*

$$-\inf_{x\in A^o} I(x) \le \liminf_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \le \limsup_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \le -\inf_{x\in \bar{A}} I(x)\,, \tag{A.1}$$

where A^o denotes the interior of A and $\bar{A}$ its closure.

Theorem 22 (Contraction Principle, see e.g. Theorem 4.2.1. in [13])**.** *If $F : X \to Y$ is a continuous map and P_ϵ satisfies a large deviation principle on X with the rate function $I(x)$, then the probability measures $Q_\epsilon := P_\epsilon F^{-1}$ satisfies a large deviation principle on Y with the rate function $J(y) = \inf_{x:F(x)=y} I(x)$.*

B Technical Proofs

Proof of Theorem 4. Let us consider OTM case for call options, that is, $K > V_0\eta^2(S_0)$. The case for the put options is similar and the proof is omitted here. First, it is easy to see that

$$\lim_{T\to 0} T\log C(T) = \lim_{T\to 0} T\log \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)^+\right], \tag{B.1}$$

if the limit exists. Next, we will show that

$$\lim_{T\to 0} T\log \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)^+\right] = \lim_{T\to 0} T\log \mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds \ge K\right), \tag{B.2}$$

if the limit exists. The equality in (B.2) can be established by considering the upper bound, i.e. lim sup on the left hand side of (B.2) and the lower bound, i.e. lim inf on the left hand side of (B.2). The argument for the lower bound is standard and is omitted here The

argument for the upper bound can be established via the following estimates. For any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)^+\right] \\
&= \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)1_{\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds \geq K}\right] \\
&\leq \left(\mathbb{E}\left[\left|\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right|^p\right]\right)^{\frac{1}{p}} \left(\mathbb{E}\left[\left(1_{\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds \geq K}\right)^q\right]\right)^{\frac{1}{q}} \\
&= \left(\mathbb{E}\left[\left|\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right|^p\right]\right)^{\frac{1}{p}} \left(\mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds \geq K\right)\right)^{\frac{1}{q}}. \qquad \text{(B.3)}
\end{aligned}$$

For any $p > 1$, we can show that

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right|^p\right] &\leq \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds + K\right)^p\right] \\
&\leq 2^{p-1}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\right)^p + K^p\right] \\
&\leq 2^{p-1}\left(\frac{1}{T}\int_0^T \mathbb{E}\left[V_s^p\eta^{2p}(S_s)\right]ds + K^p\right), \qquad \text{(B.4)}
\end{aligned}$$

where we used Jensen's inequality.

Furthermore, we can compute that

$$\frac{1}{T}\int_0^T \mathbb{E}\left[V_s^p\eta^{2p}(S_s)\right]ds \leq \frac{M_\eta^{2p}}{T}\int_0^T \mathbb{E}\left[V_s^p\right]ds, \qquad \text{(B.5)}$$

and for any $0 \leq s \leq T$,

$$\begin{aligned}
\mathbb{E}\left[V_s^p\right] &= V_0^p\mathbb{E}\left[e^{\int_0^s\left(p\mu(V_u) - \frac{p}{2}\sigma^2(V_u)\right)du + \int_0^s p\sigma(V_u)dZ_u}\right] \\
&\leq V_0^p e^{pM_\mu s + \frac{p^2}{2}M_\sigma^2 s}\mathbb{E}\left[e^{\int_0^s \frac{-p^2}{2}\sigma^2(V_u)du + \int_0^s p\sigma(V_u)dZ_u}\right] \\
&= V_0^p e^{pM_\mu s + \frac{p^2}{2}M_\sigma^2 s}. \qquad \text{(B.6)}
\end{aligned}$$

Therefore, we conclude that for any $p > 1$,

$$\mathbb{E}\left[\left|\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right|^p\right] \leq 2^{p-1}\left(M_\eta^{2p}V_0^p e^{pM_\mu T + \frac{p^2}{2}M_\sigma^2 T} + K^p\right). \qquad \text{(B.7)}$$

Hence, by (B.3) and (B.7), we showed that

$$\limsup_{T\to 0} T\log\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds-K\right)^+\right]\le\limsup_{T\to 0}\frac{T}{q}\log\mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\ge K\right). \tag{B.8}$$

Since (B.8) holds for any $q>1$, we conclude that

$$\limsup_{T\to 0} T\log\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds-K\right)^+\right]\le\limsup_{T\to 0} T\log\mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\ge K\right). \tag{B.9}$$

Therefore, we established the upper bound for (B.2). Next, let us show that the limit on the right hand side of (B.2) exists, which can be established through large deviations theory.

Under Assumptions 1 and 3, by the sample-path large deviations for small time diffusions (see for example [39] and [34]), one can see that $\mathbb{Q}(\{(\log S_{tT},\log V_{tT}),0\le t\le 1\}\in\cdot)$ satisfies a sample-path large deviation principle with the rate function:

$$\frac{1}{2(1-\rho^2)}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}-\frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2dt+\frac{1}{2}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2dt, \tag{B.10}$$

with $g(0)=\log S_0$, $h(0)=\log V_0$ and g,h being absolutely continuous and the rate function is $+\infty$ otherwise.

By an application of the contraction principle (see for example Theorem 4.2.1. in [13], restated in Theorem 22),

$$\begin{aligned}
&\lim_{T\to 0} T\log\mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\ge K\right)\\
&=\lim_{T\to 0} T\log\mathbb{Q}\left(\int_0^1 V_{tT}\eta^2(S_{tT})dt\ge K\right)\\
&=-\inf_{\substack{g(0)=\log S_0,h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}}\left\{\frac{1}{2(1-\rho^2)}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}-\frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2dt\right.\\
&\qquad\left.+\frac{1}{2}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2dt\right\}.
\end{aligned} \tag{B.11}$$

Hence, we conclude that

$$\lim_{T\to 0} T\log\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds-K\right)^+\right]$$
$$=-\inf_{\substack{g(0)=\log S_0, h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}}\left\{\frac{1}{2(1-\rho^2)}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}-\frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2 dt\right.$$
$$\left.+\frac{1}{2}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt\right\}. \tag{B.12}$$

This completes the proof. □

Proof of Proposition 6. (i) Let us first consider the OTM variance call options, i.e. $V_0\eta^2(S_0)<K$. When $\rho=0$, one can compute that

$$\mathcal{I}(S_0,V_0,K)$$
$$=\inf_{\substack{g(0)=\log S_0, h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}}\left\{\frac{1}{2}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt+\frac{1}{2}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt\right\}. \tag{B.13}$$

For this proof, we will proceed by assuming that the solutions g,h of the Euler-Lagrange equation for this variational problem exist. Some motivation for this assumption is offered by Remark 10 where we give explicit solutions for these functions.

Given h, let us define:

$$\Lambda[g]:=\frac{1}{2}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt+\lambda\left(\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt-K\right), \tag{B.14}$$

where λ is the Lagrange multiplier. The Euler-Lagrange equation gives:

$$-\frac{\eta'(e^g)e^g(g')^2}{\eta^3(e^g)e^h}+\lambda e^h 2\eta(e^g)\eta'(e^g)e^g=\frac{d}{dt}\left(\frac{g'}{\eta^2(e^g)e^h}\right)$$
$$=\frac{g''}{\eta^2(e^g)e^h}-\frac{g'(2\eta(e^g)\eta'(e^g)e^g g'e^h+\eta^2(e^g)e^h h')}{\eta^4(e^g)e^{2h}}, \tag{B.15}$$

and equation (B.15) implies that

$$\frac{\eta'(e^g)e^g(g')^2}{\eta^3(e^g)e^h}+\lambda e^h 2\eta(e^g)\eta'(e^g)e^g=\frac{g''}{\eta^2(e^g)e^h}-\frac{g'h'}{\eta^2(e^g)e^h}, \tag{B.16}$$

which is equivalent to

$$2\lambda e^h \eta^2(e^g)\eta'(e^g)e^g = \frac{g''}{\eta(e^g)e^h} - \frac{\eta'(e^g)e^g(g')^2}{\eta^2(e^g)e^h} - \frac{g'h'}{\eta(e^g)e^h}$$
$$= \frac{d}{dt}\left(\frac{g'}{\eta(e^g)e^h}\right). \tag{B.17}$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\left(\frac{g'}{\eta(e^g)e^h}\right)^2 = \frac{g'}{\eta(e^g)e^h}\frac{d}{dt}\left(\frac{g'}{\eta(e^g)e^h}\right) = 2\lambda\eta(e^g)\eta'(e^g)e^g g'. \tag{B.18}$$

Integrating the above equation from $t = 1$ to t and taking into account the transversality condition $g'(1) = 0$ gives

$$\frac{1}{2}\left(\frac{g'(t)}{\eta(e^{g(t)})e^{h(t)}}\right)^2 = \lambda\left(\eta^2(e^{g(t)}) - \eta^2(e^{g(1)})\right). \tag{B.19}$$

Therefore,

$$\frac{1}{2}\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 = \lambda\left(\eta^2(e^{g(t)})e^{h(t)} - \eta^2(e^{g(1)})e^{h(t)}\right). \tag{B.20}$$

By integrating the above equation from $t = 0$ to $t = 1$, we get

$$\int_0^1 \frac{1}{2}\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt = \lambda\left(\int_0^1 \eta^2(e^{g(t)})e^{h(t)}dt - \eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt\right)$$
$$= \lambda\left(K - \eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt\right). \tag{B.21}$$

For OTM variance call option, $\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt \geq K$, which implies that $\lambda \leq 0$. Furthermore, the assumed monotonicity of $\eta(\cdot)$ implies a monotonicity property for $g(t)$. Integrating (B.17) from t to 1 and using the transversality condition $g'(1) = 0$ gives

$$\frac{g'(t)}{\eta(e^{g(t)})e^{h(t)}} = -2\lambda\int_t^1 e^{g(s)+h(s)}\eta^2(e^{g(s)})\eta'(e^{g(s)})ds \tag{B.22}$$

By assumption $\eta'(\cdot) \leq 0$ which implies that the integral is non-positive, and thus $g'(t) \leq 0$.

From (B.19) we obtain

$$g'(t) = -\eta(e^{g(t)})e^{h(t)}\sqrt{2\lambda\left(\eta^2(e^{g(t)}) - \eta^2(e^{g(1)})\right)}. \tag{B.23}$$

This implies that

$$\frac{dg}{\eta(e^g)\sqrt{2\lambda\left(\eta^2(e^g)-\eta^2(e^{g(1)})\right)}} = -e^{h(t)}dt. \tag{B.24}$$

Therefore,

$$\int_{S_0}^{e^{g(t)}} \frac{dx}{x\eta(x)\sqrt{2\lambda\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}} = -\int_0^t e^{h(s)}ds. \tag{B.25}$$

By letting $t=1$, we can see that $g(1)$ solves the equation

$$\int_{S_0}^{e^{g(1)}} \frac{dx}{x\eta(x)\sqrt{2\lambda\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}} = -\int_0^1 e^{h(s)}ds. \tag{B.26}$$

Moreover, from (B.23), we have

$$\frac{\eta(e^{g(t)})g'(t)}{\sqrt{2\lambda\left(\eta^2(e^{g(t)})-\eta^2(e^{g(1)})\right)}} = -\eta^2(e^{g(t)})e^{h(t)}. \tag{B.27}$$

By integrating from $t=0$ to $t=1$, we obtain

$$\int_{e^{g(0)}}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\lambda\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}} = -\int_0^1 \eta^2(e^{g(t)})e^{h(t)}dt, \tag{B.28}$$

which implies that

$$\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\lambda\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}} = -K. \tag{B.29}$$

This yields that

$$\sqrt{-\lambda} = \frac{1}{-K}\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(e^{g(1)})-\eta^2(x)\right)}}. \tag{B.30}$$

By plugging (B.30) into (B.21), we get

$$\begin{aligned}
&\int_0^1 \frac{1}{2}\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt \\
&= \frac{1}{K^2}\left(\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(e^{g(1)})-\eta^2(x)\right)}}\right)^2 \left(\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt - K\right).
\end{aligned} \tag{B.31}$$

Finally, by dividing (B.26) by (B.29), we get

$$\frac{\int_{S_0}^{e^{g(1)}} \frac{dx}{x\eta(x)\sqrt{\eta^2(e^{g(1)})-\eta^2(x)}}}{\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{\eta^2(e^{g(1)})-\eta^2(x)}}} = \frac{\int_0^1 e^{h(s)}ds}{K}, \tag{B.32}$$

which determines the value of $g(1)$.

Hence, we conclude that with fixed h,

$$\begin{aligned}
&\inf_{g(0)=\log S_0, \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt \\
&= \frac{1}{K^2}\left(\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(e^{g(1)})-\eta^2(x)\right)}}\right)^2 \left(\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt - K\right),
\end{aligned} \tag{B.33}$$

where $g(1)$ is a function of $z := \int_0^1 e^{h(s)}ds$ such that

$$\begin{aligned}
&\inf_{g(0)=\log S_0, \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt \\
&= \frac{1}{K^2}\left(\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(G_c(z))-\eta^2(x)\right)}}\right)^2 \left(\eta^2(G_c(z))z - K\right),
\end{aligned} \tag{B.34}$$

where $G_c(z) := e^{g(1)}$ is the solution of the equation

$$\frac{\int_{S_0}^{G_c(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(G_c(z))-\eta^2(x)}}}{\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(G_c(z))-\eta^2(x)}}} = \frac{z}{K}. \tag{B.35}$$

Hence, we conclude that

$$\begin{aligned}
\mathcal{I}(S_0,V_0,K) = \inf_z \Bigg\{ &\frac{1}{K^2}\left(\int_{S_0}^{G_c(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(G_c(z))-\eta^2(x)\right)}}\right)^2 \left(\eta^2(G_c(z))z - K\right) \\
&+ \inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt=z} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \Bigg\}.
\end{aligned} \tag{B.36}$$

(ii) Next, let us consider solving the variational problem for the OTM put options (i.e. $K < V_0\eta^2(S_0)$) when the correlation $\rho = 0$. In this case, one can compute that

$$\begin{aligned}
&\mathcal{I}(S_0,V_0,K) \\
&= \inf_{\substack{g(0)=\log S_0, h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}} \left\{ \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \right\}.
\end{aligned} \tag{B.37}$$

Similar as in the case for call options, given h, we can write down the Euler-Lagrange equation for g. For OTM variance put option, $\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt \leq K$ which implies $\lambda \geq 0$. The assumed monotonicity property $\eta'(\cdot) \leq 0$ implies by equation (B.22) that for this case we have $g'(t) \geq 0$. Thus we have

$$g'(t) = \eta(e^{g(t)})e^{h(t)}\sqrt{2\lambda\left(\eta^2(e^{g(t)}) - \eta^2(e^{g(1)})\right)}. \tag{B.38}$$

This implies that

$$\frac{dg}{\eta(e^g)\sqrt{2\lambda\left(\eta^2(e^g) - \eta^2(e^{g(1)})\right)}} = e^{h(t)}dt. \tag{B.39}$$

Therefore,

$$\int_{S_0}^{e^{g(t)}} \frac{dx}{x\eta(x)\sqrt{2\lambda\left(\eta^2(x) - \eta^2(e^{g(1)})\right)}} = \int_0^t e^{h(s)}ds. \tag{B.40}$$

By letting $t = 1$, we can see that $g(1)$ solves the equation

$$\int_{S_0}^{e^{g(1)}} \frac{dx}{x\eta(x)\sqrt{2\lambda\left(\eta^2(x) - \eta^2(e^{g(1)})\right)}} = \int_0^1 e^{h(s)}ds. \tag{B.41}$$

Moreover, from (B.38), we have

$$\frac{\eta(e^{g(t)})g'(t)}{\sqrt{2\lambda\left(\eta^2(e^{g(t)}) - \eta^2(e^{g(1)})\right)}} = \eta^2(e^{g(t)})e^{h(t)}. \tag{B.42}$$

By integrating from $t = 0$ to $t = 1$, we obtain

$$\int_{e^{g(0)}}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\lambda\left(\eta^2(x) - \eta^2(e^{g(1)})\right)}} = \int_0^1 \eta^2(e^{g(t)})e^{h(t)}dt, \tag{B.43}$$

which implies that

$$\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\lambda\left(\eta^2(x) - \eta^2(e^{g(1)})\right)}} = K. \tag{B.44}$$

This yields that

$$\sqrt{\lambda} = \frac{1}{K}\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x) - \eta^2(e^{g(1)})\right)}}. \tag{B.45}$$

By plugging (B.45) into (B.21), we get

$$\int_0^1 \frac{1}{2}\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt$$
$$= \frac{1}{K^2}\left(\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}}\right)^2 \left(K-\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt\right). \tag{B.46}$$

Finally, by dividing (B.41) by (B.44), we get

$$\frac{\int_{S_0}^{e^{g(1)}} \frac{dx}{x\eta(x)\sqrt{\eta^2(x)-\eta^2(e^{g(1)})}}}{\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{\eta^2(x)-\eta^2(e^{g(1)})}}} = \frac{\int_0^1 e^{h(s)}ds}{K}, \tag{B.47}$$

which determines the value of $g(1)$.

Hence, we conclude that with fixed h,

$$\inf_{g(0)=\log S_0, \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt$$
$$= \frac{1}{K^2}\left(\int_{S_0}^{e^{g(1)}} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x)-\eta^2(e^{g(1)})\right)}}\right)^2 \left(K-\eta^2(e^{g(1)})\int_0^1 e^{h(t)}dt\right), \tag{B.48}$$

where $g(1)$ is a function of $z := \int_0^1 e^{h(s)}ds$ such that

$$\inf_{g(0)=\log S_0, \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt$$
$$= \frac{1}{K^2}\left(\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x)-\eta^2(G_p(z))\right)}}\right)^2 \left(K-\eta^2(G_p(z))z\right), \tag{B.49}$$

where $G_p(z) := e^{g(1)}$ is the solution of the equation

$$\frac{\int_{S_0}^{G_p(z)} \frac{dx}{x\eta(x)\sqrt{\eta^2(x)-\eta^2(G_p(z))}}}{\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{\eta^2(x)-\eta^2(G_p(z))}}} = \frac{z}{K}. \tag{B.50}$$

Hence, we conclude that

$$\mathcal{I}(S_0,V_0,K) = \inf_z \left\{\frac{1}{K^2}\left(\int_{S_0}^{G_p(z)} \frac{\eta(x)dx}{x\sqrt{2\left(\eta^2(x)-\eta^2(G_p(z))\right)}}\right)^2 \left(K-\eta^2(G_p(z))z\right)\right.$$
$$\left. + \inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt=z} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt\right\}. \tag{B.51}$$

This completes the proof. □

We give here a few details of computation for the relation appearing in Remark 10.

Proof. (i) For OTM variance call options, i.e. $V_0\eta^2(S_0) < K$, let z_c be the optimizer in (3.6). Then, from the proof of Proposition 6, $g_0(1) = \log G_c(z_c)$ and the optimal Lagrange multiplier is given by:

$$\lambda_0 = -\frac{1}{K^2}\left(\int_{S_0}^{G_c(z_c)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(G_c(z_c)) - \eta^2(x))}}\right)^2. \tag{B.52}$$

We also obtain from the proof of Proposition 6 that

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\lambda_0\left(\eta^2(x) - \eta^2(e^{g_0(1)})\right)}} = -\int_0^t e^{h_0(s)}ds. \tag{B.53}$$

For this case $\lambda_0 < 0$. As shown in the proof of Proposition 6, the assumed monotonicity of $\eta(.)$ implies that $g_0(t)$ is decreasing. Substituting here (B.30) for $\sqrt{-\lambda_0}$ this is written equivalently as

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\left(\eta^2(G_c(z_c)) - \eta^2(x)\right)}} = \frac{\int_{S_0}^{G_c(z_c)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(G_c(z_c)) - \eta^2(x))}}}{K}\int_0^t e^{h_0(s)}ds. \tag{B.54}$$

Moreover, the optimal $h_0(t)$ solves the variational problem:

$$\inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt = z_c} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt, \tag{B.55}$$

which has already been solved in [30]. For convenience we quote the solution below. Writing the optimal $h_0(t) = \log V_0 + f_0(t;z)$, we have

$$f_0'(t;z) = \begin{cases} \sqrt{-2\lambda_-}\sigma(V_0e^{f_0(t;z)})\sqrt{e^{f_0(1;z)} - e^{f_0(t;z)}}, & z > V_0\,, \lambda_- < 0 \\ -\sqrt{2\lambda_+}\sigma(V_0e^{f_0(t;z)})\sqrt{e^{f_0(t;z)} - e^{f_0(1;z)}}, & z < V_0\,, \lambda_+ > 0 \end{cases}, \tag{B.56}$$

where $\lambda_- = \frac{-1}{2}(F^{(-)}(\alpha_-))^2$ and $\lambda_+ = \frac{1}{2}(F^{(+)}(\alpha_+))^2$.

The parameters $\alpha_\pm$ and $\lambda_\pm$ are defined as follows.

a) $z \geq V_0$. For this case $\alpha_- = f_0(1;z) \geq 0$ is given by the solution of the equation

$$e^{\alpha_-} - \frac{z}{V_0} = \frac{G^{(-)}(\alpha_-)}{F^{(-)}(\alpha_-)}, \tag{B.57}$$

with

$$G^{(-)}(\alpha_-) = \int_0^{\alpha_-} \frac{\sqrt{e^{\alpha_-} - e^y}}{\sigma(V_0 e^y)} dy, \quad F^{(-)}(\alpha_-) = \int_0^{\alpha_-} \frac{1}{\sigma(V_0 e^y)} \frac{1}{\sqrt{e^{\alpha_-} - e^y}} dy\,. \tag{B.58}$$

b) $z \le V_0$. For this case $\alpha_+ = -f_0(1;z) \ge 0$ is the solution of the equation

$$\frac{z}{V_0} - e^{-\alpha_+} = \frac{G^{(+)}(\alpha_+)}{F^{(+)}(\alpha_+)}, \tag{B.59}$$

with

$$G^{(+)}(\alpha_+) = \int_0^{\alpha_+} \frac{\sqrt{e^{-y} - e^{-\alpha_+}}}{\sigma(V_0 e^{-y})} dy, \quad F^{(+)}(\alpha_+) = \int_0^{\alpha_+} \frac{1}{\sigma(V_0 e^{-y})} \frac{1}{\sqrt{e^{-y} - e^{-\alpha_+}}} dy\,, \tag{B.60}$$

Finally, we can solve (B.56) to obtain that when $z > V_0$

$$\int_0^{f_0(t;z)} \frac{dy}{\sigma(V_0 e^y)\sqrt{e^{\alpha_-} - e^y}} = F^{(-)}(\alpha_-)t, \tag{B.61}$$

and when $z < V_0$,

$$\int_0^{f_0(t;z)} \frac{dy}{\sigma(V_0 e^y)\sqrt{e^y - e^{-\alpha_+}}} = -F^{(+)}(\alpha_+)t. \tag{B.62}$$

(ii) For OTM variance put options, i.e. $V_0\eta^2(S_0) > K$, let z_p be the optimizer in (3.9). Then, from the proof of Proposition 6, $g_0(1) = \log G_p(z_p)$ and the optimal Lagrange multiplier is given by:

$$\lambda_0 = \frac{1}{K^2}\left(\int_{S_0}^{G_p(z_p)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(x) - \eta^2(G_p(z_p)))}}\right)^2. \tag{B.63}$$

We also obtain from the proof of Proposition 6 that

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\lambda_0\left(\eta^2(x) - \eta^2(e^{g_0(1)})\right)}} = \int_0^t e^{h_0(s)} ds. \tag{B.64}$$

For this case $\lambda_0 > 0$ and $g_0(t)$ is increasing.

Using (B.45) for $\sqrt{\lambda_0}$ this is written equivalently as

$$\int_{S_0}^{e^{g_0(t)}} \frac{dx}{x\eta(x)\sqrt{2\left(\eta^2(x) - \eta^2(G_p(z_p))\right)}} = \frac{\int_{S_0}^{G_p(z_p)} \frac{\eta(x)dx}{x\sqrt{2(\eta^2(x) - \eta^2(G_p(z_p)))}}}{K} \int_0^t e^{h_0(s)} ds. \tag{B.65}$$

Moreover, the optimal $h_0(t)$ solves the variational problem:

$$\inf_{h(0)=\log V_0, \int_0^1 e^{h(t)}dt = z_p} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt, \tag{B.66}$$

and the optimal $h_0(t) = \log V_0 + f_0(t;z_p)$ where $f_0(t;z)$ is defined in the proof of (i). This completes the proof. □

Proof of Proposition 11. One can compute that

$$\begin{aligned}
\Lambda_\rho[g,h] &= \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&\qquad - \frac{2\rho}{2(1-\rho^2)}\int_0^1 \frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\frac{h'(t)}{\sigma(e^{h(t)})} dt \\
&\leq \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&\qquad + \frac{|\rho|}{2(1-\rho^2)}\int_0^1 \left[\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 + \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2\right] dt \\
&= \frac{1+|\rho|}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1+|\rho|}{2(1-\rho^2)}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&= \frac{1+|\rho|}{1-\rho^2}\Lambda_0[g,h]. \qquad \text{(B.67)}
\end{aligned}$$

Hence, we conclude that

$$\mathcal{I}_\rho(S_0,V_0,K) \leq \frac{1+|\rho|}{1-\rho^2}\mathcal{I}_0(S_0,V_0,K) = \frac{1}{1-|\rho|}\mathcal{I}_0(S_0,V_0,K), \qquad \text{(B.68)}$$

where $\mathcal{I}_0(S_0,V_0,K)$ is computed in closed form in Proposition 6.

Similarly, one can compute that

$$\begin{aligned}
\Lambda_\rho[g,h] &\geq \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 dt + \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \\
&\qquad - \frac{|\rho|}{2(1-\rho^2)}\int_0^1 \left[\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2 + \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2\right] dt \\
&= \frac{1-|\rho|}{1-\rho^2}\Lambda_0[g,h], \qquad \text{(B.69)}
\end{aligned}$$

which implies that

$$\mathcal{I}_\rho(S_0,V_0,K) \geq \frac{1-|\rho|}{1-\rho^2}\mathcal{I}_0(S_0,V_0,K) = \frac{1}{1+|\rho|}\mathcal{I}_0(S_0,V_0,K), \qquad \text{(B.70)}$$

where $\mathcal{I}_0(S_0,V_0,K)$ is computed in closed form in Proposition 6. This completes the proof. □

Proof of Proposition 12. We recall from (3.24) that

$$\mathcal{I}_\rho(S_0,V_0,K)=\inf_{\substack{g(0)=\log S_0,h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}}\Lambda_\rho[g,h], \tag{B.71}$$

where $\Lambda_\rho[g,h]$ is defined in (3.25). Since g_0,h_0 are the optimal solutions for the variational problem (3.24) when $\rho=0$, they satisfy the constraints $g_0(0)=\log S_0$, $h_0(0)=\log V_0$ and $\int_0^1 e^{h_0(t)}\eta^2(e^{g_0(t)})dt=K$. Therefore, we conclude that

$$\mathcal{I}_\rho(S_0,V_0,K)\le\Lambda_\rho[g_0,h_0], \tag{B.72}$$

and this completes the proof. □

Proof of Proposition 13. By letting $g'(t)\equiv 0$ in (3.24), we get $g(t)=\log S_0$ for every $0\le t\le 1$ and

$$\mathcal{I}_\rho(S_0,V_0,K)\le\frac{1}{2(1-\rho^2)}\inf_{h(0)=\log V_0,\int_0^1 e^{h(t)}\eta^2(S_0)dt=K}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2dt. \tag{B.73}$$

By using the definition of $\mathcal{J}(\cdot,\cdot)$ in (3.8), we complete the proof. □

Proof of Proposition 14. We proceed in a similar way to the proof of Theorem 4. Under Assumptions 1 and 3, by the sample-path large deviations for small time diffusions [39] and [34]), one can see that $\mathbb{Q}(\{(\log S_{tT},\log V_{tT}),0\le t\le 1\}\in\cdot)$ in the $\rho=\pm1$ limit satisfies a sample-path large deviation principle with the rate function

$$J[g,h]=\frac{1}{2}\int_0^1\left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}\right)^2dt, \tag{B.74}$$

with $g(0)=\log S_0$, $h(0)=\log V_0$ and g,h being absolutely continuous, and since both S_t,V_t are driven by a common Brownian motion, $g(t)$ and $h(t)$ are related as

$$\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}}=\pm\frac{h'(t)}{\sigma(e^{h(t)})},\quad 0\le t\le 1, \tag{B.75}$$

where the signs correspond to $\rho=\pm1$, and the rate function is $+\infty$ otherwise.

By an application of the contraction principle we get

$$\lim_{T\to0}T\log\mathbb{Q}\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\ge K\right)=-\inf_{\substack{g(0)=\log S_0,h(0)=\log V_0\\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}}\left\{\frac{1}{2}\int_0^1\left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2dt\right\}, \tag{B.76}$$

where the infimum is also subject to the constraint (B.75). Hence, we conclude that the rate function for $\rho = \pm 1$ is

$$\mathcal{I}_{\pm 1}(S_0, V_0, K) = - \inf_{\substack{g(0)=\log S_0, h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}} \left\{ \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt \right\}, \tag{B.77}$$

where the infimum is also subject to the constraint (B.75). The variational problem can be simplified by eliminating $g(t)$ using (B.75).

This relation implies that

$$\int_0^t \frac{g'(s)}{\eta(e^{g(s)})}ds = \int_0^t \frac{\pm\sqrt{e^{h(s)}}h'(s)}{\sigma(e^{h(s)})}ds, \qquad 0 \le t \le 1, \tag{B.78}$$

which is equivalent to

$$\int_{\log S_0}^{g(t)} \frac{dx}{\eta(e^x)} = \int_{\log V_0}^{h(t)} \frac{\pm\sqrt{e^x}dx}{\sigma(e^x)}, \qquad 0 \le t \le 1, \tag{B.79}$$

where we used the constraints $g(0) = \log S_0$ and $h(0) = \log V_0$. We can further compute that this is equivalent to

$$\int_{\log S_0}^{g(t)} \frac{e^x dx}{e^x\eta(e^x)} = \int_{\log V_0}^{h(t)} \frac{\pm e^x dx}{\sqrt{e^x}\sigma(e^x)}, \qquad 0 \le t \le 1, \tag{B.80}$$

which is equivalent to

$$\int_{S_0}^{e^{g(t)}} \frac{dx}{x\eta(x)} = \int_{V_0}^{e^{h(t)}} \frac{\pm dx}{\sqrt{x}\sigma(x)}, \qquad 0 \le t \le 1. \tag{B.81}$$

Therefore, given h, the optimal g is given by

$$e^{g(t)} = \mathcal{F}_{\pm}(e^{h(t)}), \qquad 0 \le t \le 1, \tag{B.82}$$

where $\mathcal{F}_{\pm}(\cdot)$ is defined as:

$$\int_{S_0}^{\mathcal{F}_{\pm}(x)} \frac{dy}{y\eta(y)} = \int_{V_0}^{x} \frac{\pm dy}{\sqrt{y}\sigma(y)}, \tag{B.83}$$

for any $x > 0$. Hence, we conclude that

$$\mathcal{I}_{\pm 1}(S_0, V_0, K) = \inf_{\substack{h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(\mathcal{F}_{\pm}(e^{h(t)}))dt=K}} \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt. \tag{B.84}$$

This completes the proof. □

Proof of Proposition 17. We recall from (3.24)-(3.25) that

$$\mathcal{I}_\rho(S_0, V_0, K) = \inf_{\substack{g(0)=\log S_0, h(0)=\log V_0 \\ \int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt=K}} \Lambda_\rho[g,h], \tag{B.85}$$

where

$$\Lambda_\rho[g,h] := \frac{1}{2(1-\rho^2)}\int_0^1 \left(\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}} - \frac{\rho h'(t)}{\sigma(e^{h(t)})}\right)^2 dt + \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt. \tag{B.86}$$

In particular,

$$\mathcal{I}_\rho(S_0, V_0, K) \leq \frac{1}{2}\int_0^1 \left(\frac{h'(t)}{\sigma(e^{h(t)})}\right)^2 dt, \tag{B.87}$$

for any g, h that satisfies: $g(0) = \log S_0$, $h(0) = \log V_0$, $\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt = K$ and

$$\frac{g'(t)}{\eta(e^{g(t)})\sqrt{e^{h(t)}}} = \frac{\rho h'(t)}{\sigma(e^{h(t)})}, \qquad 0 \leq t \leq 1. \tag{B.88}$$

By following the proof of Proposition 14, we can compute that

$$e^{g(t)} = \mathcal{F}_\rho(e^{h(t)}), \qquad 0 \leq t \leq 1, \tag{B.89}$$

where $\mathcal{F}_\rho$ is defined as:

$$\int_{S_0}^{\mathcal{F}_\rho(x)} \frac{dy}{y\eta(y)} = \int_{V_0}^{x} \frac{\rho dy}{\sqrt{y}\sigma(y)}, \qquad \text{for any } x > 0. \tag{B.90}$$

Since this holds for any h that satisfies the constraints $\int_0^1 e^{h(t)}\eta^2(\mathcal{F}_\rho(e^{h(t)}))dt = K$ and $h(0) = \log V_0$, by taking the infimum over h, the proof is complete. □

Proof of Proposition 19. We start with an expansion for the functions g, h in powers of $x := \log\left(\frac{K}{V_0\eta_0^2}\right)$ of the form

$$g(t) = g_0(t) + xg_1(t) + O(x^2), \qquad h(t) = h_0(t) + xh_1(t) + O(x^2), \tag{B.91}$$

We expand also the Lagrange multiplier as $\lambda = \lambda_0 + x\lambda_1 + O(x^2)$ as $x \to 0$.

The zero-th order terms in these expansions are $g_0(t) \equiv \log S_0$, $h_0(t) \equiv \log V_0$ such that $g_0'(t) \equiv 0$ and $h_0'(t) \equiv 0$. The transversality conditions $g'(1) = h'(1) = 0$ are satisfied if and only if we have $g_k'(1) = h_k'(1) = 0$ for all $k \geq 1$. Also, the boundary conditions $g(0) = \log S_0, h(0) = \log V_0$ imply that one must have $g_k(0) = h_k(0) = 0$ for all $k \geq 1$.

The constraint $\int_0^1 e^{h(t)}\eta^2(e^{g(t)})dt = K$ relates $g_k(t), h_k(t)$ to all $g_j(t), h_j(t)$ of lower order $0 \le j < k$. This is written equivalently as

$$\int_0^1 V_0 e^{xh_1(t)+x^2h_2(t)+O(x^3)}\eta^2\left(S_0 e^{xg_1(t)+x^2g_2(t)+O(x^3)}\right)dt = V_0\eta^2(S_0)e^x, \tag{B.92}$$

as $x \to 0$. Expanding in x and selecting terms of the same power of x on both sides gives the constraints

$$O(x): \int_0^1 h_1(t)dt + 2\tilde{\eta}_1\int_0^1 g_1(t)dt = 1\,, \tag{B.93}$$

$$O(x^2): \int_0^1 \left(h_2(t) + 2\tilde{\eta}_1 g_2(t) + \frac{1}{2}(h_1(t))^2 + (2\tilde{\eta}_2 + \tilde{\eta}_1^2)(g_1(t))^2 + 2\tilde{\eta}_1 g_1(t)h_1(t)\right)dt = \frac{1}{2}\,, \tag{B.94}$$

and so on, where we denoted $\tilde{\eta}_k := \eta_k/\eta_0$.

We substitute the expansions (B.91) into the Euler-Lagrange equations (3.27)-(3.28) and expand in x. Let us consider the terms of given order in x resulting from this expansion.

Order $O(x^0)$. At order $O(x^0)$, the equation (3.27) gives $\lambda_0 V_0 = 0$ which gives $\lambda_0 = 0$. Both sides of the equation (3.28) vanish identically at this order.

Order $O(x)$. At order $O(x)$, the two equations become

$$\frac{d}{dt}\left(\frac{1}{1-\rho^2}\frac{g_1'(t)}{\eta_0^2 V_0} - \frac{\rho}{1-\rho^2}\frac{h_1'(t)}{\eta_0\sqrt{V_0}\sigma_0}\right) = 2\lambda_1 V_0\eta_0\eta_1, \tag{B.95}$$

and

$$\frac{d}{dt}\left(\frac{1}{1-\rho^2}\frac{h_1'(t)}{\sigma_0^2} - \frac{\rho}{1-\rho^2}\frac{g_1'(t)}{\eta_0\sqrt{V_0}\sigma_0}\right) = \lambda_1 V_0\eta_0^2, \tag{B.96}$$

with the constraints that $g_1(0) = 0$, $h_1(0) = 0$ and $\int_0^1 h_1(t)dt + \frac{2\eta_1}{\eta_0}\int_0^1 g_1(t)dt = 1$, and the transversality condition gives $g_1'(1) = h_1'(1) = 0$. We can re-write (B.95)-(B.96) as

$$\frac{g_1''(t)}{\eta_0^2 V_0} - \frac{\rho h_1''(t)}{\eta_0\sqrt{V_0}\sigma_0} = 2\lambda_1 V_0\eta_0\eta_1(1-\rho^2), \tag{B.97}$$

$$\frac{h_1''(t)}{\sigma_0^2} - \frac{\rho g_1''(t)}{\eta_0\sqrt{V_0}\sigma_0} = \lambda_1 V_0\eta_0^2(1-\rho^2), \tag{B.98}$$

which implies that

$$g_1''(t) = \lambda_1\left(2V_0^2\eta_0^3\eta_1 + V_0^{3/2}\eta_0^3\rho\sigma_0\right), \qquad g_1(0) = g_1'(1) = 0, \tag{B.99}$$

$$h_1''(t) = \lambda_1\left(\sigma_0^2 V_0\eta_0^2 + 2\rho\sigma_0 V_0^{3/2}\eta_0^2\eta_1\right), \qquad h_1(0) = h_1'(1) = 0, \tag{B.100}$$

with $\int_0^1 h_1(t)dt + \frac{2\eta_1}{\eta_0}\int_0^1 g_1(t)dt = 1$. The equations (B.99)-(B.100) can be integrated using the boundary condition $g_1'(1) = h'(1) = 0$ to give

$$g_1'(t) = \lambda_1 \left(2\sqrt{V_0}\eta_1 + \rho\sigma_0\right) V_0^{3/2}\eta_0^3(t-1), \tag{B.101}$$

$$h_1'(t) = \lambda_1 \left(\sigma_0 + 2\rho\sqrt{V_0}\eta_1\right) \sigma_0 V_0 \eta_0^2 (t-1)\,. \tag{B.102}$$

Integrating (B.101)-(B.102) again using the boundary conditions $g_1(0) = h_1(0) = 0$ gives

$$g_1(t) = \frac{1}{2}\lambda_1 \left(2\sqrt{V_0}\eta_1 + \rho\sigma_0\right) V_0^{3/2}\eta_0^3(t^2-2t), \tag{B.103}$$

$$h_1(t) = \frac{1}{2}\lambda_1 \left(\sigma_0 + 2\rho\sqrt{V_0}\eta_1\right) \sigma_0 V_0 \eta_0^2 (t^2-2t). \tag{B.104}$$

The constant λ_1 is determined from the first normalization condition (B.93)

$$\int_0^1 h_1(t)dt + \frac{2\eta_1}{\eta_0}\int_0^1 g_1(t)dt = -\frac{1}{2}\lambda_1 \left(\sigma_0^2 V_0\eta_0^2 + 4\rho\sigma_0 V_0^{3/2}\eta_0^2\eta_1 + 4V_0^2\eta_0^2\eta_1^2\right)\frac{2}{3} = 1, \tag{B.105}$$

which gives $\lambda_1 = -\frac{3}{V_0\eta_0^2(\sigma_0^2+4\rho\sigma_0 V_0^{1/2}\eta_1+4V_0\eta_1^2)}$. We conclude that

$$g_1(t) = \frac{3}{2}\frac{(2V_0^{1/2}\eta_1 + \rho\sigma_0)V_0^{1/2}\eta_0}{\sigma_0^2 + 4\rho\sigma_0 V_0^{1/2}\eta_1 + 4V_0\eta_1^2}(2t-t^2), \tag{B.106}$$

$$h_1(t) = \frac{3}{2}\frac{\sigma_0^2 + 2\rho\sigma_0 V_0^{1/2}\eta_1}{\sigma_0^2 + 4\rho\sigma_0 V_0^{1/2}\eta_1 + 4V_0\eta_1^2}(2t-t^2). \tag{B.107}$$

Finally, by plugging (B.91) into (3.25), it follows from (3.24) that

$$
\begin{aligned}
&\mathcal{I}_\rho(S_0, V_0, V_0\eta^2(S_0)e^x) \\
&= \frac{x^2}{2(1-\rho^2)} \int_0^1 \left(\frac{g_1'(t)}{\eta(S_0)\sqrt{V_0}} - \frac{\rho h_1'(t)}{\sigma_0} \right)^2 dt + \frac{x^2}{2} \int_0^1 \left(\frac{h_1'(t)}{\sigma_0} \right)^2 dt + O(x^3) \\
&= \frac{x^2\lambda_1^2}{2(1-\rho^2)} \int_0^1 \left(\frac{2V_0^2\eta_0^3\eta_1 + V_0^{3/2}\eta_0^3\rho\sigma_0}{\eta(S_0)\sqrt{V_0}} - \frac{\rho}{\sigma_0} \left(\sigma_0^2 V_0 \eta_0^2 + 2\rho\sigma_0 V_0^{3/2}\eta_0^2\eta_1 \right) \right)^2 (1-t)^2 dt \\
&\quad + \frac{x^2\lambda_1^2}{2} \int_0^1 \left(\frac{\sigma_0^2 V_0 \eta_0^2 + 2\rho\sigma_0 V_0^{3/2}\eta_0^2\eta_1}{\sigma_0} \right)^2 (1-t)^2 dt + O(x^3) \\
&= \frac{x^2\lambda_1^2}{6(1-\rho^2)} \left(2V_0^{3/2}\eta_0^2\eta_1 - 2\rho^2 V_0^{3/2}\eta_0^2\eta_1 \right)^2 + \frac{x^2\lambda_1^2}{6} \left(\sigma_0 V_0 \eta_0^2 + 2\rho V_0^{3/2}\eta_0^2\eta_1 \right)^2 + O(x^3) \\
&= \frac{x^2\lambda_1^2}{6} \left(\sigma_0^2 V_0^2 \eta_0^4 + 4\rho\sigma_0 V_0^{5/2}\eta_0^4\eta_1 + 4V_0^3\eta_0^4\eta_1^2 \right) + O(x^3) \\
&= \frac{3x^2}{2} \frac{\sigma_0^2 V_0^2 \eta_0^4 + 4\rho\sigma_0 V_0^{5/2}\eta_0^4\eta_1 + 4V_0^3\eta_0^4\eta_1^2}{\left(\sigma_0^2 V_0 \eta_0^2 + 4\rho\sigma_0 V_0^{3/2}\eta_0^2\eta_1 + 4V_0^2\eta_0^2\eta_1^2 \right)^2} + O(x^3) \\
&= \frac{3}{2\left(\sigma_0^2 + 4\rho\sigma_0 V_0^{1/2}\eta_1 + 4V_0\eta_1^2 \right)} x^2 + O(x^3),
\end{aligned} \tag{B.108}
$$

as $x \to 0$. This completes the proof for the $O(x^2)$ term in the rate function.

Order $O(x^2)$. The $O(x^2)$ terms in the expansion of the Euler-Lagrange equations give a system of two linear equations in $g_2''(t), h_2''(t)$. They are solved with the solutions

$$
g_2''(t) = a + bt + ct^2\,, \qquad h_2''(t) = \bar{a} + \bar{b}t + \bar{c}t^2\,, \tag{B.109}
$$

with coefficients a, b, c and $\bar{a}, \bar{b}, \bar{c}$ which depend on known parameters σ_i, η_j and the unknown λ_2.

Taking into account the boundary conditions at $t = 1$, the equations (B.109) can be integrated as

$$
g_2'(t) = (t-1)\left(a + \frac{1}{2}b(t+1) + \frac{1}{3}c(t^2+t+1) \right)\,, \tag{B.110}
$$

$$
h_2'(t) = (t-1)\left(\bar{a} + \frac{1}{2}\bar{b}(t+1) + \frac{1}{3}\bar{c}(t^2+t+1) \right)\,. \tag{B.111}
$$

Integrating again (B.110)-(B.111) using the boundary condition at $t = 0$ gives

$$
g_2(t) = \frac{1}{2}at(t-2) + \frac{1}{6}bt(t^2-3) + \frac{1}{12}ct(t^3-4)\,, \tag{B.112}
$$

$$
h_2(t) = \frac{1}{2}\bar{a}t(t-2) + \frac{1}{6}\bar{b}t(t^2-3) + \frac{1}{12}\bar{c}t(t^3-4)\,. \tag{B.113}
$$

The coefficient λ_2 is determined from the normalization condition (B.94) which can be re-written as

$$\begin{aligned}&\int_0^1 (h_2(t) + 2\tilde{\eta}_1 g_2(t))\, dt \\ &= \frac{1}{2} - \int_0^1 \left(\frac{1}{2}(h_1(t))^2 + (\tilde{\eta}_1^2 + 2\tilde{\eta}_2)(g_1(t))^2 + 2\tilde{\eta}_1 g_1(t) h_1(t)\right) dt\,. \end{aligned} \tag{B.114}$$

The integral on the right-hand side of (B.114) depends only on the functions $g_1(t), h_1(t)$ which have been determined in the previous step; see (B.106)-(B.107). Thus, this condition introduces a linear constraint on the coefficients $a, b, c, \bar{a}, \bar{b}, \bar{c}$ which is used to solve for λ_2.

The final expressions for the coefficients in (B.109) are rather lengthy so we give them in Appendix C.

Substituting the expressions for $g_2'(t), h_2'(t)$ from (B.110)-(B.111) into the $O(x^3)$ term in the expansion of the rate function (3.26) yields the stated result. □

Proof of Theorem 20. We only provide the proof for the ATM call option. The case for the put option is similar. First of all, it is easy to see that

$$\left| C(T) - \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_t \eta^2(S_t) dt - K\right)^+\right]\right| = O(T), \tag{B.115}$$

as $T \to 0$, which follows from the estimate:

$$\begin{aligned}&\left| C(T) - \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_t \eta^2(S_t) dt - K\right)^+\right]\right| \\ &\le \left|1 - e^{-rT}\right| \cdot \left|\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_t \eta^2(S_t) dt - K\right)^+\right]\right| \\ &\le \left|1 - e^{-rT}\right| \cdot \left(\frac{1}{T}\int_0^T \mathbb{E}[V_t \eta^2(S_t)] dt + K\right) \\ &\le \left|1 - e^{-rT}\right| \cdot \left(\frac{1}{T}\int_0^T V_0 e^{(M_\mu + \frac{1}{2}M_\sigma^2)t} M_\eta^2 dt + K\right), \end{aligned} \tag{B.116}$$

where we used the fact that $|\eta(\cdot)| \le M_\eta$ and the bound $\mathbb{E}[V_t] \le V_0 e^{M_\mu t + \frac{1}{2}M_\sigma^2 t}$ for every $t \ge 0$ from (B.6).

It has been shown in Theorem 3.2. [29] that under Assumptions 1, 2 and $\max_{0\le t\le T} \mathbb{E}[(S_t)^4] = O(1)$ as $T \to 0$, we have that uniformly in $0 \le t \le T$,

$$\mathbb{E}\left[\left|S_t - \hat{S}_t\right|^2\right] = O\left(T^{3/2}\right), \qquad \mathbb{E}\left[\left|V_t - \hat{V}_t\right|\right] = O(T), \tag{B.117}$$

where

$$\hat{S}_t = S_0 + S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_t + \rho Z_t\right), \tag{B.118}$$

$$\hat{V}_t = V_0 + V_0\sigma(V_0)Z_t. \tag{B.119}$$

Since $x \mapsto x^+$ is 1-Lipschitz, we can compute that

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \hat{V}_s\eta^2(\hat{S}_s)ds - K\right)^+\right] - \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)^+\right]\right| \\
&\le \mathbb{E}\left|\frac{1}{T}\int_0^T \hat{V}_s\eta^2(\hat{S}_s)ds - \frac{1}{T}\int_0^T V_s\eta^2(S_s)ds\right| \\
&\le \frac{1}{T}\int_0^T \mathbb{E}\left|\hat{V}_s\eta^2(\hat{S}_s) - V_s\eta^2(S_s)\right| ds.
\end{aligned} \tag{B.120}$$

Moreover, uniformly in $0 \le s \le T$,

$$\mathbb{E}\left|\hat{V}_s\eta^2(\hat{S}_s) - V_s\eta^2(S_s)\right| \le \mathbb{E}\left|\hat{V}_s\eta^2(\hat{S}_s) - \hat{V}_s\eta^2(S_s)\right| + \mathbb{E}\left|\hat{V}_s\eta^2(S_s) - V_s\eta^2(S_s)\right|. \tag{B.121}$$

Under the assumption that $|\eta(\cdot)| \le M_\eta$ uniformly and $\eta(\cdot)$ is ℓ_η-Lipschitz, we have

$$\begin{aligned}
\mathbb{E}\left|\hat{V}_s\eta^2(\hat{S}_s) - V_s\eta^2(S_s)\right| &\le 2M_\eta\ell_\eta\mathbb{E}\left[|\hat{V}_s|\cdot\left|\hat{S}_s - S_s\right|\right] + M_\eta^2\mathbb{E}\left|\hat{V}_s - V_s\right| \\
&\le 2M_\eta\ell_\eta\left(\mathbb{E}\left[|\hat{V}_s|^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left|\hat{S}_s - S_s\right|^2\right]\right)^{1/2} + M_\eta^2\mathbb{E}\left|\hat{V}_s - V_s\right| \\
&= O(T^{3/4}),
\end{aligned} \tag{B.122}$$

as $T \to 0$ by applying (B.117) and the simple bound $\mathbb{E}|\hat{V}_s|^2 \le 2V_0^2 + 2V_0^2\sigma^2(V_0)T$ uniformly in $0 \le t \le T$.

Therefore, the call option can be approximated by (with $K = V_0\eta^2(S_0)$):

$$\left|\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \hat{V}_s\eta^2(\hat{S}_s)ds - K\right)^+\right] - \mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds - K\right)^+\right]\right| = O(T^{3/4}), \tag{B.123}$$

as $T \to 0$.

Next, we can compute that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \hat{V}_s\eta^2(\hat{S}_s)ds - V_0\eta^2(S_0)\right)^+\right] \\
&= \mathbb{E}\Bigg[\Bigg(\frac{1}{T}\int_0^T (V_0 + V_0\sigma(V_0)Z_s) \\
&\qquad \cdot \eta^2\left(S_0 + S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right)ds - V_0\eta^2(S_0)\Bigg)^+\Bigg].
\end{aligned} \tag{B.124}$$

Since $x \mapsto x^+$ is 1-Lipschitz, we have

$$\begin{aligned}
&\left| \mathbb{E}\left[\left(\frac{1}{T}\int_0^T (V_0 + V_0\sigma(V_0)Z_s)\,\eta^2\left(S_0 + S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right) ds - V_0\eta^2(S_0)\right)^+ \right] \right. \\
&\quad - \mathbb{E}\left[\left(\frac{1}{T}\int_0^T \left(V_0\eta^2(S_0) + V_0 2\eta(S_0)\eta'(S_0)S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \eta^2(S_0)V_0\sigma(V_0)Z_s \right) ds - V_0\eta^2(S_0) \right)^+ \right] \Bigg| \\
&\le \mathbb{E}\left[\frac{1}{T}\int_0^T \Big| (V_0 + V_0\sigma(V_0)Z_s)\left(\eta^2\left(S_0 + S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right)\right. \right. \\
&\qquad \left. \left. - \eta^2(S_0) - (\eta^2(S_0))' S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right) \Big| ds \right] \\
&\quad + \mathbb{E}\left[\frac{1}{T}\int_0^T \left| V_0\sigma(V_0)Z_s(\eta^2(S_0))' S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right| ds \right]. \qquad \text{(B.125)}
\end{aligned}$$

Under the assumption that $\sup_{x\in\mathbb{R}^+} |(\eta^2)''(x)| < \infty$, there exists some constant $C > 0$, such that

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{T}\int_0^T \Big| (V_0 + V_0\sigma(V_0)Z_s)\left(\eta^2(S_0 + S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right)\right. \\
&\qquad \left. \left. - \eta^2(S_0) - (\eta^2(S_0))' S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)\right) \Big| ds \right] \\
&\le \frac{C}{T}\int_0^T \mathbb{E}\left[\left| (V_0 + V_0\sigma(V_0)Z_s)\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)^2 \right| \right] ds \\
&\le \frac{C}{T}\int_0^T \left(\mathbb{E}\left[(V_0 + V_0\sigma(V_0)Z_s)^2\right]\right)^{1/2} \left(\mathbb{E}\left[\left(\sqrt{1-\rho^2}W_s + \rho Z_s\right)^4\right]\right)^{1/2} ds \\
&= \frac{C}{T}\int_0^T \left(V_0^2 + V_0^2\sigma^2(V_0)s\right)^{1/2}\left(3s^2\right)^{1/2} ds = O(T), \qquad \text{(B.126)}
\end{aligned}$$

as $T \to 0$. In addition, we can compute that

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{T}\int_0^T \left|V_0\sigma(V_0)Z_s(\eta^2(S_0))'S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s+\rho Z_s\right)\right| ds\right] \\
&= \left(V_0\sigma(V_0)(\eta^2(S_0))'S_0\eta(S_0)\sqrt{V_0}\right)^2 \frac{1}{T}\int_0^T \mathbb{E}\left[\left|Z_s\left(\sqrt{1-\rho^2}W_s+\rho Z_s\right)\right|\right] ds \\
&\leq \left(V_0\sigma(V_0)(\eta^2(S_0))'S_0\eta(S_0)\sqrt{V_0}\right)^2 \\
&\qquad \cdot \frac{1}{T}\int_0^T \left(\mathbb{E}\left[(Z_s)^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\sqrt{1-\rho^2}W_s+\rho Z_s\right)^2\right]\right)^{1/2} ds \\
&= \left(V_0\sigma(V_0)(\eta^2(S_0))'S_0\eta(S_0)\sqrt{V_0}\right)^2 \frac{1}{T}\int_0^T \sqrt{s}\sqrt{s}ds = O(T),
\end{aligned} \tag{B.127}$$

as $T \to 0$.

Finally, we can compute that

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\frac{1}{T}\int_0^T \Bigg(V_0\eta^2(S_0)+V_0(\eta^2(S_0))'S_0\eta(S_0)\sqrt{V_0}\left(\sqrt{1-\rho^2}W_s+\rho Z_s\right) \\
&\qquad\qquad + \eta^2(S_0)V_0\sigma(V_0)Z_s\Bigg)ds - V_0\eta^2(S_0)\Bigg)^+\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\frac{1}{T}\int_0^T \left(\sqrt{1-\rho^2}W_s+\rho Z_s\right) ds \\
&\qquad\qquad + \eta^2(S_0)V_0\sigma(V_0)\frac{1}{T}\int_0^T Z_s ds\Bigg)^+\Bigg].
\end{aligned} \tag{B.128}$$

It is easy to check that $\frac{1}{T}\int_0^T W_s ds$ and $\frac{1}{T}\int_0^T Z_s ds$ are independently and identically

distributed as $\mathcal{N}\left(0, \frac{T}{3}\right)$ (see e.g. [30]). Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\frac{1}{T}\int_0^T\left(\sqrt{1-\rho^2}W_s+\rho Z_s\right)ds+\eta^2(S_0)V_0\sigma(V_0)\frac{1}{T}\int_0^T Z_s ds\right)^+\right]\\
&=\mathbb{E}\Bigg[\Bigg(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\sqrt{1-\rho^2}\frac{1}{T}\int_0^T W_s ds\\
&\qquad\qquad+\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\rho+\eta^2(S_0)V_0\sigma(V_0)\right)\frac{1}{T}\int_0^T Z_s ds\Bigg)^+\Bigg]\\
&=\Bigg(\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\sqrt{1-\rho^2}\right)^2\\
&\qquad+\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\rho+\eta^2(S_0)V_0\sigma(V_0)\right)^2\Bigg)^{1/2}\frac{\sqrt{T}}{\sqrt{3}}\mathbb{E}\left[X^+\right]\\
&=\Bigg(\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\sqrt{1-\rho^2}\right)^2\\
&\qquad+\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\rho+\eta^2(S_0)V_0\sigma(V_0)\right)^2\Bigg)^{1/2}\frac{\sqrt{T}}{\sqrt{3}}\frac{1}{\sqrt{2\pi}},
\end{aligned}
\tag{B.129}
$$

where $X\sim\mathcal{N}(0,1)$.

Hence, we conclude that, for ATM case,

$$
\begin{aligned}
&\lim_{T\to 0}\frac{1}{\sqrt{T}}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T V_s\eta^2(S_s)ds-V_0\eta^2(S_0)\right)^+\right]\\
&=\frac{1}{\sqrt{6\pi}}\Bigg(\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\sqrt{1-\rho^2}\right)^2\\
&\qquad\qquad+\left(2V_0\eta^2(S_0)\eta'(S_0)S_0\sqrt{V_0}\rho+\eta^2(S_0)V_0\sigma(V_0)\right)^2\Bigg)^{1/2}\\
&=\frac{1}{\sqrt{6\pi}}\sqrt{4V_0^3\eta^4(S_0)(\eta'(S_0))^2S_0^2+\eta^4(S_0)V_0^2\sigma^2(V_0)+4\rho\eta^4(S_0)\eta'(S_0)S_0V_0^{5/2}\sigma(V_0)}.
\end{aligned}
\tag{B.130}
$$

This completes the proof. □

C Coefficients of $g_2(t), h_2(t)$

We give in this Appendix the coefficients appearing in the functions $g_2(t), h_2(t)$ relevant for the $O(x^3)$ term in the rate function. The details of proof are given in the proof of Proposition 19. Recall that these functions are given by

$$g_2(t) = \frac{1}{2}at(t-2) + \frac{1}{6}bt(t^2-3) + \frac{1}{12}ct(t^3-4)\,, \tag{C.1}$$

$$h_2(t) = \frac{1}{2}\bar{a}t(t-2) + \frac{1}{6}\bar{b}t(t^2-3) + \frac{1}{12}\bar{c}t(t^3-4)\,. \tag{C.2}$$

The coefficients of $g_2(t)$ in (C.1) can be expressed as an expansion in the leading volatility-of-volatility coefficient σ_0 as follows:

$$a := \frac{1}{10D^3}3\eta_0\sqrt{V_0}\cdot\sum_{j=0}^{5}c_j\sigma_0^j \quad b := -\frac{9}{2D^2}\sum_{j=0}^{3}d_j\sigma_0^j\,, \quad c := \frac{9}{4D^2}\sum_{j=0}^{3}d_j\sigma_0^j\,, \tag{C.3}$$

where we denote $D := \sigma_0^2 + 4\eta_1\sigma_0\rho\sqrt{V_0} + 4\eta_1^2V_0$ and the coefficients in (C.3) are

$$c_0 := 64\eta_1^3(14\eta_1^2 + 9\eta_0\eta_2)V_0^{5/2}\,, \tag{C.4}$$

$$c_1 := 144\eta_1^3\rho^2\sigma_1V_0^{3/2} + 16\eta_1^2(149\eta_1^2 + 54\eta_0\eta_2)\rho V_0^2\,, \tag{C.5}$$

$$c_2 := 144\eta_1^2\rho\sigma_1V_0 + 72\eta_1^2\rho^3\sigma_1V_0 + 640\eta_1^3V_0^{3/2} + 8\eta_1(227\eta_1^2 + 54\eta_0\eta_2)\rho^2V_0^{3/2}\,, \tag{C.6}$$

$$c_3 := 36\eta_1\sigma_1\sqrt{V_0} + 72\eta_1\rho^2\sigma_1\sqrt{V_0} + 864\eta_1^2\rho V_0 + 4(91\eta_1^2 + 18\eta_0\eta_2)\rho^3V_0\,, \tag{C.7}$$

$$c_4 := 18\rho\sigma_1 + 86\eta_1\sqrt{V_0} + 212\eta_1\rho^2\sqrt{V_0}\,, \tag{C.8}$$

$$c_5 := 28\rho\,, \tag{C.9}$$

and

$$d_0 := 8\eta_0\eta_1V_0^2(5\eta_1^2 + 2\eta_0\eta_2)\,, \tag{C.10}$$

$$d_1 := 4\eta_0\eta_1\rho^2\sigma_1V_0 + 8\eta_0(8\eta_1^2 + \eta_0\eta_2)\rho V_0^{3/2}\,, \tag{C.11}$$

$$d_2 := 2\eta_0\rho\sigma_1\sqrt{V_0} + 16\eta_0\eta_1V_0 + 16\eta_0\eta_1\rho^2V_0\,, \tag{C.12}$$

$$d_3 := 5\eta_0\rho\sqrt{V_0}\,. \tag{C.13}$$

The coefficients of $h_2(t)$ in (C.2) have a similar expansion in σ_0 of the form:

$$\bar{a} := \frac{1}{10D^3}3\sigma_0\cdot\sum_{j=0}^{5}\bar{c}_j\sigma_0^j \quad \bar{b} := -\frac{9}{D^2}\sigma_0\sum_{j=0}^{3}\bar{d}_j\sigma_0^j\,, \quad \bar{c} := \frac{9}{2D^2}\sigma_0\sum_{j=0}^{3}\bar{d}_j\sigma_0^j\,, \tag{C.14}$$

where the coefficients in (C.14) are given by:

$$\bar{c}_0 := 32\left(15\eta_1^4\rho^2\sigma_1 V_0^2 + 13\eta_1^5\rho V_0^{5/2} + 18\eta_0\eta_1^3\eta_2\rho V_0^{5/2}\right), \tag{C.15}$$

$$\begin{aligned}\bar{c}_1 := 16\Big(&30\eta_1^3\rho\sigma_1 V_0^{3/2} + 39\eta_1^3\rho^3\sigma_1 V_0^{3/2} - 2\eta_1^4 V_0^2 + 18\eta_0\eta_1^2\eta_2 V_0^2 \\ &+ 76\eta_1^4\rho^2 V_0^2 + 36\eta_0\eta_1^2\eta_2\rho^2 V_0^2\Big),\end{aligned} \tag{C.16}$$

$$\begin{aligned}\bar{c}_2 := 8\Big(&15\eta_1^2\sigma_1 V_0 + 102\eta_1^2\rho^2\sigma_1 V_0 + 66\eta_1^3\rho V_0^{3/2} + 36\eta_0\eta_1\eta_2\rho V_0^{3/2} \\ &+ 91\eta_1^3\rho^3 V_0^{3/2} + 18\eta_0\eta_1\eta_2\rho^3 V_0^{3/2}\Big),\end{aligned} \tag{C.17}$$

$$\bar{c}_3 := 4\left(87\eta_1\rho\sigma_1\sqrt{V_0} + 20\eta_1^2 V_0 + 137\eta_1^2\rho^2 V_0 + 18\eta_0\eta_2\rho^2 V_0\right), \tag{C.18}$$

$$\bar{c}_4 := 4\left(12\sigma_1 + 37\eta_1\rho\sqrt{V_0}\right), \tag{C.19}$$

$$\bar{c}_5 := 13\,, \tag{C.20}$$

and

$$\bar{d}_0 := 4\eta_1\rho\left(3\eta_1\rho\sigma_1 + 2\eta_1^2\sqrt{V_0} + 2\eta_0\eta_2\sqrt{V_0}\right)V_0\,, \tag{C.21}$$

$$\bar{d}_1 := 2\left(7\eta_1\rho\sigma_1\sqrt{V_0} + 7\eta_1^2\rho^2 V_0 + 2\eta_0\eta_2\rho^2 V_0\right), \tag{C.22}$$

$$\bar{d}_2 := 4\sigma_1 + 7\eta_1\rho\sqrt{V_0}\,, \tag{C.23}$$

$$\bar{d}_3 := 1\,. \tag{C.24}$$